SEMIGROUPS OF MATRICES OF INTERMEDIATE GROWTH

FERRAN CEDÓ AND JAN OKNIŃSKI

Abstract. Finitely generated linear semigroups over a field K that have intermediate growth are considered. New classes of such semigroups are found and a conjecture on the equivalence of the subexponential growth of a finitely generated linear semigroup S and the nonexistence of free noncommutative subsemigroups in S, or equivalently the existence of a nontrivial identity satisfied in S, is stated. This 'growth alternative' conjecture is proved for linear semigroups of degree 2, 3 or 4. Certain results supporting the general conjecture are obtained. As the main tool, a new combinatorial property of groups is introduced and studied.

1. Introduction

Let $S = \langle g_1, \ldots, g_m \rangle$ be a finitely generated semigroup. The growth function $d_S : \mathbb{N} \longrightarrow \mathbb{N}$ of S is obtained by defining $d_S(n)$ as the number of elements of S that can be presented as words of length not exceeding n in the generators $g_1, \ldots, g_m$. The growth of S is the equivalence class of d_S for the relation $\sim$ defined on the set of possible growth functions by the condition: $f \sim g$ if $f(n) \leq g(cn)$ and $g(n) \leq f(cn)$ for some $c > 0$ and all sufficiently big positive integers n. This is independent of the choice of the generating set of S. We refer to [7] for the basic facts on the theory of growth of algebras, semigroups and groups. Gromov proved that the class of groups of polynomial growth coincides with the class of finitely generated nilpotent-by-finite groups, [3]. On the other hand, after Golod's construction of a counterexample to the general Burnside problem, it is not hard to see that there exist finitely generated periodic groups of exponential growth (see [13], pages 413-415). Clearly, such groups do not have any free noncommutative subsemigroup. Recall that the growth of a finitely generated group G can also be intermediate, that is, not polynomial and not exponential. This was first shown by Grigorchuk, who later proved that

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the growth of such a group exceeds $e^{\sqrt{n}}$, [2]. On the other hand, finitely generated groups of matrices over a field either have a polynomial growth or contain a free noncommutative subsemigroup. The latter is a consequence of Tits alternative [19] and of a theorem of Rosenblatt, [15].

Let $R = \langle g, e\rangle$ be the subsemigroup of the full linear (multiplicative) monoid $M_3(\mathbb{Q})$ generated by $g = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Using the classical theory of partitions [5], it was first shown in [11], see also [12], that R has intermediate growth. Moreover, every nonempty intersection of R with a maximal subgroup of the multiplicative monoid $M_3(\mathbb{Q})$ is contained in an infinite cyclic group. Actually, $R \subseteq T = \langle g\rangle \cup I$, where $I = \mathcal{M}(H, X, X; P)$ for a completely 0-simple semigroup I over an infinite cyclic group H, where X is a countable infinite set and P is a sandwich matrix (for the definition see [1] or Section 3). Here I consists of matrices of rank 2, so it is an ideal in T.

Another example of the latter type is the semigroup $Q = \langle h, f\rangle \subseteq M_2(\mathbb{Q})$, where $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $f = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, considered later in [10]. However, in this example, $Q \cap f M_2(\mathbb{Q}) f$ generates an infinitely generated subgroup of the maximal subgroup $f M_2(\mathbb{Q}) f \setminus \{0\} \cong \mathbb{Q}^*$ of $M_2(\mathbb{Q})$. An exact rate of growth of Q was determined in [8]. Namely, the growth function is equivalent to $e^{\sqrt{n/\log n}}$. So, this is also in contrast with Grigorchuk's result on the growth of groups.

It is known that a linear semigroup $S \subseteq M_n(K)$ over a field K satisfies a semigroup identity if and only if every nontrivial intersection $G \cap S$ with a maximal subgroup G of the multiplicative monoid $M_n(K)$ generates a nilpotent-by-finite subgroup of G. Moreover, if the field K is finitely generated, the latter is equivalent to the fact that S does not have free noncommutative subsemigroups, see [12, Theorem 6.11]. In particular, the semigroup R defined above satisfies an identity and it was the first example of a semigroup of intermediate growth that satisfies an identity and is linear. This is clearly in contrast with the case of linear groups. In this context, we notice also that an example of a group G that contains a free nonabelian subgroup and is generated by a finite subset $\{a_1, \dots, a_n\}$ such that the semigroup $\langle a_1, \dots, a_n\rangle$ satisfies a nontrivial identity has been recently constructed in [6].

In this paper, we study the growth of finitely generated linear semigroups $S \subseteq M_n(K)$ over a field K. A general 'growth alternative' conjecture is proposed, which asserts that S has subexponential growth if and only if S has no free noncommutative subsemigroups. The problem leads in a natural way to the study of the behavior of the subexponential growth property under ideal extensions of semigroups of certain special types. This, in turn, leads to a new combinatorially defined property of groups that is introduced in Section 2 and becomes the main tool in the paper. The conjecture is confirmed in case $n \leq 4$ in Section 3. Certain results supporting the general case are obtained in Section 4, where we also discuss certain natural related problems. Finally, in Section 5, new examples of semigroups of intermediate growth are constructed and used to establish the equivalence of some of the proposed conjectures. All this shows, rather unexpectedly, that there is an abundance of linear semigroups of intermediate growth that satisfy nontrivial identities.

It is worth mentioning that semigroups of arbitrarily large subexponential growth have been constructed in [16]. A very nice recent result of Shneerson shows that, in nonperiodic varieties of semigroups that satisfy identities of certain types, every finitely generated semigroup has subexponential growth, [17]. However, this type of identities does not apply to the general identities satisfied by linear semigroups. We refer to [17] for the bibliography on several other results on growth, considered in the context of varieties of semigroups. Recall also that some partial results on the growth of linear semigroups, with an emphasis on polynomial growth, can be found in [12], which is also our main reference for the theory of linear semigroups. For the necessary background on semigroup theory we refer to [1].

2. Subexponential property for sequences in groups

We start with a combinatorial property for sequences of elements of a group, which will play a crucial role for the main techniques and results of the paper. It will be mainly considered in the context of nilpotent-by-finite groups.

Definition 2.1. *Let G be a group. Let $b(1), b(2), \ldots$ be a sequence of positive integers. For a sequence*

$$g_{1,1}, g_{1,2}, \ldots, g_{1,b(1)}, g_{2,1}, g_{2,2}, \ldots, g_{2,b(2)}, \ldots \tag{1}$$

of elements of G, define the set

$$T_n = \{g_{i_1,j_1} g_{i_2,j_2} \cdots g_{i_s,j_s} \in G \mid s \geq 1,\ i_1 + \cdots + i_s \leq n\}.$$

Let $g(n) = |T_n|$ and $f(n) = \sum_{i=1}^{n} b(i)$.

We say that G satisfies the subexponential property for sequences if for every sequence (1):

$$\limsup_{n\to\infty} f(n)^{\frac{1}{n}} \leq 1 \implies \limsup_{n\to\infty} g(n)^{\frac{1}{n}} \leq 1.$$

We say that G satisfies the subexponential weak property for sequences if for every sequence (1):

$$f \text{ has polynomial growth } \Longrightarrow g \text{ has subexponential growth.}$$

Recall that the latter means that there is no $c > 1$ such that $g(n) \geq c^n$ for sufficiently big n. Note that, for a monotone increasing function $f\colon \mathbb{N} \longrightarrow \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of nonnegative real numbers, the condition

$$\limsup_{n\to\infty} f(n)^{\frac{1}{n}} \leq 1$$

implies that f has subexponential growth. Recall also that f has polynomial growth if and only if there exist positive integers d and m such that $f(n) \leq n^d$ for all $n \geq m$. Therefore, we distinguish the following types of growth: polynomial, intermediate (that is, subexponential but not polynomial) and exponential.

In order to give simplest examples, we first need the following combinatorial observation.

Lemma 2.2. *Let $\{\mathcal{C}_n\}_{n=1}^{\infty}$ be a family of finite pairwise disjoint sets. Let $T = \bigcup_{n=1}^{\infty} \mathcal{C}_n$. Let $l\colon T \longrightarrow \mathbb{N}$ be defined by $l(t) = n$ if and only if $t \in \mathcal{C}_n$. Let $\leq$ be a well-order on T. Let a_n be the number of all finite ordered sequences $t_1 \leq t_2 \leq \cdots \leq t_k$ of elements in T such that $\sum_{j=1}^{k} l(t_j) = n$. Let b_n be the number of elements in $\mathcal{C}_n$. Then*

$$\prod_{m=1}^{\infty} (1 - x^m)^{-b_m} = 1 + \sum_{n=1}^{\infty} a_n x^n.$$

Proof. Let $\mathcal{L}_n$ be the set of all finite ordered sequences

$$t_1 \leq t_2 \leq \cdots \leq t_k$$

of elements in T such that $\sum_{j=1}^{k} l(t_j) = n$. Let $\mathcal{F}_n = \{f\colon T \longrightarrow \{0, 1, \ldots, n\}\}$. We define $\varphi\colon \mathcal{L}_n \longrightarrow \mathcal{F}_n$ by $\varphi(t_1, \ldots, t_k)(t) = m$ where m is the number of times t appears in the sequence $(t_1, \ldots, t_k)$. It is clear that φ is injective. Hence a_n is the number of all $f \in \mathcal{F}_n$ such that $\sum_{t\in T} l(t)f(t) = n$.

Consider the set of commuting indeterminates $X = \{x_t \mid t \in T\}$. Let

$$g = \prod_{t\in T} (1 - x_t^{l(t)})^{-1} = \prod_{t\in T} (\sum_{i=0}^{\infty} x_t^{il(t)}) \in \mathbb{Z}[\![X]\!].$$

It is easy to see that there is a one-to-one correspondence between the set of all $f \in \mathcal{F}_n$ such that $\sum_{t\in T} l(t)f(t) = n$, and the set of all monomials of total degree n in the support of g. Let ψ be the homomorphism from $\mathbb{Z}[\![X]\!]$

to $\mathbb{Z}[\![x]\!]$ such that $\psi(x_t) = x$ for all $t \in T$. Then

$$\psi(g) = \prod_{m=1}^{\infty} (1 - x^m)^{-b_m} = 1 + \sum_{n=1}^{\infty} a_n x^n.$$

∎

The following result is due to M. K. Smith [18].

Lemma 2.3. *Let $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a monotone increasing function. Let $b(n) = f(n) - f(n-1)$ for all positive integers n. Let $(a(n))$ be the sequence that satisfies*

$$\prod_{m=1}^{\infty} (1 - t^m)^{-b(m)} = \sum_{n=0}^{\infty} a(n) t^n \in \mathbb{Z}[\![t]\!].$$

Let $g\colon \mathbb{N} \longrightarrow \mathbb{N}$ be the function defined by $g(n) = \sum_{m=0}^{n} a(m)$.

If $\limsup_{n\to\infty} f(n)^{\frac{1}{n}} \leq 1$ then $\limsup_{n\to\infty} g(n)^{\frac{1}{n}} \leq 1$, and thus g has subexponential growth. ∎

A more precise relation between the asymptotics of the sequences $(b(n))$ and $(a(n))$ can be found in [14].

We can now derive our first consequence of Lemma 2.2 and Lemma 2.3.

Corollary 2.4. *Every abelian group satisfies the subexponential property for sequences.*

Proof. Let G be an abelian group. Let $b(1), b(2), \ldots$ be a sequence of positive integers and let

$$g_{1,1}, g_{1,2}, \ldots, g_{1,b(1)}, g_{2,1}, g_{2,2}, \ldots, g_{2,b(2)}, \ldots \tag{2}$$

be a sequence of elements in G. Define the set

$$T_n = \{g_{i_1,j_1} g_{i_2,j_2} \cdots g_{i_s,j_s} \in G \mid s \geq 1,\ i_1 + \cdots + i_s \leq n\}.$$

Let $g(n) = |T_n|$ and $f(n) = \sum_{i=1}^{n} b(i)$. Suppose that

$$\limsup_{n\to\infty} f(n)^{\frac{1}{n}} \leq 1.$$

By Lemma 2.2, with $\mathcal{C}_n = \{g_{n,1}, g_{n,2}, \ldots, g_{n,b(n)}\}$ and the well-order on $\bigcup_{n=1}^{\infty} \mathcal{C}_n$ determined by the sequence (2), we have

$$\prod_{m=1}^{\infty} (1 - x^m)^{-b(m)} = 1 + \sum_{n=1}^{\infty} a(n) x^n,$$

where $a(n)$ is the number of all finite ordered sequences

$$g_{i_1,j_1} \leq g_{i_2,j_2} \leq \cdots \leq g_{i_s,j_s}$$

of elements in $\bigcup_{n=1}^{\infty} \mathcal{C}_n$ such that $i_1 + \cdots + i_s = n$.

Since G is abelian, $g(n) \leq 1 + \sum_{m=1}^{n} a(m)$. By Lemma 2.3,

$$\limsup_{n\to\infty} g(n)^{\frac{1}{n}} \leq 1.$$

Hence G satisfies the subexponential property for sequences. ∎

Clearly, if a group G satisfies the subexponential (weak) property for sequences, then every subgroup of G also satisfies this property. In this case, if H is a normal subgroup of G, then G/H satisfies the subexponential (weak) property for sequences. The converse can be proved if the index of H in G is finite. In particular, this allows us to extend the assertion of Corollary 2.4 to the class of abelian-by-finite groups.

Lemma 2.5. *Let G be a group. Let H be a normal subgroup of finite index in G. Then G satisfies the subexponential (weak) property for sequences if and only if H satisfies this property.*

Proof. Suppose that H satisfies the subexponential (weak) property for sequences. Let $b(1), b(2), \dots$ be a sequence of positive integers and let

$$g_{1,1}, g_{1,2}, \dots, g_{1,b(1)}, g_{2,1}, g_{2,2}, \dots, g_{2,b(2)}, \dots$$

be a sequence of elements in G. Let

$$T_n = \{g_{i_1,j_1} g_{i_2,j_2} \cdots g_{i_s,j_s} \in G \mid s \geq 1,\ i_1 + \cdots + i_s \leq n\}.$$

Define $g(n) = |T_n|$ and $f(n) = \sum_{i=1}^{n} b(i)$. Suppose that

$$\limsup_{n\to\infty} f(n)^{\frac{1}{n}} \leq 1$$

(respectively, f has polynomial growth).

Let $x_1, x_2, \dots, x_r$ be a complete set of left coset representatives for H in G. We may assume that $x_1 = 1$. Thus we have $G = x_1 H \cup \cdots \cup x_r H$. Given $i, j \in \{1, \dots, r\}$,

$$x_i x_j \in x_{k(i,j)} H,$$

for some $k(i,j) \in \{1, \dots, r\}$. Let $h_{i,j} = x_{k(i,j)}^{-1} x_i x_j \in H$. For all $g_{i,j}$ in the sequence, there exist $g'_{i,j} \in H$ and $x_{f(i,j)} \in \{x_1, \dots, x_r\}$ such that $g_{i,j} = x_{f(i,j)} g'_{i,j}$.

Let $a_{p,q,t,i,j} = h_{p,q} x_t^{-1} g'_{i,j} x_t$. Let

$$\mathcal{C}_n = \{a_{p,q,t,n,j} \mid p, q, t \in \{1, \dots, r\},\ j = 1, \dots, b(n)\}.$$

Then $|\mathcal{C}_n| \leq r^3 b(n)$. Define also

$$T'_n = \{a_{p_1,q_1,t_1,i_1,j_1} \cdots a_{p_s,q_s,t_s,i_s,j_s} \mid s \geq 1,\ i_1 + \cdots + i_s \leq n\}$$

and $g_1(n) = |T'_n|$. Since H satisfies the subexponential (weak) property for sequences, we have $\limsup_{n\to\infty} g_1(n)^{\frac{1}{n}} \leq 1$ (respectively, g_1 has subexponential growth).

We claim that $T_n \subseteq x_1T'_n \cup \cdots \cup x_rT'_n$. Let $t = g_{i_1,j_1}g_{i_2,j_2}\cdots g_{i_s,j_s} \in T_n$, (with $i_1 + \cdots + i_s \leq n$). If $s = 1$ then

$$t = g_{i_1,j_1} = x_{f(i_1,j_1)}g'_{i_1,j_1} = x_{f(i_1,j_1)}a_{1,1,1,i_1,j_1} \in x_{f(i_1,j_1)}T'_n.$$

Suppose that $s > 1$ and

$$g_{i_2,j_2}\cdots g_{i_s,j_s} = x_s a_{p_2,q_2,t_2,i_2,j_2}\cdots a_{p_s,q_s,t_s,i_s,j_s} \in x_sT'_n$$

for some $s \in \{1,\ldots,r\}$. Then

$$\begin{aligned} t &= g_{i_1,j_1}x_s a_{p_2,q_2,t_2,i_2,j_2}\cdots a_{p_s,q_s,t_s,i_s,j_s} \\ &= x_{f(i_1,j_1)}g'_{i_1,j_1}x_s a_{p_2,q_2,t_2,i_2,j_2}\cdots a_{p_s,q_s,t_s,i_s,j_s} \\ &= x_{f(i_1,j_1)}x_s x_s^{-1}g'_{i_1,j_1}x_s a_{p_2,q_2,t_2,i_2,j_2}\cdots a_{p_s,q_s,t_s,i_s,j_s} \\ &= x_{k(f(i_1,j_1),s)}h_{f(i_1,j_1),s}x_s^{-1}g'_{i_1,j_1}x_s a_{p_2,q_2,t_2,i_2,j_2}\cdots a_{p_s,q_s,t_s,i_s,j_s} \\ &= x_{k(f(i_1,j_1),s)}a_{f(i_1,j_1),s,s,i_1,j_1}a_{p_2,q_2,t_2,i_2,j_2}\cdots a_{p_s,q_s,t_s,i_s,j_s}. \end{aligned}$$

Hence, by induction we get $T_n \subseteq x_1T'_n \cup \cdots \cup x_rT'_n$, as claimed.

It follows that $g(n) \leq rg_1(n)$. Therefore,

$$\limsup_{n\to\infty} g(n)^{\frac{1}{n}} \leq 1 \quad \text{(respectively, } g \text{ has subexponential growth.)}$$

This completes the proof of the lemma. ∎

Corollary 2.6. *Every abelian-by-finite group satisfies the subexponential property for sequences.*

In order to provide more examples (that will be also crucial for linear semigroups of degree not exceeding 4, in view of Lemma 3.3), we first need some results on nilpotent groups of class 2.

Lemma 2.7. *Let H be a free nilpotent group of class 2 on generators $x_1, x_2, \ldots$. Suppose that $w = x_{i_1}\cdots x_{i_n}, w' = x_{i'_1}\cdots x_{i'_m} \in H$ for some $n, m \geq 1$ and some positive integers i_j, i'_j. If $i, j \geq 1$ then define $a_{i,j}$ for the word w (and similarly $a'_{i,j}$ for the word w') as the number of all pairs k, l such that $i = i_k, j = i_l$ and $k < l$. Then $w = w'$ if and only if $a_{i,j} = a'_{i,j}$ for all $1 \leq j \leq i$. In particular, in this case $n = m$.*

Proof. This is a consequence of the so called collecting process, and of the uniqueness of the presentation of elements of H in terms of basic products of basic commutators, see [4, Chapter 11].

Assume that $a_{i,j} = a'_{i,j}$ for all $1 \leq j \leq i$ and choose minimal j which is of the form $j = i_k$. We may bring all copies of x_j in front of the word w. Then $w = x_j^{r_j}\overline{w}\prod_{j\leq i}[x_i,x_j]^{a_{i,j}}$, where r_j is the multiplicity of x_j in w, and $\overline{w}$ is obtained from w by erasing all copies of x_j. This is because H is nilpotent of class 2. Similarly we obtain $w' = x_j^{r'_j}\overline{w'}\prod_{j\leq i}[x_i,x_j]^{a'_{i,j}}$. Notice that $r_j(r_j-1)/2 = a_{j,j}$, so $r_j = r'_j$. The equalities $a_{i,j} = a'_{i,j}$ imply that $\prod_{j\leq i}[x_i, x_j]^{a_{i,j}} = \prod_{j\leq i}[x_i, x_j]^{a'_{i,j}}$. So, by induction on $n+m$ applied to $\overline{w}, \overline{w'}$, it follows that $\overline{w} = \overline{w'}$. Thus, we easily get $w = w'$, as desired. Since $r_{i_1} + \cdots + r_{i_s} = n$, where $x_{i_1}, \ldots, x_{i_s}$ are all the different generators involved in w, it follows also that $n = m$. Conversely, if $w = w'$, then $a_{i,j} = a'_{i,j}$ for all i, j is a consequence of [4, Theorem 11.2.4]. ∎

Theorem 2.8. *Let $b(1), b(2), \ldots$ be a sequence of positive integers and let H be the free nilpotent group of class 2 on generators*

$$x_{1,1}, x_{1,2}, \ldots, x_{1,b(1)}, x_{2,1}, x_{2,2}, \ldots, x_{2,b(2)}, \ldots.$$

Let $T_n = \{x_{i_1,j_1} x_{i_2,j_2} \cdots x_{i_s,j_s} \in H \mid s \geq 1,\ i_1 + \cdots + i_s \leq n\}$. Let $g(n) = |T_n|$. If the function b has polynomial growth, then the function g has subexponential growth. So H satisfies the subexponential weak property for sequences.

Proof. By the proof of Lemma 2.7, the elements w of T_n are of the form

$$\begin{aligned} w &= x_{1,1}^{r_{1,1}} x_{1,2}^{r_{1,2}} \cdots x_{n,b(n)}^{r_{n,b(n)}} [x_{1,2}, x_{1,1}]^{a_{1,2,1,1}} \\ &\quad \cdots [x_{n,b(n)}, x_{1,1}]^{a_{n,b(n),1,1}} \cdots [x_{n,b(n)}, x_{n,b(n)-1}]^{a_{n,b(n),n,b(n)-1}}, \end{aligned} \tag{3}$$

for some nonnegative integers $r_{i,j}$ and $a_{i,j,i',j'}$, such that $\sum_{i=1}^n i(\sum_{j=1}^{b(i)} r_{i,j}) \leq n$. Furthermore, $a_{i,j,i',j'} \leq r_{i,j} r_{i',j'} \leq n^2$. We call (3) the basic form of w. By Lemmas 2.2 and 2.3, the function λ measuring the number $\lambda(n)$ of sequences of nonnegative integers $(r_{1,1}, r_{1,2}, \ldots, r_{n,b(n)})$ such that $\sum_{i=1}^n i(\sum_{j=1}^{b(i)} r_{i,j}) \leq n$ has subexponential growth.

Let $w = x_{i_1,j_1} x_{i_2,j_2} \cdots x_{i_s,j_s} \in T_n$. Let $r_{i,j}(w)$ be the degree in $x_{i,j}$ of the word $x_{i_1,j_1} x_{i_2,j_2} \cdots x_{i_s,j_s}$. For all $(i,j) > (i',j')$, with respect to the lexicographical order, let $a_{i,j,i',j'}(w)$ be the number of all pairs k, l such that $i = i_k$, $j = j_k$, $i' = i_l$, $j' = j_l$ and $k < l$ in the word w. Suppose that (3) is the basic form of w. Then, by the proof of Lemma 2.7, $r_{i,j}(w) = r_{i,j}$ and $a_{i,j,i',j'}(w) = a_{i,j,i',j'}$.

Since $b(n)$ has polynomial growth, with no loss of generality we may assume that $b(n) = (2n)^d$ for some positive integer d. Let $r_{1,1}, \ldots, r_{n,b(n)}$ be nonnegative integers such that $\sum_{i=1}^n i(\sum_{j=1}^{b(i)} r_{i,j}) \leq n$. Let

$$T(r_{1,1}, \ldots, r_{n,b(n)}) = \{w \in T_n \mid r_{i,j}(w) = r_{i,j}, \forall j = 1, \ldots, b(i), \forall i = 1, \ldots, n\}.$$

Let $w = x_{i_1,j_1} \cdots x_{i_s,j_s} \in T(r_{1,1}, \ldots, r_{n,b(n)})$. Let w' denote the word obtained from w by erasing all the $x_{i,j}$ with $i \geq n^{\frac{1}{d^2+3d+3}}$. Let w'' denote the word obtained from w by erasing all the $x_{i,j}$ with $i < n^{\frac{1}{d^2+3d+3}}$. Define the sets $T'(r_{1,1}, \ldots, r_{n,b(n)}) = \{w' \mid w \in T(r_{1,1}, \ldots, r_{n,b(n)})\}$

and $T''(r_{1,1},\dots,r_{n,b(n)}) = \{w'' \mid w \in T(r_{1,1},\dots,r_{n,b(n)})\}$. Since $\sum_{i=1}^{n} i(\sum_{j=1}^{b(i)} r_{i,j}) \le n$, it follows that

$$n^{\frac{1}{d^2+3d+3}} \sum_{n^{\frac{1}{d^2+3d+3}} \le i \le n} \sum_{j=1}^{b(i)} r_{i,j} \le \sum_{n^{\frac{1}{d^2+3d+3}} \le i \le n} i \left(\sum_{j=1}^{b(i)} r_{i,j} \right) \le n.$$

Therefore

$$\sum_{n^{\frac{1}{d^2+3d+3}} \le i \le n} \sum_{j=1}^{b(i)} r_{i,j} \le n^{1-\frac{1}{d^2+3d+3}} = n^{\frac{d^2+3d+2}{d^2+3d+3}}.$$

Hence

$$|T''(r_{1,1},\dots,r_{n,b(n)})| \le \left\lceil n^{\frac{d^2+3d+2}{d^2+3d+3}} \right\rceil !. \tag{4}$$

Note that $a_{i,j,i',j'}(w') = a_{i,j,i',j'}(w)$ for all $(i,j) > (i',j')$ such that $1 \le i, i' < n^{\frac{1}{d^2+3d+3}}$. Note also that

$$\sum_{i=1}^{n} b(i) \le \int_0^{n+1} 2^d t^d dt = \frac{2^d (n+1)^{d+1}}{d+1}. \tag{5}$$

The number of all quadruples (i,j,i',j') such that $(i,j) > (i',j')$, $1 \le j \le b(i)$, $1 \le j' \le b(i')$ and $1 \le i, i' < n^{\frac{1}{d^2+3d+3}}$ is less than

$$\frac{\left(\sum_{i=1}^{n^{\frac{1}{d^2+3d+3}}} b(i) \right)^2}{2} \le \frac{4^d \left(n^{\frac{1}{d^2+3d+3}} + 1 \right)^{2(d+1)}}{2(d+1)^2} < \frac{4^{2d+1} n^{\frac{2(d+1)}{d^2+3d+3}}}{2},$$

where the first inequality follows from (5). Since $a_{i,j,i',j'}(w) \le n^2$, looking at the basic form of the elements w', we thus get that

$$|T'(r_{1,1},\dots,r_{n,b(n)})| < n^{4^{2d+1} n^{\frac{2(d+1)}{d^2+3d+3}}}. \tag{6}$$

In order to determine an element $w \in T(r_{1,1},\dots,r_{n,b(n)})$, it is sufficient to know w', w'' and all $a_{i,j,i',j'}(w)$, for $n^{\frac{1}{d^2+3d+3}} \le i \le n$ and $1 \le i' < n^{\frac{1}{d^2+3d+3}}$, such that $r_{i,j} r_{i',j'} \ne 0$.

Let $k(n)$ be the nonnegative integer satisfying

$$\sum_{i=1}^{k(n)} ib(i) \le n < \sum_{i=1}^{k(n)+1} ib(i). \tag{7}$$

Let $\mu = |\{(i,j) \mid r_{i,j} \neq 0\}|$. Note that

$$\mu < \sum_{i=1}^{k(n)+1} b(i). \tag{8}$$

Since

$$\frac{2^d n^{d+2}}{d+2} = \int_0^n 2^d t^{d+1} dt \leq \sum_{i=1}^{n} i b(i),$$

from (7) it follows that $\frac{2^d k(n)^{d+2}}{d+2} \leq n$. Thus

$$k(n) \leq \frac{((d+2)n)^{\frac{1}{d+2}}}{2^{\frac{d}{d+2}}} \leq ((d+2)n)^{\frac{1}{d+2}}. \tag{9}$$

By (5), (8) and (9), we have

$$\begin{aligned} \mu &< \frac{2^d(k(n)+2)^{d+1}}{d+1} \leq \frac{2^d(((d+2)n)^{\frac{1}{d+2}}+2)^{d+1}}{d+1} \\ &< 2^d 3^{d+1}((d+2)n)^{\frac{d+1}{d+2}}. \end{aligned} \tag{10}$$

On the other hand, the number of different pairs (i',j'), with $1 \leq i' < n^{\frac{1}{d^2+3d+3}}$ and $1 \leq j' \leq b(i')$, is less than

$$\sum_{i=1}^{n^{\frac{1}{d^2+3d+3}}} b(i) \leq \frac{2^d\left(n^{\frac{1}{d^2+3d+3}}+1\right)^{d+1}}{d+1} < 2^{2d+1} n^{\frac{d+1}{d^2+3d+3}}, \tag{11}$$

where the first inequality follows from (5). Let

$$\mu' = |\{(i,j) \mid n^{\frac{1}{d^2+3d+3}} \leq i \leq n \text{ and } r_{i,j} \neq 0\}|.$$

Let

$$\mu'' = |\{(i,j,i',j') \mid n^{\frac{1}{d^2+3d+3}} \leq i \leq n,\ 1 \leq i' < n^{\frac{1}{d^2+3d+3}} \text{ and } r_{i,j}r_{i',j'} \neq 0\}|.$$

By (11),

$$\mu'' \leq 2^{2d+1} n^{\frac{d+1}{d^2+3d+3}} \mu'.$$

Since $a_{i,j,i',j'}(w) \leq n^2$, the number of all lexicographically ordered sequences $(a_{i,j,i',j'}(w))$, with $n^{\frac{1}{d^2+3d+3}} \leq i \leq n$ and $1 \leq i' < n^{\frac{1}{d^2+3d+3}}$, such that $r_{i,j}r_{i',j'} \neq 0$, obtained from the elements $w \in T(r_{1,1},\dots,r_{n,b(n)})$, is less than or equal to

$$n^{2\mu''} \leq n^{2^{2d+2} n^{\frac{d+1}{d^2+3d+3}} \mu'} \leq n^{2^{2d+2} n^{\frac{d+1}{d^2+3d+3}} \mu}.$$

In view of (10),

$$2^{2d+2} n^{\frac{d+1}{d^2+3d+3}} \mu < 2^{2d+2} n^{\frac{d+1}{d^2+3d+3}} 2^d 3^{d+1}((d+2)n)^{\frac{d+1}{d+2}},$$

and

$$\frac{d+1}{d^2+3d+3}+\frac{d+1}{d+2}=\frac{d^3+5d^2+9d+5}{d^3+5d^2+9d+6}.$$

Therefore

$$\begin{aligned}
&|T(r_1,\dots,r_n)|\le\\
&\le\quad |T'(r_1,\dots,r_n)|\cdot|T''(r_1,\dots,r_n)|\cdot n^{2^{2d+2}n^{\frac{d+1}{d^2+3d+3}}\mu}\\
&<\quad n^{4^{2d+1}n^{\frac{2(d+1)}{d^2+3d+3}}}\cdot\left[n^{\frac{d^2+3d+2}{d^2+3d+3}}\right]!\cdot n^{2^{3d+2}3^{d+1}(d+2)^{\frac{d+1}{d+2}}n^{\frac{d^3+5d^2+9d+5}{d^3+5d^2+9d+6}}},\\
&\qquad\text{(by (4) and (6))}.
\end{aligned}$$

Thus we get

$$g(n)=$$

$$|T_n|\le\lambda(n)\cdot n^{4^{2d+1}n^{\frac{2(d+1)}{d^2+3d+3}}}\cdot\left[n^{\frac{d^2+3d+2}{d^2+3d+3}}\right]!\cdot n^{2^{3d+2}3^{d+1}(d+2)^{\frac{d+1}{d+2}}n^{\frac{d^3+5d^2+9d+5}{d^3+5d^2+9d+6}}}.$$

Hence g has subexponential growth. ∎

The following is now an immediate consequence of Lemma 2.5.

Corollary 2.9. *Every group that is a finite extension of a nilpotent group of class 2 satisfies the subexponential weak property for sequences.*

3. Linear semigroups of degree 2, 3 and 4

In order to apply the results of the preceding section, we need to recall some basic facts about the structure of the full linear (multiplicative) monoid $M_n(K)$ over a field K. For this, we follow [12]. Let H be a group, X, Y nonempty sets and let $P=(p_{yx})$ be a $Y\times X$ matrix over $H\cup\{0\}$ (called a sandwich matrix). Then $\mathcal{M}(H,X,Y;P)$ denotes the corresponding semigroup of matrix type. So, this is the set consisting of the zero element θ and of all triples (h,x,y) with $h\in H, x\in X, y\in Y$, subject to the operation $(h,x,y)(h',x',y')=(hp_{yx'}h',x,y')$ if $p_{yx'}\in H$ and $(h,x,y)(h',x',y')=\theta$ otherwise. For every nonnegative integer $j\le n$ define $M_j=\{a\in M_n(K)\mid \operatorname{rank}(a)\le j\}$. It is well-known that $M_j, j=0,1,\dots,n$, are the only ideals of the monoid $M_n(K)$. Moreover, every Rees factor $M_j/M_{j-1}, j=1,\dots,n$, is a completely 0-simple semigroup. In other words, it is isomorphic to a semigroup of matrix type whose sandwich matrix has no zero rows or columns. The maximal subgroups of the monoid $M_n(K)$ are of the form $G_e=eM_n(K)e\setminus M_{r-1}$, where $e=e^2\in M_n(K), e\neq 0$, and $r=\operatorname{rank}(e)$. Hence $G_e\cong GL_r(K)$.

Let S be a subsemigroup of $M_n(K)$. Put $S_j = M_j \cap S$ and

$$T_j = \{a \in S_j \mid \text{ the ideal of } S \text{ generated by } a \text{ does not intersect maximal subgroups of } M_n(K) \text{ contained in } M_j \setminus M_{j-1}\}.$$

By [12, Theorem 3.5],

$$S_0 \subseteq T_1 \subseteq S_1 \subseteq T_2 \subseteq S_2 \subseteq \cdots \subseteq S_{n-1} = T_n \subseteq S_n = S$$

are ideals of S (if nonempty). Moreover,

(1) every T_j/S_{j-1} is a nilpotent ideal of S/S_{j-1},
(2) every S_j/T_j is a 0-disjoint union of finitely many subsemigroups $U_{j1}, \ldots, U_{jn_j}$ of completely 0-simple semigroups $J_{ji} = \mathcal{M}(G_{ji}, X_{ji}, Y_{ji}; P_{ji}) \subseteq M_j/M_{j-1}$, $i = 1, \ldots, n_j$, over subgroups G_{ji} of $M_n(K)$ contained in $M_j \setminus M_{j-1}$; furthermore every U_{ji} is an ideal of S_j/T_j.

Here we use the convention that $T/\emptyset = T$ and only nonempty factors are considered in conditions (1) and (2). Moreover, every U_{ji} is of a rather special type (referred to as a uniform subsemigroup of J_{ji}). In particular, for every maximal subgroup H of J_{ji}, the subgroup generated by $H \cap U_{ji}$ is equal to H. Clearly, $H \cong G_{ji}$ embeds into $GL_j(K)$. Actually, $S_1 \subseteq M_1$ and the latter is a completely 0-simple semigroup over the group $K^* = GL_1(K)$.

Furthermore, if S is finitely generated, then S does not have free noncommutative subsemigroups if and only if every nontrivial intersection $G \cap S$ with a maximal subgroup G of the monoid $M_n(K)$ generates a nilpotent-by-finite subgroup of G. So, in terms of the ideal chain described above, this means that all groups G_{ij} are nilpotent-by-finite. The latter is also equivalent to the fact that S satisfies a nontrivial identity, [12, Theorem 6.11].

Because of the ideal chain discussed above, it is clear that in order to control the growth of $S \subseteq M_n(K)$ one has to consider ideal extensions of the appropriate types. The first type creates no problem.

Lemma 3.1. *Assume that I is a nilpotent ideal of a finitely generated semigroup S with zero. Then S and S/I have growth of the same type.*

Proof. Suppose first that $I^2 = 0$. Assume that $S = \langle a_1, \ldots, a_m \rangle$. Define the set

$$A = \{a_{i_1} \cdots a_{i_n} \in I \mid a_{i_2} \cdots a_{i_n} \notin I, a_{i_1} \cdots a_{i_{n-1}} \notin I, n \geq 1\}.$$

Let n be a positive integer and let $a = a_{i_1} \cdots a_{i_q} \in I$ for some $q \leq n$. Since $I^2 = 0$, we may write

$$a = (a_{i_1} \cdots a_{i_{k-1}})(a_{i_k} \cdots a_{i_m})(a_{i_{m+1}} \cdots a_{i_q})$$

for some $k \leq m$ such that $a_{i_1} \cdots a_{i_{k-1}} \notin I$ and $a_{i_{m+1}} \cdots a_{i_q} \notin I$ but $a_{i_k} \cdots a_{i_m} \in A$. Moreover $a_{i_k} \cdots a_{i_m} = a_{i_k}(a_{i_{k+1}} \cdots a_{i_m})$ and $a_{i_{k+1}} \cdots a_{i_m} \notin$

I. Therefore the growth functions $d_S(n), d_{S/I}(n)$ of S and S/I corresponding to the given set of generators of S (and their images in S/I) satisfy

$$d_{S/I}(n) \leq d_S(n) \leq d_{S/I}(n) + d_{S/I}(n-1)^3 m \leq d_{S/I}(n)^3(m+1).$$

Hence the types of growth of S and S/I are the same. The assertion now follows by an easy induction on the nilpotency index of the ideal I. ∎

In view of the above observation, the problem of characterizing linear semigroups of subexponential growth leads naturally to the case where S has an ideal I such that S/I has subexponential growth and $I \subseteq J = \mathcal{M}(G, X, Y; P)$ for a completely 0-simple semigroup J.

Let S be a semigroup generated by a set $A = \{g_1, \ldots, g_m\}$. We say that an element $s \in S$ has length n in the generators $g_1, \ldots, g_m$ if s can be expressed as a word of length n in these generators and not as a word of smaller length. Let $l_A(s)$ denote the length of $s \in S$ in the generators $g_1, \ldots, g_m$. Let $S(A, n) = \{s \in S \mid l_A(s) \leq n\}$ and $d_{S,A}(n) = |S(A, n)|$. Milnor proved that $\lim_{n\to\infty} d_{S,A}(n)^{\frac{1}{n}}$ always exists (see [9]). Furthermore S has exponential growth if and only if $\lim_{n\to\infty} d_{S,A}(n)^{\frac{1}{n}} > 1$.

Our first main result reads as follows.

Theorem 3.2. *Let S be a finitely generated semigroup. Suppose that S has an ideal I which is a 0-disjoint union of finitely many ideals $I_1, \ldots, I_m$ of S such that I_i is a subsemigroup of a semigroup of matrix type $\mathcal{M}(G_i, X_i, Y_i; P_i)$ over a group G_i, $i = 1, \ldots, m$. If S/I has subexponential (respectively, polynomial) growth and the groups G_i satisfy the subexponential (respectively, the subexponential weak) property for sequences, then S has subexponential growth.*

Proof. Suppose that S/I has subexponential (respectively, polynomial) growth and the groups G_i satisfy the subexponential (respectively, the subexponential weak) property for sequences.

Let $S = \langle g_1, \ldots, g_r \rangle$. Let $l_1(s)$ denote the length of $s \in S$ in the generators $g_1, \ldots, g_r$. Let $A = \{g_1, \ldots, g_r\}$. For any positive integer n, let

$$\begin{aligned} \mathcal{C}_n \quad = \quad & \{s \in I \mid l_1(s) = n \text{ and there exist } g_{i_1}, \ldots, g_{i_n} \in A \\ & \text{such that } s = g_{i_1} \cdots g_{i_n} \text{ and } g_{i_1} \cdots g_{i_{n-1}} \notin I\}. \end{aligned}$$

We know that $I = \bigcup_{i=1}^m I_i$ is a 0-disjoint union of some subsemigroups I_i of $\mathcal{M}(G_i, X_i, Y_i; P_i)$. Thus, the nonzero elements of I are of the form (h, x, y), with $h \in G_i$, $x \in X_i$ and $y \in Y_i$ for some $i = 1, \ldots, m$, and $(h, x, y) \cdot (h', x', y') = 0$ if $(h, x, y) \in I_i$, $(h', x', y') \in I_j$ and $i \neq j$. Let

$$X_i^{(n)} = \{x \in X_i \mid \exists (h, x, y) \in \bigcup_{j=1}^n \mathcal{C}_j\},$$

$$Y_i^{(n)} = \{y \in Y_i \mid \exists (h,x,y) \in \bigcup_{j=1}^{n} \mathcal{C}_j\}.$$

Let $P_i^{(n)}$ be the corresponding $Y_i^{(n)} \times X_i^{(n)}$ submatrix of P_i. Let

$$P'_{i,n} = \{h \in G_i \mid h \text{ is an entry of } P_i^{(n)}\}.$$

Let $B = A \setminus I$. Then $B \cup \{0\}$ is a set of generators of S/I. Let $l_2(s)$ denote the length of $s \in S/I$ in the generators in $B \cup \{0\}$. Clearly, $l_2(s) = l_1(s)$ for all $s \in S \setminus I$. Let $f(n)$ be the number of all elements $s \in S/I$ such that $l_2(s) \leq n$. Note that for every integer $n > 1$ we have $|\bigcup_{j=1}^{n} \mathcal{C}_j| \leq rf(n-1)$,

$$|P'_{i,n}| \leq |X_i^{(n)}| \cdot |Y_i^{(n)}| \tag{12}$$

and

$$|X_i^{(n)}|, |Y_i^{(n)}| \leq |\bigcup_{j=1}^{n} \mathcal{C}_j| = \sum_{j=1}^{n} |\mathcal{C}_j| \leq rf(n-1), \tag{13}$$

because the sets $\mathcal{C}_j$ are pairwise disjoint.

For $t = (h,x,y) \in I$, let $\bar{t} = h$. Let $\mathcal{C}_{i,n} = \mathcal{C}_n \cap I_i$. Let $\mathcal{C}'_{i,n} = \{\bar{s}p \mid s \in \mathcal{C}_{i,n},\ p \in P'_{i,n} \cup \{1\}\}$. Let $b_i(n)$ be the number of all elements in $\mathcal{C}'_{i,n}$. For every integer $n > 1$, we have

$$\sum_{j=1}^{n} b_i(j) \leq \left(1 + |P'_{i,n}|\right) \cdot \sum_{j=1}^{n} |\mathcal{C}_{i,j}| \leq rf(n-1) + r^3 f(n-1)^3,$$

where the second inequality follows from (12) and (13). Since S/I has subexponential (respectively, polynomial) growth, $\lim_{n\to\infty} f(n)^{\frac{1}{n}} \leq 1$ (respectively, f has polynomial growth). Let

$$T_{i,n} = \{\overline{s_1}p_1\overline{s_2}p_2 \cdots \overline{s_k}p_k \mid s_z \in \bigcup_{j=1}^{n} \mathcal{C}_{i,j},\ p_z \in P'_{i,n} \cup \{1\} \text{ and } \sum_{z=1}^{k} l_1(s_z) \leq n\},$$

$$T_n = \bigcup_{i=1}^{m} T_{i,n}.$$

Let $g_i(n) = |T_{i,n}|$. Since the groups G_i satisfy the subexponential (respectively, the subexponential weak) property for sequences, all functions $g_i, i = 1, \ldots, m$, have subexponential growth. Let $g(n) = |T_n|$. Since $g(n) \leq \sum_{i=1}^{m} g_i(n)$, the function g also has subexponential growth.

Let $S(A,n) = \{s \in S \mid l_1(s) \leq n\}$. Let $s \in S(A,n) \cap I$ be such that $l_1(s) = n' \leq n$. Then there exist $a_1, \ldots, a_{n'} \in A$ such that $s = a_1 \cdots a_{n'}$. Since $s \in I_i$ for some i, there exist positive integers

$$1 \leq j_1 < j_2 < \cdots < j_q \leq n'$$

such that $a_1 \cdots a_{j_1} \in \mathcal{C}_{i,j_1}$ and $a_{j_k+1} \cdots a_{j_{k+1}} \in \mathcal{C}_{i,j_{k+1}-j_k}$ for all $k = 1, \ldots, q-1$, and $a_{j_q+1} \cdots a_{n'} \notin I$. Let $s_1 = a_1 \cdots a_{j_1}$ and $s_{k+1} = a_{j_k+1} \cdots a_{j_{k+1}}$ for all $k = 1, \ldots, q-1$. Thus there exist $p_1, \ldots, p_{q-1} \in P'_{i,n}$, $x \in X_i^{(n)}$ and $y \in Y_i^{(n)}$ such that

$$s = s_1 \cdots s_q a_{j_{q+1}} \cdots a_{n'} = (\overline{s_1} p_1 \overline{s_2} p_2 \cdots \overline{s_{q-1}} p_{q-1} \overline{s_q}, x, y) a_{j_q+1} \cdots a_{n'}. \tag{14}$$

Hence, the number of elements $\overline{s_1} p_1 \overline{s_2} p_2 \cdots \overline{s_{q-1}} p_{q-1} \overline{s_q}$ that can be obtained in (14) is less than or equal to $g_i(n)$. Since $a_{j_q+1} \cdots a_{n'} \in S \setminus I$, the number of such elements that can be obtained in (14) is less than or equal to $f(n)$. Thus

$$|S(A,n) \cap I| \leq \sum_{i=1}^{m} g_i(n) |X_i^{(n)}| \cdot |Y_i^{(n)}| f(n).$$

Since $S(A,n) = (S(A,n) \setminus I) \cup (S(A,n) \cap I)$ and $f(n) \geq |S(A,n) \setminus I|$, we have

$$|S(A,n)| \leq f(n) + \sum_{i=1}^{m} g_i(n) |X_i^{(n)}| \cdot |Y_i^{(n)}| f(n) \leq f(n) + g(n) r^2 f(n-1)^2 f(n),$$

where the second inequality follows from (13). Therefore S has subexponential growth. ∎

In order to apply this to linear semigroups of small degrees, we need the following observation, which seems to be well-known.

Lemma 3.3. *Let K be a field. Let G be a nilpotent subgroup of the multiplicative monoid $M_n(K)$, with $n > 1$. Then there exists a nilpotent subgroup N of class $< n$ of G such that $[G : N] < \infty$.*

Proof. We may assume that K is algebraically closed. Let $e \in G$ be the unity of G. Then there exists $g \in GL_n(K)$ such that

$$g^{-1} e g = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right),$$

where $I_r \in M_r(K)$ is the identity matrix and r is the rank of e. Thus $G \cong g^{-1} G g$ is isomorphic to a subgroup of $GL_r(K)$. Therefore we may assume that G is a nilpotent subgroup of $GL_n(K)$. By [20, Theorem 5.11 and Lemma 5.2], we may also assume that G is a closed connected nilpotent subgroup of $GL_n(K)$. By [20, Theorem 14.22], $G = G_u \times G_d$, where $G_u = \{a \in G \mid a \text{ is unipotent}\}$ is a closed connected subgroup of G and G_d is a closed connected diagonalizable subgroup of G. In particular, G_d is abelian and G_u is conjugate to a unipotent triangular subgroup of $GL_n(K)$ and hence G is nilpotent of class $n-1$ at most. ∎

We can now apply Theorem 3.2 to linear semigroups of degrees not exceeding 4.

Theorem 3.4. *Let K be a field. Let n be an integer such that $1 \leq n \leq 4$. Assume that $S \subseteq M_n(K)$ is a finitely generated semigroup. Then S has subexponential growth if and only if S has no free noncommutative subsemigroups, or equivalently, S satisfies a nontrivial identity.*

Proof. Clearly, we may assume that $n = 4$. Suppose that S has no free noncommutative subsemigroups. By the comments at the beginning of Section 3, S has a finite ideal chain $S_1 \subseteq T_2 \subseteq S_2 \subseteq T_3 \subseteq S_3 = T_4 \subseteq S_4 = S$ such that each of S_2/T_2 and S_3/T_3 (if nonempty) is a 0-disjoint union of finitely many ideals that are subsemigroups of completely 0-simple semigroups of the form $\mathcal{M}(G_i, X_i, Y_i; P_i)$. Moreover, we may assume that the groups G_i are nilpotent-by-finite and the groups arising from S_j/T_j embed into $GL_j(K)$, for $j = 2, 3$. By Lemma 3.3, these groups are abelian-by-finite for $j = 2$ and they have nilpotent of class at most 2 subgroups of finite index if $j = 3$. Also S_1 embeds into a semigroup of the form $\mathcal{M}(K^*, X, Y; P) \cong M_1/M_0$. Moreover $S \setminus S_3$ (if nonempty) generates a nilpotent-by-finite subgroup G of $GL_4(K)$. Furthermore, the factors T_2/S_1 and T_3/S_2 are nilpotent (if nonempty).

Since S is finitely generated and G is of polynomial growth, it follows that S/S_3 has polynomial growth. The groups arising from S_3/T_2 satisfy the subexponential weak property for sequences by Theorem 2.8. Therefore, Theorem 3.2 implies that S/T_3 has subexponential growth. From Lemma 3.1 it then follows that S/S_2 has subexponential growth. Notice that, in view of Corollary 2.6, the groups G_i arising from S_2/T_2 and the group K^* satisfy the subexponential property for sequences. Hence, using Theorem 3.2, followed by Lemma 3.1, and again by Theorem 3.2, we get that S has subexponential growth.

Since the converse implication is clear, the result follows. ∎

Our next result is another simple consequence of Theorem 3.2 and of the ideal structure of linear semigroups.

Corollary 3.5. *Assume that $S \subseteq M_n(K)$ is a finitely generated linear semigroup such that, for every maximal subgroup H of the monoid $M_n(K)$, the intersection $S \cap H$ generates an abelian-by-finite group, if nonempty. Then S has subexponential growth.*

Proof. As in the proof of Theorem 3.4, the assertion follows from Theorem 3.2, Lemma 3.1, and Corollary 2.6. ∎

4. Growth alternative conjecture

The results of the preceding section motivate the following conjecture.

Conjecture 4.1. *Let K be a field. Let $S \subseteq M_n(K)$ be a finitely generated linear semigroup. Then S has subexponential growth if and only if S has no free noncommutative subsemigroups.*

Recall that the latter is equivalent to the fact that S satisfies a nontrivial identity. As mentioned before, it is clear that if S has a free noncommutative subsemigroup then S has exponential growth.

In view of our approach via natural ideal chains in S, one might even expect that a result more general than the above conjecture is true: if a finitely generated semigroup S has a finite ideal chain such that every factor of the chain is either nilpotent or it embeds into a completely 0-simple semigroup over a nilpotent-by-finite group, then S has subexponential growth. Then Lemma 3.1 can be used to reduce the problem to the case where I is an ideal of S such that $I \subseteq \mathcal{M}(G, X, Y; P)$ for a nilpotent-by-finite group G and S/I has subexponential growth. Thus, the next conjecture is stronger than Conjecture 4.1.

Conjecture 4.2. *Let S be a finitely generated semigroup. Suppose that S has an ideal I such that S/I has subexponential growth. If I is a subsemigroup of a semigroup of matrix type $\mathcal{M}(G, X, Y; P)$ over a nilpotent-by-finite group G, then S has subexponential growth.*

In view of the proof of Theorem 3.4, it is clear that, if one can generalize Corollary 2.6 to nilpotent-by-finite groups, then Conjecture 4.1 would be true. Thus the following conjecture seems natural.

Conjecture 4.3. *Nilpotent groups satisfy the subexponential property for sequences.*

In Section 5 we will see that Conjectures 4.3 and 4.2 actually are equivalent.

We do not know whether Conjecture 4.3 is true for nilpotent groups of class 2. This case will be now studied in more detail.

Let G be a free nilpotent group of class 2 on generators $x_1, x_2, \ldots$ Let $w = x_{i_1} \cdots x_{i_n}$, $w' = x_{i'_1} \cdots x_{i'_m} \in G$. As a consequence of Lemma 2.7, if $i_1, \ldots, i_n$ are n different positive integers, then $w = w'$ if and only if $n = m$ and $i_j = i'_j$ for all $j = 1, \ldots, n$. Thus, in view of Conjecture 4.3, one can ask the following question.

Let $b(1), b(2), \ldots$ be a sequence of positive integers and let H be the free group on free generators

$$x_{1,1}, x_{1,2}, \ldots, x_{1,b(1)}, x_{2,1}, x_{2,2}, \ldots, x_{2,b(2)}, \ldots$$

Let

$$\begin{aligned} T'_n &= \{x_{i_1,j_1}x_{i_2,j_2}\cdots x_{i_s,j_s} \in H \mid i_1+\cdots+i_s \leq n \\ &\quad \text{and } x_{i_1,j_1}, x_{i_2,j_2}, \ldots, x_{i_s,j_s} \text{ are different } \}. \end{aligned}$$

Let $g(n) = |T'_n|$ and $f(n) = \sum_{i=1}^{n} b(i)$. Is it true that, if

$$\limsup_{n\to\infty} f(n)^{\frac{1}{n}} \leq 1,$$

then $g(n)$ has subexponential growth?

The answer to the above question is positive if $b(1) \leq b(2) \leq \ldots$ are positive integers such that $b(m+n) \leq b(m)b(n)$ for all m, n. In order to see this, we need some preparatory results.

Lemma 4.4. *Let $f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a strictly increasing differentiable function such that $f'(x)$ is continuous for all $x > 0$. Suppose that there exists $\delta > 0$ such that $f'(x) \geq \delta$ for all $x > 0$ and*

$$\lim_{x\to+\infty} \frac{f(x)}{c^x} = 0 \quad \forall c > 1.$$

Let $k\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the function defined by

$$\int_0^{k(x)} tf'(t)dt = x.$$

(Note that $k(x)$ is well-defined because $tf'(t) > 0$ for $t > 0$ and $\int_0^{+\infty} tf'(t)dt = +\infty$). Then:

(i) *for all $c > 1$,*

$$\lim_{x\to+\infty} \frac{e^{f(k(x))}}{c^x} = 0,$$

(ii) $\lim_{x\to+\infty} \frac{f(k(x))}{x} = 0$,

(iii) $\lim_{x\to+\infty} \frac{\ln x}{k(x)} = 0$.

Proof. (i) Since $\int_0^{k(x)} tf'(t)dt = x$, by differentiation we have,

$$k'(x)k(x)f'(k(x)) = 1. \tag{15}$$

Let $c > 1$. Note that $k(x)$ is a strictly increasing function and $\lim_{x\to+\infty} k(x) = +\infty$. Thus there exists x_0 such that $\frac{1}{\ln c} < k(x)$ and

$$\begin{aligned} x \ln c &= \ln c \int_0^{\frac{1}{\ln c}} tf'(t)dt + \int_{\frac{1}{\ln c}}^{k(x)} \ln c \cdot tf'(t)dt \\ &\geq \int_{\frac{1}{\ln c}}^{k(x)} f'(t)dt = f(k(x)) - f\left(\frac{1}{\ln c}\right), \end{aligned} \tag{16}$$

for all $x > x_0$. Therefore

$$\begin{aligned}
\lim_{x\to+\infty} \frac{e^{f(k(x))}}{c^x} &= \lim_{x\to+\infty} \frac{e^{f(k(x))}}{e^{x\ln c}} \\
&= \lim_{x\to+\infty} \frac{k'(x)f'(k(x))e^{f(k(x))}}{\ln c \cdot e^{x\ln c}} \quad \text{(by l'Hôpital)} \\
&= \lim_{x\to+\infty} \frac{e^{f(k(x))-x\ln c}}{k(x)\ln c} \quad \text{(by (15))} \\
&\leq \lim_{x\to+\infty} \frac{e^{f(\frac{1}{\ln c})}}{k(x)\ln c} \quad \text{(by (16))},
\end{aligned}$$

and thus

$$\lim_{x\to+\infty} \frac{e^{f(k(x))}}{c^x} = 0.$$

(ii) Let $\varepsilon > 0$. For $c = e^{\varepsilon}$, we have by (i) that

$$\lim_{x\to+\infty} \frac{e^{f(k(x))}}{e^{x\varepsilon}} = \lim_{x\to+\infty} e^{f(k(x))-x\varepsilon} = 0.$$

Hence $\lim_{x\to+\infty}(f(k(x)) - x\varepsilon) = -\infty$. Thus there exists x_0 such that

$$f(k(x)) - x\varepsilon < 0,$$

for all $x > x_0$. Therefore $\lim_{x\to+\infty} \frac{f(k(x))}{x} = 0$.
(iii) Let $F(x) = \int_0^x tf'(t)dt$. Clearly $F(x)$ is a strictly increasing function and for all $c > 1$,

$$\lim_{x\to+\infty} \frac{F(x)}{c^x} \leq \lim_{x\to+\infty} \frac{x(f(x) - f(0))}{c^x} = 0,$$

because $\lim_{x\to+\infty} \frac{f(x)}{c^x} = 0$ for all $c > 1$. Since $k(x) = F^{-1}(x)$, we have

$$\lim_{x\to+\infty} \frac{x}{c^{k(x)}} = 0,$$

for all $c > 1$. Since $\frac{x}{c^{k(x)}} = e^{\ln x - k(x)\ln c}$,

$$\lim_{x\to+\infty} (\ln x - k(x)\ln c) = -\infty.$$

Thus, as in (ii), it is easy to see that

$$\lim_{x\to+\infty} \frac{\ln x}{k(x)} = 0.$$

∎

Proposition 4.5. *Let $f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a strictly increasing differentiable function such that $f'(x)$ is continuous for all $x > 0$. Suppose that there exists $\delta > 0$ such that $f'(x) \geq \delta$ for all $x > 0$ and*

$$\lim_{x\to+\infty} \frac{f(x)}{c^x} = 0 \quad \forall c > 1.$$

Let $k\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the function defined by

$$\int_0^{k(x)} tf'(t)dt = x.$$

Then for all $c > 1$,

$$\lim_{x\to+\infty} \frac{x^{f(k(x))}}{c^x} = 0.$$

Proof. Let $M > 0$ and $c > 1$. By l'Hôpital, we have

$$\begin{aligned}
\lim_{x\to+\infty} \frac{f(k(x))\ln x}{x\ln c - M} &= \lim_{x\to+\infty} \frac{f(k(x))}{x\ln c} + \lim_{x\to+\infty} \frac{k'(x)f'(k(x))\ln x}{\ln c} \\
&= \lim_{x\to+\infty} \frac{f(k(x))}{x\ln c} + \lim_{x\to+\infty} \frac{\ln x}{k(x)\ln c} \quad \text{(by (15))} \\
&= 0 \quad \text{(by Lemma 4.4)}.
\end{aligned}$$

Therefore, there exists x_0 such that

$$\frac{f(k(x))\ln x}{x\ln c - M} < 1 \quad \text{and} \quad x\ln c > M,$$

for all $x > x_0$. Hence

$$f(k(x))\ln x < x\ln c - M,$$

and thus

$$f(k(x))\ln x - x\ln c < -M,$$

for all $x > x_0$. It follows that $\lim_{x\to+\infty}(f(k(x))\ln x - x\ln c) = -\infty$, and therefore

$$\lim_{x\to+\infty} \frac{x^{f(k(x))}}{c^x} = \lim_{x\to+\infty} e^{f(k(x))\ln x - x\ln c} = 0.$$

∎

Now we can settle a special case of the question raised before Lemma 4.4.

Theorem 4.6. *Let H be the free group on free generators*

$$x_{1,1}, x_{1,2}, \dots, x_{1,b(1)}, x_{2,1}, x_{2,2}, \dots, x_{2,b(2)}, \dots,$$

where $b(1) \leq b(2) \leq \dots$ are positive integers such that $b(m+n) \leq b(m)b(n)$ for all m, n. Let $T_n = \{x_{i_1,j_1}x_{i_2,j_2}\cdots x_{i_s,j_s} \in H \mid s \geq 1,\ i_1 + \cdots + i_s \leq n$ and $x_{i_1,j_1}, x_{i_2,j_2}, \dots, x_{i_s,j_s}$ are different$\}$. Let $g(n) = |T_n|$ and $f(n) =$

$\sum_{i=1}^{n} b(i)$. *If* $\limsup_{n\to\infty} f(n)^{\frac{1}{n}} \le 1$ *then* $\limsup_{n\to\infty} g(n)^{\frac{1}{n}} \le 1$, *and thus* g *has subexponential growth.*

Proof. Let $\overline{H} = H/H'$, where H' is the commutator subgroup of H, and let $\overline{T}_n$ be the image of T_n under the natural map $H \longrightarrow \overline{H}$. Let $\mu(n)$ be the maximum length of the elements of T_n. Let $\lambda(n) = |\overline{T}_n|$. Then clearly $g(n) \le \lambda(n) \cdot \mu(n)!$. By Corollary 2.4, $\limsup_{n\to\infty} \lambda(n)^{\frac{1}{n}} \le 1$.

Let $k(n)$ be the nonnegative integer such that

$$\sum_{i=1}^{k(n)} ib(i) \le n < \sum_{i=1}^{k(n)+1} ib(i). \tag{17}$$

Note that

$$\mu(n) \le \sum_{i=1}^{k(n)+1} b(i) = f(k(n)+1). \tag{18}$$

We extend the function b to a function, that we also denote by b, from $\mathbb{R}^+$ to $\mathbb{R}^+$, by defining $b(0) = b(1)$ and $b(x) = b(i)(1-x+i)+b(i+1)(x-i)$ for every nonnegative integer i and for all $i < x \le i+1$. Note that $b\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous monotone increasing and $b(x) \ge 1$ for all x. Let $f_1(x) = \int_0^x b(t)dt$. Clearly

$$f_1(n) = \sum_{i=1}^{n} \frac{b(i-1)+b(i)}{2} = f(n) + \frac{b(0)-b(n)}{2} < f(n) + \frac{b(0)}{2} < f_1(n+1).$$

Thus, by the hypothesis on f, $\limsup_{n\to\infty} f_1(n)^{\frac{1}{n}} \le 1$ (for $n \in \mathbb{N}$). Note that for all $x > 1$ we have $f_1(x) > f_1(1) = b(1) \ge 1$, and

$$\begin{aligned} f_1(x)^{\frac{1}{x}} &\le f_1([x]+1)^{\frac{1}{[x]}} \le (f_1([x]) + b([x]+1))^{\frac{1}{[x]}} \\ &\quad \text{(since } 1 < f_1([x]+1) \le f_1([x]) + b([x]+1)) \\ &\le (f_1([x]) + b(1)b([x]))^{\frac{1}{[x]}} \le (f_1([x])(1+2b(1)))^{\frac{1}{[x]}} \\ &\quad \text{(by the assumption on } b \text{ and since } b([x]) \le 2f_1([x])). \end{aligned}$$

Hence $\limsup_{x\to+\infty} f_1(x)^{\frac{1}{x}} \le 1$, and thus

$$\lim_{x\to+\infty} \frac{f_1(x)}{c^x} = 0 \quad \forall c > 1. \tag{19}$$

Let $k_1 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the function defined by

$$\int_0^{k_1(x)} tb(t)dt = x.$$

Then, for every positive integer n,

$$n = \int_0^{k_1(n)} tb(t)dt < \int_0^{[k_1(n)]+1} tb(t)dt \leq \sum_{i=1}^{[k_1(n)]+1} ib(i).$$

By (17), we have that $k(n) \leq [k_1(n)] \leq k_1(n)$ for every positive integer n. In view of (18), this implies that

$$\mu(n) \leq f(k(n)+1) \leq f_1(k(n)+2) \leq f_1(k_1(n)+2). \tag{20}$$

On the other hand, since $b(m+n) \leq b(m)b(n)$ for all positive integers m, n, and $b(0) = b(1) \geq 1$, it follows that $b(x+2) \leq b(x)b(2)$ for all $x \geq 0$, and thus

$$\begin{aligned} f_1(k_1(n)+2) &= \int_0^{k_1(n)+2} b(t)dt \leq \int_0^2 b(t)dt + \int_2^{k_1(n)+2} b(t-2)b(2)dt \\ &= f_1(2) + b(2)f_1(k_1(n)). \end{aligned}$$

Since $\lim_{x\to+\infty} k_1(x) = +\infty$, there exists n_0 such that $k_1(n) > 2$ for all $n \geq n_0$. Hence, for all $n \geq n_0$,

$$f_1(k_1(n)+2) \leq (1+b(2))f_1(k_1(n)),$$

and, by (20),

$$\mu(n)! \leq \mu(n)^{\mu(n)} \leq ((1+b(2))f_1(k_1(n)))^{(1+b(2))f_1(k_1(n))}.$$

By (19) and Lemma 4.4, $\lim_{n\to\infty} \frac{f_1(k_1(n))}{n} = 0$. Thus there exists an integer $n_1 > n_0$ such that $f_1(k_1(n)) < n$ for all $n > n_1$. Therefore

$$\mu(n)! \leq ((1+b(2))n)^{(1+b(2))f_1(k_1(n))},$$

for all $n > n_1$. Since, by (19) and Proposition 4.5, $\lim_{x\to+\infty} \frac{x^{f_1(k_1(x))}}{c^x} = 0$ for all $c > 1$, this implies that $\lim_{n\to\infty} \frac{\mu(n)!}{c^n} = 0$ for all $c > 1$, and thus

$$\limsup_{n\to\infty} (\mu(n)!)^{\frac{1}{n}} \leq 1.$$

Therefore, the first paragraph of the proof implies that

$$\limsup_{n\to\infty} g(n)^{\frac{1}{n}} \leq 1$$

and the result follows. ∎

5. Examples and more about the conjectures

We conclude with some examples of semigroups of intermediate growth arising from ideal extensions of simplest types that were used in our considerations in the preceding sections. Also, we show that Conjectures 4.2 and 4.3 are equivalent.

In order to prove this equivalence we need the following construction.

Example 5.1. *Let T be a semigroup. Then there exists a semigroup $S = T \cup I$, a disjoint union, where T is a subsemigroup of S, $I = \mathcal{M}(G, X, Y; P)$ is a completely 0-simple ideal of S and G is the free group on $\{x_t \mid t \in T \setminus \{1\}\}$. Furthermore, there exists an idempotent $e \in I$ such that the subsemigroup $S' = \{eu_1eu_2 \cdots eu_ke \mid u_1, u_2, \dots, u_k \in T\}$ of S is isomorphic to the subsemigroup $\{1\} \cup \langle x_t \mid t \in T \setminus \{1\}\rangle$ of G.*

Proof. Let T be a semigroup. Let $I = \mathcal{M}(G, X, Y; P)$ be the semigroup of matrix type over the free group G on $\{x_t \mid t \in T \setminus \{1\}\}$, with $X = Y = T$, if T has unity, or $X = Y = T \cup \{1\}$, if T has no unity, and with the sandwich matrix $P = (p_{u,v})$, where

$$p_{u,v} = \begin{cases} 1 & \text{if } u = 1 \text{ or } v = 1 \\ x_u^{-1} x_{uv} x_v^{-1} & \text{if } u \neq 1 \text{ and } v \neq 1. \end{cases}$$

Let S be the disjoint union $S = T \cup I$. Let x_1 denote the unity of G. We extend the operations of the semigroups T and I to an operation in S by defining $t(g, u, v) = (x_{tu} x_u^{-1} g, tu, v)$ and $(g, u, v)t = (g x_v^{-1} x_{vt}, u, vt)$, for all $t \in T$ and $(g, u, v) \in I$. We claim that S with this operation is a semigroup. Let $t, t' \in T$ and $(g, u, v), (g', u', v') \in I$. We have

$$\begin{aligned} t(t'(g,u,v)) &= t(x_{t'u} x_u^{-1} g, t'u, v) = (x_{tt'u} x_u^{-1} g, tt'u, v) = (tt')(g,u,v), \\ ((g,u,v)t)t' &= (g x_v^{-1} x_{vt}, u, vt)t' = (g x_v^{-1} x_{vtt'}, u, vtt') = (g,u,v)(tt'), \\ (t(g,u,v))t' &= (x_{tu} x_u^{-1} g, tu, v)t' = (x_{tu} x_u^{-1} g x_v^{-1} x_{vt'}, tu, vt') \\ &= t(g x_v^{-1} x_{vt'}, u, vt') = t((g,u,v)t'), \end{aligned}$$

$$\begin{aligned} ((g,u,v)(g',u',v'))t &= (g p_{v,u'} g', u, v')t = (g p_{v,u'} g' x_{v'}^{-1} x_{v't}, u, v't) \\ &= (g,u,v)(g' x_{v'}^{-1} x_{v't}, u', v't) = (g,u,v)((g',u',v')t), \\ t((g,u,v)(g',u',v')) &= t(g p_{v,u'} g', u, v') = (x_{tu} x_u^{-1} g p_{v,u'} g', tu, v') \\ &= (x_{tu} x_u^{-1} g, tu, v)(g',u',v') = (t(g,u,v))(g',u',v'), \\ ((g,u,v)t)(g',u',v') &= (g x_v^{-1} x_{vt}, u, vt)(g',u',v') = (g x_v^{-1} x_{vt} p_{vt,u'} g', u, v'), \\ (g,u,v)(t(g',u',v')) &= (g,u,v)(x_{tu'} x_{u'}^{-1} g', tu', v') = (g p_{v,tu'} x_{tu'} x_{u'}^{-1} g', u, v'). \end{aligned}$$

Note that

$$x_v^{-1} x_{vt} p_{vt,u'} = \begin{cases} x_v^{-1} x_{vt} & \text{if } vt = 1 \text{ or } u' = 1 \\ x_v^{-1} x_{vtu'} x_{u'}^{-1} & \text{if } vt \neq 1 \text{ and } u' \neq 1 \end{cases}$$

and

$$p_{v,tu'}x_{tu'}x_{u'}^{-1} = \begin{cases} x_{tu'}x_{u'}^{-1} & \text{if } v=1 \text{ or } tu'=1 \\ x_v^{-1}x_{vtu'}x_{u'}^{-1} & \text{if } v\neq 1 \text{ and } tu'\neq 1. \end{cases}$$

Hence, if $vt \neq 1$, $u' \neq 1$, $v \neq 1$ and $tu' \neq 1$, then

$$((g,u,v)t)(g',u',v') \quad = \quad (g,u,v)(t(g',u',v')). \tag{21}$$

If $vt = 1$, $v \neq 1$ and $tu' \neq 1$, then $x_v^{-1}x_{vt} = x_v^{-1}x_{vtu'}x_{u'}^{-1}$, and thus (21) holds. If $vt = 1$ and $v = 1$, then $t = 1$ and $x_v^{-1}x_{vt} = x_{tu'}x_{u'}^{-1}$, and thus (21) holds. If $vt = 1$ and $tu' = 1$, then $v = u'$ and $x_v^{-1}x_{vt} = x_{tu'}x_{u'}^{-1}$, and thus (21) holds. If $u' = 1$, $v \neq 1$ and $tu' \neq 1$, then $x_v^{-1}x_{vt} = x_v^{-1}x_{vtu'}x_{u'}^{-1}$, and thus (21) holds. If $u' = 1$ and $v = 1$, then $x_v^{-1}x_{vt} = x_{tu'}x_{u'}^{-1}$, and thus (21) holds. If $u' = 1$ and $tu' = 1$, then $t = 1$ and $x_v^{-1}x_{vt} = x_{tu'}x_{u'}^{-1}$, and thus (21) holds. If $vt \neq 1$, $u' \neq 1$ and $v = 1$, then $x_v^{-1}x_{vtu'}x_{u'}^{-1} = x_{tu'}x_{u'}^{-1}$, and thus (21) holds. If $vt \neq 1$, $u' \neq 1$ and $tu' = 1$, then $x_v^{-1}x_{vtu'}x_{u'}^{-1} = x_{tu'}x_{u'}^{-1}$, and thus (21) holds. Therefore the operation is associative. Hence S is a semigroup, as claimed, T is a subsemigroup and I is an ideal of S.

Let $e = (x_1, 1, 1) \in I$. Let $S' = \{eu_1eu_2\cdots eu_ke \mid u_1, u_2, \ldots, u_k \in T\}$. Since $e^2 = e$, S' is a subsemigroup of S. Note that

$$\begin{aligned}(eu_1)(eu_2)\cdots(eu_k)e &= (x_{u_1},1,u_1)(x_{u_2},1,u_2)\cdots(x_{u_k},1,u_k)(x_1,1,1)\\ &= (x_{u_1}x_{u_2}\cdots x_{u_k},1,1),\end{aligned}$$

for all $u_1, u_1, \ldots, u_k \in T$. Now it is easy to see that S' is isomorphic to the subsemigroup $\{x_1\} \cup \langle x_t \mid t \in T \setminus \{1\}\rangle$ of G. ∎

Assume that T is a semigroup (not necessarily a monoid), G is a group and $\chi\colon T \setminus \{1\} \longrightarrow G$ is a mapping. Then we can construct, as above, a semigroup $S(T, G, \chi) = T \cup I$, a disjoint union, such that T is a subsemigroup of $S(T, G, \chi)$, and $I = \mathcal{M}(G, X, Y; P)$ is the semigroup of matrix type over G with $X = Y = T$, if T has unity, or $X = Y = T \cup \{1\}$, if T has no unity, and with the sandwich matrix $P = (p_{u,v})$, where

$$p_{u,v} = \begin{cases} 1 & \text{if } u=1 \text{ or } v=1 \\ \chi(u)^{-1}\chi(uv)\chi(v)^{-1} & \text{if } u\neq 1 \text{ and } v\neq 1. \end{cases}$$

This is accomplished by defining $t(g,u,v) = (\chi(tu)\chi(u)^{-1}g, tu, v)$ and $(g,u,v)t = (g\chi(v)^{-1}\chi(vt), u, vt)$, for all $t \in T$ and $(g,u,v) \in I$, where $\chi(1)$ denotes the unity of G. Then I is a completely 0-simple ideal of $S(T, G, \chi)$ and $e = (1,1,1) \in I$ is an idempotent such that the subsemigroup $S'(T, G, \chi) = \{eu_1eu_2\cdots eu_ke \mid u_1, u_2, \ldots, u_k \in T\}$ of $S(T, G, \chi)$ is isomorphic to the subsemigroup $\{1\} \cup \langle \chi(t) \mid t \in T \setminus \{1\}\rangle$ of G.

Theorem 5.2. *Conjecture 4.2 is true if and only if Conjecture 4.3 is true.*

Proof. Suppose that Conjecture 4.3 is true. Then, by Lemma 2.5, nilpotent-by-finite groups satisfy the subexponential property for sequences. Thus, Theorem 3.2 implies that Conjecture 4.2 is true.

Conversely, suppose that Conjecture 4.2 is true. Let H be a nilpotent group of class m. Let $b(1), b(2), \dots$ be a sequence of positive integers and

$$h_{1,1}, h_{1,2}, \dots, h_{1,b(1)}, h_{2,1}, h_{2,2}, \dots, h_{2,b(2)}, \dots \tag{22}$$

a sequence of elements in H. Define the set

$$T_n = \{h_{i_1,j_1} h_{i_2,j_2} \cdots h_{i_s,j_s} \in H \mid s \geq 1,\ i_1 + \cdots + i_s \leq n\}.$$

Let $g(n) = |T_n|$ and $f(n) = \sum_{i=1}^{n} b(i)$. Suppose that $\limsup_{n\to\infty} f(n)^{\frac{1}{n}} \leq 1$.

By [16, Theorem 1.1], there exists a 2-generated semigroup $T = \langle a, b\rangle$ whose growth is intermediate and larger than the growth of f. In fact, we may assume that there exists a positive integer n_0 such that $d_{T,\{a,b\}}(n) > f(n)$ for all $n \geq n_0$ (see [16]). Let S be the free semigroup on generators $y_1, \dots, y_{f(n_0)}$. Consider the ideal J of S generated by all products $y_i y_j$, and set $\overline{S} = S/J$. We denote by $\bar{s}$ the image of s under the natural projection $\pi\colon S \longrightarrow \overline{S}$. Thus, $\overline{S} = \langle \bar{y}_1, \dots, \bar{y}_{f(n_0)}\rangle$. Let $T^1 = T \cup \{1\}$ and $\overline{S}^1 = \overline{S} \cup \{1\}$. Let $A_1 = \{1, a, b\}$ and $B_1 = \{1, \bar{y}_1, \dots, \bar{y}_{f(n_0)}\}$. Let $T' = T^1 \times T^1 \times \overline{S}^1$ and $A = A_1 \times A_1 \times B_1$. Thus $T' = \langle A\rangle$. Since $\overline{S}^1 = \{0, 1, \bar{y}_1, \dots, \bar{y}_{f(n_0)}\}$,

$$d_{T',A}(n) \leq d_{T^1,A_1}(n)^2 (f(n_0) + 2).$$

Since T has subexponential growth, T^1 also has subexponential growth. Hence T' has subexponential growth.

Since T^1 is an infinite finitely generated semigroup, for every positive integer n there exists an element $w_n \in T^1$ of length n in the generators from A_1. Let

$$C_n = \{w \in T^1 \mid w \text{ has length } n \text{ in the generators from } A_1\}.$$

Then $D = \{(w_n, 1, \bar{y}_j) \mid j = 1, \dots, f(n_0)\}$, $D_{n+1} = C_{n+1} \times \{1\} \times \{1\}$ and $D_i = (C_i \setminus \{1\}) \times (C_{n+1-i} \setminus \{1\}) \times \{1\}$, for $i = 1, \dots, n$, are disjoint subsets of elements of T' of length $n+1$ in the generators from A. Hence

$$\begin{aligned} & d_{T',A}(n+1) - d_{T',A}(n) \leq \\ \geq \quad & |D| + |D_{n+1}| + \sum_{i=1}^{n} |D_i| \\ = \quad & f(n_0) + |C_{n+1}| + \sum_{i=1}^{n} |C_i \setminus \{1\}| \cdot |C_{n+1-i} \setminus \{1\}| \\ \geq \quad & f(n_0) - 1 + \sum_{i=1}^{n+1} |C_i| \\ & \quad (\text{since } C_k = C_k \setminus \{1\}, \text{ for } k > 1) \\ = \quad & f(n_0) - 1 + d_{T^1,A_1}(n+1) \\ = \quad & f(n_0) + d_{T,\{a,b\}}(n+1) \\ > \quad & f(n+1) \geq b(n+1), \end{aligned}$$

for all $n \geq 1$. Furthermore, $d_{T',A}(1) > f(n_0) \geq b(1)$.

Let G be the free nilpotent group of class m on generators $\{x_t \mid t \in T' \setminus \{1\}\}$. Let $\chi\colon T' \setminus \{1\} \longrightarrow G$ be the map defined by $\chi(t) = x_t$ for all $t \in T' \setminus \{1\}$. Let $S(T', G, \chi)$ be the semigroup constructed as above. Define S_1 as the subsemigroup of $S(T', G, \chi)$ generated by the set $C = A \cup \{e\}$, where $e = (1,1,1) \in S(T', G, \chi) \setminus T'$. Let $l(t)$ denote the length of $t \in T'$ in the generators from A. Let

$$T'_n = \{x_{t_1} x_{t_2} \cdots x_{t_s} \in G \mid l(t_1) + \cdots + l(t_s) \leq n\}$$

and $g_1(n) = |T'_n|$. Since $d_{T',A}(1) > f(n_0) \geq b(1)$ and $d_{T',A}(n+1) - d_{T',A}(n) > b(n+1)$, for all $n \geq 1$, we have that $g_1(n) \geq g(n)$ for all n. Since

$$\psi\colon S'(T', G, \chi) \longrightarrow \{1\} \cup \langle x_t \mid t \in T' \setminus \{1\} \rangle,$$

defined by $\psi(et_1et_2 \cdots et_se) = x_{t_1} x_{t_2} \cdots x_{t_s}$, is an isomorphism, we obtain

$$|T'_n| = |\{et_1et_2 \cdots et_se \in S'(T', G, \chi) \mid l(t_1) + \ldots + l(t_s) \leq n\}| \leq d_{S_1,C}(2n+1).$$

Since Conjecture 4.2 is true, $d_{S_1,C}$ has subexponential growth. Therefore

$$\lim_{n \to +\infty} d_{S_1,C}(n)^{\frac{1}{n}} \leq 1.$$

Hence

$$\begin{aligned} \limsup_{n \to +\infty} g(n)^{\frac{1}{n}} & \leq \limsup_{n \to +\infty} g_1(n)^{\frac{1}{n}} \leq \limsup_{n \to +\infty} d_{S_1,C}(2n+1)^{\frac{1}{n}} \\ & \leq \limsup_{n \to +\infty} d_{S_1,C}(3n)^{\frac{1}{n}} \leq \limsup_{n \to +\infty} d_{S_1,C}(n)^{\frac{3}{n}} \leq 1. \end{aligned}$$

The result follows. ∎

Let $T = \langle g \rangle$ be an infinite cyclic semigroup, G a free nilpotent group of class 2 on generators $x_1, x_2, \ldots$, and let $\chi\colon T \longrightarrow G$ be defined by $\chi(g^i) = x_i$. Then the semigroup $S(T, G, \chi)$ is a simple example of a semigroup satisfying the hypotheses of Theorem 3.2. We continue with another construction of a semigroup of intermediate growth that has the form $S = T \cup I$, a disjoint union, where T is an infinite cyclic semigroup and I is an ideal, satisfying the hypotheses of Theorem 3.2.

Example 5.3. *There exists a semigroup of the form $S = \langle g \rangle \cup I$, a disjoint union, where $I = \mathcal{M}(H, \mathbb{Z}, \mathbb{Z}; Q)$ is a completely 0-simple ideal of S over a free nilpotent of class 2 group H, such that:*

(i) *$e = (1, 1, 1) \in I$ is an idempotent,*
(ii) *$e\langle g, e\rangle e$ generates a subgroup of I isomorphic to H,*
(iii) *$\langle g, e\rangle$ has intermediate growth.*

Proof. Let G be the free group on free generators $x_1, x_2, \ldots$. First we construct a monoid of the form $T = \langle g, g^{-1}\rangle \cup J$, where $J = \mathcal{M}(G, \mathbb{Z}, \mathbb{Z}; P)$ is a completely 0-simple semigroup. We interpret elements of J as $\mathbb{Z} \times \mathbb{Z}$-matrices with at most one nonzero entry, chosen from G. Hence (g, i, j) denotes the matrix with $g \in G$ in position (i, j). We need to define the sandwich matrix $P = (p_{ij})$ and the action of the cyclic group generated by g on J. Define $z_1 = x_1, z_n = x_{n-1}^{-1}x_n$ for $n > 1$ and $z_n = 1$ for $n \leq 0$. Let $A = (a_{ij})$ be the $\mathbb{Z} \times \mathbb{Z}$-matrix with entries in $G \cup \{0\}$ such that $a_{ij} = z_i$ if $j = i + 1$ and $a_{ij} = 0$ otherwise. We shall find P such that $A \circ P = P \circ A$, where $\circ$ stands for the usual matrix multiplication (notice that these products make sense because of the form of A). Then we define $g^k a = A^k \circ a$ and $ag^k = a \circ A^k$ for $a \in J$ and $k \in \mathbb{Z}$. Thus, for every $b \in J$ we have $(a \circ A^k)b = (a \circ A^k) \circ P \circ b = a \circ P \circ (A^k \circ b) = a(A^k \circ b)$ because A and P commute. The remaining conditions needed for the associativity of the operation in T follow immediately. Next, notice that the condition $A \circ P = P \circ A$ is equivalent to

$$z_i p_{i+1,j} = p_{i,j-1} z_{j-1} \text{ for every } i, j \in \mathbb{Z}. \tag{23}$$

We claim that P can be chosen so that $p_{i1} = 1$ for every $i \in \mathbb{Z}$. Indeed, we have chosen one column of P (the column with index 1). Then relations (23) allow us to determine uniquely all other entries of P. (These relations allow to determine the entire diagonal consisting of all entries p_{rs} of P such that $r - s = i - 1$, knowing only p_{i1}.) So we have determined a semigroup structure on T, extending the structure of J. Now, consider the natural homomorphism $\phi : G \longrightarrow H$ onto the free nilpotent of class 2 group on free generators also denoted by $x_1, x_2, \ldots$. Then we have an induced homomorphism $J \longrightarrow I = \mathcal{M}(H, \mathbb{Z}, \mathbb{Z}, Q)$, obtained by mapping every $(g_{ij}) \in J$ to $(\phi(g_{ij}))$, and defining the entries of the sandwich matrix Q by the rule $q_{ij} = \phi(p_{ij})$ if $p_{ij} \in G$ (notice that all entries of P are nonzero). It is easy to see that this determines an onto homomorphism $T \longrightarrow \langle g, g^{-1}\rangle \cup I$.

Let $e = (e_{ij}) \in I$ be the matrix with the only nonzero entry $e_{11} = 1$. It is clear that $e^2 = e$, which proves (i). For $n \geq 1$ write $A^n = (h_{ij}^{(n)})$. Then, for all $i \in \mathbb{Z}$, we have

$$h_{ij}^{(n)} = 0 \text{ if } j \neq i + n,$$

and

$$h_{i,i+n}^{(n)} = a_{i,i+1}a_{i+1,i+2} \cdots a_{i+n-1,i+n} = z_i z_{i+1} \cdots z_{i+n-1}.$$

In particular,

$$h_{1,1+n}^{(n)} = z_1 z_2 \cdots z_n = x_1(x_1^{-1}x_2) \cdots (x_{n-1}^{-1}x_n) = x_n.$$

If t is a positive integer, then

$$eg^t e = (eg^t)e = (x_t, 1, t+1)(1, 1, 1) = (x_t p_{t+1,1}, 1, 1) = (x_t, 1, 1).$$

Therefore the subgroup of I generated by $e\langle g, e\rangle e$ is isomorphic to H, and this proves (ii).

Note that, for every positive integer n and every $i_1, i_2, \ldots, i_k$ such that $k + 1 + \sum_{j=1}^{k} i_j = n$, the only nonzero entry of $u = eg^{i_1}eg^{i_2}e \cdots eg^{i_k}e$ is equal to $x_{i_1} \cdots x_{i_k}$. Moreover, u has length n in e, g. Now, by Theorem 2.8, it follows easily that $\langle g, e\rangle$ has intermediate growth, and this proves (iii). ∎

Let $\mathbb{Z}_{\leq 1}, \mathbb{Z}_{\geq 1}$ be the sets of integers ≤ 1 and ≥ 1, respectively. It is easy to see that $I_+ = \mathcal{M}(H, \mathbb{Z}_{\leq 1}, \mathbb{Z}_{\geq 1}; Q_+)$, where Q_+ is the corresponding submatrix of Q, is a completely 0-simple subsemigroup of I. Moreover, for every $m \in \mathbb{Z}_{\leq 1}$ and $n \in \mathbb{Z}_{\geq 1}$, the semigroup $\langle g, e\rangle$ contains an element of the form $(h, m, n), h \in H$. Therefore, $\langle g, e\rangle \cap I$ is a uniform subsemigroup of I_+ in the sense of [12].

Our final aim is to show that examples of similar types can be constructed within the class of linear semigroups. Namely, we find a matrix realization of a semigroup of intermediate growth arising from Example 5.1.

Example 5.4. *Let*

$$h = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & y & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \subseteq M_5(K),$$

where $K = \mathbb{Q}(x, y)$ is the field of rational functions in the indeterminates x and y. Let $S = \langle h, f\rangle$. Then $S = \langle h\rangle \cup J$, where J is the ideal consisting *of matrices of rank* 3 *and it is a subsemigroup of a completely* 0*-simple semigroup* $\mathcal{M}(H,X,Y;P)$*, for a nilpotent group* H *of class* 2*. Moreover, if* $T=\langle g\rangle$ *is an infinite cyclic semigroup,* G *a free nilpotent group of class* 2 *on generators* $x_1,x_2,\ldots$*, and* $\chi\colon T\longrightarrow G$ *is defined by* $\chi(g^i)=x_i$ *for* $i=1,2,\ldots$*, then* $\langle h^3,f\rangle\cong\langle g,e\rangle\subseteq S(T,G,\chi)$*, where* $e=(1,1,1)\in S(T,G,\chi)$*. In particular,* S *has intermediate growth.*

Proof. It is clear that the set J of all matrices of rank 3 in S forms an ideal of S and $S=\langle h\rangle\cup J$. As explained at the beginning of Section 3, from the general structure theorem for linear semigroups it then follows that J embeds into a completely 0-simple semigroup $\mathcal{M}(H,X,Y;P)$ and H can be identified with the group generated by $S\cap fM_5(K)f$. The latter is isomorphic to a unipotent nonabelian subgroup of $GL_3(K)$. Thus H is nilpotent of class 2. By Theorem 2.8 and Theorem 3.2, S has subexponential growth.

Let n be a positive integer. Then, using induction, it is easy to check that h^n is of the form

$$h^n=\begin{pmatrix}1 & a_n & f_n & * & k_n\\ 0 & x^n & a_n & * & *\\ 0 & 0 & 1 & b_n & g_n\\ 0 & 0 & 0 & y^n & b_n\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

where $a_n=1+x+\cdots+x^{n-1}, b_n=1+y+\cdots+y^{n-1}$ and $f_n=a_1+\cdots+a_{n-1}, g_n=b_1+\cdots+b_{n-1}$ and k_n is a polynomial in x,y. Consider any element of the form $w=fh^{i_1}fh^{i_2}f\cdots fh^{i_k}f$. Clearly $w=(fh^{i_1}f)(fh^{i_2}f)\cdots(fh^{i_k}f)$ and hence

$$w=\begin{pmatrix}1 & 0 & p_w & 0 & z_w\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & q_w\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix},$$

where $p_w=f_{i_1}+\cdots+f_{i_k}$. Since f_j is a monic polynomial with $\deg(f_j)=j-2$ for $j\geq 2$, the $(1,3)$ entry p_w of w determines the exponents $i_1,\ldots,i_k$ (taking their multiplicities into account). In particular, $fh^{3i}f, i=1,2,\ldots$, are independent modulo the commutator subgroup H' of H.

It is easy to see that the commutator $[fh^if,fh^jf]$ in the group H is of the form $1+e_{15}(f_ig_j-g_if_j)$, where e_{15} is the corresponding matrix unit. The leading term of the polynomial $f_ig_j-g_if_j$ is $x^{i-2}y^{j-2}-y^{i-2}x^{j-2}$. Therefore $[fh^{3i}f,fh^{3j}f], i>j\geq 1$, are independent in the abelian group H'. Thus, from Lemma 2.7 it follows that the map $fh^{3i}f\mapsto x_i$, $i=1,2,\ldots$, extends to an injective homomorphism from the semigroup F generated by $fh^{3i}f, i=$

$1, 2, \ldots$, to the free nilpotent group G of class 2 on free generators $x_1, x_2, \ldots$. Since $F \subseteq H$ has a group of right quotients because H is nilpotent, this implies that $fh^{3i}f, i = 1, 2, \ldots$, are free generators of a free nilpotent group of class 2. Since $eg^ie, i = 1, 2, \ldots$, also have this property by Example 5.1, it follows that the map $eg^ie \mapsto fh^{3i}f$ extends to an isomorphism $e\langle g, e\rangle e \longrightarrow f\langle h^3, f\rangle f$. Suppose that $h^iwh^j = h^pvh^q$ for some $w, v \in H$ and some integers i, j, p, q. Then $w = h^{p-i}vh^{q-j} = h^{p-i}(vh^{q-j}f)$ and comparing the $(2, 3)$-entries of these matrices we get $p = i$. A symmetric argument yields $q = j$. This implies that $\langle h^3, f\rangle = \bigcup_{i,j=1}^{\infty} h^{3i}Wh^{3j}$, where $W = f\langle h^3, f\rangle f$, is a disjoint union. Since $g^iGg^j, i, j \geq 1$, are also disjoint, it follows that the rules:

$$g^i(eg^{i_1}eg^{i_2}e \cdots eg^{i_k}e)g^j \mapsto h^{3i}(fh^{3i_1}fh^{3i_2}f \cdots fh^{3i_k}f)h^{3j} \text{ and } g^m \mapsto h^{3m},$$

for any nonnegative integers $i, j, k, i_1, \ldots, i_k, m$, determine a bijection $\pi\colon \langle g, e\rangle \longrightarrow \langle h^3, f\rangle$. It is then clear that π is an isomorphism. ∎

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Ferran Cedó
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona), Spain

Jan Okniński
Institute of Mathematics
Warsaw University
02-097 Warsaw, Poland