# TRAVERSO'S ISOGENY CONJECTURE FOR $p$-DIVISIBLE GROUPS

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*To Carlo Traverso, for his 60th birthday*

ABSTRACT. Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $c, d \in \mathbb{N}$. Let $j := \lceil \frac{cd}{c+d} \rceil \in \mathbb{N}$. Let $H$ be a $p$-divisible group of codimension $c$ and dimension $d$ over $k$. We show that the isogeny class of $H$ is uniquely determined by the truncation of $H$ of level $j$ (i.e., by $H[p^j]$). This proves Traverso's isogeny conjecture for $p$-divisible groups.

## 1. INTRODUCTION

Let $p \in \mathbb{N}$ be a prime. Let $k$ be an algebraically closed field of characteristic $p$. Let $c, d \in \mathbb{N}$. Let $r := c + d$. Let $H$ be a $p$-divisible group over $k$ of codimension $c$ and dimension $d$; its height is $r$. Let $j := \lceil \frac{cd}{r} \rceil \in \mathbb{N}$. It is well-known (see [Ma], [Tr1, Thm. 3], [Tr2, Thm. 1], [Va, Cor. 1.3], and [Oo3, Cor. 1.7]) that there exists a minimal number $n_H \in \mathbb{N}$ such that $H$ is uniquely determined up to isomorphism by $H[p^{n_H}]$ (i.e., if $H'$ is a $p$-divisible group over $k$ such that $H'[p^{n_H}]$ is isomorphic to $H[p^{n_H}]$, then $H'$ is isomorphic to $H$). For instance, we have $n_H \leq cd + 1$ (cf. [Tr1, Thm. 3]). This implies that there exists a minimal number $b_H \in \mathbb{N}$ such that the *isogeny class* of $H$ is uniquely determined by $H[p^{b_H}]$. We call $b_H$ the *isogeny cutoff* of $H$. Traverso speculated the following (cf. [Tr3, §40, Conj. 5]):

**Conjecture 1.1.** *The isogeny cutoff $b_H$ is bounded above by $\lceil \frac{cd}{r} \rceil$ i.e., $b_H \leq \lceil \frac{cd}{r} \rceil$.*

The goal of this paper is to prove the Conjecture:

**Theorem 1.2.** *Let $c, d \in \mathbb{N}$. Let $r := c + d$ and $j := \lceil \frac{cd}{r} \rceil$. The isogeny class of a $p$-divisible group $H$ over $k$ of codimension $c$ and dimension $d$ is uniquely determined by its truncation $H[p^j]$, and thus we have $b_H \leq j$.*

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In Section 2, we introduce notations and basic invariants which pertain to $p$-divisible groups over $k$ and which allow us to obtain a practical upper bound of $b_H$ (see Corollary 2.13). In Section 3, we prove Theorem 1.2; see Example 2.10 for a simpler proof in the particular case when $H$ is isosimple. If the $a$-number of $H$ is at most 1, then Theorem 1.2 is in essence due to Traverso (cf. [Tr3, Thm. of §21, and §40]; see Theorem 3.1). Example 3.2 points out that Theorem 1.2 is optimal.

## 2. Estimates of the isogeny cutoff $b_H$

Let $W(k)$ be the ring of Witt vectors with coefficients in $k$. Let $\sigma$ be the Frobenius automorphism of $W(k)$ induced from $k$. Let $(M,\phi)$ be the (contravariant) Dieudonné module of $H$. Thus $M$ is a free $W(k)$-module of rank $r$ and $\phi: M \to M$ is a $\sigma$-linear endomorphism such that we have $pM \subseteq \phi(M)$. Let $\vartheta: M \to M$ be the $\sigma^{-1}$-linear Verschiebung map of $\phi$; we have $\phi\vartheta = \vartheta\phi = p1_M$.

The Dieudonné–Manin classification of $F$-isocrystals over $k$ (see [Di, Thms. 1 and 2], [Ma, Ch. II, §4], [De, Ch. IV]) states that:

- there exists $m \in \mathbb{N}$ such that we have a direct sum decomposition $(M[\frac{1}{p}],\phi) = \oplus_{i=1}^m (M_i,\phi)$ into simple $F$-isocrystals over $k$, and
- there exist numbers $c_i, d_i \in \mathbb{N} \cup \{0\}$ which satisfy the inequality $r_i := d_i + c_i > 0$, which are relative prime (i.e., $g.c.d.(c_i,d_i) = 1$), and for which there exists a $B(k)$-basis for $M_i$ formed by elements fixed by $p^{-d_i}\phi^{r_i}$.

The unique slope of $(M_i,\phi)$ is $\alpha_i := \frac{d_i}{r_i} \in [0,1] \cap \mathbb{Q}$. To fix ideas, we assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m$. Each $i \in \{1,\ldots,m\}$ gives a slope $\alpha_i$ with multiplicity $r_i$. The Newton polygon $\mathcal{N}_H$ of $H$ is a continuous, piecewise linear, upward convex function $\mathcal{N}_H : [0,r] \to [0,d]$ which for all $i \in \{1,\ldots,m\}$ has slope $\alpha_i$ on the interval $[\sum_{\ell=1}^{i-1} r_\ell, \sum_{\ell=1}^{i} r_\ell]$. As the field $k$ is algebraically closed, the isogeny class of $H$ is uniquely determined by the Newton polygon $\mathcal{N}_H$. The finite set $\mathcal{N}_{c,d}$ of Newton polygons we thus obtain by varying $H$, can be partially ordered as follows: we say that $\mathcal{N}_1$ lies above (resp. strictly above) $\mathcal{N}_2$ if for all $t \in [0,r]$ we have $\mathcal{N}_1(t) \geq \mathcal{N}_2(t)$ (resp. if for all $t \in [0,r]$ we have $\mathcal{N}_1(t) \geq \mathcal{N}_2(t)$ and moreover $\mathcal{N}_1 \neq \mathcal{N}_2$). This partial order is convenient when studying the variation of the Newton polygon in families. We will use the notation $\mathcal{N}_\bullet$ to denote the Newton polygon of $\bullet$, where $\bullet$ is a $p$-divisible group over a field that contains $k$.

Let $H_{c_i,d_i}$ be the $p$-divisible group over $k$ that has the following two properties: (i) it has codimension $c_i$, dimension $d_i$, and unique Newton polygon slope $\alpha_i$, and (ii) its endomorphism ring $\mathrm{End}(H_{c_i,d_i})$ is the maximal order in the simple $\mathbb{Q}_p$-algebra $\mathrm{End}(H_{c_i,d_i}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of invariant $\alpha_i \in \mathbb{Q}/\mathbb{Z}$. Let $H_0 := \prod_{i=1}^m H_{c_i,d_i}$. Let $(M_0,\phi_0)$ be the Dieudonné module of $H_0$.

**Definition 2.1.** ([Oo4, Def. (1.1)]) *We say that a p-divisible group $H$ over $k$ is minimal if it is isomorphic to the p-divisible group $H_0$ determined uniquely from the Newton polygon $\mathcal{N}_H$ of $H$.*

The $p$-divisible group $H_0$ is the *unique* (up to isomorphism) minimal $p$-divisible over $k$ that is isogenous to $H$.

**Theorem 2.2.** *A minimal p-divisible group $H$ over $k$ is determined up to isomorphism by $H[p]$.*

*Proof:* For the isoclinic case (i.e., when the pairs $(c_i, d_i)$ do not depend on $i$) we refer to [Va1, Exa. 3.3.6]. For the general case we refer to either [Oo4, Thm. 1.2] or [Va2, Thm. 1.8]. □

**Definition 2.3.** *By the minimal height $q_H$ of a p-divisible group $H$ over $k$ we mean the smallest non-negative integer $q_H$ such that there exists an isogeny $\psi_0 : H \to H_0$ to a minimal p-divisible group, whose kernel $Ker(\psi_0)$ is annihilated by $p^{q_H}$.*

**Definition 2.4.** *By the a-number $a_H$ of a p-divisible group $H$ over $k$ we mean the number $\dim_k(\boldsymbol{\alpha}_p, H[p]) = \dim_k(M/\phi(M) + \vartheta(M)) \in \mathbb{N} \cup \{0\}$, where $\boldsymbol{\alpha}_p$ is the local-local group scheme of order p.*

See [Oo2, Prop. 2.8] for a proof of the following specialization trick.

**Proposition 2.5.** *For every p-divisible group $H$ over $k$, there exists a p-divisible group $\mathcal{H}$ over $k[[x]]$ that has the following two properties:*

**(i)** *its fibre over $k$ is $H$;*

**(ii)** *if $\overline{k((x))}$ is an algebraic closure of $k((x))$, then $\mathcal{H}_{\overline{k((x))}}$ has the same Newton polygon as $H$ and its a-number is at most 1.*

**Definition 2.6.** *By the weak isogeny cutoff of a p-divisible group $H$ over $k$ we mean the smallest number $\tilde{b}_H \in \mathbb{N}$ such that the following two properties hold:*

**(i)** *if $H'$ is a p-divisible group over $k$ such that $H'[p^{\tilde{b}_H}]$ is isomorphic to $H[p^{\tilde{b}_H}]$, then its Newton polygon $\mathcal{N}_{H'}$ is not strictly above $\mathcal{N}_H$;*

**(ii)** *there exists a p-divisible group $\mathcal{H}$ over $k[[x]]$ that has the following three properties:*

**(ii.a)** *its fibre over $k$ is $H$;*

**(ii.b)** *the fibre $\mathcal{H}_{k((x))}$ over $k((x))$ has the same Newton polygon as $H$;*

**(ii.c)** *the isogeny cutoff of $\mathcal{H}_{\overline{k((x))}}$ is at least $\tilde{b}_H$ and the a-number of $\mathcal{H}_{\overline{k((x))}}$ is at most 1.*

Note that property (i) holds for any level greater or equal to $b_H$. Thus the existence of a minimal number $\tilde{b}_H \in \mathbb{N}$ is implied by Proposition 2.5.

**Fact 2.7.** *If $H^t$ is the Cartier dual of $H$, then $q_H = q_{H^t}$, $b_H = b_{H^t}$, and $\tilde{b}_H = \tilde{b}_{H^t}$.*

*Proof:* This follows from Cartier duality (the Cartier dual of $H_{c_i,d_i}$ is $H_{d_i,c_i}$). □

**Lemma 2.8.** *The isogeny cutoff $b_H$ is the smallest natural number such that for every element $g \in \boldsymbol{GL}_M(W(k))$ congruent to $1_M$ modulo $p^{b_H}$, the Dieudonné module $(M, g\phi)$ is isogenous to $(M, \phi)$.*

*Proof:* Let $t \in \mathbb{N}$. The arguments proving either [Va1, Cor. 3.2.3] or [NV, Thm. 2.2 (a)] show that: (i) if $g \in \boldsymbol{GL}_M(W(k))$ is congruent to $1_M$ modulo $p^t$, then every $p$-divisible group $H_g$ over $k$ whose Dieudonné module is isomorphic to $(M, g\phi)$, has the property that $H_g[p^t]$ is isomorphic to $H[p^t]$, and (ii) if a $p$-divisible group $H_g$ over $k$ has the property that $H_g[p^t]$ is isomorphic to $H[p^t]$, then there exists $g \in \boldsymbol{GL}_M(W(k))$ congruent to $1_M$ modulo $p^t$ and such that the Dieudonné module of $H_g$ is isomorphic to $(M, g\phi)$. The Lemma follows from these two statements and the very definition of $b_H$. □

**Lemma 2.9.** *For every $p$-divisible group $H$ over $k$, the isogeny cutoff $b_H$ is bounded by the minimal height $q_H$ plus 1 i.e., we have $b_H \leq q_H + 1$.*

*Proof:* Let $\psi : H \to H_0$ be an isogeny whose kernel is annihilated by $p^{q_H}$. Let $(M_0, \phi_0) \hookrightarrow (M, \phi)$ be the monomorphism associated to the isogeny $\psi$; we will identify $M_0$ with its image under this monomorphism. We have $p^{q_H} M \subseteq M_0 \subseteq M$. Let $g \in \boldsymbol{GL}_M(W(k))$ be congruent to $1_M$ modulo $p^{q_H+1}$. We write $g = 1_M + p^{q_H+1}e$, where $e \in \mathrm{End}(M)$. We have $p^{q_H+1}e(M_0) \subseteq p^{q_H+1}e(M) \subseteq p^{q_H+1}M \subseteq pM_0$. Thus we have $g \in \boldsymbol{GL}_{M_0}(W(k))$ and moreover $g$ is congruent to $1_{M_0}$ modulo $p$. Therefore the reductions modulo $p$ of the two triples $(M_0, \phi, \vartheta)$ and $(M_0, g\phi, \vartheta g^{-1})$ coincide. As the Dieudonné module $(M_0, \phi)$ is minimal (being isomorphic to $(M_0, \phi_0)$), from Theorem 2.2 we get that it is isomorphic to $(M_0, g\phi)$. The isogeny class of $(M, g\phi)$ (resp. of $(M, \phi)$) is the same as of $(M_0, g\phi)$ (resp. of $(M_0, \phi)$). From the last two sentences we get that the Dieudonné modules $(M, g\phi)$ and $(M, \phi)$ are isogenous. From this and Lemma 2.8, we get that $b_H \leq q_H + 1$. □

**Example 2.10.** Suppose that $m = 1$ i.e., $H$ is an isosimple $p$-divisible group. Let $\theta_0 : H_0 \to H_0$ be an endomorphism such that $\mathrm{End}(H_0) = W(\mathbb{F}_{p^r})[\theta_0]$ and $\theta_0^r = p$, cf. [Ma, Ch. III, §4, 1] and [dJO, Lem. 5.4]. From [dJO, 5.8] we get that there exist inclusions $(M_0, \phi_0) \hookrightarrow (M, \phi) \hookrightarrow$

$(\theta_0^{-(c-1)(d-1)}M_0, \phi_0)$ between Dieudonné modules over $k$. Let $\psi_0 : H \to H_0$ be the isogeny defined by the first inclusion $(M_0, \phi_0) \hookrightarrow (M, \phi)$. Its kernel is annihilated by $\theta_0^{(c-1)(d-1)} = p^{\frac{(c-1)(d-1)}{r}}$ and therefore we have $q_H \leq \lceil \frac{(c-1)(d-1)}{r} \rceil = j - 1$. Thus $b_H \leq j$, cf. Lemma 2.9.

**Remark 2.11.** The function $f : (0, \infty) \times (0, \infty) \to (0, \infty)$ defined by the rule $f(x, y) = \frac{xy}{x+y}$ is subadditive i.e., for all $x_1, x_2, y_1, y_2 \in (0, \infty)$, we have an inequality $f(x_1, y_1) + f(x_2, y_2) \leq f(x_1 + x_2, y_1 + y_2)$. But the function $g(x, y) := \lceil f(x, y) \rceil$ is not subadditive. Due to this and the plus 1 part of Lemma 2.9, our attempts to use Lemma 2.9 in order to prove Theorem 1.2 by induction on $m \in \mathbb{N}$, failed. On the other hand, in most cases Lemma 2.9 provides better upper bounds of $b_H$ than those provided by the following Proposition.

**Proposition 2.12.** *For every $p$-divisible group $H$ over $k$, the isogeny cutoff $b_H$ is bounded by the weak isogeny cutoff $\tilde{b}_H$ i.e., we have $b_H \leq \tilde{b}_H$.*

*Proof:* Let $\overline{k((x))}$ be an algebraic closure of $k((x))$. Let $\mathcal{H}$ be a $p$-divisible group over $k[[x]]$ of constant Newton polygon such that its fibre over $k$ is $H$ and the isogeny cutoff $b$ of $\mathcal{H}_{\overline{k((x))}}$ is at most $\tilde{b}_H$, cf. property (ii) of Definition 2.6. We recall that the statement that $\mathcal{H}$ has constant Newton polygon means that the Newton polygons of $H$ and $\mathcal{H}_{\overline{k((x))}}$ coincide i.e., $\mathcal{N}_H = \mathcal{N}_{\mathcal{H}_{\overline{k((x))}}}$. For the proof of this Proposition it is irrelevant what the $a$-number of $\mathcal{H}_{\overline{k((x))}}$ is.

Let $H'$ be a $p$-divisible group over $k$ such that $H'[p^{\tilde{b}_H}] = H[p^{\tilde{b}_H}]$. Based on [Il, Thm. 4.4. f)] we get that for all $n \in \mathbb{N}$, there exists a $p$-divisible group $\mathcal{H}'_n$ over $\mathrm{Spec}(k[[x]]/(x^n))$ that lifts $H'$ and such that $\mathcal{H}'_n[p^{\tilde{b}_H}] = \mathcal{H}[p^{\tilde{b}_H}] \times_{k[[x]]} k[[x]]/(x^n)$. Based on loc. cit., we can assume that $\mathcal{H}'_{n+1}$ restricted to $\mathrm{Spec}(k[[x]]/(x^n))$ is $\mathcal{H}'_n$. Therefore the $\mathcal{H}'_n$'s glue together to define a $p$-divisible group $\mathcal{H}'^{\mathrm{f}}$ over the formal scheme $\mathrm{Spf}(k[[x]])$. We recall that the categories of $p$-divisible groups over $\mathrm{Spf}(k[[x]])$ and $\mathrm{Spec}(k[[x]])$ are naturally equivalent, cf. [dJ, Lem. 2.4.4]. Thus let $\mathcal{H}'$ be the $p$-divisible group over $\mathrm{Spec}(k[[x]])$ that corresponds naturally to $\mathcal{H}'^{\mathrm{f}}$.
The $p$-divisible group $\mathcal{H}'$ lifts $H'$ and we have

$$\mathcal{H}'[p^{\tilde{b}_H}] = \text{proj.lim.}_{n\to\infty}\mathcal{H}'_n[p^{\tilde{b}_H}] = \mathcal{H}[p^{\tilde{b}_H}].$$

As $\mathcal{H}'[p^{\tilde{b}_H}] = \mathcal{H}[p^{\tilde{b}_H}]$ and $b \leq \tilde{b}_H$, from the very definition of $b$, we get that $\mathcal{H}'_{\overline{k((x))}}$ has the same Newton polygon as $\mathcal{H}_{\overline{k((x))}}$; thus $\mathcal{N}_{\mathcal{H}'_{\overline{k((x))}}} = \mathcal{N}_H$. As the Newton polygons go up under specialization (see [De, Ch. IV, §7, Thm.]), we conclude that the Newton polygon $\mathcal{N}_{H'}$ of $H'$ is above the Newton polygon $\mathcal{N}_H$ of $H$. But $\mathcal{N}_{H'}$ is not strictly above $\mathcal{N}_H$, cf. property

(i) of Definition 2.6. From the last two sentences we get that $\mathcal{N}_{H'} = \mathcal{N}_H$. This implies that $b_H \leq \tilde{b}_H$. □

From Lemma 2.10 and Proposition 2.12 we get:

**Corollary 2.13.** *For every p-divisible group H over k, we have the following inequality* $b_H \leq \min\{\tilde{b}_H, q_H + 1\}$.

## 3. The proof of Theorem 1.2

In this Section we prove Theorem 1.2. We begin by proving the following particular case of Theorem 1.2 which in essence is due to Traverso (cf. [Tr3, Thm. of §21, and §40]).

**Theorem 3.1.** *Suppose that* $a_H \leq 1$. *Then we have* $b_H \leq j$.

*Proof:* We first recall how to compute the Newton polygon $\mathcal{N}_H$ of $H$ (cf. [Ma], [De, Ch. IV, Lem. 2, pp. 82–84], [Tr3, §21, Thm.], and [Oo1, Lem. 2.6]). Let $v_p : W(k) \to \mathbb{N} \cup \{0\}$ be the normalized valuation of $W(k)$; thus $v_p(p) = 1$. Let $x \in M$ be such that its reduction modulo $p$ generates the $k$-vector space $M/\phi(M) + \vartheta(M)$. Let $a_0, a_1, \ldots, a_c, b_1, \ldots, b_d \in W(k)$ be such that the map

$$\psi := \sum_{i=0}^{c} a_{c-i}\phi^i + \sum_{\ell=1}^{d} b_\ell \vartheta^\ell : M \to M$$

annihilates $x$. It is easy to see that we can choose $x$ such that $v_p(a_0) = v_p(b_d) = 0$ and $\{x, \phi(x), \ldots, \phi^{c-1}(x), \vartheta(x), \ldots, \vartheta^d(x)\}$ is a $W(k)$-basis for $M$. For $\ell \in \{1, \ldots, d\}$ let $a_{c+\ell} := p^\ell b_\ell$; thus $a_i$ is well defined for all $i \in \{0, \ldots, r\}$. Let

$$Q_x(t) := \sum_{i=0}^{r} v_p(a_i) t^{r-i} = \sum_{i=0}^{r} v_p(\sigma^d(a_i)) t^{r-i} \in \mathbb{Z}[t].$$

We recall that the Newton polygon of $Q_x(t)$ is the greatest continuous, piecewise linear, upward convex function $\mathcal{N}_x : [0, r] \to [0, d]$ with the property that for all $i \in \{0, \ldots, r\}$ we have $\mathcal{N}_x(i) \leq v_p(a_i)$. Then we have

$$\mathcal{N}_H = \mathcal{N}_x. \tag{1}$$

To check Formula (1) we view $M$ as a $W(k)[F]$-module, where $F \cdot \lambda = \lambda^\sigma F$ and where $F$ acts on $M$ as $\phi$ does. The $W(k)[F]$-module $M$ is isogenous to the $W(k)[F]$-module $M' := W(k)[F]/W(k)[F]\psi'$, where

$$\psi' = F^d \psi := \sum_{i=0}^{r} \sigma^d(a_i) F^{r-i}.$$

But the Newton polygon of the $W(k)[F]$-module $M'$ is $\mathcal{N}_x$, cf. [De, Ch. IV, Lemma 2, pp. 82–84]. Thus Formula (1) holds.

We recall that $j = \lceil \frac{cd}{r} \rceil$. As $c, d > 0$, we have $j \geq 1$. Let $g \in \boldsymbol{GL}_M(W(k))$ be congruent to $1_M$ modulo $p^j$. Let $H_g$ be a $p$-divisible group over $k$ whose Dieudonné module is isomorphic to $(M, g\phi)$. As $j \geq 1$, $H_g[p]$ is isomorphic to $H[p]$ and therefore $a_{H_g} = a_H \leq 1$. Based on Lemma 2.8, to prove the Theorem it suffices to show that $\mathcal{N}_{H_g}$ is the same as $\mathcal{N}_H$. By replacing $(\phi, \vartheta)$ with $(g\phi, \vartheta g^{-1})$, the map $\psi : M \to M$ which annihilates $x$ gets replaced by another map

$$\psi_g := \sum_{i=0}^{c} a_{g,c-i}\phi^i + \sum_{\ell=1}^{d} b_{g,\ell}\vartheta^\ell : M \to M$$

which annihilates $x$, where $a_{g,c-i}$ is congruent to $a_{c-i}$ modulo $p^j$ and where $b_{g,\ell}$ is congruent to $b_\ell$ modulo $p^j$. For $\ell \in \{1, \ldots, d\}$ let $a_{g,c+\ell} := p^\ell b_{g,\ell}$; thus $a_{g,i}$ is well defined for all $i \in \{0, \ldots, r\}$ and it is congruent to $a_i$ modulo $p^{j+\min\{0,i-c\}}$. The pair $(Q_x(t), \mathcal{N}_x)$ gets replaced by the pair $(Q_{g,x}(t), \mathcal{N}_{g,x})$, where $Q_{g,x}(t) := \sum_{i=0}^{r} v_p(a_{g,i})t^{r-i}$ and where $\mathcal{N}_{g,x}$ is the Newton polygon of $Q_{g,x}(t)$. As $\mathcal{N}_x$ is upward convex and as $\mathcal{N}_x(0) = 0$ and $\mathcal{N}_x(r) = d$, we have $\mathcal{N}_x(t) \leq \frac{dt}{r}$ for all $t \in [0, r]$. Thus $j = \lceil \mathcal{N}_x(c) \rceil \geq \mathcal{N}_x(c) \geq \mathcal{N}_x(i)$ for all $i \in \{0, \ldots, c\}$. As $d < r$, for $i \in \{c+1, \ldots, r\}$ we have $j + i - c \geq \frac{di}{r}$. From the last two sentences we get that $j + \min\{0, i-c\} \geq \frac{di}{r} \geq \mathcal{N}_x(i)$ for all $i \in \{0, \ldots, r\}$. From this and the fact that for all $i \in \{0, \ldots, r\}$ the elements $a_{g,i}$ and $a_i$ are congruent modulo $p^{j+\min\{0,i-c\}}$, we get that $\mathcal{N}_{g,x} = \mathcal{N}_x$. From this and Formula (1) we get that $\mathcal{N}_{H_g} = \mathcal{N}_H$. □

3.1. **End of the proof of Theorem 1.2.** Based on Proposition 2.12, to prove Theorem 1.2 it suffices to show that $\tilde{b}_H \leq j$. We recall that $\mathcal{N}_{c,d}$ is the set of Newton polygons of $p$-divisible groups over $k$ of codimension $c$ and dimension $d$. Let $\mathcal{D}_H$ be the class of $p$-divisible groups of codimension $c$ and dimension $d$ over $k$ whose Newton polygons are strictly above $\mathcal{N}_H$. We prove the inequality $\tilde{b}_H \leq j$ by decreasing induction on $\mathcal{N}_H \in \mathcal{N}_{c,d}$. Thus we can assume that for every $\bullet \in \mathcal{D}_H$ we have $\tilde{b}_\bullet \leq j$; as $b_\bullet \leq \tilde{b}_\bullet$ (cf. Proposition 2.12) we also have $b_\bullet \leq j$.

To show that $\tilde{b}_H \leq j$, let $\mathcal{H}$ be a $p$-divisible group over $k[[x]]$ such that the properties (ii.a) to (ii.c) of Definition 2.6 hold. Let $b$ be the isogeny cutoff of $\mathcal{H}_{\overline{k((x))}}$. From the very definition of $\tilde{b}_H$ we get that

$$\tilde{b}_H \leq \max\{b, b_\bullet | \bullet \in \mathcal{D}_H\}.$$

As $b \leq j$ (cf. Theorem 3.1 applied to $\mathcal{H}_{\overline{k((x))}}$) and as $b_\bullet \leq j$ for all $\bullet \in \mathcal{D}_H$ (cf. previous paragraph), we have $\tilde{b}_H \leq j$. This ends the induction.

Thus $\tilde{b}_H \leq j$ and therefore $b_H \leq j$. □

**Example 3.2.** Obviously Theorem 1.2 is optimal if $j = 1$. Suppose that $j \geq 2$. Let $s \in \mathbb{N}$ be the smallest number such that $r$ divides $cd - s$; we have $j - 1 = \frac{cd-s}{r}$. We assume that there exists an element $x \in M$ such that the $r$-tuple $(e_1, \ldots, e_r) := (x, \phi(x), \ldots, \phi^{c-1}(x), \vartheta^d(x), \ldots, \vartheta(x))$ is an ordered $W(k)$-basis for $M$ and we have an equality $\vartheta^d(x) = \phi^c(x)$. Thus $\phi(e_i) = e_{i+1}$ if $i \in \{1, \ldots, c\}$ and $\phi(e_i) = pe_{i+1}$ if $i \in \{c+1, \ldots, r\}$. We have $\phi^r(e_i) = p^d e_i$ and therefore all Newton polygon slopes of $H$ are $\frac{d}{r}$; thus $m = g.c.d.(c, r)$. Replacing the equality $\vartheta^d(x) = \phi^c(x)$ by the equality $\tilde{\vartheta}^d(x) = \tilde{\phi}^c(x) - p^{j-1}x$, we get a $p$-divisible group $\tilde{H}$ over $k$ whose Dieudonné module $(M, \tilde{\phi})$ is such that $\tilde{\phi}(e_i) = e_{i+1}$ if $i \in \{1, \ldots, c-1\}$, $\tilde{\phi}(e_c) = p^{j-1}e_1 + e_{c+1}$, and $\tilde{\phi}(e_i) = pe_{i+1}$ if $i \in \{c+1, \ldots, r\}$. As $(\tilde{\phi}, \tilde{\vartheta})$ and $(\phi, \vartheta)$ are congruent modulo $p^{j-1}$, $\tilde{H}[p^{j-1}]$ is isomorphic to $H[p^{j-1}]$. We have $\mathcal{N}_H(t) = \frac{dt}{r}$ but the piecewise linear function $\mathcal{N}_{\tilde{H}}(t)$ changes slope at $c$; more precisely we have $\mathcal{N}_{\tilde{H}}(c) = j - 1$ (cf. Formula (1) applied to $\tilde{H}$ and to the polynomial $\tilde{Q}_x(t) := t^r + (j-1)t^d + d$). Thus the Newton polygon slopes of $\tilde{H}$ are $\frac{j-1}{c}$ (with multiplicity $c$) and $1 - \frac{j-1}{d}$ (with multiplicity $d$) and therefore are different from $\frac{d}{r}$. This implies that $b_H \geq j$ and therefore (cf. Theorem 1.2) we have $b_H = j$.

**Example 3.3.** Suppose that $c = d$ and that $H$ is as in Example 3.2. Then $q_H = \frac{c-1}{2}$ (cf. [NV, Rmk. 3.3]) and $j = \lceil \frac{c}{2} \rceil$. If $c$ is odd, then $b_H = j = \frac{c+1}{2} = q_H + 1$. Thus the inequality $b_H \leq q_H + 1$ (see Lemma 2.9) is optimal in general.

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