HOPF BIFURCATION OF A DELAY DIFFERENTIAL EQUATION WITH TWO DELAYS

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Dedicated to Professor Dr. Ioan A. Rus on his 70th birthday.

Abstract. We consider a delay differential equation with two delays. The Hopf bifurcation of this equation is investigated together with the stability of the bifurcated periodic solution, its period and the bifurcation direction. Finally, three applications are given.

1. Introduction

Hopf [9] was the first who state a theorem concerning the bifurcation of periodic solutions from a singular point of an ordinary differential equation. Many generalizations to infinite–dimensional systems have been given (see [12] for references). As far as we know, the first statement similar to this theorem for retarded functional–differential equations was given by Chow and Mallet–Paret in a course at Brown University in 1974, see [7]. Efficient procedures for determining the stability and the amplitude of the bifurcating periodic orbit using a method of averaging have been given by Chow and Mallet–Paret [4]. The global existence of a Hopf bifurcation as a function of initial data and the period has been discussed by Chow and Mallet–Paret [5] and Nussbaum [14]. The interest on the periodic orbits of a delay differential equation has increased strongly these last years, see for instance [2], [3], [13], [15]–[17].

In this paper we study the delay differential equation of the form

$$\begin{aligned}\dot{x}(t) \quad = \quad & -(a\pi+\mu)[x(t-1)+x(t-2)+G_2(x(t),x(t-1),x(t-2))]\cdot \\ & [1+G_1(x(t),x(t-1),x(t-2))],\end{aligned} \tag{1}$$

where $9/100 \leq a \leq \sqrt{3}/9$, and $G_1(x,y,z)$ and $G_2(x,y,z)$ are analytic functions in a neighborhood of $\mathbf{0} \in \mathbb{R}^3$, starting with terms of degree at least 1 and 2 respectively. We prove that equation (1) exhibits a Hopf bifurcation and we discuss for distinct functions G_1 and G_2 about the period, the

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stability of the bifurcated periodic orbit and its direction in the parameter space.

The periodic orbits of the delay differential equation (1) for the particular case $a = \sqrt{3}/9$, $G_1(x, y, z) = -x^2$ and $G_2(x, y, z) = 0$ were studied by Jones [10], and Kaplan and Yorke [11]. The Hopf bifurcation for this particular delay differential equation has been analyzed in Hassard, Kazarinoff and Wan [8].

2. Analysis of the equation

Consider the delay differential equation (1). We define the function

$$F(x, y, z, \mu) = -(a\pi + \mu)[y + z + G_2(x, y, z)][1 + G_1(x, y, z)].$$

Clearly it satisfies

(i) F is analytic in (x, y, z) and μ in a neighborhood of $(\mathbf{0}, 0) \in \mathbb{R}^3 \times \mathbb{R}$,
(ii) $F(\mathbf{0}, \mu) = 0$ for μ in an open interval containing 0.

Additionally we assume that

(iii) $\mathbf{0} \in \mathbb{R}^3$ is an isolated stationary zero of F for μ in an open interval containing the zero.

The linearization of (1) about $x = 0$ is

$$\dot{x}(t) = -(a\pi + \mu)(x(t-1) + x(t-2)), \tag{2}$$

and its characteristic equation is given by

$$-(a\pi + \mu)(e^{-\lambda} + e^{-2\lambda}) - \lambda = 0, \tag{3}$$

for more details see [8, 6] or Appendix 2.

Proposition 1. *At $\mu = 0$ equation (3) has exactly two pure imaginary roots and no roots with positive real part.*

Proof: The straight line $u = -\lambda/(a\pi)$ does not intersect the curve $u = e^{-\lambda} + e^{-2\lambda}$ for $\lambda \geq 0$. Thus, at $\mu = 0$, equation (3) has no positive real roots.

In order that equation (3) has a pure imaginary root $\lambda = i\omega$ with $\omega > 0$ at $\mu = 0$ it must be that its real and imaginary parts are zero, or equivalently that

$$\cos\omega + \cos 2\omega = 2\cos\frac{\omega}{2}\cos\frac{3\omega}{2} = 0, \tag{4}$$

and

$$\omega = a\pi(\sin\omega + \sin 2\omega). \tag{5}$$

Equation (5) can be written in the following form

$$\sin\frac{3\omega}{2}\cos\frac{\omega}{2}=\frac{\omega}{2a\pi} \tag{6}$$

If $\cos(\omega/2)=0$, equation (6) is not satisfied. Thus $\cos(3\omega/2)=0$. That means $3\omega/2\in\{\pi/2+k\pi | k\in\mathbb{Z}\}$. In this case, $\sin(3\omega/2)=\pm1$. Using the graphics of the functions $\cos(\omega/2)$ and $\pm\omega/(2a\pi)$, it is easy to see that for $a\in[9/100,\sqrt{3}/9]$, the equation $\cos(\omega/2)=\omega/(2a\pi)$ has exactly one solution, $\omega_0\in[17\pi/100,33\pi/100]$, and that the equation $\cos(\omega/2)=-\omega/(2a\pi)$ has no solution for $\omega>0$.

It remains to prove that equation (3) has no complex roots with positive real part for $\mu=0$. Suppose $\lambda=\alpha+i\omega$, $\alpha>0$ is a solution of (3). Then $\alpha-i\omega$ is also a solution. So we can consider $\omega>0$. We note that λ is a root of (3) at $\mu=0$ if and only if

$$\alpha=-a\pi(e^{-\alpha}\cos\omega+e^{-2\alpha}\cos 2\omega) \tag{7}$$

and

$$\omega=a\pi(e^{-\alpha}\sin\omega+e^{-2\alpha}\sin 2\omega). \tag{8}$$

If $\alpha>0$, (8) implies that $|\ \omega\ |\leq 2a\pi$. Therefore we only need to look for solutions of (7)–(8) for $\omega\in(0,2a\pi]$. Now (7) implies

$$-\frac{\alpha}{a\pi\cos\omega}=e^{-\alpha}+\frac{\cos 2\omega}{\cos\omega}e^{-2\alpha}. \tag{9}$$

Since $\cos 2t/\cos t\geq -1$ if $t\in(0,\pi/3]$, the right–hand side of (9) is positive. When $1/6<a\leq\sqrt{3}/9$, we need only to look for solutions of (7)–(8) with $\omega\in(\pi/3,2a\pi)$. But

$$K(\omega)=\omega-a\pi e^{-\alpha}\sin\omega-a\pi e^{-2\alpha}\sin 2\omega$$

has a positive derivative for such ω and $K(\pi/3)>0$. Hence (3) has no complex roots with positive real parts for $\mu=0$. ∎

Theorem 2. *The delay differential equation* (1) *satisfying (iii) has a family of Hopf periodic solutions bifurcating from the origin at* $\mu=0$.

Proof: Clearly from (i), (ii) and (iii) the delay differential equation (1) satisfies the conditions (a) and (b) of Theorem 5 from Appendix 1.

Derivating the characteristic equation (3) with respect to μ we get

$$\lambda'(\mu)=(a\pi+\mu)(e^{-\lambda(\mu)}+2e^{-2\lambda(\mu)})\lambda'(\mu)-(e^{-\lambda(\mu)}+e^{-2\lambda(\mu)}).$$

Evaluating the above equation at $\mu=0$ and noticing that $\lambda(0)=i\omega_0$ we obtain

$$\lambda'(0)=-\frac{e^{-i\omega_0}+e^{-2i\omega_0}}{1-a\pi(e^{-i\omega_0}+2e^{-2i\omega_0})},$$

or equivalently

$$\lambda'(0) = \frac{2\cos(\frac{\omega_0}{2})^2(1+3a\pi-2\cos\omega_0)+i\sin\omega_0(1+a\pi+2\cos\omega_0)}{1+5a^2\pi^2-2a\pi[(1-2a\pi)\cos\omega_0+2\cos 2\omega_0]}$$
$$= \alpha'(0)+i\omega'(0).$$

Hence, we obtain that

$$\alpha'(0) = \frac{2\cos(\frac{\omega_0}{2})^2(1+3a\pi-2\cos\omega_0)}{1+5a^2\pi^2-2a\pi[(1-2a\pi)\cos\omega_0+2\cos 2\omega_0]}.$$

Substituting ω_0 with $2v$ and noticing that $\cos v = v/a\pi$ we get

$$\alpha'(0) = \frac{2(3a^2\pi^2(1+a\pi)-4v^2)v^2}{a^4\pi^4(-1+a\pi)^2+4a^3\pi^3(7+2a\pi)v^2-32\pi v^4}.$$

Replacing $2v$ with ω_0, we have

$$\alpha'(0) = \frac{(3a^2\pi^2(1+a\pi)-\omega_0^2)\omega_0^2}{2a^4\pi^4(-1+a\pi)^2+2a^3\pi^3(7+2a\pi)\omega_0^2-4\pi\omega_0^4}.$$

In the following we prove that $\alpha'(0)$ is positive. Since the denominator of $\alpha'(0)$ is positive (because is the the square module of a complex number) it is enough to show that the numerator is positive, i.e. $3a^2\pi^2(1+a\pi)-\omega_0^2>0$. This is obvious because the derivative of $h(a) = 3a^2\pi^2(1+a\pi)-\omega_0^2$ is positive for $a \in [9/100, \sqrt{3}/9]$ and since $\omega_0 = 0.544648$ for $a = 9/100$, we have that $h(9/100) = 0.0110011 > 0$.

In particular the transversality condition $\alpha'(0) > 0$ of statement (c) of Theorem 5 is satisfied. Consequently, by Proposition 1, assumptions (c) and (d) of Theorem 5 of Appendix 1 hold. In short, we can apply the Hopf bifurcation theory to equation (1) at $\mu = 0$. Therefore, a family of Hopf periodic solutions bifurcates from the origin at $\mu = 0$. ∎

If we are in the assumptions of Theorem 2 due to Theorem 5 of Appendix 1, the following result is satisfied.

Theorem 3. *For the delay differential equation* (1) *satisfying (iii) the following statements hold.*

(a) *There is an $\varepsilon_0 > 0$ and an analytic function $\mu(\varepsilon) = \mu_2\varepsilon^2 + O(\varepsilon^3)$ for $0 < \varepsilon < \varepsilon_0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there exists a periodic solution $p_\varepsilon(t)$ occuring for $\mu = \mu(\varepsilon)$.*

(b) *The period $T(\varepsilon) = 2\pi[1 + \tau_2\varepsilon^2 + O(\varepsilon^3)]/\omega_0$ of $p_\varepsilon(t)$ is an analytic function.*

(c) *The periodic solution $p_\varepsilon(t)$ is orbitally asymptotically stable if $\beta_2 < 0$, and unstable if $\beta_2 > 0$.*

Now we shall compute μ_2, τ_2 and β_2 for some function G_1 and G_2.

For the delay differential equation (1) we have that $r = 2$. Now we follow Appendix 2. The function from the Riesz representation theorem is $d\eta(\theta,\mu) = -(a\pi+\mu)[\delta(\theta+2)+\delta(\theta+1)]$. For equation (1) it follows that A and R are

$$A(\mu)(\phi) = \begin{cases} \dfrac{d\phi}{d\theta} & \text{if} \quad -2 \leq \theta < 0, \\ \\ -(a\pi+\mu)[\phi(-1)+\phi(-2)] & \text{if} \quad \theta = 0, \end{cases} \tag{10}$$

and

$$R(\phi) = \begin{cases} 0 & \text{if} \quad -2 \leq \theta < 0, \\ \\ f(\phi,\mu) & \text{if} \quad \theta = 0, \end{cases}$$

where

$$\begin{aligned} f(\phi,\mu) \quad = \quad & -(a\pi+\mu)[G_2(\phi(0),\phi(-1),\phi(-2))+ \\ & (\phi(-1)+\phi(-2))G_1(\phi(0),\phi(-1),\phi(-2))+ \\ & G_2(\phi(0),\phi(-1),\phi(-2))G_1(\phi(0),\phi(-1),\phi(-2))]. \end{aligned} \tag{11}$$

We define $q(\theta) = e^{i\omega_0\theta}$ and $q^*(\theta) = De^{i\omega_0\theta}$.

Lemma 4. *Normalizing q and q^* by the condition $\langle q^*, q\rangle = 1$ we obtain that $D = ReD + ImD$ where*

$$ReD = \frac{2a^3\pi^3 - 2a^4\pi^4 + 7a^2\pi^2\omega_0^2 - 2\omega_0^4}{2a^3\pi^3(a\pi-1)^2 + 2a^2\pi^2(7+2a\pi)\omega_0^2 - 4\omega_0^4},$$

$$ImD = \frac{-3a^2\pi^2\omega_0\sqrt{4a^2\pi^2-\omega_0^2} + 2\omega^3\sqrt{4a^2\pi^2-\omega_0^2}}{2a^3\pi^3(a\pi-1)^2 + 2a^2\pi^2(7+2a\pi)\omega_0^2 - 4\omega_0^4}.$$

Proof: Since

$$\begin{aligned} \langle q^*, q\rangle \quad &= \quad \bar{q}^*(0)q(0) - \int_{\theta=-2}^{0}\left(\int_{\xi=0}^{\theta}\bar{q}^*(\xi-\theta)q(\xi)d\xi\right)d\eta(\theta) \\ &= \quad \bar{D} - \int_{\theta=-2}^{0}\left(\int_{\xi=0}^{\theta}\bar{D}e^{-i\omega_0(\xi-\theta)}e^{i\omega_0\xi}d\xi\right)d\eta(\theta) \\ &= \quad \bar{D} + a\pi\bar{D}\int_{\theta=-2}^{0}\left(\int_{\xi=0}^{\theta}e^{i\omega_0\theta}d\xi\right)(\delta(\theta+1)+\delta(\theta+2))d\theta \\ &= \quad \bar{D} + a\pi\bar{D}\int_{\theta=-2}^{0}\theta e^{i\omega_0\theta}(\delta(\theta+1)+\delta(\theta+2))d\theta \\ &= \quad \bar{D}\left(1 - a\pi(e^{-i\omega_0} + 2e^{-2i\omega_0})\right), \end{aligned}$$

we obtain

$$\begin{aligned}\bar{D} &= \left(1 - a\pi(e^{-i\omega_0} + 2e^{-2i\omega_0})\right)^{-1} \\ &= (1 - a\pi(\cos\omega_0 + 2\cos 2\omega_0) + ia\pi(\sin\omega_0 + 2\sin 2\omega_0))^{-1}.\end{aligned}$$

Rationalizing the above equation, we get

$$\bar{D} = \frac{1 - a\pi(\cos\omega_0 + 2\cos 2\omega_0) - ia\pi(\sin\omega_0 + 2\sin 2\omega_0)}{1 + 5a^2\pi^2 - 2a\pi[(1-2a\pi)\cos\omega_0 + 2\cos 2\omega_0]}.$$

Taking into account that $\cos(\omega_0/2) = \omega_0/(2a\pi)$, we obtain the value of D. ∎

From the definition of z and w in Appendix 2, and (24) we have

$$\begin{aligned}\dot{z}(t) &= i\omega_0 z(t) + \bar{D}f(w(z(t),\bar{z}(t),0) + 2\mathrm{Re}(z(t)q(0)),0) \\ &= i\omega_0 z(t) + \bar{D}f_0(z(t),\bar{z}(t)) \qquad (12) \\ &= i\omega_0 z + g(z,\bar{z}).\end{aligned}$$

Next step is to solve system (29) and find the coefficients w_{ij} for $i+j=2$. From the equations (26) and (27) we have for $\theta \in [-2,0)$ that

$$H(z,\bar{z},\theta) = -\bar{D}f_0 q(\theta) - D\bar{f}_0\bar{q}(\theta) = -gq(\theta) - \bar{g}\bar{q}(\theta).$$

Using the expression for H obtained in (28) and for g given in (30), we get

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \qquad (13)$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \qquad (14)$$

Using the definition of A, the first equation of (29) and (13) we have

$$\begin{aligned}\dot{w}_{20}(\theta) &= A(\mu)(w_{20}(\theta)) \\ &= 2i\omega_0 w_{20}(\theta) - H_{20}(\theta) \\ &= 2i\omega_0 w_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).\end{aligned}$$

Noticing that $q(\theta) = e^{i\omega_0\theta}$, we solve the linear differential equation in the variable $w_{20}(\theta)$, and we obtain

$$w_{20}(\theta) = \frac{i}{\omega_0}g_{20}e^{i\omega_0\theta} + \frac{i}{3\omega_0}\bar{g}_{02}e^{-i\omega_0\theta} + ce^{2i\omega_0\theta}, \qquad (15)$$

where c is a complex constant.

Similarly, using the definition of A, the second equation of (29) and (14) we obtain

$$\dot{w}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta),$$

which leads to

$$w_{11}(\theta) = -\frac{i}{\omega_0}g_{11}e^{i\omega_0\theta} + \frac{i}{\omega_0}\bar{g}_{11}e^{-i\omega_0\theta} + d, \qquad (16)$$

where d is also a complex constant.
In the following we have to find the constants c and d. From equations (26), (27) and (12), for $\theta = 0$ we have that

$$\begin{aligned} H(z,\bar{z},0) &= -\bar{D}f_0 q(0) - D\bar{f}_0\bar{q}(0) + f_0 \\ &= -gq(0) - \bar{g}\bar{q}(0) + \frac{\bar{q}^*(0)f_0}{\bar{q}^*(0)} \\ &= -g - \bar{g} + \frac{g}{\bar{D}}. \end{aligned}$$

Using the expression for H obtained in (28) and for g obtained in (30), we get

$$H_{20}(0) = -g_{20} - \bar{g}_{02} + \frac{g_{20}}{\bar{D}}, \tag{17}$$

$$H_{11}(0) = -g_{11} - \bar{g}_{11} + \frac{g_{11}}{\bar{D}}. \tag{18}$$

From the definition of A (see (10)), the first two equations of (29), (17) and (18), for $\theta = 0$ we have that

$$\begin{aligned} -a\pi(w_{20}(-1) + w_{20}(-2)) &= 2i\omega_0 w_{20}(0) - H_{20}(0) \\ &= 2i\omega_0 w_{20}(0) - g_{20} - \bar{g}_{02} + \bar{D}^{-1}g_{20}, \end{aligned}$$

and

$$\begin{aligned} -a\pi(w_{11}(-1) + w_{11}(-2)) &= -H_{11}(0) \\ &= -g_{11} - \bar{g}_{11} + \bar{D}^{-1}g_{11}. \end{aligned}$$

Substituting w_{20} and w_{11} in the above equations we obtain

$$\begin{aligned} &-a\pi(\frac{i}{\omega_0}g_{20}e^{-i\omega_0} + \frac{i}{3\omega_0}\bar{g}_{02}e^{i\omega_0} + ce^{-2\omega_0} + \frac{i}{\omega_0}g_{20}e^{-2i\omega_0} + \frac{i}{3\omega_0}\bar{g}_{02}e^{2i\omega_0}) + ce^{-4\omega_0}) \\ &= -2(g_{20} + \frac{1}{3}\bar{g}_{02} - ic\,\omega_0) - g_{20} - \bar{g}_{02} + \bar{D}^{-1}g_{20}, \end{aligned}$$

and

$$\begin{aligned} -a\pi \quad & (-\frac{i}{\omega_0}g_{11}e^{-i\omega_0} + \frac{i}{\omega_0}\bar{g}_{11}e^{i\omega_0} + d - \frac{i}{\omega_0}g_{11}e^{-2i\omega_0} + \frac{i}{\omega_0}\bar{g}_{11}e^{2i\omega_0} + d) \\ = \quad & -g_{11} - \bar{g}_{11} + \bar{D}^{-1}g_{11}. \end{aligned}$$

Solving these two equations for c and d, we get

$$c = -\frac{\frac{ia\pi}{\omega_0}(g_{20}e^{-i\omega_0} + \frac{1}{3}\bar{g}_{02}e^{i\omega_0} + g_{20}e^{-2i\omega_0} + \frac{1}{3}\bar{g}_{02}e^{2i\omega_0}) - 3g_{20} - \frac{5}{3}\bar{g}_{02} + \bar{D}^{-1}g_{20}}{a\pi e^{-2i\omega_0} + a\pi e^{-4i\omega_0} + 2i\omega_0},$$

and

$$d = \frac{\frac{ia\pi}{\omega_0}(g_{11}e^{-i\omega_0} - \bar{g}_{11}e^{i\omega_0} + g_{11}e^{-2i\omega_0} - \bar{g}_{11}e^{2i\omega_0}) + g_{11} + \bar{g}_{11} - \bar{D}^{-1}g_{11}}{2a\pi}.$$

From the definition of w we have that $x_t(\theta) = w(t,\theta) + zq(\theta) + \bar{z}\bar{q}(\theta)$. Replacing w with the expression obtained in (23), we find that

(19)
$$x_t(\theta) = ze^{i\omega_0\theta} + \bar{z}e^{-i\omega_0\theta} + w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + O(|\, z, \bar{z}\,|^3).$$

Taking into account (11), (12) and (19), we can find $g_{20}, g_{11}, g_{02}, g_{21}$ and

$$\mu_2 = \frac{1}{2\omega_0\alpha'(0)}(-\omega_0\mathrm{Re}(g_{21}) + \mathrm{Im}(g_{20})\mathrm{Re}(g_{11}) + \mathrm{Im}(g_{11})\mathrm{Re}(g_{20})),$$

$$\tau_2 = -\frac{1}{6\omega_0^2}[-\mathrm{Im}(g_{02})^2 - 6\mathrm{Im}(g_{11})^2 - 3\mathrm{Im}(g_{11})\mathrm{Im}(g_{20}) - \mathrm{Re}(g_{02})^2 - 6\mathrm{Re}(g_{11})^2$$
$$+ 3\mathrm{Re}(g_{11})\mathrm{Re}(g_{20}) + 3\mathrm{Im}(g_{21})\omega_0] - \frac{1}{\omega_0}\mu_2\omega'(0)$$

and

$$\beta_2 = \frac{1}{\omega_0}(\omega_0\mathrm{Re}(g_{21}) - \mathrm{Im}(g_{20})\mathrm{Re}(g_{11}) - \mathrm{Im}(g_{11})\mathrm{Re}(g_{20})).$$

3. Applications

In this section we give some applications of the theory developed in Section 2 .

3.1. **Example 1.**

We consider the delay differential equation (1) with $9/100 \leqslant a \leqslant \sqrt{3}/9$, $G_1 = -mx$ and $G_2 = mxy + mxz + nxz^2$ where m, n are arbitrary real numbers. Thus equation (1) has a Hopf periodic orbit for the value of the parameter $\mu = \mu_2\varepsilon^2 + O(\varepsilon^3)$ having period $T = \dfrac{2\pi}{\omega_0}(1 + \tau_2\varepsilon^2) + O(\varepsilon^3)$, where ω_0 is the imaginary part of the pure imaginary root of the characteristic equation of delay differential equation (1). This periodic orbit is asymptotically stable if $\beta_2\varepsilon^2 + O(\varepsilon^3) < 0$ and unstable if $\beta_2\varepsilon^2 + O(\varepsilon^3) > 0$. In this case μ_2, τ_2 and β_2 are given by

$$\begin{aligned}\mu_2 &= \frac{1}{6\pi a - 4\cos(w_0) + 2}(an\pi\left(5\pi^2a^2 + 2\pi(2a\pi - 1)\cos(w_0)a\right.\\ &- \left.4\pi\cos(2w_0)a + 1\right)\sec^2\left(\frac{w_0}{2}\right)(\mathrm{Re}D(\cos(4w_0) + 2) - \mathrm{Im}D\sin(4w_0))),\end{aligned}$$

$$\begin{aligned}
\tau_2 &= \frac{1}{w_0(-3\pi a + 2\cos(w_0) - 1)}(an\pi \sec\left(\frac{w_0}{2}\right)(6a\mathrm{Im}D\pi\cos\left(\frac{w_0}{2}\right) \\
&- 2\mathrm{Im}D\cos\left(\frac{3w_0}{2}\right) - \mathrm{Im}D\cos\left(\frac{5w_0}{2}\right) + a\mathrm{Im}D\pi\cos\left(\frac{7w_0}{2}\right) \\
&+ 2a\mathrm{Im}D\pi\cos\left(\frac{9w_0}{2}\right) + 2a\mathrm{Re}D\pi\sin\left(\frac{w_0}{2}\right) + 2\mathrm{Re}D\sin\left(\frac{3w_0}{2}\right) \\
&- \mathrm{Re}D\sin\left(\frac{5w_0}{2}\right) + a\mathrm{Re}D\pi\sin\left(\frac{7w_0}{2}\right) + 2a\mathrm{Re}D\pi\sin\left(\frac{9w_0}{2}\right))),
\end{aligned}$$

and

$$\beta_2 = -2an\pi(\mathrm{Re}D(\cos(4w_0)+2) - \mathrm{Im}D\sin(4w_0)),$$

where $\mathrm{Re}D$ and $\mathrm{Im}D$ are given in Lemma 4.

3.2. **Example 2.** We consider the delay differential equation (1) for $a = \sqrt{3}/9$. It follows that $\omega_0 = \pi/3$. We take $G_1 = 0$ and $G_2 = mx^2 + ny^2 + pz^2$ with m, n, p arbitrary real numbers. Here μ_2, τ_2 and β_2 are

$$\begin{aligned}
\mu_2 &= \\
&= \frac{1}{\mu_{21}}(2\sqrt{3}(-59049\pi^7(34234m^2 + 1200nm - 15917pm - 15917n^2 \\
&- 133336p^2 - 132136np) - 3486784401\sqrt{3}(8m^2 + 9nm + 5pm + 5n^2 \\
&- 5p^2 + 4np) - 117649(2m^2 + 12nm + 11pm + 11n^2 + 28p^2 + 40np)\pi^{13} \\
&+ 151263\sqrt{3}(16m^2 + 5nm - 3pm - 3n^2 - 49p^2 - 44np)\pi^{12} \\
&+ 194481(62m^2 + 8nm - 23pm - 23n^2 - 224p^2 - 216np)\pi^{11} \\
&+ 3695139\sqrt{3}(8m^2 + 9nm + 5pm + 5n^2 - 5p^2 + 4np)\pi^{10} \\
&+ 13931190(2m^2 + 12nm + 11pm + 11n^2 + 28p^2 + 40np)\pi^9 \\
&- 275562\sqrt{3}(964m^2 - 131nm - 613pm - 613n^2 - 4249p^2 - 4380np)\pi^8 \\
&- 531441\sqrt{3}(9208m^2 + 5497nm + 893pm + 893n^2 - 20341p^2 - 14844np)\pi^6 \\
&- 9565938(1778m^2 + 944nm + 55pm + 55n^2 - 4280p^2 - 3336np)\pi^5 \\
&- 15943230\sqrt{3}(1468m^2 + 1203nm + 469pm + 469n^2 - 2263p^2 - 1060np)\pi^4 \\
&- 43046721(1138m^2 + 848nm + 279pm + 279n^2 - 2008p^2 - 1160np)\pi^3 \\
&- 129140163\sqrt{3}(344m^2 + 335nm + 163pm + 163n^2 - 371p^2 - 36np)\pi^2 \\
&- 387420489(122m^2 + 108nm + 47pm + 47n^2 - 164p^2 - 56np)\pi)),
\end{aligned}$$

where

$$\begin{aligned}
\mu_{21} &= 117(27 + 3\sqrt{3}\pi + 7\pi^2)(27\sqrt{3} + \pi(9 + 7\sqrt{3}\pi))^2 \cdot \\
&\cdot (19683 + \pi(6561\sqrt{3} + \pi(17496 \\
&+ \pi(3483\sqrt{3} + 7\pi(648 + 7\pi(9\sqrt{3} + 7\pi))))));
\end{aligned}$$

$$
\begin{aligned}
\tau_2 &= \\
&= \frac{1}{\tau_{21}}(-2(8m^2+9nm+5pm+5n^2-5p^2+4np)(-31381059609\sqrt{3} \\
&+ \pi(-66248903619+\pi(-65861483130\sqrt{3}+\pi(-91043814915 \\
&+ \pi(-47939698287\sqrt{3}+\pi(-46193914602+\pi(-15740750979\sqrt{3} \\
&+ \pi(-10381699935+7\pi(-323883765\sqrt{3}+7\pi(-17281674 \\
&+ 7\pi(-187353\sqrt{3}+7\pi(11421+7\pi(810\sqrt{3}+7\pi(45+7\sqrt{3}\pi))))))))))))))),
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_{21} &= \\
&= 13\pi(729+\pi(162\sqrt{3}+\pi(405+7\pi(6\sqrt{3}+7\pi))))^2(19683+\pi(6561\sqrt{3} \\
&+ \pi(17496+\pi(3483\sqrt{3}+7\pi(648+7\pi(9\sqrt{3}+7\pi))))));
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_2 &= \\
&= \frac{1}{\beta_{21}}(6\pi(59049\pi^7(34234m^2+1200nm-15917pm-15917n^2 \\
&- 133336p^2-132136np)+3486784401\sqrt{3}(8m^2+9nm+5pm+5n^2 \\
&- 5p^2+4np)+117649(2m^2+12nm+11pm+11n^2+28p^2+40np)\pi^{13} \\
&- 151263\sqrt{3}(16m^2+5nm-3pm-3n^2-49p^2-44np)\pi^{12} \\
&- 194481(62m^2+8nm-23pm-23n^2-224p^2-216np)\pi^{11} \\
&- 3695139\sqrt{3}(8m^2+9nm+5pm+5n^2-5p^2+4np)\pi^{10} \\
&- 13931190(2m^2+12nm+11pm+11n^2+28p^2+40np)\pi^9 \\
&+ 275562\sqrt{3}(964m^2-131nm-613pm-613n^2-4249p^2-4380np)\pi^8 \\
&+ 531441\sqrt{3}(9208m^2+5497nm+893pm+893n^2-20341p^2-14844np)\pi^6 \\
&+ 9565938(1778m^2+944nm+55pm+55n^2-4280p^2-3336np)\pi^5 \\
&+ 15943230\sqrt{3}(1468m^2+1203nm+469pm+469n^2-2263p^2-1060np)\pi^4 \\
&+ 43046721(1138m^2+848nm+279pm+279n^2-2008p^2-1160np)\pi^3 \\
&+ 129140163\sqrt{3}(344m^2+335nm+163pm+163n^2-371p^2 \\
&- 36np)\pi^2+387420489(122m^2+108nm+47pm+47n^2 \\
&- 164p^2-56np)\pi)),
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_{21} &= 13(729+\pi(162\sqrt{3}+\pi(405+7\pi(6\sqrt{3}+7\pi))))^2(19683 \\
&+ \pi(6561\sqrt{3}+\pi(17496+\pi(3483\sqrt{3}+7\pi(648+7\pi(9\sqrt{3}+7\pi)))))).
\end{aligned}
$$

3.3. **Example 3.** We consider the delay differential equation (1) for $9/100 \leqslant a \leqslant \sqrt{3}/9$, $G_1 = -x$ and $G_2 = xy + xz + x^3$. Thus, μ_2, τ_2 and β_2 are given by

$$\mu_2 = \frac{6a\mathrm{Re}D\pi\left(5\pi^2a^2 + 2\pi(2a\pi - 1)\cos(w_0)a - 4\pi\cos(2w_0)a + 1\right)}{(6\pi a - 4\cos(w_0) + 2)(\cos(w_0) + 1)},$$

$$\begin{aligned}\tau_2 \;&=\; -\frac{1}{w_0(3\pi a - 2\cos(w_0) + 1)}(3a\pi\sec\left(\frac{w_0}{2}\right)(3a\mathrm{Im}D\pi\cos\left(\frac{w_0}{2}\right) \\ &- \;\mathrm{Im}D\cos\left(\frac{3w_0}{2}\right) + \mathrm{Re}D(\pi a + 2\cos(w_0) + 1)\sin\left(\frac{w_0}{2}\right))),\end{aligned}$$

and

$$\beta_2 = -6a\mathrm{Re}D\pi,$$

where $\mathrm{Re}D$ and $\mathrm{Im}D$ are given in Lemma 4.

4. Appendix 1: The Hopf bifurcation Theorem

The following result can be found in [8].

Theorem 5. *We consider the delay differential equation*

$$\dot{x}(t) = \frac{dx(t)}{dt} = F(x(t), x(t - r_1), \ldots, x(t - r_{n-1}), \mu) \tag{20}$$

If

(a) *F is analytic in x and μ in a neighborhood of $(\mathbf{0}, 0)$ in $\mathbb{R}^n \times \mathbb{R}$,*
(b) *$F(\mathbf{0}, \mu) = 0$ for μ in an open interval containing 0, and $x(t) = 0$ is an isolated stationary solution of (20),*
(c) *the characteristic equation of (20) has a pair of complex conjugate eigenvalues λ and $\bar{\lambda}$ such that $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$ where $\omega(0) = \omega_0 > 0$, $\alpha(0) = 0$, $\alpha'(0) \neq 0$,*
(d) *the remaining eigenvalues of the characteristic equation have strictly negative real parts,*

then the delay differential equation (20) has a family of Hopf periodic solutions. More precisely, there is an $\varepsilon_0 > 0$ and an analytic function $\mu(\varepsilon) = \sum_{i=2}^{\infty} \mu_i \varepsilon^i$ for $0 < \varepsilon < \varepsilon_0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there exists a periodic solution $p_\varepsilon(t)$ occurring for $\mu = \mu(\varepsilon)$. If $\mu(\varepsilon)$ is not identically zero, the first nonvanishing coefficient μ_i has an even subscript, and there is an $\varepsilon_1 \in (0, \varepsilon_0]$ such that $\mu(\varepsilon)$ is either strictly positive or strictly negative for $\varepsilon \in (0, \varepsilon_1)$. For each $L > 2\pi/\omega_0$ there is a neighborhood V of $x = 0$ and an open interval I containing 0 such that for any $\mu \in I$ the only nonconstant periodic solutions of the delay differential equation (20) with period less then L which lie in V are members of the family $p_\varepsilon(t)$

for values of a satisfying $\mu(\varepsilon) = \mu$, $\varepsilon \in (0, \varepsilon_0)$. *For* $0 < \varepsilon < \varepsilon_0$ *the period* $T(\varepsilon) = 2\pi \left[1 + \sum_{i=2}^{\infty} \tau_i \varepsilon^i\right] / \omega_0$ *of* $p_{\varepsilon(t)}$ *is an analytic function. Exactly two of the Floquet exponents of* $p_\varepsilon(t)$ *approach* 0 *as* $\varepsilon \searrow 0$. *One is* 0 *for* $\varepsilon \in (0, \varepsilon_0)$, *and the other is an analytic function* $\beta(\varepsilon) = \sum_{i=2}^{\infty} \beta_i \varepsilon^i$ *for* $0 < \varepsilon < \varepsilon_0$. *The periodic solution* $p_\varepsilon(t)$ *is orbitally asymptotically stable if* $\beta(\varepsilon) < 0$, *and unstable if* $\beta(\varepsilon) > 0$.

5. Appendix 2: The algorithm for computing the period and the stability of the Hopf periodic orbit

The following algorithm also follows from [8]. Since the algorithm for computing the period and the stability of the Hopf periodic orbit does not depend from the fact that there is one or more delays, we describe it for one delay differential equation with a unique delay.

Consider the autonomous equation

$$\begin{aligned} &\frac{dx(t)}{dt} = L_\mu x_t + f(x_t(\cdot), \mu), \quad t > 0,\ \mu \in \mathbb{R}, \\ &\dot{\mu} = 0, \end{aligned} \tag{21}$$

where for some $r > 0$

$$x_t(\theta) = x(t + \theta), \quad x : [-r, 0] \to \mathbb{R}, \quad \theta \in [-r, 0].$$

We denote by $C[-r, 0]$ the set of all continuous functions from $[-r, 0]$ to $\mathbb{R}$. In $C[-r, 0]$ we put the topology of the supremum. Then we define C as the set of all continuous functions from $C[-r, 0]$ to $\mathbb{R}$. An orbit corresponding to a solution $x(t)$ of (21) is a curve in C traced out by the family of functions $x(\cdot)$, $(x_t(\theta) = x(t + \theta))$ as t rangers over $(0, \infty)$; the orbit of a periodic solution is a closed curve in C. The individual periodic orbits will belong to slices C_μ, (μ–constant) of C.

Here $L_\mu : C[-r, 0] \to \mathbb{R}$ and the operator $f(\cdot, \mu) : C[-r, 0] \to \mathbb{R}$ contains the nonlinear terms, beginning with at least quadratic terms, i.e.

$$f(0, \mu) = 0, \quad f'_x(0, \mu) = 0.$$

For simplicity we consider that $f(\cdot, \mu)$ is analytic and L_μ depends analytically on the bifurcation parameter μ for $|\ \mu\ |$ small.

In the following we transform the linear problem $\dot{x} = L_\mu x_t$. By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-r, 0]$ such that for all $\phi \in C[-r, 0]$,

$$L_\mu \phi = \int_{-r}^{0} \phi(\theta) d\eta(\theta, \mu).$$

In particular

$$L_\mu x_t = \int_{-r}^{0} x(t+\theta)d\eta(\theta,\mu).$$

For example, if $L_\mu x_t = \mu x(t-1)$, then $d\eta(\theta,\mu) = \mu\delta(\theta+1)$, where $\delta(\theta)$ is the Dirac delta function. The choice $d\eta(\theta,\mu) = \mu\delta(\theta)$ corresponds to the ordinary differential equation $\dot{x} = \mu x$.

We make the usual Hopf assumption on the spectrum $\sigma(\mu) = = \left\{\lambda \mid \lambda - L_\mu e^{\lambda\theta} = 0\right\}$ of L_μ; namely

(1) there exists a pair of complex, simple eigenvalues $\lambda(\mu)$ and $\bar{\lambda}(\mu)$ such that $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$, where α and ω are real and $\alpha(0) = 0$, $\omega(0) = \omega_0 > 0$, and $\alpha'(0) \neq 0$ (the transversality hypothesis);
(2) all the other elements of $\sigma(0)$ have negative real parts.

We identify $\lambda(\mu)$ by choosing $\lambda(0) = i\omega_0$. Of course, $\lambda - L_\mu e^{\lambda\theta} = 0$ is the *characteristic equation* of (21).

Next we define for $\phi \in C^1[-r,0]$

$$A(\mu)(\phi) = \begin{cases} \dfrac{d\phi}{d\theta} & \text{if} \quad -r \le \theta < 0, \\ \int_{-r}^{0} \phi(s)d\eta(s,\mu) = L_\mu\phi & \text{if} \quad \theta = 0, \end{cases}$$

and

$$R(\phi) = \begin{cases} 0 & \text{if} \quad -r \le \theta < 0, \\ f(\phi,\mu) & \text{if} \quad \theta = 0. \end{cases}$$

Then since $dx_t/d\theta = dx_t/dt$, (21) can be written as

$$\dot{x}_t = A(\mu)(x_t) + R(x_t), \tag{22}$$

which is a more mathematically pleasing one because this equation involves a single unknown variable x_t rather than both x and x_t.

We shall obtain explicit expressions only for μ_2, τ_2 and β_2 (see Appendix 1). We define $q(\theta)$ to be the eigenvector for $A(0)$ corresponding to $\lambda(0)$; namely $A(0)(q(\theta)) = i\omega_0 q(\theta)$. The adjoint operator $A^*(0)$ is defined by

$$A^*(0)(\alpha(s)) = \begin{cases} \dfrac{d\alpha}{ds} & \text{if} \quad 0 < s \le r, \\ \int_{-r}^{0} \alpha(-t)d\eta(t,0) & \text{if} \quad s = 0. \end{cases}$$

We shall henceforth simply right A for $A(0)$, A^* for $A^*(0)$, $\eta(s)$ for $\eta(s,0)$, etc. Since $A(q(\theta)) = \lambda(0)q(\theta) = i\omega_0 q(\theta)$, $\bar{\lambda}(0)$ is an eigenvalue for A^*, and $A^*(q^*) = -i\omega_0 q^*$ for some nonzero vector q^*.

To construct coordinates to describe C_0 near $\mathbf{0} \in \mathbb{R}^3$, we need an inner product. For $\psi \in C[0,r]$ and $\phi \in C[-r,0]$ that is defined by

$$\langle \psi, \phi \rangle = \bar{\psi}(0)\phi(0) - \int_{\theta=-r}^{0} \left(\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)\phi(\xi) d\xi \right) d\eta(\theta),$$

Then, as usual, $\langle \psi, A\phi \rangle = \langle A^*\psi, \phi \rangle$ for $(\phi, \psi) \in D(A) \times D(A^*)$, where $D(A)$ denotes the definition domain of A. We normalize q and q^* by the condition $\langle q^* \rangle = 1$. Of course $\langle q^*, \bar{q} \rangle = 0$, since $i\omega_0$ is a simple eigenvalue for A. For each $x \in D(A)$, we may than associate the pair (z, w) where $z = \langle q^*, x \rangle$ and $w = x - zq - \bar{z}\bar{q} = x - 2\mathrm{Re}(zq)$.

For x_t a solution of (22) at $\mu = 0$, we define $z(t) = \langle q^*, x_t \rangle$ and then define $w(t,\theta) = x_t(\theta) - 2\mathrm{Re}(z(t)q(t))$. On the manifold C_0, $w(t,\theta) = w(z(t), \bar{z}(t), \theta)$ where

$$w(z, \bar{z}, \theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + w_{30}(\theta)\frac{z^3}{6} + \ldots. \tag{23}$$

In effect, z and $\bar{z}$ are local coordinates for C_0 in C in the directions of q^* and $\bar{q}^*$. Note that w is real if x_t is; we shall deal with real solutions only. It is easy to see that $\langle q^*, w \rangle = 0$.

Now, for solutions $x_t \in C_0$ of (22), $\langle q^*, \dot{x}_t \rangle = \langle q^*, A(x_t) + R(x_t) \rangle$ or, since $\mu = 0$,

$$\begin{aligned} \dot{z}(t) &= \langle \dot{q}^*, x_t \rangle + \langle \dot{q}^*, A(x_t) + R(x_t) \rangle \\ &= i\omega_0 z(t) + \langle \dot{q}^*, A(x_t) \rangle + \langle \dot{q}^*, R(x_t) \rangle \\ &= i\omega_0 z(t) + \bar{q}^*(0) f(x_t, 0) \\ &= i\omega_0 z(t) + \bar{q}^*(0) f(w(z(t), \bar{z}(t), 0) + 2\mathrm{Re}(z(t)q(0)), 0) \\ &= i\omega_0 z(t) + \bar{q}^*(0) f_0(z(t), \bar{z}(t)), \end{aligned} \tag{24}$$

which we write in abbreviated form as

$$\dot{z} = i\omega_0 z + g(z, \bar{z}). \tag{25}$$

Our next object is to expand g in powers of z and $\bar{z}$; and then to obtain from this expansion the values of μ_2, τ_2 and β_2. First, it is required to derive equations for the coefficient $w_{ij}(\theta)$. We write $\dot{w} = \dot{x}_t - \dot{z}q - \dot{\bar{z}}\bar{q}$ and use (25) and (22) to obtain

$$\dot{w} = \begin{cases} A(w) - 2\mathrm{Re}(\bar{q}^*(0) \cdot f_0 q(\theta)) & \text{if} \quad -r \le \theta < 0, \\ A(w) - 2\mathrm{Re}(\bar{q}^*(0) \cdot f_0 q(0)) + f_0 & \text{if} \quad \theta = 0, \end{cases} \tag{26}$$

which we rewrite as

$$\dot{w} = A(w) + H(z, \bar{z}, \theta), \tag{27}$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \dots . \tag{28}$$

On the other hand, on C_0 near the origin $\dot{w} = w_z\dot{z} + w_{\bar{z}}\dot{\bar{z}}$. Using (23) and (25) to replace w_z and $\dot{z}$ and their conjugates by their power series expansions (which involve the w_{ij}), we get a second expression for $\dot{w}$. We equate this to the right–hand side of (27). The result is an equation from which we can derive equations for the n–vectors $w_{ij}(\theta)$, $(i + j = 2, 3, \dots)$. These are

$$\begin{aligned} 2i\omega_0 w_{20}(\theta) - A(w_{20}(\theta)) &= H_{20}(\theta), \\ -A(w_{11}(\theta)) &= H_{11}(\theta), \\ -2i\omega_0 w_{02}(\theta) - A(w_{02}(\theta)) &= H_{02}(\theta), \\ \dots \end{aligned} \tag{29}$$

Now the H_{ij} with $i + j = 2$ do not involve any of the w_{ij} with $i + j > 2$. Further, by hypothesis $2i\omega_0$ and 0 are not eigenvalues of A. Thus, the first three equations (29) can be solved for w_{20}, w_{11} and $w_{02} = \bar{w}_{20}$. At each stage the equations for w_{ij} $(i+j \leq k+1)$ only involve via the H_{ij} coefficients w_{ij} with $i + j \leq k$. Hence the equations (29) can be solved successively for the w_{ij}. Only the values of w_{ij} $(i+j = 2)$ are needed to compute μ_2, τ_2 and β_2. If μ_{2k}, τ_{2k} and β_{2k} are desired for some $k > 1$, then μ must not be set equal to 0 in the previous analysis.

Once the w_{ij} are determined, the differential equation (25) for z can be written as

$$\dot{z} = i\omega_0 z + g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \dots , \tag{30}$$

where the coefficients g_{ij} for $i + j \leq 3$ may be computed by expanding the expression

$$\bar{q}^*(0) \cdot f\left(zq(\theta) + \bar{z}\bar{q}(\theta) + w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2}\right). \tag{31}$$

The coefficient $c_1(0)$ of the Poincaré normal form (see [8], pp. 25–36, 45–51 and [1], chapters 5 and 6) is given in terms of these g_{ij} by formula

$$c_1(0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2 \mid g_{11} \mid^2 - \frac{1}{3} \mid g_{02} \mid^2\right) + \frac{g_{21}}{2}.$$

The following formulas give us the values of μ_2, τ_2 and β_2

$$\mu_2 = -\frac{\mathrm{Re}(c_1(0))}{\alpha'(0)}, \tag{32}$$

$$\tau_2 = -\frac{1}{\omega_0}\left[\mathrm{Im}(c_1(0)) + \mu_2\omega'(0)\right], \tag{33}$$

$$\beta_2 = 2\mathrm{Re}(c_1(0)). \tag{34}$$

For more details see [8] pages 29, 31 and 44, respectively. We recall that $\mu = \mu_2\varepsilon^2 + O(\varepsilon^3)$, $T(\varepsilon) = 2\pi(1 + \tau_2\varepsilon^2 + O(\varepsilon^3))/\omega_0$, and that the Hopf periodic orbit is orbitally asymptotically stable if $\beta_2 < 0$, and unstable if $\beta_2 > 0$.

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References

[1] V. Arnol'd, *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*, Mir, Moscow, 1984.

[2] P. Bi and M. Han, *Bifurcation of periodic solutions of delay differential equation with two delays*, J. Math. Anal. Appl. **284** (2003), 548–563.

[3] P. Bi, M. Han and Y. Wu, *Existence and bifurcation of periodic solutions of three–dimensional delay differential equations*, J. Bifurcation and Chaos **14** (2004), 3921–3929.

[4] S.-N. Chow and J. Mallet–Paret, *Integral averaging and bifurcation*, J. Differential Eqns. **26** (1977), 112–159.

[5] S.-N. Chow and J. Mallet–Paret, *The Fuller index and global Hopf bifurcation*, J. Differential Eqns. **29** (1978), 66–85.

[6] K. Gopalsamy, *Stability and oscilations in delay differential equations of population dynamics*, Mathematics and Its Applications **74**, Kluwer Academic Publisher, London, 1992.

[7] J.K. Hale, S.M.V. Lunel, *Introduction to functional differential eqations*, Appl. Math. Sciences **99**, Springer–Verlag, New York, 1993.

[8] B.D. Hassard, N.D. Kazarinoff and Y–H. Wan, *Theory and applications of Hopf bifurcation*, London Math. Soc. Lecture Note Series **41**, Cambridge University Press, Cambridge, 1981.

[9] E. Hopf, *Abzweigung einer periodischen lösung eines differential systems*, Berichen Math. Phys. Kl. Säch. Akad. Wiss. Leipzig **94** (1942), 1–22.

[10] G.S. Jones, *Periodic motions in Banach space and applications to functional differential equations*, Contrib. Diff. Eqns. **3** (1964), 75–106.

[11] J.L. Kaplan and J.A. Yorke, *Ordinary differential equations which yield periodic solutions of differential delay equations*, J. Math. Anal. Appl. **48** (1974), 317–324.

[12] J.E. Marsden and M.F. MacCracken, *The Hopf bifurcation and its applications*, Appl. Math. Sci. **19**, Springer–Verlag, 1976.

[13] X. Meng and J. Wei, *Stability and bifurcation of mutual system with time delay*, Chaos, Solitons and Fractals **21** (2004), 729–740.

[14] R.G. Nussbaum *A Hopf bifurcation theorem for retarded functional differential equations*, Trans. Amer. Math. Soc. **238** (1978), 139–164.

[15] L.P. Shayer and S.A. Campbell, *Stability, bifurcation, and multistability in a system of two coupled neurons with multiple time delays*, SIAM Appl. Math. **61** (2000), 673–700.

[16] Y. Song, M. Han and J. Wei, *Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays*, Physica D **200** (20059), 185–204.

[17] L. Wang and X. Zou, *Hopf bifurcation in bidirectional associative memory neural networks with delays: analysis and computation*, J. Comput. Appl. Math. **167** (2004), 73–90.

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