

POLYNOMIAL AND LINEARIZED NORMAL FORMS FOR ALMOST PERIODIC DIFFERENTIAL SYSTEMS

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ABSTRACT. For almost periodic differential systems $\dot{x} = \varepsilon f(x, t, \varepsilon)$ with $x \in \mathbb{C}^n$, $t \in \mathbb{R}$ and $\varepsilon > 0$ small enough, we get a polynomial normal form in a neighborhood of a hyperbolic singular point of the system $\dot{x} = \varepsilon \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x, t, 0) dt$, if its eigenvalues are in the Poincaré domain. The normal form linearizes if the real part of the eigenvalues are non-resonant.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Normal form theory has a long history. The basic idea of simplifying ordinary differential equations through changes of variables can be found in the work of Poincaré [13]. Recently this theory has been developed very rapidly since it plays a very important role in the study of bifurcation, stability and so on. Usually, normal form theory is applied to simplifying a nonlinear system in the neighborhood of a reference solution, which is almost exclusively assumed to be a singular point (sometimes a periodic solution). For an outline of normal form theory, we mentioned the work of Bibikov [3], Dulac [6], Sternberg [16], Chen [4], Takens [17], and many others.

Nowadays, due to the construction of spectral theory for many kinds of non-autonomous linear systems [5, 14, 1] and detailed analysis on normal form transformation operators [1, 18], many classic theorems in normal form theory for autonomous cases can be extended to more general systems, such as [10, 11] for random dynamical systems and [15] for difference systems.

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Following this way the purpose of our paper is to study normal forms for almost periodic differential systems used in the averaging theory.

There are several known equivalent definitions of almost periodic functions. Here we choose the one given by Bochner, which is very direct and useful in its applications to differential equations. Let $f(x, t) \in C(D \times \mathbb{R}, \mathbb{C}^n)$, where D is an open set in \mathbb{C}^n (more generally, a separable Banach space), and assume that for any sequence $\{h_k\}$ of real numbers, there exists a subsequence $\{h_{k_j}\}$ such that $\{f(x, t + h_{k_j})\}$ converges uniformly on $S \times \mathbb{R}$, where S is any compact set in D . Then we say $f(x, t)$ is *almost periodic* in t uniformly for $x \in D$. Moreover, for the fixed x let

$$\alpha(\gamma, f) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x, t) e^{-\gamma t \sqrt{-1}} dt,$$

the set

$$\Gamma(f) = \{\gamma \in \mathbb{R} : \alpha(\gamma, f) \neq 0\}$$

is called the set of *Fourier exponents* of f . The module generated by $\Gamma(f)$ is defined as the *module* $m(f)$ of f . For instant, if f is periodic of period $2\pi/\omega$, then $\{m(f) = n\omega, n = 0, \pm 1, \dots\}$.

Suppose that $x \in \mathbb{C}^n$ and that $\varepsilon \geq 0$ is a real parameter, then we say $f \in \mathcal{F}(D, [0, \infty))$, if

- (i) the function $f: D \times \mathbb{R} \times [0, \infty) \rightarrow \mathbb{C}^n$ is continuous,
- (ii) $f(x, t, \varepsilon)$ is almost periodic in t uniformly with respect to x in compact sets of D for each fixed ε ,
- (iii) $f(x, t, \varepsilon)$ is analytic with respect to $x \in D$ for fixed t and ε ,
- (iv) $f(x, t, \varepsilon) \rightarrow f(x, t, 0)$ as $\varepsilon \rightarrow 0$ uniformly for t in $(-\infty, \infty)$, x in compact sets.

We associate to the system of differential equations

$$(1) \quad \dot{x} = \varepsilon f(x, t, \varepsilon),$$

the averaged system

$$(2) \quad \dot{x} = \varepsilon f_0(x),$$

where

$$f_0(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x, t, 0) dt.$$

By the classical averaging theorem (see [8, 7]), if $x_0 \in D$ is a hyperbolic singular point of system (2), then system (1) has an almost periodic solution $x^*(t, \varepsilon) \rightarrow x_0$ uniformly as $\varepsilon \rightarrow 0$.

Let $U_\delta = \{x \in \mathbb{C}^n : \|x\| \leq \delta\}$ and A be a n -square matrix. Set $\lambda(A) = (\lambda_1, \dots, \lambda_n)$ be the eigenvalues of A . Denote by Re the real part of

a complex. Without loss of generality, we can assume $\operatorname{Re}\lambda_1 \leq \dots \leq \operatorname{Re}\lambda_n$. We say that the eigenvalues $\lambda(A)$ are in the *Poincaré domain* if $\operatorname{Re}\lambda_n < 0$ or $\operatorname{Re}\lambda_1 > 0$. The following conditions are called *resonant conditions*:

$$(3) \quad \sum_{i=1}^n k_i \operatorname{Re}\lambda_i - \operatorname{Re}\lambda_j = 0,$$

where $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ and $j = 1, \dots, n$. As usual \mathbb{Z}_+ denotes the set of non-negative integers. We say A is *non-resonant*, if $\lambda(A)$ possess no resonant conditions for any $|k| = \sum_{i=1}^n k_i \geq 2$ and $j = 1, \dots, n$. Let $x = (x_1, \dots, x_n)$ be the coordinates with respect to the standard basis $\{e_1, \dots, e_n\}$ of \mathbb{C}^n in which the matrix A is in the Jordan Normal Form (JNF, for short) with diagonal elements $(\lambda_1, \dots, \lambda_n)$. Then a monomial $x^k e_j$ with $|k| \geq 1$ and j is called a *resonant monomial* if and only if (3) holds.

We do to system (1) the change of variables $x \mapsto y$ given by

$$(4) \quad x = y + x^*(t, \varepsilon),$$

then the system becomes

$$(5) \quad \dot{y} = \varepsilon F(y, t, \varepsilon),$$

where $F(0, t, \varepsilon) \equiv 0$ and $F(y, t, \varepsilon) = f(y + x^*(t, \varepsilon), t, \varepsilon) - f(x^*(t, \varepsilon), t, \varepsilon)$ satisfies the same conditions as the function $f(x, t, \varepsilon)$.

First we deal with formal normal forms of system (5). Let \mathcal{P} be the set of continuous functions, which possess that $f(t, \varepsilon) \in \mathcal{P}$ if f is almost periodic in t and $f(t, \varepsilon) \rightarrow f(t, 0)$ uniformly for $t \in \mathbb{R}$ as $\varepsilon \rightarrow 0$. Then denote the set of almost period formal Taylor series by

$$\mathcal{W} := \left\{ f = \sum f_k^j(t, \varepsilon) x^k e_j \mid f_k^j \in \mathcal{P}, k \in \mathbb{Z}_+^n, j = 1, \dots, n \right\}.$$

Consider the following formal system

$$(6) \quad \dot{y} = \varepsilon \tilde{F}(y, t, \varepsilon),$$

where $\tilde{F} \in \mathcal{W}$, $\tilde{F}(0, t, \varepsilon) \equiv 0$ and $\mathcal{D}_y \tilde{F}(0, t, 0) = B$ is a constant. Here \mathcal{D}_y denotes the calculation to get the Jacobian matrix with respect to y .

Theorem 1. *Assume B is in the JNF. Then for any positive integer $N > 0$ there exists $\varepsilon_0 = \varepsilon_0(N) > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ under the coordinates substitution $y = z + w_N(z, t, \varepsilon)$, where $w_N \in \mathcal{W}$ is a polynomial of degree N with respect to z , $\mathcal{D}_z w_N(0, t, \varepsilon) \rightarrow 0$ uniformly for $t \in \mathbb{R}$ as $\varepsilon \rightarrow 0$ and $m(w_N) \subset m(\operatorname{Jet}_{y=0}^N \tilde{F})$, system (6) becomes*

$$(7) \quad \dot{z} = \varepsilon Q_N(z, t, \varepsilon) + \varepsilon R(z, t, \varepsilon),$$

where Q_N and $R \in \mathcal{W}$, Q_N is a polynomial of degree N in z and only contains resonant monomials with respect to B , $R(z, t, \varepsilon) = O(\|z\|^{N+1})$.

The N -th iteration system, i.e.,

$$\dot{z} = \varepsilon Q_N(z, t, \varepsilon) = \varepsilon \tilde{B}(t, \varepsilon)z + \cdots,$$

is called the N -th order B normal form of system (7). In particular, we note that by our definition $\tilde{B}(t, \varepsilon)$ is in the block diagonal form corresponding to the different real parts of $\lambda(B)$. Next we go back to system (5) to study the polynomial and linear normal forms in the Poincaré domain.

Theorem 2. *Let $A = \mathcal{D}_x f^0(x_0)$. If $\lambda(A)$ is in the Poincaré domain, then there exists $\varepsilon_0 > 0$ and $\delta > 0$ independent of ε such that for $0 < \varepsilon \leq \varepsilon_0$ under the change of coordinates $y = z + w(z, t, \varepsilon)$, where $w \in \mathcal{F}(U_\delta, [0, \varepsilon_0))$, $\mathcal{D}_z w(0, t, \varepsilon) \rightarrow 0$ uniformly for $t \in \mathbb{R}$ as $\varepsilon \rightarrow 0$ and $m(w) \subset m(F)$, system (5) becomes*

$$(8) \quad \dot{z} = \varepsilon P_d(z, t, \varepsilon),$$

where $P_d \in \mathcal{F}(U_\delta, [0, \varepsilon_0))$ is a polynomial of degree d with respect to z , $\text{Jet}_{z=0}^d P_d = \text{Jet}_{y=0}^d F$, the integer $d \leq \max\{\text{Re}\lambda_1/\text{Re}\lambda_n, \text{Re}\lambda_n/\text{Re}\lambda_1\}$. In addition, if the real parts of $\lambda(A)$ are non-resonant, then $d = 1$; i.e.

$$\dot{z} = \varepsilon \mathcal{D}_y F(0, t, \varepsilon)z.$$

Collecting results of both theorems, we obtain the corollary.

Corollary 3. *If the allowed change of coordinates in Theorem 2 has the form $y = Sz + w(z, t, \varepsilon)$, where $\det S \neq 0$, then P_d can be chosen as the d -th order B normal form of (5), where B is the JNF of A .*

On one hand under the view of normal form theory, there are some obvious difference between our theorem and other theorems. Normally, if the linear part of original system is not an autonomous one, dichotomy spectrum will be applied to character non-resonant conditions. However, how to calculate the exact dichotomy spectrum is still an unsolved technical problem. Whereas for our theorem the resonant condition is given out by the eigenvalues of fixed point of averaged system and it can be verified easily. In this sense, our theorem is more straightforward and clearer. Moreover, if averaging systems are degenerated to the systems independent on the time t , i.e., autonomous systems with small parameter ε , our theorem coincides with the classic one. So our paper also can be seen as the extension of the work in [9] to special non-autonomous systems.

On the other hand under the view of averaging methods, our work gives a new aspect to characterize the close relationship between system (1) and

the averaged system. Instead of concerning about the approaching of trajectories of original and corresponding averaged systems in finite time interval, we pay more attention to the long time behavior of solutions. In fact, in Fink's book [7] (pp.266), he regards system (1) as a perturbation of the linear system at the singular point of the averaged system under a stronger condition. Here using our methods, by a more reasonable condition we can obtain a better result, i.e., the linearization and polynomialization of the original system.

The paper is organized as follows. In Section 2 a series lemmas and some useful facts are collected. In Section 3 and 4 we present the proof of Theorem 1 and Theorem 2, respectively.

2. PRELIMINARY RESULTS

In this section we introduce some basic definitions and lemmas, which are important for our proof. First, we describe some of the hyperbolic properties of a non-autonomous system. By studying the exponential dichotomy we can characterize the asymptotic speed of the solutions tending to infinity, which implies a close relationship with the unique bounded solution of a non-autonomous system.

Let $\Phi(t)$ be the fundamental matrix such that $\Phi(0) = I$ (as usual I denotes the identity matrix) for the linear differential equation

$$(9) \quad \dot{x} = A(t)x,$$

where the $n \times n$ coefficient matrix $A(t)$ is continuous in \mathbb{R} . Equation (9) is said to possess an *exponential dichotomy* if there exists a *projection* P , that is, a matrix P such that $P^2 = P$, and positive constants K and α such that

$$\begin{aligned} \|\Phi(t)P\Phi^{-1}(s)\| &\leq Ke^{-\alpha(t-s)}, & t \geq s, \\ \|\Phi(t)(I-P)\Phi^{-1}(s)\| &\leq Ke^{-\alpha(s-t)}, & s \geq t. \end{aligned}$$

Then, the following lemma show the toughness of the exponential dichotomy. A detailed proof can be found in [5].

Lemma 4. *Suppose the linear differential system (9) has an exponential dichotomy and the $n \times n$ coefficient matrix $B(t)$ is continuous in \mathbb{R} . If*

$$\beta = \sup_{t \in \mathbb{R}} \|B(t)\| < \alpha/(4K^2),$$

then the perturbed system

$$(10) \quad \dot{y} = (A(t) + B(t))y$$

also has an exponential dichotomy

$$\begin{aligned}\|\Phi(t)Q\Phi^{-1}(s)\| &\leq (5/2)K^2e^{-(\alpha-2K\beta)(t-s)}, & t \geq s, \\ \|\Phi(t)(I-Q)\Phi^{-1}(s)\| &\leq (5/2)K^2e^{-(\alpha-2K\beta)(s-t)}, & s \geq t,\end{aligned}$$

where $\Phi(t)$ is the fundamental matrix of (10) with the initial condition $\Phi(0) = I$ and the projection Q has the same null space as the projection P .

It is well known that the exponential dichotomy implies the existence of a unique bounded solution, which is almost periodic for almost periodic systems. Here, the statement of the next lemma is from [5], see also [8, 7].

Lemma 5. *For the non-homogeneous equation*

$$\dot{x} = A(t)x + f(t),$$

where $A(t)$ and $f(t)$ are almost periodic functions, if the corresponding homogeneous equation $\dot{x} = A(t)x$ has an exponential dichotomy on \mathbb{R} , then there exists a unique almost periodic solution ψ of that non-homogeneous equation which satisfies $m(\psi) \subset m(A(t)x + f(t))$.

The following is a strong version of the Gronwall type integral inequality, which comes from a result of Sardarly (1965), and its proof can be found in [2].

Lemma 6. *Let $u(t)$, $a(t)$, $b(t)$ and $q(t)$ be continuous functions in $J = [\alpha, \beta]$, let $c(t, s)$ be a continuous function for $\alpha \leq s \leq t \leq \beta$, let $b(t)$ and $q(t)$ be non-negative in J and suppose that*

$$u(t) \leq a(t) + \int_{\alpha}^t (q(t)b(s)u(s) + c(t, s))ds, \text{ for all } t \in J.$$

Then for all $t \in J$ we have that

$$u(t) \leq a(t) + \int_{\alpha}^t c(t, s)ds + q(t) \int_{\alpha}^t b(s) \left[a(s) + \int_{\alpha}^s c(s, \tau)d\tau \right] e^{\int_s^t b(\tau)q(\tau)d\tau} ds.$$

3. PROOF OF THEOREM 1

In this section we consider formal normal forms of system (6). The whole proof consists of two parts. First, due to Palmer's work [12], the linear part can be changed into the block diagonal form because of the roughness of exponential dichotomy. Then following the classical way we can simplify higher order terms in the original system.

Consider the linear homogeneous system

$$(11) \quad \dot{y} = \varepsilon(B + Q(t, \varepsilon))y,$$

where B is a constant and in JNF with diagonal elements $(\lambda_1, \dots, \lambda_n)$, $Q \in \mathcal{P}$, $Q(t, \varepsilon) \rightarrow 0$ uniformly for $t \in \mathbb{R}$ as $\varepsilon \rightarrow 0$.

Lemma 7. *There exists a coordinate substitution $z = y + \varepsilon S(t, \varepsilon)y$ such that system (11) can be changed into its 1-th B normal form, where $S, \partial S/\partial t \in \mathcal{P}$ and $m(S) \subset m(Q)$.*

Proof. Collecting all the blocks with same real parts of eigenvalues together, we can assume $B = \text{diag}(B_1, \dots, B_s)$. Denotes μ_i the same real parts of $\lambda(B_i)$ for $i = 1, \dots, s$. In addition, we assume $\mu_1 > \dots > \mu_s$. Set $\gamma_i = (\mu_i + \mu_{i+1})/2$ for $i = 1, \dots, s-1$ and

$$\delta = \min\{(\mu_i - \mu_{i+1})/4 \mid i = 1, \dots, s-1\}.$$

Let $B_{\gamma_1} = B - \gamma_1 I$ and $P_1 = \text{diag}(I_1, 0, \dots, 0)$. By our condition there exists $\varepsilon_1 > 0$ such that

$$\sup_{\mathbb{R} \times [0, \varepsilon_1]} \|\tilde{B}(t, \varepsilon)\| \leq \delta/36.$$

Therefore, from Lemma 3 of [12] we know that the system

$$\dot{y} = \varepsilon(B - \gamma_1 I + Q(t, \varepsilon))y$$

can be changed into the block diagonal form by the coordinates substitution $z = y + H_1(t, \varepsilon)y$ for $0 < \varepsilon \leq \varepsilon_1$, where H_1 with $m(H_1) \subset m(Q)$ satisfies the following integral equation

$$\begin{aligned} H_1(t) = & \varepsilon \int_{-\infty}^t e^{\varepsilon B_{\gamma_1} t} (I - P_1) e^{-\varepsilon B_{\gamma_1} s} (I - H_1(s)) Q(s, \varepsilon) (I \\ & + H_1(s)) e^{\varepsilon B_{\gamma_1} s} P_1 e^{-\varepsilon B_{\gamma_1} t} ds \\ & - \varepsilon \int_t^{\infty} e^{\varepsilon B_{\gamma_1} t} P_1 e^{-\varepsilon B_{\gamma_1} s} (I - H_1(s)) Q(s, \varepsilon) (I \\ & + H_1(s)) e^{\varepsilon B_{\gamma_1} s} (I - P_1) e^{-\varepsilon B_{\gamma_1} t} ds, \end{aligned}$$

and $\partial H_1/\partial t$ is almost periodic in t for fixed ε . Moreover, $H_1(t, \varepsilon) \rightarrow 0$ uniformly for $t \in \mathbb{R}$ as $\varepsilon \rightarrow 0$ by the contracting mapping principle. Obviously, this coordinate substitution also changes system (11) into the block diagonal form. Finally, using the matrix block technique and $s-1$ times doing in the similar way, we can get the result. \square

After changing the linear part into a block diagonal form, we can use classical methods to simplify higher order terms. Assume $\tilde{B}(t, \varepsilon) = \text{diag}(\tilde{B}_1(t, \varepsilon), \dots, \tilde{B}_s(t, \varepsilon))$, $\tilde{B}_i(t, \varepsilon) \rightarrow B_i$ uniformly for $t \in \mathbb{R}$ and $i = 1, \dots, s$ as $\varepsilon \rightarrow 0$, $B = \text{diag}(B_1, \dots, B_s)$ is in the JNF and the real parts of the eigenvalues of each blocks are same.

Lemma 8. Denote by $H_{k,n}(\mathbb{C}^n)$ the linear space of the n -dimensional vector having as components homogeneous polynomials in n variables of degree k with complex coefficients. Define a (t, ε) -depending linear operator $L_k^{\tilde{B}}$ on $H_{k,n}(\mathbb{C}^n)$ as follows,

$$(12) \quad L_k^{\tilde{B}} h = \tilde{B}(t, \varepsilon)h - \mathcal{D}_x h \tilde{B}(t, \varepsilon)x,$$

for $h(x) \in H_{k,n}(\mathbb{C}^n)$. Then by choosing a convenient basis of $H_{k,n}(\mathbb{C}^n)$ independent on t and ε , the matrix presentation of the operator $L_k^{\tilde{B}}$ has form

$$Q_k(t, \varepsilon) = \begin{pmatrix} Q_k^h(t, \varepsilon) & 0 \\ 0 & Q_k^0(t, \varepsilon) \end{pmatrix},$$

where $Q_k(t, \varepsilon) \rightarrow \tilde{Q}_k = \text{diag}(\tilde{Q}_k^h, \tilde{Q}_k^0)$ uniformly for $t \in \mathbb{R}$ as $\varepsilon \rightarrow 0$, \tilde{Q}_k is a constant, $\text{Re}\lambda(\tilde{Q}_k^h) \neq 0$ and $\text{Re}\lambda(\tilde{Q}_k^0) = 0$.

Proof. Let B_i be a t_i -square matrix. Set $t_0 = 0$ and $r_i = \sum_{j=1}^i t_{j-1}$, then we

write all the vectors in block form according to the block decomposition of $\tilde{B}(t, \varepsilon)$. That is, $x = (\tilde{x}_1, \dots, \tilde{x}_s)$, $m = (\tilde{m}_1, \dots, \tilde{m}_s) \in \mathbb{Z}_+^n$, where $\tilde{x}_i = (x_{r_i+1}, \dots, x_{r_{i+1}})$, $\tilde{m}_i = (m_{r_i+1}, \dots, m_{r_{i+1}})$. Using the same block decomposition for e_j and denoting the unique non-zero block component by \tilde{e}_l , we obtain $e_j = (\dots, \tilde{e}_l, \dots)$, where $r_l + 1 \leq j \leq r_{l+1}$. For any $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{Z}_+^s$ satisfying $|\mu| = k$ and $p = 1, \dots, s$, we can define a linear subspace of $H_{n,k}(\mathbb{C})$ by

$$E_{\mu,p} = \text{Span}\{x^m e_j \mid |\tilde{m}_i| = \mu_i, r_p + 1 \leq j \leq r_{p+1}\}.$$

By direct computation, if $x^m e_j \in E_{\mu,p}$, we obtain

$$L_k^{\tilde{B}} x^m e_j = x^m \tilde{B}_l(t, \varepsilon) \tilde{e}_l - \sum_{i=1}^s \mathcal{D}_{x_i} x^m \tilde{B}_i \tilde{x}_i \tilde{e}_l \in E_{\mu,p}.$$

So the linear subspace $E_{\mu,p}$ is $L_k^{\tilde{B}}$ invariant, which means that the matrix representation can be in the block diagonal form by choosing the convenient basis of $H_{n,k}(\mathbb{C}^n)$.

Since $\tilde{B}(t, \varepsilon) \rightarrow B$ uniformly for $t \in \mathbb{R}$ as $\varepsilon \rightarrow 0$ and every entry of the matrix representation of the linear operator $L_k^{\tilde{B}}(\varepsilon, t)$ is the multiplication and addition of the entries of $\tilde{B}(t, \varepsilon)$, we have the uniform convergence

$$\sup_{\mathbb{R}} \|L_k^{\tilde{B}}(\varepsilon, t) - L_k^B\| \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Since B is in JNF, it is well known that in the lexicographic ordering of the basis $\{x^m e_j \mid |m| = k, j = 1, \dots, n\}$ of $H_{n,k}(\mathbb{C}^n)$, the matrix representation

of L_k^B has the lower triangle form. See also [3] for more details. By an easy computation, we have

$$(13) \quad L_k^B x^m e_j = \left(\lambda_j - \sum_{i=1}^n m_i \lambda_i \right) x^m e_j + \dots$$

Since $E_{\mu,p}$ is L_k^B -invariant, the restriction of lexicographic ordering on $E_{\mu,p}$ preserves the form of the operator $L_k^B|_{E_{\mu,p}}$ as a lower triangle matrix representation with the diagonal elements mentioned in (13). So

$$\operatorname{Re}\lambda(L_k^B|_{E_{\mu,p}}) = \left\{ \operatorname{Re}\lambda_j - \sum_{i=1}^n m_i \operatorname{Re}\lambda_i \mid x^m e_j \in E_{\mu,p} \right\},$$

which have a close relationship with our definition of resonant conditions. Moreover, we know that the real parts of the eigenvalues of $L_k^B|_{E_{\mu,p}}$ are the same, because all the eigenvalues of B_i have the same real parts. Collecting all the blocks with zero real parts of eigenvalues together, then we get the block $Q_k^0(t, \varepsilon)$, which corresponds to all the resonant monomials in $H_{n,k}(\mathbb{C}^n)$. This completes the proof of the lemma. \square

Proof of Theorem 1. By Lemma 7, Theorem 1 holds for $N = 1$. Assume for $N = q - 1 \geq 2$ the theorem is still valid. Now consider system (6) in a special form

$$(14) \quad \dot{y} = \varepsilon \tilde{B}(t, \varepsilon) y + \sum_{i=2}^q \varepsilon P_i(y, t, \varepsilon) + H.O.T.,$$

where $P_i \in \mathcal{W}$ is a homogeneous polynomial of degree i with respect to z , $\tilde{B}(t, \varepsilon)$ satisfies the same condition mentioned before Lemma 8 and $m(P_q) \subset m(Jet_{y=0}^q \tilde{F})$. As usual H.O.T. denotes higher order terms with respect to y .

By the change of variable $y = z + \tilde{w}_q(z, t, \varepsilon)$, where $\tilde{w}_q(z, t, \varepsilon)$ is a homogeneous polynomial of degree m with respect to z , system (14) becomes

$$\dot{z} = \varepsilon \tilde{B}(t, \varepsilon) z + \varepsilon \sum_{i=2}^{q-1} P_i(y, t, \varepsilon) + G_q(z, t, \varepsilon) + H.O.T.,$$

where $G_q = L_q^{\varepsilon \tilde{B}} \tilde{w}_q + \varepsilon P_q(z, t, \varepsilon) - \partial \tilde{w}_q / \partial t$ and $L_q^{\varepsilon \tilde{B}}$ is similarly defined as in (12). From Lemma 8, we know that by choosing a convenient basis of $H_{n,q}(\mathbb{C}^n)$ the matrix presentation of $L_q^{\varepsilon \tilde{B}}$ is in block diagonal form. For the simplicity of notations, we identify vectors \tilde{w}_q , P_q and G_q with their

presentations under that basis of $H_{k,n}(\mathbb{C}^n)$, respectively. Then we obtain that $\tilde{w}_q = (\tilde{w}_q^h, \tilde{w}_q^0)$, $P_q = (P_q^h, P_q^0)$, $G_q = (G_q^h, G_q^0)$ and

$$G_q^* = \varepsilon Q_q^*(t, \varepsilon) \tilde{w}_q^* + \varepsilon P_q^*(t, \varepsilon) - \frac{d\tilde{w}_q^*}{dt},$$

where Q_q^* is the same as in Lemma 8 and $*$ = h or 0 . By Lemma 8, there are two cases. If $x^m e_j$ is a resonant monomial with respect to B , then its coefficient lies in one component of the vector P_q^0 . Thus we set $\tilde{w}_q^0 \equiv 0$, then $G_q^0 = \varepsilon P_q^0$. Otherwise, we set $G_q^h \equiv 0$. Then we obtain

$$(15) \quad \frac{d\tilde{w}_q^h}{dt} = \varepsilon Q_q^h(t, \varepsilon) \tilde{w}_q^h + \varepsilon P_q^h(t, \varepsilon).$$

Let $\varepsilon \rightarrow 0$, then $Q_q^h(t, \varepsilon) \rightarrow \tilde{Q}_q^h$ uniformly for $t \in \mathbb{R}$ and \tilde{Q}_q^h is a constant with $\operatorname{Re} \lambda(\tilde{Q}_q^h) \neq 0$. So the system

$$\frac{dz}{dt} = \varepsilon \tilde{Q}_q^h z$$

admits an exponential dichotomy together with positive constants K and $\varepsilon \alpha$, where α is independent on ε . Thus there exists $\varepsilon_0 > 0$ such that

$$\sup_{\mathbb{R}} \|Q_q^h(\varepsilon, t) - \tilde{Q}_q^h\| = \beta \leq \alpha/(4K^2), \quad 0 \leq \varepsilon \leq \varepsilon_0.$$

By Lemma 4, system

$$\frac{dz}{dt} = \varepsilon Q_q^h(t, \varepsilon) z$$

also has an exponential dichotomy with constant $(5/2)K^2$ and $\varepsilon(\alpha - 2K\beta)$. Therefore, system (15) has a unique almost periodic solution $\tilde{w}_q^h(t, \varepsilon)$, which satisfies $\tilde{w}_q^h \in \mathcal{W}$ and $m(\tilde{w}_q^h) \subset m(\operatorname{Jet}_{y=0}^q \tilde{F})$. Finally, by induction hypothesis we complete the proof of the theorem. \square

4. PROOF OF THEOREM 2

The next five lemmas prepare the proof of Theorem 2.

Lemma 9. *Let $\varepsilon_1 > 0$, there exists a function $\varepsilon u(z, t, \varepsilon) \in \mathcal{F}(U_\delta, [0, \varepsilon_1))$, $\varepsilon u \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly on $U_\delta \times \mathbb{R}$, such that the transformation of variables $y = z + \varepsilon u(z, t, \varepsilon)$ is invertible for $0 < \varepsilon \leq \varepsilon_1$ and transforms system (5) into*

$$(16) \quad \dot{z} = \varepsilon F_0(z) + \varepsilon \tilde{F}(z, t, \varepsilon),$$

where

$$F_0(z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F(z, t, 0) dt$$

is analytic in $z \in U_\delta$ and $F_0(0) = 0$, $\tilde{F} \in \mathcal{F}(U_\delta, [0, \varepsilon_1))$, $\tilde{F}(z, t, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly for $(z, t) \in U_\delta \times \mathbb{R}$ and $\tilde{F}(0, t, \varepsilon) \equiv 0$. Furthermore, $m(\tilde{F}) \subset m(F)$.

Proof. For fixed $(x, t) \in U_\delta \times \mathbb{R}$, we define

$$u(z, t, \varepsilon) = \int_{-\infty}^t e^{-\varepsilon(t-s)} Z(z, s) ds,$$

where $Z(z, t) = F(z, t, 0) - F_0(z)$. By Lemmas 4 and 5, it can be seen as the unique almost periodic solution of the system

$$\dot{x} = -\varepsilon x + Z(z, t)$$

for a fixed $z \in U_\delta$. So $u(z, t, \varepsilon)$ is analytic in $z \in U_\delta$ for fixed $(t, \varepsilon) \in \mathbb{R} \times [0, \varepsilon_1)$ and so also F_0 and \tilde{F} are analytic. The rest of the proof can be seen in [5, 7]. \square

Now using Taylor expansion in z system (16) can be written as

$$(17) \quad \dot{z} = \varepsilon \tilde{A}(t, \varepsilon) z + \varepsilon G(z, t, \varepsilon),$$

where $\tilde{A}(t, \varepsilon) \rightarrow A = \partial f_0(x_0)/\partial x$ uniformly for $t \in \mathbb{R}$ as $\varepsilon \rightarrow 0$ and $G = O(\|z\|^2)$ as $z \rightarrow 0$ uniformly for $t \in \mathbb{R}$ and fixed $\varepsilon \in (0, \varepsilon_0)$.

Let $v(x, y, \varepsilon)$ and $w(x, y, \varepsilon)$ be two families of ε -depending continuous skew-product vector fields defined on $\tilde{D} = D \times \mathbb{R} \subset \mathbb{C}^n \times \mathbb{R}$ and analytic in $x \in D$ for fixed y and ε , where $\varepsilon \in (0, \varepsilon_0)$ is the parameter. More precisely, $v = (v_1(x, y, \varepsilon), v_2(y, \varepsilon))$ and $w = (w_1(x, y, \varepsilon), v_2(y, \varepsilon))$, where $v_2 \in C(\mathbb{R} \times (0, \varepsilon_0), \mathbb{R})$, v_1 and $w_1 \in C(\tilde{D} \times (0, \varepsilon_0), \mathbb{C}^n)$ are analytic in $x \in D$ for fixed y and ε . Moreover, we say v and w are *analytically equivalent* if there exists an ε -depending coordinate substitution $z = u(x, y, \varepsilon)$, which changes one vector field into the other, where $u \in C(\tilde{D})$, $\varepsilon \in (0, \varepsilon_0)$ is the parameter and u is analytic in $x \in D$ for fixed y and ε . Let

$$R = w - v, \quad v_s = v + sR,$$

then $v_0 = v$, $v_1 = w$. Consider the ε -depending vector field on the $\tilde{D} \times \Delta$

$$V(x, y, \varepsilon, s) = (v_s(x, y, \varepsilon), 0), \quad s \in \Delta,$$

where $\Delta = \{x \in \mathbb{C} : \|x\| \leq 2\}$.

Lemma 10. Assume there exists an ε -depending vector field

$$U(x, y, \varepsilon, s) = (h(x, y, \varepsilon, s), 1), \quad (x, y, s) \in \tilde{D} \times \Delta,$$

satisfying

$$(18) \quad [h, v_s] = R,$$

where $h \in C(\tilde{D} \times \Delta)$ is analytic on $D \times \Delta$ for fixed y and ε , $\varepsilon \in (0, \varepsilon_0)$ is the parameter and $[\cdot, \cdot]$ is the Lie bracket taken with respect to the variables x and y . Let $\tilde{D}_0, \tilde{D}_1 \subset \tilde{D}$ be two domains satisfying

$$g_U^1(\tilde{D}_0 \times \{0\}) = \tilde{D}_1 \times \{1\},$$

where g_U^1 is the time-1 map defined by the of vector field U , then the two vector field $v|_{\tilde{D}_0}$ and $w|_{\tilde{D}_1}$ are analytically equivalent.

Proof. Note that the set $\{s = \text{constant}\}$ is invariant under the vector field V . Moreover, the homological equation (18) implies $[U, V] \equiv 0$, where $[\cdot, \cdot]$ is the Lie bracket taken with respect to the variables x, y and s . Together with the condition that g_U^1 maps $\tilde{D}_0 \times \{0\}$ into $\tilde{D}_1 \times \{1\}$, it follows

$$g_U^1 \circ g_{V|_{s=0}}^t = g_{V|_{s=1}}^t \circ g_U^1.$$

Thus we complete the proof by the differentiability on the initial values and the fact that $V|_{s=0} = (v, 0)$ and $V|_{s=1} = (w, 0)$. \square

Lemma 11. *The function*

$$h(x, y, \varepsilon, s) = - \int_0^\infty X^{-1}(t; x, y, \varepsilon, s) \cdot R \circ g^t(x, y, \varepsilon, s) dt$$

is a formal solution of the homological equation (18), where $g^t(x, y, \varepsilon, s)$ is the time- t map defined by the vector field v_s and the matrix solution $X(t; x, y, \varepsilon, s)$ is defined as

$$X(t; x, y, \varepsilon, s) = \frac{\partial g^t(x, y, \varepsilon, s)}{\partial(x, y)}.$$

Proof. For simplicity of notation, we fix ε, s and denote $\bar{x} = (x, y)$, $g_{v_s}^t \bar{x} = g^t(x, y, \varepsilon, s)$ and $X(t; \bar{x}) = X(t; x, y, \varepsilon, s)$. Let $h^\tau := (g_{v_s}^\tau)_* h$ which is defined as

$$h^\tau(\bar{x}) = (X(t, \bar{x})h) \circ g_{v_s}^{-\tau}(\bar{x}) = X(\tau; g_{v_s}^{-\tau} \bar{x})h(g_{v_s}^{-\tau} \bar{x}).$$

Since we have

$$\begin{aligned} g_{v_s}^\tau \bar{x} &= x + \tau v_s(\bar{x}) + o(\tau), \\ X(\tau; \bar{x}) &= I + \tau \frac{\partial v_s(\bar{x})}{\partial \bar{x}} + o(\tau). \end{aligned}$$

It follows that

$$\begin{aligned} h^\tau(\bar{x}) &= h \circ g_{v_s}^{-\tau} \bar{x} + \tau \left(\frac{\partial v_s}{\partial \bar{x}} h \right) \circ g_{v_s}^{-\tau} \bar{x} + o(\tau) \\ &= h(\bar{x}) - \tau \frac{\partial h(\bar{x})}{\partial \bar{x}} v_s(\bar{x}) + \tau \frac{\partial v_s(\bar{x})}{\partial \bar{x}} h(\bar{x}) + o(\tau), \end{aligned}$$

which means

$$\left. \frac{dh^\tau}{d\tau} \right|_{\tau=0} = [v_s, h].$$

Again by definition, we have

$$\begin{aligned} h^\tau &= - \int_0^\infty X^{-1}(-\tau; \bar{x}) X^{-1}(t; g_{v_s}^{-\tau} \bar{x}) R(g_{v_s}^{-\tau+t} \bar{x}) dt \\ &= - \int_0^\infty X(t - \tau; \bar{x}) R(g_{v_s}^{t-\tau} \bar{x}) dt \\ &= - \int_{-\tau}^\infty X^{-1}(t; \bar{x}) R(g_{v_s}^t \bar{x}) dt. \end{aligned}$$

So

$$\left. \frac{dh^\tau}{d\tau} \right|_{\tau=0} = -X^{-1}(0, \bar{x}) R(g_{v_s}^0 \bar{x}) = -R(\bar{x}).$$

This completes the proof of this lemma. \square

Let $\lambda(A) = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ be the eigenvalues of the matrix $A = \partial f_0(x_0)/\partial x$. Assume $\operatorname{Re} \lambda_1 \leq \dots \leq \operatorname{Re} \lambda_n < 0$. Now we consider the following system

$$(19) \quad \dot{x} = \varepsilon \tilde{A}(t, \varepsilon)x + \varepsilon \tilde{G}(x, t, \varepsilon) + s\varepsilon r(x, t, \varepsilon),$$

where $\tilde{A}(t, \varepsilon) \rightarrow A$ uniformly for $t \in \mathbb{R}$ as $\varepsilon \rightarrow 0$ and it is almost periodic in t for a fixed ε , \tilde{G} and $r \in \mathcal{F}(U_\delta, [0, \varepsilon_0))$, $\tilde{G}(z, t, 0) = r(z, t, 0) = 0$. Moreover, \tilde{G} is a polynomial of degree $\tilde{d} > \operatorname{Re} \lambda_1 / \operatorname{Re} \lambda_n$ with respect to x , $\tilde{G}(x, t, \varepsilon) = J \varepsilon t_{x=0}^{\tilde{d}} G(x, t, \varepsilon)$, $r(x, t, \varepsilon) = G(x, t, \varepsilon) - \tilde{G}(x, t, \varepsilon)$ and $s \in \Delta = \{x \in \mathbb{C} : \|x\| \leq 2\}$.

Lemma 12. *Let $W(x, t, \varepsilon, s) = \tilde{G}(x, t, \varepsilon) + sr(x, t, \varepsilon)$. Consider the following system*

$$(20) \quad \frac{dx}{dt} = \varepsilon \tilde{A}(t + y, \varepsilon)x + \varepsilon W(x, t + y, \varepsilon, s),$$

where y is a real parameter, \tilde{A} , \tilde{G} and r are just defined before the statement of the lemma. Let $\mathcal{G}^t(x, y, \varepsilon)$ be the solution of system (20) with the initial condition $\mathcal{G}^0(x, y, \varepsilon) = x$. Then there exists $\varepsilon_1 > 0$ and $\delta_1 > 0$ independent of ε such that $\mu(\varepsilon_1, \delta_1) \rightarrow 0$ as $(\varepsilon_1, \delta_1) \rightarrow 0$ and for a fixed $(y, \varepsilon) \in \mathbb{R} \times (0, \varepsilon_1)$ the following statements hold.

- (a) $\|r(x, t, \varepsilon)\| \leq C\|x\|^{\tilde{d}+1}$ for all $(x, t) \in U_{\delta_0} \times \mathbb{R}$.
- (b) $\|\mathcal{G}(t; x, y, \varepsilon)\| \leq C e^{\varepsilon(\operatorname{Re} \lambda_n + \mu(\varepsilon_1, \delta_1))t}$ for all $(t, x) \in [0, \infty) \times U_{\delta_1}$, $s \in \Delta$.
- (c) $\|\partial_x^{-1} \mathcal{G}(t; x, y, \varepsilon)\| \leq C e^{\varepsilon(-\operatorname{Re} \lambda_1 + \mu(\varepsilon_1, \delta_1))t}$ for all $(t, x) \in [0, \infty) \times U_{\delta_1}$, $s \in \Delta$.

Proof. By the Cauchy's integral representation

$$\begin{aligned}\partial_x^k W(x, t, \varepsilon, s) : &= \frac{\partial^{|k|} W(x_1, \dots, x_n, t, \varepsilon, s)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \\ &= \frac{k!}{(2\pi\sqrt{-1})^n} \int_{\gamma} \frac{W(z, t, \varepsilon, s) dz}{(z-x)^{k+e}},\end{aligned}$$

where $|k| = \sum_{i=1}^n k_i$, $e = (1, \dots, 1) \in \mathbb{Z}_+^n$ and $\gamma = \{z : |z_i| = r - \chi, i = 1, \dots, n\}$ for $0 < \chi < 1$, $\partial_x^k W(x, t, \varepsilon, s)$ is an almost periodic function in the variable t uniformly for $\|x\| \leq \delta_0 < \delta$ and $k \in \mathbb{Z}_+^n$. Furthermore, the following norm estimations are valid for $0 < \delta_0 < \delta/3$

$$\begin{aligned}\sup_{U_{\delta_0} \times \mathbb{R}} \|\partial_x W(x, t, \varepsilon, s)\| &= \rho \leq \frac{C_1 M}{\delta} \delta_0, \\ \|r(x, t, \varepsilon)\| &\leq \frac{C_1 M (\tilde{d}+1)^n (\tilde{d}+1)!}{\delta^{\tilde{d}+1}} \|x\|^{\tilde{d}+1}, \quad (x, t) \in U_{\delta_0} \times \mathbb{R},\end{aligned}$$

where C_1 is a constant depending on n , $\sup_{U_{\delta} \times \mathbb{R}} \|W\| = M < \infty$ and $\partial_x W$ is the Jacobian matrix of W with respect to the variable x . This completes the proof of statement (a).

Consider the linear part of system (20) $\dot{x} = \varepsilon \tilde{A}(t+y, \varepsilon)x$. Let $\Phi(t)$ be its fundamental matrix with the initial condition $\Phi(0) = I$ and denote $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$. Then by Lemma 4, we have

$$\|\Phi(t+y, y)\| \leq (5/2)K^2 e^{\varepsilon(\operatorname{Re}\lambda_n + 2K\beta)t},$$

where $\beta = \sup_{\mathbb{R}} \|\tilde{A}(t, \varepsilon) - A\| < -\operatorname{Re}\lambda_n/2K$ for $0 < \varepsilon < \varepsilon_2$. Now we can rewrite system (20) as the integral equation

$$\mathcal{G}(t; x, y, \varepsilon) = \Phi(t+y, y)x + \int_0^t \varepsilon \Phi(t+y, v+y)W(\mathcal{G}(v; x, y, \varepsilon), v+y, \varepsilon, s) dv,$$

Since $\rho = \sup_{U_{\delta_0} \times \mathbb{R}} \|\partial_x W(x, t, \varepsilon, s)\|$, we have

$$\begin{aligned}\|\mathcal{G}(t; x, y, \varepsilon)\| &\leq (5/2)K^2 \left(e^{\varepsilon(\operatorname{Re}\lambda_n + 2K\beta)t} + \varepsilon \rho \int_0^t e^{\varepsilon(\operatorname{Re}\lambda_n + 2K\beta)(t-v)} \|\mathcal{G}(v; x, y, \varepsilon)\| dv \right) \\ &\leq (5/2)K^2 e^{\varepsilon(\operatorname{Re}\lambda_n + 2K\beta)t} + (5/2)K^2 \varepsilon \rho \int_0^t \|\mathcal{G}(v; x, y, \varepsilon)\| dv.\end{aligned}$$

Then, by Lemma 6, the strong type Gronwall inequality, we obtain

$$\|\mathcal{G}(t; x, y, \varepsilon)\| \leq (5/2)K^2 e^{\varepsilon(\operatorname{Re}\lambda_n + 2K\beta + (5/2)K^2\rho)t}, \quad \text{for all } t \geq 0, \quad s \in \Delta.$$

So this proves statement (b) for $\mu(\varepsilon_1, \delta_1) = 2K\beta + (5/2)K^2\rho$ and $C_2 = (5/2)K^2$.

Now we study the Jaccobian matrix $\partial_x \mathcal{G}(t; x, y, \varepsilon)$. By take derivative with respect to x in system (20), we can get the matrix differential equation

$$\begin{aligned} \frac{d}{dt} \partial_x^{-1} \mathcal{G}(t; x, y, \varepsilon) &= -\varepsilon \partial_x^{-1} \mathcal{G}(t; x, y, \varepsilon) (\tilde{A}(t + y, \varepsilon) + \\ &\quad \partial_x W(\mathcal{G}(t; x, y, \varepsilon), t + y, \varepsilon, s)), \end{aligned}$$

which can also be written as the matrix integral equation

$$\begin{aligned} \partial_x^{-1} \mathcal{G}(t; x, y, \varepsilon) &= \Phi(y, t + y) \\ &\quad - \int_0^t \varepsilon \partial_x^{-1} \mathcal{G}(v; x, y, \varepsilon) \partial_x W(\mathcal{G}(v; x, y, \varepsilon, s), v + y, \varepsilon) \Phi(v + y, t + y) dv. \end{aligned}$$

Here, $\Phi(s, t)$ can be seen as the fundamental solution of linear system

$$\frac{d}{dt} \Phi(s, t) = -\Phi(s, t) \varepsilon \tilde{A}(t, \varepsilon).$$

So, again by Lemma 4, we have

$$\|\Phi(s + y, t + y)\| \leq (5/2) K^2 e^{\varepsilon(-\operatorname{Re} \lambda_1 + 2K\beta)(t-s)}, \quad t \geq s,$$

which means that

$$\begin{aligned} \|\partial_x^{-1} \mathcal{G}(t; x, y, \varepsilon)\| &\leq (5/2) K^2 \left(e^{\varepsilon(-\operatorname{Re} \lambda_1 + 2K\beta)t} \right. \\ &\quad \left. + \varepsilon \rho \int_0^t e^{\varepsilon(-\operatorname{Re} \lambda_1 + 2K\beta)(t-v)} \|\partial_x^{-1} \mathcal{G}(v; x, y, \varepsilon)\| dv \right). \end{aligned}$$

Therefore, again using Lemma 6, we have

$$\begin{aligned} \|\partial_x^{-1} \mathcal{G}(t; x, y, \varepsilon)\| &\leq \left((5/2) K^2 + \frac{(5/2) K^2 \rho}{-\operatorname{Re} \lambda_1 + 2K\beta + (5/2) K^2 \rho} \right) e^{\varepsilon(-\operatorname{Re} \lambda_1 + 2K\beta + (5/2) K^2 \rho)t} \\ &= C_3 e^{\varepsilon(-\operatorname{Re} \lambda_1 + \mu(\varepsilon_1, \delta_1))t}. \end{aligned}$$

Thus taking $C = \max\{C_1 M(d+1)^n(d+1)!/\delta^{d+1}, C_2, C_3\}$, $\varepsilon_1 = \varepsilon_2$ and $\delta_1 = \delta_0$, we get statement (c). \square

Lemma 13. *Let $f(x, t)$ be a continuous function, which is almost periodic in the variable t uniformly for x in any compact set D and satisfies*

$$\|f(x_1, t) - f(x_2, t)\| \leq L \|x_1 - x_2\|.$$

Consider the non-autonomous system

$$\dot{x} = f(x, t + y),$$

where y is a real parameter. Let $g^t(x, y)$ be the solution with initial condition $g^0(x, y) = x$, then $g^t(x, y)$ is almost periodic in y for a fixed t and $m(g^t) \subset m(f)$.

Proof. The solution g^t satisfies the integral equation

$$g^t(x, y) = x + \int_0^t f(g^s(x, y), s + y) ds.$$

Note that

$$\begin{aligned} A &= \int_0^t \|f(g^s(x, y + \alpha_n), s + y + \alpha_n) - f(g^s(x, y + \alpha_m), s + y + \alpha_m)\| ds \\ &\leq t \sup_{(x, t) \in D \times \mathbb{R}} \|f(x, t + \alpha_n) - f(x, t + \alpha_m)\|, \end{aligned}$$

and

$$\begin{aligned} B &= \int_0^t \|f(g^s(x, y + \alpha_n), s + y + \alpha_m) - f(g^s(x, y + \alpha_m), s + y + \alpha_m)\| ds \\ &\leq L \int_0^t \|g^s(x, y + \alpha_n) - g^s(x, y + \alpha_m)\| ds. \end{aligned}$$

Thus we have

$$\|g^t(x, y + \alpha_n) - g^t(x, y + \alpha_m)\| \leq A + B.$$

By Lemma 6, we get

$$\|g^t(x, y + \alpha_n) - g^t(x, y + \alpha_m)\| \leq te^{Lt} \sup_{(x, t) \in D \times \mathbb{R}} \|f(x, t + \alpha_n) - f(x, t + \alpha_m)\|,$$

which means $g^t(x, y)$ is almost periodic in the variable y for a fixed t , and by the definition $m(g^t) \subset m(f)$. \square

Proof of Theorem 2. First we do the change of variables (4), and we get system (5), which can be transformed into system (17) using Lemma 9. In order to eliminate the discrepancy of order great than \tilde{d} , we study system (19) instead of system (17). Considering the corresponding autonomous system of system (19) in higher dimension, we get

$$\dot{z} = \varepsilon \tilde{A}(y, \varepsilon)z + \varepsilon W(z, y, \varepsilon, s), \quad \dot{y} = 1,$$

where W is the function defined in Lemma 12. Let $\tilde{W}(z, y, \varepsilon, s) = \varepsilon \tilde{A}(y, \varepsilon)z + \varepsilon W(z, y, \varepsilon, s)$. Applying Lemma 10 to this system, we get the homological equation (18) for $v_s = (\tilde{W}, 1)$, $h = (h_1, h_2)$ and $R = (\varepsilon r, 0)$. By Lemma 11, it has the formal solution,

$$\begin{aligned} h(x, y, \varepsilon, s) &= - \int_0^\infty X^{-1}(t; x, y, \varepsilon, s) \cdot R \circ g^t(x, y, \varepsilon, s) dt \\ &= \begin{pmatrix} -\varepsilon \int_0^\infty \partial_x^{-1} \mathcal{G}(t; x, y, \varepsilon) \cdot r(\mathcal{G}(t; x, y, \varepsilon), t + y, \varepsilon) dt \\ 0 \end{pmatrix}, \end{aligned}$$

where \mathcal{G} is defined in Lemma 12. By the norm estimation of Lemma 12, we know that the increasing of the norm of the integrand of h is control by a exponential function $\varepsilon C e^{\varepsilon \eta t}$, $t \geq 0$, where $\eta = (\tilde{d} + 1)\text{Re}\lambda_n - \text{Re}\lambda_1 + (\tilde{d} +$

$2)\mu(\varepsilon_1, \delta_1)$. Since $\tilde{d} > \operatorname{Re}\lambda_n/\operatorname{Re}\lambda_1$, we can choose ε_1 and δ_1 small enough such that $\eta < 0$. So h converges with the maximum norm. Therefore, by the differentiability of the solutions with respect to the initial value and parameter, the time-1 map w of h is analytic in the domain U_{δ_1} with t and ε fixed. Moreover, by the result of Lemma 13 and similar arguments, h is an almost periodic function in y uniformly for $x \in U_{\delta_1}$ with fixed ε , and so the same occurs for the time-1 map w .

In addition, when the real parts of $\lambda(A)$ are non-resonant, we can apply Theorem 1 to eliminate any finite order of W with respect to the variable x . By the same arguments, we get the final result of the theorem. \square

Proof of Corollary 3. As above we do the change of variables (4), and we get system (5), which can be transformed into system (17) using Lemma 9. Applying Theorem 1, we can obtain the N -th order normal form for any fixed $N > 0$. By Theorem 2, we complete the proof of the theorem. \square

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