DISTORTION OF HAUSDORFF MEASURES AND IMPROVED PAINLEVÉ REMOVABILITY FOR QUASIREGULAR MAPPINGS

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ABSTRACT. The classical Painlevé theorem tells that sets of zero length are removable for bounded analytic functions, while (some) sets of positive length are not. For general K-quasiregular mappings in planar domains the corresponding critical dimension is $\frac{2}{K+1}$. We show that when $K > 1$, unexpectedly one has improved removability. More precisely, we prove that sets E of σ-finite Hausdorff $\frac{2}{K+1}$-measure are removable for bounded K-quasiregular mappings. On the other hand, $\dim(E) = \frac{2}{K+1}$ is not enough to guarantee this property. We also study absolute continuity properties of pull-backs of Hausdorff measures under K-quasiconformal mappings, in particular at the relevant dimensions 1 and $\frac{2}{K+1}$. For general Hausdorff measures $\mathcal{H}^t$, $0 < t < 2$, we reduce the absolute continuity properties to an open question on conformal mappings, see Conjecture 2.3.

1. INTRODUCTION

A homeomorphism $\phi : \Omega \to \Omega'$ between planar domains $\Omega, \Omega' \subset \mathbb{C}$ is called K-*quasiconformal* if it belongs to the Sobolev space $W^{1,2}_{loc}(\Omega)$ and satisfies the *distortion inequality*

$$\max_\alpha |\partial_\alpha \phi| \leq K \min_\alpha |\partial_\alpha \phi| \quad \text{a.e. in } \Omega \qquad (1.1)$$

It has been known since the work of Ahlfors [3] that quasiconformal mappings preserve sets of zero Lebesgue measure. It is also well known that they preserve sets of zero Hausdorff dimension, since K-quasiconformal mappings are Hölder continuous with exponent $1/K$, see [20]. However, these maps do not preserve Hausdorff dimension in general, and it was in the work

of the first author [4] where the precise bounds for the distortion of dimension were given. For any compact set E with dimension t and for any K-quasiconformal mapping ϕ we have

$$\frac{1}{K}\left(\frac{1}{t}-\frac{1}{2}\right) \leq \frac{1}{\dim(\phi(E))} - \frac{1}{2} \leq K\left(\frac{1}{t}-\frac{1}{2}\right) \tag{1.2}$$

Furthermore, these bounds are optimal, that is, equality may occur in either estimate.

The fundamental question we study in this work is whether the estimates (1.2) can be improved to the level of Hausdorff measures $\mathcal{H}^t$. In other words, if ϕ is a planar K-quasiconformal mapping, $0 < t < 2$ and $t' = \frac{2Kt}{2+(K-1)t}$, we ask whether it is true that

$$\mathcal{H}^t(E) = 0 \;\; \Rightarrow \;\; \mathcal{H}^{t'}(\phi(E)) = 0, \tag{1.3}$$

or put briefly, $\phi_*\mathcal{H}^{t'} \ll \mathcal{H}^t$. Note that the above classical results of Ahlfors and Mori assert that this is true when $t = 0$ or $t = 2$. In fact [4], for the Lebesgue measure one has even precise quantitative bounds

$$|\phi(E)| \leq C\, |E|^{\frac{1}{K}},$$

a result which also leads to the sharp Sobolev regularity, $\phi \in W^{1,p}_{loc}(\mathbb{C})$ for every $p < \frac{2K}{K-1}$.

As a first main result of this paper we prove (1.3) for $t = \frac{2}{K+1}$, i.e. for the case of image dimension $t' = 1$.

Theorem 1.1. *Let ϕ be a planar K-quasiconformal mapping, and let E be a compact set. Then,*

$$\mathcal{M}^1(\phi(E)) \leq C\left(\mathcal{M}^{\frac{2}{K+1}}(E)\right)^{\frac{K+1}{2K}} \tag{1.4}$$

As a consequence,

$$\mathcal{H}^{\frac{2}{K+1}}(E) = 0 \;\; \Rightarrow \;\; \mathcal{H}^1(\phi(E)) = 0$$

Here $\mathcal{M}^t$ denotes t-dimensional Hausdorff content. As one of the key points in proving Theorem 1.1 we show that for planar quasiconformal mappings h which are *conformal* in the complement $\mathbb{C} \setminus E$ the inequality (1.4) improves strongly: such mappings h essentially preserve the 1-dimensional Hausdorff content of the compact set E,

$$\mathcal{M}^1(h(E)) \leq C_K\, \mathcal{M}^1(E) \tag{1.5}$$

The constant C_K depends only on K if h is normalized at ∞, requiring $h(z) = z + \mathcal{O}(1/z)$. For the area the corresponding estimate was shown in [4]. In fact, as we will see later, a counterpart of (1.5) for the t-dimensional Hausdorff content $\mathcal{M}^t$ is the only missing detail for proving the absolute

continuity $\phi^* \mathcal{H}^{t'} \ll \mathcal{H}^t$ for general t. Towards solving (1.3) we conjecture that actually

$$\mathcal{M}^t(h(E)) \leq C \, \mathcal{M}^t(E), \qquad 0 < t \leq 2$$

whenever $E \subset \mathbb{C}$ compact and h is normalized and conformal in $\mathbb{C} \setminus E$ admitting a K-quasiconformal extension to $\mathbb{C}$. For a more detailed discussion and other formulations see Section 2.

The reason our methods work only in the special case of dimension $t = 1$ is that the content $\mathcal{M}^1$ is equivalent to a suitable BMO-capacity [29]. For dimensions $1 < t < 2$, we do have interpolating estimates but unfortunately we have to settle for the Riesz capacities. We have

$$C_{\alpha,t}(h(E)) \leq C \, C_{\alpha,t}(E)$$

for any $t \in (1, 2)$ and $\alpha = \frac{2}{t} - 1$. This fact has consequences towards the absolute continuity of Hausdorff measures under quasiconformal mappings, but these bounds are not strong enough for (1.3) when $1 < t' < 2$.

Recall that f is a K-*quasiregular mapping* in a domain $\Omega \subset \mathbb{C}$ if $f \in W^{1,2}_{loc}(\Omega)$ and f satisfies the distortion inequality (1.1). When $K = 1$, this class agrees with the class of analytic functions on Ω. The classical *Painlevé problem* consists of giving metric and geometric characterizations of those sets E that are removable for bounded analytic functions. Here Painlevé's theorem tells us that sets of zero length are removable, while Ahlfors [2] showed that no set of Hausdorff dimension > 1 has this property. For the related BMO-problem Kaufman [15] proved that the condition $\mathcal{H}^1(E) = 0$ is a precise characterization for removable singularities of BMO analytic functions. Thus for analytic removability, dimension 1 is the critical point both for L^∞ and BMO. However, the solution to the original Painlevé problem lies much deeper and was only recently achieved by Tolsa ([27],[28]) in terms of curvatures of measures. Under the assumption that $\mathcal{H}^1(E)$ is finite, Painlevé's problem was earlier solved by G. David [11], who showed that a set E of positive and finite length is removable for bounded analytic functions if and only if it is purely unrectifiable. Furthermore, the countable semiadditivity of analytic capacity, due to Tolsa [27], asserts that this result remains true if we only assume $\mathcal{H}^1(E)$ to be σ-finite.

It is now natural to approach the Painlevé problem for K-quasiregular mappings. We say that a compact set E is *removable for bounded K-quasiregular mappings*, or simply K-*removable*, if for every open set $\Omega \supset E$, every bounded K-quasiregular mapping $f : \Omega \setminus E \to \mathbb{C}$ admits a K-quasiregular extension to Ω. In this definition, as in the analytic setting, we may replace $L^\infty(\Omega)$ by $BMO(\Omega)$ to get a close variant of the problem.

The sharpness of the bounds in equation (1.2) determines the index $\frac{2}{K+1}$ as the critical dimension in both the L^∞ and BMO quasiregular removability problems. In fact, Iwaniec and Martin previously conjectured [14] that in $\mathbb{R}^n$, $n \geq 2$, sets with Hausdorff measure $\mathcal{H}^{\frac{n}{K+1}}(E) = 0$ are removable for bounded K-quasiregular mappings. A preliminary positive answer for $n = 2$ was described in [7]. Generalizing this, in the present work we show that surprisingly, for $K > 1$ one can do even better: we have the following improved Painlevé removability.

Theorem 1.2. *Let $K > 1$ and suppose E is any compact set with*

$$\mathcal{H}^{\frac{2}{K+1}}(E) \quad \sigma - finite.$$

Then E is removable for all bounded K-quasiregular mappings.

The theorem fails for $K = 1$, since for instance the line segment $E = [0, 1]$ is not removable.

For the converse direction, the work [4] finds for every $t > \frac{2}{K+1}$ non-K-removable sets with $\dim(E) = t$. We make an improvement also here and construct compact sets with dimension precisely equal to $\frac{2}{K+1}$ yet not removable for some bounded K-quasiregular mappings. For details see Theorem 5.1.

The above theorems are closely connected via the classical Stoilow factorization, which tells [7], [16] that in planar domains K-quasiregular mappings are precisely the maps f representable in the form $f = h \circ \phi$, where h is analytic and ϕ is K-quasiconformal. Indeed, the first step in proving Theorem 1.2 will be to show that for a general K-quasiconformal mapping ϕ one has

$$\mathcal{H}^{\frac{2}{K+1}}(E) \ \sigma\text{-finite} \ \Rightarrow \ \mathcal{H}^1(\phi(E)) \ \sigma\text{-finite}$$

However, this conclusion will not be enough since there are rectifiable sets of finite length, such as $E = [0, 1]$, that are non-removable for bounded analytic functions. Therefore, in addition, we need to establish that such 'good' sets of positive analytic capacity actually behave better also under quasiconformal mappings. That is, we show that up to a set of zero length,

$$F \quad 1\text{-rectifiable} \ \Rightarrow \ \dim(\phi(F)) > \frac{2}{K+1}$$

For details and a precise formulation see Corollary 3.2.

The paper is structured as follows. In Section 2 we deal with the quasiconformal distortion of Hausdorff measures and of other set functions. In

Section 3 we study the quasiconformal distortion of 1-rectifiable sets. Section 4 gives the proof for the improved Painlevé removability theorem for K-quasiregular mappings and other related questions. Finally in section 5 we describe a construction of non-removable sets.

2. Absolute Continuity

There are several natural ways to normalize the quasiconformal mappings $\phi : \mathbb{C} \to \mathbb{C}$. In this work we mostly use the *principal K-quasiconformal* mappings, i.e. mappings that are conformal outside a compact set and are normalized by $\phi(z) - z = O\left(\frac{1}{|z|}\right)$ as $|z| \to \infty$.

It is shown in the work [4] of the first author that for all K-quasiconformal mappings $\phi : \mathbb{C} \to \mathbb{C}$,

$$|\phi(E)| \leq C\,|E|^{1/K} \tag{2.1}$$

where C is a constant that depends on the normalizations. By scaling we may always arrange

$$\mathrm{diam}(\phi(E)) = \mathrm{diam}(E) \leq 1 \tag{2.2}$$

and then $C = C(K)$ depends only on K. In order to achieve the result (2.1), one first reduces to the case where the set E is a finite union of disks. Secondly, applying Stoilow factorization methods the mapping ϕ is written as $\phi = h \circ \phi_1$, where both $h, \phi_1 : \mathbb{C} \to \mathbb{C}$ are K-quasiconformal mappings, such that ϕ_1 is conformal on E and h is conformal in the complement of the set $F = \phi_1(E)$. Here one obtains the right conclusion for ϕ_1,

$$|\phi_1(E)| \leq C\,|E|^{\frac{1}{K}}$$

by including ϕ_1 in a holomorphic family of quasiconformal mappings. Further, one shows in [4, p. 50] that under the special assumption where h is conformal outside of F, we have

$$|h(F)| \leq C\,|F| \tag{2.3}$$

where the constant C still depends only on K.

In searching for absolute continuity properties of other Hausdorff measures under quasiconformal mappings, such a decomposition seems unavoidable, and this leads one to look for counterparts of (2.3) for Hausdorff measures $\mathcal{H}^t$ or Hausdorff contents $\mathcal{M}^t$. Here we establish the result for the dimension $t = 1$.

Lemma 2.1. *Suppose $E \subset \mathbb{C}$ is a compact set, and let $\phi : \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping, such that ϕ is conformal on $\mathbb{C} \setminus E$. Then,*

$$\mathcal{M}^1(\phi(E)) \simeq \mathcal{M}^1(E)$$

with constants depending only on K.

In order to prove this result some background is needed. The space of functions of bounded mean oscillation, BMO, is invariant under quasiconformal changes of variables [24]. More precisely, if ϕ is a K-quasiconformal mapping and $f \in BMO(\mathbb{C})$, then $f \circ \phi \in BMO(\mathbb{C})$ with BMO-norm

$$\|f \circ \phi\|_* \leq C(K)\, \|f\|_*$$

The space $BMO(\mathbb{C})$ gives rise to a capacity,

$$\gamma_0(F) = \sup |f'(\infty)|$$

where the supremum runs over all functions $f \in BMO(\mathbb{C})$ with $\|f\|_* \leq 1$, that are holomorphic on $\mathbb{C} \setminus E$ and satisfy $f(\infty) = 0$. Here $f'(\infty) = \lim_{|z| \to \infty} z\,(f(z) - f(\infty))$. Observe that in this situation $\bar\partial f$ defines a distribution supported on F, and actually $|\langle \bar\partial f, 1 \rangle| = |f'(\infty)|$. It turns out [29] that for any compact set E we have

$$\gamma_0(E) \simeq \mathcal{M}^1(E). \tag{2.4}$$

According to the theorem of Kaufman [15], in the class of functions $f \in BMO(\mathbb{C})$ holomorphic on $\mathbb{C} \setminus E$ every f admits a holomorphic extension to the whole plane if and only if $\mathcal{M}^1(E) = 0$. That is, γ_0 characterizes those compact sets which are removable for BMO holomorphic functions. Because of these equivalences, to prove Lemma 2.1 it suffices to show that $\gamma_0(\phi(E)) \simeq \gamma_0(E)$.

Proof of Lemma 2.1. Suppose that $f \in BMO(\mathbb{C})$ is a holomorphic mapping of $\mathbb{C} \setminus E$ such that $\|f\|_* \leq 1$ and $f(\infty) = 0$. Then the function $g = f \circ \phi^{-1}$ is in $BMO(\mathbb{C})$ and $\|g\|_* \leq C(K)$. On the other hand, g is holomorphic on $\mathbb{C} \setminus \phi(E)$, and since ϕ is a principal K-quasiconformal mapping, $g(\infty) = 0$ and

$$|g'(\infty)| = \lim_{|z| \to \infty} |z\,g(z)| = \lim_{|w| \to \infty} |\phi(w)\,f(w)| = |f'(\infty)|.$$

Hence, $\gamma_0(E) \leq C(K)\,\gamma_0(\phi(E))$. The converse inequality follows by symmetry, since also the inverse ϕ^{-1} is a principal mapping,. $\qquad\square$

This lemma is a first step towards the results on absolute continuity, as presented in the following reformulation of Theorem 1.1.

Theorem 2.2. *Let E be a compact set and $\phi : \mathbb{C} \to \mathbb{C}$ K-quasiconformal, normalized by (2.2). Then*

$$\mathcal{M}^1(\phi(E)) \leq C \left(\mathcal{M}^{\frac{2}{K+1}}(E) \right)^{\frac{K+1}{2K}}$$

where the constant $C = C(K)$ depends only on K. In particular, if $\mathcal{H}^{\frac{2}{K+1}}(E) = 0$ then $\mathcal{H}^1(\phi(E)) = 0$.

Proof. There is no restriction if we assume $E \subset \mathbb{D}$. We can also assume that ϕ is a principal K-quasiconformal mapping, conformal outside $\mathbb{D}$. Now, given $\varepsilon > 0$, there is a finite covering of E by open disks $D_j = D(z_j, r_j)$, $j = 1, ..., n$, such that

$$\sum_{j=1}^{n} r_j^{\frac{2}{K+1}} \leq \mathcal{M}^{\frac{2}{K+1}}(E) + \varepsilon$$

Denote $\Omega = \cup_{j=1}^{n} D_j$. As in [4], we use a decomposition $\phi = h \circ \phi_1$, where both ϕ_1, h are principal K-quasiconformal mappings. Moreover, we may require that ϕ_1 is conformal in $\Omega \cup (\mathbb{C} \setminus \mathbb{D})$ and that h is conformal outside $\phi_1(\overline{\Omega})$.

By Lemma 2.1, we see that

$$\mathcal{M}^1(\phi(E)) \leq \mathcal{M}^1(\phi(\Omega)) = \mathcal{M}^1(h \circ \phi_1(\Omega)) \leq C \, \mathcal{M}^1(\phi_1(\Omega))$$

Hence the problem has been reduced to estimating $\mathcal{M}^1(\phi_1(\Omega))$. For this, K-quasidisks have area comparable to the square of the diameter,

$$\text{diam}(\phi_1(D_j)) \simeq |\phi_1(D_j)|^{1/2} = \left(\int_{D_j} J(z, \phi_1) \, dA(z) \right)^{\frac{1}{2}}$$

with constants which depend only on K. Thus, using Hölder estimates twice, we obtain

$$\sum_{j=1}^{n} \text{diam}(\phi_1(D_j)) \leq C(K) \left(\sum_{j=1}^{n} \int_{D_j} J(z, \phi_1)^p dA(z) \right)^{\frac{1}{2p}} \left(\sum_{j=1}^{n} |D_j|^{\frac{p-1}{2p-1}} \right)^{1 - \frac{1}{2p}}$$

as long as $J(z, \phi_1)^p$ is integrable. But here we are in the special situation of [6, Lemma 5.2]. Namely, as ϕ_1 is conformal in the subset Ω, we may take $p = \frac{K}{K-1}$ and apply [6] to obtain

$$\sum_{j=1}^{n} \int_{D_j} J(z, \phi_1)^p dA(z) = \int_{\Omega} J(z, \phi_1)^p dA(z) \leq \pi$$

With the above choice of p one has $\frac{p-1}{2p-1} = \frac{1}{K+1}$. Hence we get

$$\sum_{j=1}^{n} \text{diam}(\phi_1(D_j)) \leq C(K) \left(\sum_{j=1}^{n} r_j^{\frac{2}{K+1}} \right)^{\frac{K+1}{2K}} \leq C(K) \left(\mathcal{M}^{\frac{2}{K+1}}(E) + \varepsilon \right)^{\frac{K+1}{2K}}$$

$$(2.5)$$

But $\cup_j \phi_1(D_j)$ is a covering of $\phi_1(\Omega)$, so that actually we have

$$\mathcal{M}^1(\phi(E)) \leq C\mathcal{M}^1(\phi_1(\Omega)) \leq C(K) \left(\mathcal{M}^{\frac{2}{K+1}}(E) + \varepsilon \right)^{\frac{K+1}{2K}}$$

Since this holds for every $\varepsilon > 0$, the result follows. $\qquad\square$

At this point we want to emphasize that for a general quasiconformal mapping ϕ we have $J(z,\phi) \in L^p_{loc}$ only for $p < \frac{K}{K-1}$. The improved borderline integrability $(p = \frac{K}{K-1})$ under the extra assumption that $\phi_{|\Omega}$ is conformal was shown in [6, Lemma 5.2]. This phenomenon was crucial for our argument, since we are studying Hausdorff measures rather than dimension. Actually, the same procedure shows that inequality (2.5) works in a much more general setting. That is, still under the special assumption that ϕ_1 is conformal in $\cup_{j=1}^n D_j$, we have for any $t \in [0, 2]$

$$\left(\sum_{j=1}^n \mathrm{diam}(\phi_1(D_j))^d \right)^{\frac{1}{d}} \leq C(K) \left(\sum_{j=1}^n \mathrm{diam}(D_j)^t \right)^{\frac{1}{t} \frac{1}{K}} \tag{2.6}$$

where $d = \frac{2Kt}{2+(K-1)t}$. On the other hand, another key point in our proof was the estimate

$$\mathcal{M}^1(h(E)) \leq C\,\mathcal{M}^1(E),$$

valid whenever h is a principal K-quasiconformal mapping which is conformal outside E. We believe that finding the counterpart to this estimate is crucial for understanding distortion of Hausdorff measures under quasiconformal mappings. We make the following

Conjecture 2.3. *Suppose we are given a real number $d \in (0,2]$. Then for any compact set $E \subset \mathbb{C}$ and for any principal K-quasiconformal mapping h which is conformal on $\mathbb{C} \setminus E$, we have*

$$\mathcal{M}^d(h(E)) \simeq \mathcal{M}^d(E) \tag{2.7}$$

with constants that depend on K and d only.

One may also formulate a convenient discrete variant, which is actually stronger than Conjecture 2.3.

Question 2.4. *Suppose we are given a real number $d \in (0,2]$ and a finite number of disjoint disks $D_1, ..., D_n$. If a mapping h is conformal on $\mathbb{C} \setminus \cup_{j=1}^n D_j$ and admits a K-quasiconformal extension to $\mathbb{C}$, is it then true that*

$$\sum_{j=1}^n diam\,(h(D_j))^d \simeq \sum_{j=1}^n diam(D_j)^d \tag{2.8}$$

with constants that depend only on K and d ?

We already know that (2.7) is true for $d = 1$ and $d = 2$; however for Question 2.4 we know a proof only at $d = 2$. An affirmative answer to Conjecture 2.3, combined with the optimal integrability bound proving (2.6), would provide the absolute continuity of $\phi^* \mathcal{H}^d$ with respect to $\mathcal{H}^t$, where $d = \frac{2Kt}{2+(K-1)t}$, $0 \leq t \leq 2$ and $K \geq 1$. Therefore, (2.7) would have important consequences in the theory of quasiconformal mappings.

The positive answer to (2.7) for the dimension $d = 1$ was based on the equivalence (2.4) and the invariance of BMO. Actually more is true: the space VMO, equal to the BMO-closure of uniformly continuous functions, is quasiconformally invariant as well. We may also describe VMO as consisting of functions $f \in BMO$ for which

$$\lim \frac{1}{|B|} \int_B |f - f_B| = 0$$

as $|B| + \frac{1}{|B|} \to \infty$. As we now see, the invariance of VMO has interesting consequences.

Theorem 2.5. *Let $E \subset \mathbb{C}$ be a compact set, and $\phi : \mathbb{C} \to \mathbb{C}$ a K-quasiconformal mapping. If $\mathcal{H}^{\frac{2}{K+1}}(E)$ is finite (or even σ-finite), then $\mathcal{H}^1(\phi(E))$ is σ-finite.*

This result may be equivalently expressed in terms of the lower Hausdorff content. To understand this alternative formulation of Theorem 2.5, we first need some background. A *measure function* is a continuous non-decreasing function $h(t)$, $t \geq 0$, such that $\lim_{t \to 0} h(t) = 0$. If h is a measure function and $F \subset \mathbb{C}$ we set

$$\mathcal{M}^h(F) = \inf \sum_j h(\delta_j)$$

where the infimum is taken over all countable coverings of F by disks of diameter δ_j. When $h(t) = t^\alpha$, $\alpha > 0$, $\mathcal{M}^h(F) = \mathcal{M}^\alpha(F)$ equals the α-dimensional Hausdorff content of F. Moreover, the content $\mathcal{M}^\alpha$ and the measure $\mathcal{H}^\alpha$ have the same zero sets. We will denote by $\mathcal{F} = \mathcal{F}_d$ the class of measure functions $h(t) = t^d \varepsilon(t)$, $0 \leq \varepsilon(t) \leq 1$, such that $\lim_{t \to 0} \varepsilon(t) = 0$. The lower d-dimensional Hausdorff content of F is then defined by

$$\mathcal{M}^d_*(F) = \sup_{h \in \mathcal{F}_d} \mathcal{M}^h(F)$$

One has $\mathcal{M}^d_* \leq \mathcal{M}^d$ but it can happen that $\mathcal{M}^d_*(F) = 0 < \mathcal{M}^d(F)$. For instance, if F is the segment $[0,1]$ in the plane, then $\mathcal{M}^1_*(F) = 0$ but $\mathcal{M}^1(F) = 1$. An old result of Sion and Sjerve [26] in geometric measure theory asserts that $\mathcal{M}^d_*(F) = 0$ if and only if F is a countable union of sets with finite d-dimensional Hausdorff measure. For a disk B, $\mathcal{M}^d_*(B) =$

$\mathcal{M}^d(B)$, and for open sets U, $\mathcal{M}_*^d(U) \simeq \mathcal{M}^d(U)$. We may now reformulate Theorem 2.5 as follows.

Theorem 2.6. *Let $E \subset \mathbb{C}$ be a compact set, and $\phi : \mathbb{C} \to \mathbb{C}$ a principal K-quasiconformal mapping. If $\mathcal{M}_*^{\frac{2}{K+1}}(E) = 0$, then $\mathcal{M}_*^1(\phi(E)) = 0$.*

For the proof, for any bounded set $F \subset \mathbb{C}$ define first

$$\gamma_*(F) = \sup |f'(\infty)| \qquad (2.9)$$

where the supremum is taken over all functions $f \in VMO$, with $\|f\|_* \leq 1$, which are holomorphic on $\mathbb{C} \setminus F$ and satisfy $f(\infty) = 0$. Again here we may replace $|f'(\infty)|$ by $|\langle \bar{\partial} f, 1 \rangle|$. The VMO invariance leads to the following analogue of Lemma 2.1.

Lemma 2.7. *Let E be a compact set. For any principal K-quasiconformal mapping $\phi : \mathbb{C} \to \mathbb{C}$, conformal on $\mathbb{C} \setminus E$, we have*

$$\gamma_*(\phi(E)) \simeq \gamma_*(E).$$

Proof. Consider $f \in VMO$ which is analytic in $\mathbb{C} \setminus \phi(E)$ and $f(\infty) = 0$. Set $g = f \circ \phi$. Then $g \in VMO$, g is analytic on $\mathbb{C} \setminus E$, $\|g\|_* \leq C \|f\|_*$ and $|g'(\infty)| = |f'(\infty)|$ since ϕ is a principal K-quasiconformal mapping. Consequently $\gamma_*(\phi(E)) \leq C \gamma_*(E)$. $\qquad \square$

It was shown by Verdera that this VMO capacity is essentially the 1-dimensional lower content.

Lemma 2.8 ([29], p. 288). *For any compact set E, $\mathcal{M}_*^1(E) \simeq \gamma_*(E)$.*

With these tools we are ready to prove Theorem 2.6.

Proof of Theorem 2.6. Naturally, the argument is similar to that in Theorem 2.2. Without loss of generality, we may assume that $E \subset \mathbb{D}$ and that ϕ is a principal K-quasiconformal mapping. Furthermore, we may assume that $\mathcal{H}^{\frac{2}{K+1}}(E)$ is finite, and for any δ we have a finite family of disks D_i such that $E \subset \cup_i D_i$, $\sum_i \operatorname{diam}(D_i)^{\frac{2}{K+1}} \leq \mathcal{H}^{\frac{2}{K+1}}(E) + 1$ and $\operatorname{diam}(D_i) < \delta$. Set $\Omega = \cup_i D_i$. Again, we have a decomposition $\phi = \phi_2 \circ \phi_1$, where both ϕ_1 and ϕ_2 are principal K-quasiconformal mappings, and where we may require that ϕ_1 is conformal in $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup \Omega$, and ϕ_2 is conformal outside $\phi_1(\overline{\Omega})$. Thus,

$$\mathcal{M}_*^1(\phi(E)) \leq \mathcal{M}_*^1(\phi(\overline{\Omega}))$$

By Lemma 2.8, the lower content can be replaced by the VMO capacity,

$$\mathcal{M}_*^1(\phi(\overline{\Omega})) \leq C \gamma_*(\phi(\overline{\Omega}))$$

Since ϕ_2 is conformal outside of $\phi_1(\overline{\Omega})$, from Lemma 2.7 we obtain

$$\gamma_*(\phi(\overline{\Omega})) = \gamma_*(\phi_2 \circ \phi_1(\overline{\Omega})) \simeq \gamma_*(\phi_1(\overline{\Omega})) \leq C \mathcal{M}_*^1(\phi_1(\overline{\Omega}))$$

where the last inequality uses again Lemma 2.8. It hence remains to estimate $\mathcal{M}^1_*(\phi_1(\overline{\Omega}))$. For this, take $h \in \mathcal{F}$, $h(t) = t\,\varepsilon(t)$, and argue as in Theorem 2.2. Since K-quasiconformal mappings are Hölder continuous with exponent $1/K$,

$$\mathcal{M}^h(\phi_1(\overline{\Omega}))$$
$$\leq \sum_j \operatorname{diam}(\phi_1(D_j))\varepsilon(\operatorname{diam}(\phi_1(D_j))) \ \leq \ \varepsilon(C_K\,\delta^{1/K}) \sum_j \operatorname{diam}(\phi_1(D_j))$$

$$\leq \varepsilon(C_K\,\delta^{1/K}) \sum_j \left(\int_{D_j} J(z,\phi_1)^{\frac{K}{K-1}}\,dm(z) \right)^{\frac{K-1}{2K}} |D_j|^{1/2K}$$

$$\leq \varepsilon(C_K\,\delta^{1/K})\,C_K \left(\sum_j \operatorname{diam}(D_j)^{\frac{2}{K+1}} \right)^{\frac{K+1}{2K}}$$

$$\leq \varepsilon(C_K\,\delta^{1/K})\,C_K \left(\mathcal{H}^{\frac{2}{K+1}}(E) + 1 \right)^{\frac{K+1}{2K}}$$

Finally, taking $\delta \to 0$ we get $\mathcal{M}^h(\phi(E)) = 0$. This holds for any $h \in \mathcal{F}$, and the Theorem follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

One might think of extending the preceeding results from the critical index $\frac{2}{K+1}$ to arbitrary ones by using other capacities that behave like a Hausdorff content. For instance, the capacity γ_α, associated to analytic functions with the $\operatorname{Lip}(\alpha)$ norm [21], satisfies

$$\mathcal{M}^{1+\alpha}(E) \simeq \gamma_\alpha(E)$$

but unfortunately, the space $\operatorname{Lip}(\alpha)$ is not invariant under a quasiconformal change of variables. Thus, other procedures are needed. It turns out that the homogeneous Sobolev spaces provide suitable tools, basically since $\dot{W}^{1,2}(\mathbb{C})$ is invariant under quasiconformal mappings. Here recall that for $0 < \alpha < 2$ and $p > 1$, the homogeneous Sobolev space $\dot{W}^{\alpha,p}(\mathbb{C})$ is defined as the space of Riesz potentials

$$f = I_\alpha * g$$

where $g \in L^p(\mathbb{C})$ and $I_\alpha(z) = \frac{1}{|z|^{2-\alpha}}$. The norm is given by $\|f\|_{\dot{W}^{\alpha,p}(\mathbb{C})} = \|g\|_p$. When $\alpha = 1$, $\dot{W}^{1,p}(\mathbb{C})$ agrees with the space of functions f whose first order distributional derivatives are given by $L^p(\mathbb{C})$ functions. Let $f \in \dot{W}^{1,2}(\mathbb{C})$ and let ϕ be a K-quasiconformal mapping on $\mathbb{C}$. Defining $g = f \circ \phi$

we have

$$\int_{\mathbb{C}} |Dg(z)|^2 dA(z) \leq \int_{\mathbb{C}} |Df(\phi(z))|^2 |D\phi(z)|^2 \, dA(z)$$

$$\leq K \int_{\mathbb{C}} |Df(\phi(z))|^2 \, J(z,\phi) \, dA(z)$$

$$= K \int_{\mathbb{C}} |Df(w)|^2 \, dA(w)$$

so that $g \in \dot{W}^{1,2}(\mathbb{C})$. In other words, every K-quasiconformal mapping ϕ induces a bounded linear operator

$$T : \dot{W}^{1,2}(\mathbb{C}) \to \dot{W}^{1,2}(\mathbb{C}), \qquad T(f) = f \circ \phi$$

with norm depending only on K. As we have mentioned before, this operator T is also bounded on $BMO(\mathbb{C})$ [24]. Moreover, Reimann and Rychener [25, p.103] proved that $\dot{W}^{\frac{2}{q},q}(\mathbb{C})$, $q > 2$, may be represented as a complex interpolation space between $BMO(\mathbb{C})$ and $\dot{W}^{1,2}(\mathbb{C})$. It follows that T is bounded on the Sobolev spaces $\dot{W}^{\frac{2}{q},q}(\mathbb{C}), q > 2$. More precisely, there exists a constant $C = C(K,q)$ such that

$$\|f \circ \phi\|_{\dot{W}^{\frac{2}{q},q}(\mathbb{C})} \leq C\|f\|_{\dot{W}^{\frac{2}{q},q}(\mathbb{C})} \tag{2.10}$$

for any K-quasiconformal mapping ϕ on $\mathbb{C}$. These invariant function spaces provide us with related invariant capacities. Recall (e.g. [1, pp.34 and 46]) that for any pair $\alpha > 0$, $p > 1$ with $0 < \alpha p < 2$, one defines the Riesz capacity of a compact set E by

$$\dot{C}_{\alpha,p}(E) = \sup\{\mu(E)^p\}$$

where the supremum runs over all positive measures μ supported on E, such that $\|I_\alpha * \mu\|_q \leq 1$, $\frac{1}{p} + \frac{1}{q} = 1$. We get an equivalent capacity if we replace positive measures μ by distributions T supported on E, $\|I_\alpha * T\|_q \leq 1$, and take the supremum of $|\langle T, 1 \rangle|^p$.

To see the connection with equation (2.10) consider the set functions

$$\gamma_{1-\alpha,q}(E) = \sup\{|f'(\infty)|; f \text{ analytic in } \mathbb{C} \setminus E, \|f\|_{\dot{W}^{1-\alpha,q}} \leq 1 \text{ and } f(\infty) = 0\}$$

Observe again that $|f'(\infty)| = |\langle \overline{\partial} f, 1 \rangle|$ where this action must be understood in the sense of distributions. With this terminology we have

Lemma 2.9. *Suppose that E is a compact subset of the plane. Then, for any $p \in (1,2)$,*

$$\dot{C}_{\alpha,p}(E)^{1/p} \simeq \gamma_{1-\alpha,q}(E)$$

where $\alpha = \frac{2}{p} - 1$ and $q = \frac{p}{p-1}$.

Proof. On one hand, let μ be an admissible measure for $\dot{C}_{\alpha,p}$. Then, $I_\alpha * \mu$ is in L^q with norm at most 1. Define $f = \frac{1}{z} * \mu$. Clearly, f is analytic outside E, $f(\infty) = 0$ and $f'(\infty) = \mu(E)$. Moreover, up to multiplicative constants,

$$\widehat{f}(\xi) \simeq \frac{1}{\xi}\, \widehat{\mu}(\xi) = \frac{\overline{\xi}}{|\xi|}\, \frac{1}{|\xi|}\, \widehat{\mu}(\xi) = \widehat{R}\, \widehat{I_1}\, \widehat{\mu}$$

and consequently we can write

$$f = \frac{1}{z} * \mu = R(I_1 * \mu) = I_{1-\alpha} * R(I_\alpha * \mu)$$

where R is a Calderón-Zygmund operator and $\|f\|_{\dot{W}^{1-\alpha,q}} = \|R(I_\alpha * \mu)\|_q \lesssim \|I_\alpha * \mu\|_q$.

For the converse, let $f = I_{1-\alpha} * g$ be an admissible function for $\gamma_{1-\alpha,q}$. We have that, up to a multiplicative constant, $T = \overline{\partial} f$ is an admissible distribution for $\dot{C}_{\alpha,p}$ because

$$I_\alpha * T = R^t(g)$$

where R^t is the transpose of R. Thus, $\dot{C}_{\alpha,p}(E)^{1/p} \geq |\langle T, 1 \rangle| = |f'(\infty)|$ and the proof is complete. □

We end up with new quasiconformal invariants built on the Riesz capacities.

Theorem 2.10. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping of the plane, which is conformal on $\mathbb{C} \setminus E$. Let $1 < p < 2$ and $\alpha = \frac{2}{p} - 1$. Then*

$$\dot{C}_{\alpha,p}(\phi(E)) \simeq \dot{C}_{\alpha,p}(E)$$

with constants that depend only on K and p.

Proof. By the preceding Lemma it suffices to show that $\gamma_{1-\alpha,q}(\phi(E)) \leq C_K \gamma_{1-\alpha,q}(E)$.

Let f be an admissible function for $\gamma_{1-\alpha,q}(\phi(E))$. This means that f is holomorphic on $\mathbb{C} \setminus \phi(E)$, $f(\infty) = 0$ and that $\|f\|_{\dot{W}^{1-\alpha,q}} \leq 1$. Then, we consider the function $g = f \circ \phi$. Clearly, $\overline{\partial}(f \circ \phi) = 0$ outside E and $g(\infty) = 0$. Moreover, for $\alpha = \frac{2}{p} - 1$ we have $1 - \alpha = \frac{2}{q}$. Hence, because of equation (2.10),

$$\|g\|_{\dot{W}^{1-\alpha,q}} \leq C_K \|f\|_{\dot{W}^{\frac{2}{q},q}} \leq C(K,q)$$

so that $\frac{1}{C(K,q)}\, g$ is an admissible function for $\gamma_{1-\alpha,q}(E)$. Hence, as ϕ is a principal K-quasiconformal mapping,

$$\gamma_{1-\alpha,q}(E) \geq \frac{1}{C(K,q)} |f'(\infty)|$$

and we may take supremum over f. □

The above theorem has direct consequences towards the absolute continuity of Hausdorff measures, but unfortunately these are slighly weaker than one would wish for. In fact, there are compact sets F such that $C_{\alpha,p}(F) = 0$ and $\mathcal{H}^h(F) > 0$, for some measure function $h(t) = t^p \, \varepsilon(t)$. Thus, Theorem 2.10 does not help for Conjecture 2.3. We have to content with the following setup:

Given $1 < d < 2$ consider the measure functions $h(t) = t^d \, \varepsilon(t)$ where

$$\int_0 \varepsilon(t)^{\frac{1}{d-1}} \frac{dt}{t} < \infty \qquad (2.11)$$

Typical examples of such functions are $h(t) = t^d |\log t|^{-s}$ or $h(t) = t^d |\log t|^{1-d} \log^{-s}(|\log t|)$ where $s > d - 1$.

Corollary 2.11. *Let E be a compact set on the plane, and $\phi : \mathbb{C} \to \mathbb{C}$ a principal K-quasiconformal mapping, conformal outside of E. Let $1 < d < 2$. Then,*

$$\mathcal{M}^h(\phi(E)) \leq C \, \mathcal{M}^d(E)$$

for any measure function $h(t) = t^d \, \varepsilon(t)$ satisfying (2.11). Moreover, if $\mathcal{H}^d(E) < \infty$ then $\mathcal{H}^h(\phi(E)) = 0$ for every such h.

Proof. By [1, Theorem 5.1.13], given a measure function h satisfying (2.11) there is a constant $C = C(h)$ with

$$\mathcal{M}^h(\phi(E)) \leq C \, C_{\alpha,d}(\phi(E)), \qquad \alpha = \frac{2}{d} - 1$$

By Theorem 2.10, $C_{\alpha,d}(\phi(E)) \leq C \, C_{\alpha,d}(E)$ and using again [1, Theorem 5.1.9] we finally have $C_{\alpha,d}(E) \leq \mathcal{M}^d(E)$. $\square$

Arguing now as in Theorems 2.2 and 2.6, we arrive at the following conclusion.

Corollary 2.12. *Let E be a compact set of the plane and suppose $\phi : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal mapping. Let $t \in (\frac{2}{K+1}, 2)$ and $d = \frac{2Kt}{2+(K-1)t}$. Then, under the normalization (2.2),*

$$\mathcal{M}^h(\phi(E)) \leq C \, C_{\alpha,d}(\phi(E)) \leq C \left(\mathcal{M}^t(E) \right)^{\frac{1}{Kt}}, \quad \alpha = \frac{2}{d} - 1$$

for any measure function h satisfying (2.11). The constant C depends only on h and K.

Here note that for $\frac{2}{K+1} < t < 2$ we always have $1 < d < 2$ in the above Corollary.

3. DISTORTION OF RECTIFIABLE SETS

In general, if ϕ is a K-quasiconformal mapping and E is a compact set, it follows from (1.2) that

$$\dim(E) = 1 \quad \Rightarrow \quad \frac{2}{K+1} \leq \dim \phi(E) \leq \frac{2K}{K+1} \qquad (3.1)$$

Here for both estimates one may find mappings ϕ and sets E such that the equality is attained, see [4]. There all examples come from non regular Cantor-type constructions. Thus the extremal distortion of Hausdorff dimension is attained, at least, by sets irregular enough. The main purpose of this section is to prove that some irregularity is also necessary. Namely, we show that quasiconformal images of 1-rectifiable sets cannot achieve the maximal distortion of dimension.

Theorem 3.1. *Suppose that $\phi : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal mapping. Let $E \subset \partial \mathbb{D}$ be a subset of the unit circle with $\dim(E) = 1$. Then we have the strict inequality*

$$\dim(\phi(E)) > \frac{2}{K+1}$$

With similar but easier argument one may also prove that for such sets E, neither can $\dim(\phi(E))$ attain the upper bound in (3.1). For details see Remark 3.7.

From this Theorem we obtain as an immediate corollary the following more general result.

Corollary 3.2. *Suppose that E is a 1-rectifiable set, and let $\phi : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping. Then there exists a subset $E_0 \subset E$ of zero length such that*

$$\dim \phi(E \setminus E_0) > \frac{2}{K+1}.$$

Recall that a set $E \subset \mathbb{C}$ is said to be 1-*rectifiable* if there exists a set E_0 of zero length such that $E \setminus E_0$ is contained in a countable union of Lipschitz curves, that is,

$$E \setminus E_0 \subset \bigcup_{j=1}^{\infty} \Phi_j([0,1])$$

where all $\Phi_j : [0,1] \to \mathbb{C}$ are Lipschitz mappings. Alternatively [18] 1-rectifiable sets can be viewed as subsets countable unions of $\mathcal{C}^1$ curves, modulo a set of zero length. In particular, for any $\varepsilon > 0$ there is a decomposition

$$E \setminus E_0' = \bigcup_{i=1}^{\infty} E_i$$

where E_0' has zero length and each E_i can be written as $E_i = f_i(F_i)$, with $f_i : \mathbb{C} \to \mathbb{C}$ a $(1 + \varepsilon)$-bilipschitz mapping and $F_i \subset \partial\mathbb{D}$. From this and Theorem 3.1 we obtain Corollary 3.2.

To prove Theorem 3.1, first some reductions may be made. Recall [16] that every K-quasiconformal mapping ϕ can be factored as $\phi = \phi_n \circ \cdots \circ \phi_1$ where each ϕ_j is K_j-quasiconformal, and $K_1 K_2 \cdots K_n = K$. In particular, given $\varepsilon > 0$, we can choose $K_j \leq 1 + \varepsilon$ for all $j = 1, ..., n$, when n is large enough. On the other hand, recall that from the distortion of Hausdorff dimension (1.2) we have

$$\frac{1}{\dim \phi(E)} - \frac{1}{2} \leq K \left(\frac{1}{\dim E} - \frac{1}{2} \right) \tag{3.2}$$

If ϕ is such that equality in (3.2) holds for E, then every factor ϕ_j above must give equality for the set $E_j = \phi_{j-1} \circ \ldots \phi_1(E)$ and $K = K_j$. In other words, if one of the mappings ϕ_j fails to satisfy the equality in (3.2), then so will ϕ. By combining these facts, we deduce that in order to prove Theorem 3.1 we can assume that $K = 1 + \varepsilon$ with $\varepsilon > 0$ as small as we wish.

For mappings with small dilatation it is possible achieve quantitative and more symmetric local distortion estimates. In particular, Theorem 3.1 will follow from the next lower bounds for compression of dimension.

Theorem 3.3. *Suppose $\phi : \mathbb{C} \to \mathbb{C}$ is $(1 + \varepsilon)$-quasiconformal and $E \subset \partial\mathbb{D}$. Then for all $\varepsilon > 0$ small enough,*

$$\dim(E) \geq 1 - c_0\,\varepsilon^2 \quad \Rightarrow \quad \dim(\phi(E)) \geq 1 - c_1\,\varepsilon^2 \tag{3.3}$$

where the constants c_0, $c_1 > 0$ are independent of ε.

Our basic strategy towards this result is to reduce it to the properties of harmonic measure and conformal mappings admitting quasiconformal extensions. Indeed, denote by μ the Beltrami coefficient of ϕ and let h be the principal solution to $\bar{\partial} h = \chi_{\mathbb{D}} \mu\, \partial h$. Then h is conformal outside the unit disk. Inside $\mathbb{D}$ it has the same dilatation μ as ϕ, and hence differs from this by a conformal factor. Consequently, we may find Riemann mappings $f : \mathbb{D} \to \Omega := \phi(\mathbb{D})$ and $g : \mathbb{D} \to \Omega' := h(\mathbb{D})$ so that

$$\phi(z) = f \circ g^{-1} \circ h(z), \quad z \in \mathbb{D} \tag{3.4}$$

Moreover, since the $(1 + \varepsilon)$-quasiconformal mapping $G = g^{-1} \circ h$ preserves the disk, reflecting across the boundary $\partial\mathbb{D}$ one may extend G to a $(1 + \varepsilon)$-quasiconformal mapping of $\mathbb{C}$. At the same time, this procedure provides both f and g with $(1 + \varepsilon)^2$-quasiconformal extensions to the entire plane $\mathbb{C}$.

As the final reduction we now find from (3.4) that for Theorems 3.1 and 3.3 it is sufficient to prove the following result.

Theorem 3.4. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a $(1+\varepsilon)$-quasiconformal mapping of $\mathbb{C}$, conformal in the disk $\mathbb{D}$. Let $A \subset \partial\mathbb{D}$. There are constants c_0, c_1 and γ_0, γ_1, independent of ε, such that for $\varepsilon \geq 0$ small enough,*

$$(i) \qquad \dim(A) \geq 1 - c_0\,\varepsilon^2 \;\Rightarrow\; \dim(f(A)) \geq 1 - c_1\,\varepsilon^2$$

and

$$(ii) \qquad \dim(A) \leq 1 - \gamma_0\,\varepsilon^2 \;\Rightarrow\; \dim(f(A)) \leq 1 - \gamma_1\,\varepsilon^2$$

Proof. The first conclusion (i) follows from Makarov's fundamental estimates for the harmonic measure [17], see also [22, p.231]. In the work [17] Makarov proves that for any conformal mapping f defined on $\mathbb{D}$, for any Borel subset $A \subset \partial\mathbb{D}$ and for every $q > 0$ we have the lower bound

$$\dim(f(A)) \geq \frac{q\,\dim(A)}{\beta_f(-q) + q + 1 - \dim(A)} \tag{3.5}$$

Here $\beta_f(p)$ stands for the integral means spectrum. That is, for a given $p \in \mathbb{R}$, $\beta_f(p)$ is the infimum of all numbers β such that

$$\int_0^{2\pi} |f'(re^{it})|^p dt = O\left(\frac{1}{(1-r)^\beta}\right) \tag{3.6}$$

as $r \to 1^-$.

We hence need estimates for $\beta_f(p)$, and here for mappings admitting K-quasiconformal extensions one has qualitively sharp bounds. Indeed, it can be shown [22, p.182] that

$$\beta_f(p) \leq 9\left(\frac{K-1}{K+1}\right)^2 p^2 \tag{3.7}$$

for any $p \in \mathbb{R}$. The constant 9 is not optimal but suffices for our purposes. Choosing $q = 1$ in (3.5) gives immediately the first claim (i).

For general conformal mappings there is no bound for expansion of dimension, i.e. there is no upper bound analogue of (3.5). Hence the proof of (ii) uses strongly the fact that mappings considered have $(1+\varepsilon)$-quasiconformal extensions. However, also here this information is easiest to use in the form (3.7).

We first need to introduce some further notation. The Carleson squares of the unit disk are defined as

$$Q_{j,k} = \left\{z \in \mathbb{D} : 2^{-k} \leq 1 - |z| < 2^{-k+1},\ 2^{-k+1}\pi j \leq \arg(z) < 2^{-k+1}\pi(j+1)\right\}$$

Given a point $z \in \mathbb{D} \setminus \{0\}$, let $Q(z)$ denote the unique Carleson square that contains z. Then it follows from Koebe's distortion Theorem and quasisymmetry [7], [16] that if $D(\xi, r)$ is a disk centered at $\xi \in \partial D$, we have

$$\text{diam}(f(D)) \simeq \text{diam}(f(Q(z))) \simeq |f'(z)|(1-|z|), \qquad \text{for} \ \ z = (1-r)\xi, \ (3.8)$$

whenever $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal mapping, conformal in $\mathbb{D}$.

Furthermore, assume we are given a family of disjoint disks $D_i = D(\xi_i, r_i)$ with centers $\xi_i \in \partial\mathbb{D}$, $i \in \mathbb{N}$, on the unit circle. Write then $z_i = (1 - r_i)\xi_i$, and for any pair of real numbers $0 < \alpha < \delta < 1$ define two subsets of indices,

$$I_g(\alpha, \delta) = \left\{ i \in \mathbb{N}; |f'(z_i)| \leq (1 - |z_i|)^{\frac{\alpha}{\delta} - 1} \right\}$$
$$I_b(\alpha, \delta) = \mathbb{N} \setminus I_g(\alpha, \delta)$$

Diameter sums over the 'good' indexes $I_g(\alpha, \delta)$ are easy to estimate. We have

$$\sum_{i \in I_g(\alpha, \delta)} \text{diam}(f(D_i))^\delta \leq C \sum_{i \in I_g(\alpha, \delta)} |f'(z_i)|^\delta (1 - |z_i|)^\delta \leq C \sum_{i \in I_g(\alpha, \delta)} (1 - |z_i|)^\alpha$$

where C depends only on K. In other words,

$$\sum_{i \in I_g(\alpha, \delta)} \text{diam}(f(D_i))^\delta \leq C \sum_{i \in I_g(\alpha, \delta)} \text{diam}(D_i)^\alpha \qquad (3.9)$$

It is well known that the integral means can be used to control the complementary indexes $I_b(\alpha, \delta)$. We give the technical details in a separate Lemma:

Lemma 3.5. *Assume that $0 < \alpha = 1 - M\varepsilon^2$, for some $M > 400$, and let $\delta = \alpha(1 + N\varepsilon^2)$, where $20\sqrt{M} < N < M$. Then*

$$\sum_{i \in I_b(\alpha, \delta)} \text{diam}(f(D_i))^\delta \leq C$$

where C is independent of D_j. Moreover, δ satisfies $\delta < 1 - \gamma\varepsilon^2$ where $\gamma = M - N > 0$.

Proof. We classify the bad indexes $I_b(\alpha, \delta)$ by defining for $k = 1, 2, ...$, and $m \in \mathbb{Z}$

$$I_m^k = \left\{ i \in I_b(\alpha, \delta); \ 2^{-k} \leq 1 - |z_i| < 2^{1-k}, \ \ 2^{-1-m} \leq |f'(z_i)|(1 - |z_i|) \leq 2^{-m} \right\}$$

and write $q_m^k = \#I_m^k$. By (3.8) $|f'(z_i)|(1 - |z_i|)$ is comparable to $\text{diam}(f(D_i))$, which is always smaller than $\text{diam}(f(3\mathbb{D}))$. On the other hand, if $i \in I_m^k$ then

$$(2^{-k})^{\frac{\alpha}{\delta}} \leq (1 - |z_i|)^{\frac{\alpha}{\delta}} < (1 - |z_i|)|f'(z_i)| \leq 2^{-m}$$

Hence the indexes m with I_m^k nonempty lie on an interval $m_0 \leq m \leq \frac{\alpha}{\delta} k$.

From Koebe we also see that if $i \in I_m^k$ then $|f'(w)|^p \sim 2^{p(k-m)}$ for every $w \in Q(z_i)$, with constants depending only on p. Combining this with (3.7) gives for any $\tau > 0$

$$q_m^k \leq C \, 2^{k\left(\frac{9}{4}\varepsilon^2 p^2 + \tau + 1 - \frac{k-m}{k}p\right)}$$

where C now depends on p and τ. We may take $p = \frac{k-m}{10k\varepsilon^2}$ and obtain

$$q_m^k \leq C \, 2^{(1+\tau)k - \frac{k}{(10\varepsilon)^2}\left(\frac{k-m}{k}\right)^2}$$

Since $\operatorname{diam}(f(D_i))$ is comparable to $|f'(z_i)|\,(1-|z_i|) \sim 2^{-m}$ for $i \in I_m^k$,

$$
\sum_{i \in I_b(\alpha,\delta)} \operatorname{diam}(f(D_i))^\delta \leq C \sum_{k=0}^{\infty} \sum_{m=m_0}^{\frac{\alpha}{\delta}k} q_m^k 2^{-m\delta} \tag{3.10}
$$

$$
\leq C \sum_{k=0}^{\infty} \sum_{m=m_0}^{\frac{\alpha}{\delta}k} 2^{k\left(1+\tau - m\frac{\delta}{k} - \frac{1}{100\varepsilon^2}\left(\frac{k-m}{k}\right)^2\right)}
$$

One now needs to ensure that the exponent $1 + \tau - m\frac{\delta}{k} - \frac{1}{100\varepsilon^2}\left(\frac{k-m}{k}\right)^2$ is negative. In particular, we want the exponent to attain its maximum at $m = \frac{\alpha}{\delta}k$, and this is satisfied if

$$\frac{\alpha}{\delta} \leq 1 - \frac{1}{2}(10\varepsilon)^2\,\delta$$

Under the assumptions of the Lemma this is easy to verify. Similarly one verifies that the specific choices of the Lemma yield the maximum value

$$1 + \tau - \alpha - \frac{1}{(10\varepsilon)^2}\left(1 - \frac{\alpha}{\delta}\right)^2 < 0$$

when τ is chosen small enough. It follows that the sum in (3.10) has a finite upper bound depending only on the constants M, N. This proves Lemma 3.5. $\qquad\square$

The dimension bounds required in part (ii) of Theorem 3.4 are now easy to establish. For every $\alpha > 1 - \gamma_0 \varepsilon^2$ we have coverings of A, consisting of families of disks $D_j = D(z_j, r_j)$ centered on $\partial\mathbb{D}$ and radius $r_j \leq \rho \to 0$ uniformly small, such that the sums $\sum_j \operatorname{diam}(D_j)^\alpha$ are uniformly bounded. On the image side, for each $\delta > 0$

$$\sum_i \operatorname{diam} f(D_i)^\delta = \sum_{i \in I_g(\alpha,\delta)} \operatorname{diam} f(D_i)^\delta + \sum_{i \in I_b(\alpha,\delta)} \operatorname{diam} f(D_i)^\delta$$

As soon as $\alpha < \delta < 1$, estimate (3.9) gives

$$\sum_{i \in I_g(\alpha,\delta)} \operatorname{diam} f(D_i)^\delta \leq C \sum_{i \in I_g(\alpha,\delta)} \operatorname{diam}(D_i)^\alpha$$

Furthermore, by Lemma 3.5, there exists an exponent $\alpha < \delta < 1 - \gamma_1\,\varepsilon^2$ such that the series

$$\sum_{i \in I_b(\alpha,\delta)} \operatorname{diam} f(D_i)^\delta$$

is bounded independently of the covering D_j. Thus the entire sum $\sum_i \operatorname{diam}(f(D_i))^\delta$ remains bounded as $\sup_i \operatorname{diam}(D_i) \to 0$. This means $\dim f(A) \leq \delta \leq 1 - \gamma_1\,\varepsilon^2$, and completes the proof of Theorem 3.4. $\qquad\square$

By symmetry, c.f. (3.4), Theorem 3.4 proves bounds also for expansion of dimension.

Corollary 3.6. *There are constants $c_0, c_1 > 0$ such that if $E \subset \partial\mathbb{D}$ and $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal with $K = 1 + \varepsilon$, then*

$$\dim(E) \leq 1 - c_0\,\varepsilon^2 \quad \Rightarrow \quad \dim(f(E)) < 1 - c_1\,\varepsilon^2$$

when $\varepsilon > 0$ is small enough.

Very recently, I. Prause [23] has obtained a different proof for Theorem 3.3 and Corollary 3.6, based in the ideas on [4] and a well known result from Becker and Pommerenke [9] which says that

$$\dim(\Gamma) \leq 1 + 37 \left(\frac{K-1}{K+1}\right)^2 \qquad\qquad (3.11)$$

for every K-quasicircle Γ.

Remark 3.7. *Similarly as the compression bound (3.3) led to Theorem 3.1, the inequality (3.11) yields improved upper estimates. We have hence the symmetric strict inequalities:*

If $\phi : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal mapping and $E \subset \partial\mathbb{D}$ with $\dim(E) = 1$, then

$$\frac{2}{K+1} < \dim(\phi(E)) < \frac{2K}{K+1}.$$

Moreover, for the dimension of quasicircles Smirnov (unpublished) has obtained the upper bound

$$\dim(\Gamma) \leq 1 + \left(\frac{K-1}{K+1}\right)^2,$$

answering a question in [4]. It is still unknown if this bound is sharp; the best known lower bounds so far [8] give curves with dimension $1 + 0.69 \left(\frac{K-1}{K+1} \right)^2$.

The arguments we have used are related to the *generalized Brennan conjecture*, which says that

$$\beta_f(p) \leq \frac{p^2}{4} \left(\frac{K-1}{K+1} \right)^2 \qquad \text{for } |p| \leq 2 \frac{K+1}{K-1}, \tag{3.12}$$

whenever f is conformal in $\mathbb{D}$ and admits a K-quasiconformal extension to $\mathbb{C}$. This connection suggests the following

Question 3.8. *Let $E \subset \mathbb{R}$ be a set with Hausdorff dimension 1, and let ϕ be a K-quasiconformal mapping. Is it true that then*

$$1 - \left(\frac{K-1}{K+1} \right)^2 \leq \dim(\phi(E)) \leq 1 + \left(\frac{K-1}{K+1} \right)^2 \tag{3.13}$$

The positive answer for the right hand side inequality follows from Smirnov's unpublished work, while the left hand side is only known up to some multiplicative constants. On the other hand, Prause [23] proves the left inequality for the mappings that preserve the unit circle $\partial \mathbb{D}$.

4. Improved Painlevé Theorems

A compact set E is said to be *removable for bounded analytic functions* if for any open set Ω with $E \subset \Omega$, every bounded analytic function on $\Omega \setminus E$ has an analytic extension to Ω. Equivalently, such sets are described by the condition $\gamma(E) = 0$, where γ is the analytic capacity

$$\gamma(E) = \sup\{|f'(\infty)| : f \in H^\infty(\mathbb{C} \setminus E), f(\infty) = 0, \|f\|_\infty = 1\}$$

Finding a geometric characterization for the sets of zero analytic capacity was a long standing problem. It was solved by G. David [11] for sets of finite length, and finally by X. Tolsa [27] in the general case. The difficulties of dealing with this question motivated the study of related problems. In particular, we have the question of determining the *removable sets for BMO analytic functions*, that is, those compact sets E such that every BMO function in the plane, holomorphic on $\mathbb{C} \setminus E$, admits an entire extension. This problem was solved by Kaufman (see [15]), who showed that a set E has this BMO-removability property if and only if $\mathcal{H}^1(E) = 0$.

For the original case of bounded functions the Painlevé condition $\mathcal{H}^1(E) = 0$ can be weakened. As is well known, there are sets E with zero analytic capacity and positive length (see [13] for an example). In fact, it is now known that among the compact sets E with $0 < \mathcal{H}^1(E) < \infty$, precisely the purely unrectifiable ones are the removable sets for bounded analytic functions [11]. Moreover, if E has positive σ-finite length, this

characterization still remains true, due to the countable semiadditivity of analytic capacity [27].

The preceeding problems can be formulated also in the K-quasiregular setting. More precisely, a set E is said to be *removable for bounded (resp. BMO) K-quasiregular mappings*, if every K-quasiregular mapping in $\mathbb{C} \setminus E$ which is in $L^\infty(\mathbb{C})$ (resp. $BMO(\mathbb{C})$) admits a K-quasiregular extension to $\mathbb{C}$. For simplicity, we use here the term *K-removable* for sets that are removable for bounded K-quasiregular mappings.

Obviously, when $K = 1$, in both situations L^∞ and BMO we recover the original analytic problem. Moreover, by means of the Stoilow factorization, one can represent any bounded K-quasiregular function as a composition of a bounded analytic function and a K-quasiconformal mapping. The corresponding result holds true also for BMO since this space, like L^∞, is quasiconformally invariant.

Therefore, when we ask ourselves if a set E is K-removable, we just need to analyze how it may be distorted under quasiconformal mappings, and then apply the known results for the analytic situation. With this basic scheme, it is shown in [4, Corollary 1.5] that every set with dimension strictly below $\frac{2}{K+1}$ is K-removable. Indeed, the precise formulas for the distortion of dimension (1.2) ensure that for such sets the K-quasiconformal images have dimension strictly smaller than 1.

Iwaniec and Martin [14] had earlier conjectured that, more generally, sets of zero $\frac{2}{K+1}$-dimensional measure are K-removable. A preliminary answer to this question was found in [7], and actually it was that argument which suggested Theorem 2.2. Using our results from above we can now prove that sets of zero $\frac{2}{K+1}$-dimensional measure are even BMO-removable.

Corollary 4.1. *Let E be a compact subset of the plane. Assume that $\mathcal{H}^{\frac{2}{K+1}}(E) = 0$. Then E is removable for all BMO K-quasiregular mappings.*

Proof. Assume that $f \in BMO(\mathbb{C})$ is K-quasiregular on $\mathbb{C} \setminus E$. Denote by μ the Beltrami coefficient of f, and let ϕ be the principal solution to $\bar{\partial}\phi = \mu\, \partial\phi$. Then, $F = f \circ \phi^{-1}$ is holomorphic on $\mathbb{C} \setminus \phi(E)$ and $F \in BMO(\mathbb{C})$. On the other hand, as we showed in Theorem 2.2, $\mathcal{H}^1(\phi(E)) = 0$. Thus, $\phi(E)$ is a removable set for BMO analytic functions. In particular, F admits an entire extension and $f = F \circ \phi$ extends quasiregularly to the whole plane. $\square$

We believe that Corollary 4.1 is sharp, in the sense that we expect a positive answer to the following

Question 4.2. *Does there exist for every $K \geq 1$ a compact set E with $0 < \mathcal{H}^{\frac{2}{K+1}}(E) < \infty$, such that E is not removable for some K-quasiregular functions in $BMO(\mathbb{C})$.*

Here we observe that by [4, Corollary 1.5], for every $t > \frac{2}{K+1}$ there exists a compact set E with dimension t, nonremovable for bounded and hence in particular nonremovable for BMO K-quasiregular mappings.

Next we return back to the problem of removable sets for bounded K-quasiregular mappings. Here Theorem 2.2 proves the conjecture of Iwaniec and Martin that sets with $\mathcal{H}^{\frac{2}{K+1}}(E) = 0$ are K-removable. However, the analytic capacity is somewhat smaller than length, and hence with Theorem 2.5 we may go even further: If a set has finite or σ-finite $\frac{2}{K+1}$-measure, then all K-quasiconformal images of E have at most σ-finite length. Such images may still be removable for bounded analytic functions, if we can make sure that the rectifiable part of these sets has zero length. But for this Theorem 3.1 provides exactly the correct tools. We end up with the following improved version of Painlevés theorem for quasiregular mappings.

Theorem 4.3. *Let E be a compact set in the plane, and let $K > 1$. Assume that $\mathcal{H}^{\frac{2}{K+1}}(E)$ is σ-finite. Then E is removable for all bounded K-quasiregular mappings.*

In particular, for any K-quasiconformal mapping ϕ the image $\phi(E)$ is purely unrectifiable.

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be bounded, and assume that f is K-quasiregular on $\mathbb{C} \setminus E$. As in Corollary 4.1 we may find the principal quasiconformal homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$, such that $F = f \circ \phi^{-1}$ is analytic in $\mathbb{C} \setminus \phi(E)$. If we can extend F holomorphically to the whole plane, we are done. Thus we have to show that $\phi(E)$ has zero analytic capacity.

By Theorem 2.5, $\phi(E)$ has σ-finite length, that is, $\phi(E) = \cup_n F_n$ where each $\mathcal{H}^1(F_n) < \infty$. A well known result due to Besicovitch (see e.g.[18, p.205]) assures that each set F_n can be decomposed as

$$F_n = R_n \cup U_n \cup B_n$$

where R_n is a 1-rectifiable set, U_n is a purely 1-unrectifiable set, and B_n is a set of zero length. Because of the semiadditivity of analytic capacity [27],

$$\gamma(F_n) \leq C\left(\gamma(R_n) + \gamma(U_n) + \gamma(B_n)\right)$$

Now, $\gamma(B_n) \leq C\,\mathcal{H}^1(B_n) = 0$ and $\gamma(U_n) = 0$ since purely 1-unrectifiable sets of finite length have zero analytic capacity [11]. On the other hand, R_n is a 1-rectifiable image, under a K-quasiconformal mapping, of a set of dimension $\frac{2}{K+1}$. Thus applying Theorem 3.1 and Corollary 3.2 to ϕ^{-1}

shows that we must have $\mathcal{H}^1(R_n) = 0$. Therefore we get $\gamma(F_n) = 0$ for each n. Again by countable semiadditivity of analytic capacity we conclude $\gamma(\phi(E)) = 0$. $\qquad\square$

As pointed out earlier, the above theorem does not hold for $K = 1$. Any 1-rectifiable set such as $E = [0,1]$ of finite and positive length gives a counterexample. In the above proof the improved distortion of 1-rectifiable sets was the decisive phenomenon allowing the result. In fact, such good behavior of rectifiable sets has further consequences. For instance, even *strictly above* the critical dimension $\frac{2}{K+1} = 1 - \frac{K-1}{K+1}$ one may find removable sets, as soon as they have enough geometric regularity.

Corollary 4.4. *There exists a constant $c \geq 1$ such that if $E \subset \partial\mathbb{D}$ is compact and*

$$\dim(E) < 1 - c\left(\frac{K-1}{K+1}\right)^2$$

then E is removable for bounded and BMO K-quasiregular mappings, $K = 1 + \varepsilon$, whenever $\varepsilon > 0$ is small enough.

Proof. This is a consequence of Corollary 3.6. If $\varepsilon > 0$ is small enough and $K = 1 + \varepsilon$, then the K-quasiconformal images of E will always have dimension strictly below 1, so that $\gamma(\phi(E)) = 0$ for each K-quasiconformal mapping ϕ. $\qquad\square$

In conjunction with Question 3.8 we have

Question 4.5. *Let $K > 1$. Is then every set $E \subset \partial\mathbb{D}$ with $\dim(E) < 1 - \left(\frac{K-1}{K+1}\right)^2$ removable for bounded and BMO K-quasiregular mappings*

5. EXAMPLES OF EXTREMAL DISTORTION

The previous sections provide a delicate analysis of distortion of 1-dimensional sets under quasiconformal mappings but still leave open the cases where $\dim(E) = \frac{2}{K+1}$ precisely but E does not have σ-finite $\frac{2}{K+1}$-measure. Hence we are faced with the natural question: Are there compact sets E, with $\dim(E) = \frac{2}{K+1}$, that are non removable for some bounded K-quasiregular mappings.

In this last section we give a positive answer and show that our results are sharp in a quite strong sense. Indeed, to compare with the analytic removability recall first that by Mattila's theorem [19], if a compact set E supports a probability measure with $\mu(B(z,r)) \leq r\,\varepsilon(r)$ and

$$\int_0 \frac{\varepsilon(t)^2}{t} dt < \infty, \tag{5.1}$$

then the analytic capacity $\gamma(E) > 0$. On the other hand, if the integral in (5.1) diverges, then there are compact sets E of vanishing analytic capacity supporting a probability measure with $\mu(B(z,r)) \leq r\,\varepsilon(r)$ [27]. In a complete analogy we prove

Theorem 5.1. *Let $K \geq 1$. Suppose $h(t) = t^{\frac{2}{K+1}}\,\varepsilon(t)$ is a measure function such that*

$$\int_0 \frac{\varepsilon(t)^{1+1/K}}{t}dt < \infty \qquad (5.2)$$

Then there is a compact set E which is not K-removable and yet supports a probability measure μ, with $\mu(B(z,r)) \leq h(r)$ for every z and $r > 0$.

In particular, whenever $\varepsilon(t)$ is chosen so that in addition for every $\alpha > 0$ we have $t^{\alpha}/\varepsilon(t) \to 0$ as $t \to 0$, then the construction gives a non-K-removable set E with $\dim(E) = \frac{2}{K+1}$.

Proof. We will construct a compact set E and a K-quasiconformal mapping ϕ such that $\mathcal{H}^h(E) \simeq 1$, and at the same time $\phi(E)$ has a positive and finite $\mathcal{H}^{h'}$-measure for some measure function $h'(t) = t\,\varepsilon'(t)$ where

$$h'(t) = t\,\varepsilon'(t) \quad \text{with} \quad \int_0^1 \frac{\varepsilon'(t)^2}{t}dt < \infty$$

Mattilas theorem shows then $\gamma(\phi(E)) > 0$, so that there exists non-constant bounded functions h holomorphic on $\mathbb{C} \setminus \phi(E)$. Thus with $f = h \circ \phi$ we see that E is not removable for bounded K-quasiregular mappings.

We will construct the K-quasiconformal mapping ϕ as the limit of a sequence ϕ_N of K-quasiconformal mappings, and E will be a Cantor-type set. To reach the optimal estimates we need to change, at every step in the construction of E, both the size and the number m_j of the generating disks.

Without loss of generality we may assume that for every $\alpha > 0$, $t^{\alpha}/\varepsilon(t) \to 0$ as $t \to 0$.

Step 1. Choose first m_1 disjoint disks $D(z_i, R_1) \subset \mathbb{D}$, $i = 1, ..., m_1$, so that

$$c_1 := m_1 R_1^2 \in (\frac{1}{2}, 1)$$

For R_1 small enough (i.e. for m_1 large enough) this is clearly possible. The function $f(t) = m_1 h(tR_1)$ is continuous with $f(0) = 0$. Moreover, for each fixed t

$$f(t) = m_1(tR_1)^{\frac{2}{K+1}}\,\varepsilon(tR_1) = \frac{\varepsilon\left(t\sqrt{c_1/m_1}\right)}{\left(t\sqrt{c_1/m_1}\right)^{\frac{2}{K+1}}}\,t^2 c_1 \ \to \ \infty$$

as $m_1 \to \infty$. Hence for any $t < 1$ we may choose m_1 so large that there exists $\sigma_1 \in (0, t)$ satisfying $m_1 \, h(\sigma_1^K R_1) = 1$. A simple calculation gives

$$m_1 \, \sigma_1 R_1 \, \varepsilon(\sigma_1^K R_1)^{\frac{K+1}{2K}} \, (c_1)^{\frac{1-K}{2K}} = 1 \qquad (5.3)$$

Next, let $r_1 = R_1$. For each $i = 1, \ldots, m_1$, let $\varphi_i^1(z) = z_i + \sigma_1^K R_1 \, z$ and, using the notation $\alpha D(z, \rho) := D(z, \alpha \rho)$, set

$$D_i := \frac{1}{\sigma_1^K} \, \varphi_i^1(\mathbb{D}) = D(z_i, r_1)$$

$$D_i' := \varphi_i^1(\mathbb{D}) = D(z_i, \sigma_1^K r_1) \subset D_i$$

As the first approximation of the mapping define

$$g_1(z) = \begin{cases} \sigma_1^{1-K}(z - z_i) + z_i, & z \in D_i' \\ \left| \frac{z - z_i}{r_1} \right|^{\frac{1}{K}-1} (z - z_i) + z_i, & z \in D_i \setminus D_i' \\ z, & z \notin \cup D_i \end{cases}$$

This is a K-quasiconformal mapping, conformal outside of $\bigcup_{i=1}^{m_1}(D_i \setminus D_i')$. It maps each D_i onto itself and D_i' onto $D_i'' = D(z_i, \sigma_1 r_1)$, while the rest of the plane remains fixed. Write $\phi_1 = g_1$.

Step 2. We have already fixed m_1, R_1, σ_1 and c_1. Consider m_2 disjoint disks of radius R_2, centered at z_j^2, $j = 1, \ldots, m_2$, uniformly distributed inside of $\mathbb{D}$, so that

$$c_2 = m_2 \, R_2^2 > \frac{1}{2}$$

Then repeat the above procedure and choose m_2 so large that the equation

$$m_1 \, m_2 \, h(\sigma_1^K \sigma_2^K R_1 R_2) = 1$$

has a unique solution $\sigma_2 \in (0, 1)$, as small as we wish. Then,

$$m_1 m_2 \, \sigma_1 \sigma_2 R_1 R_2 \, \varepsilon(\sigma_1^K \sigma_2^K R_1 R_2)^{\frac{K+1}{2K}} \, (c_1 c_2)^{\frac{1-K}{2K}} = 1$$

Denote $r_2 = R_2 \, \sigma_1 r_1$ and $\varphi_j^2(z) = z_j^2 + \sigma_2^K R_2 \, z$, and define the auxiliary disks

$$D_{ij} = \phi_1 \left(\frac{1}{\sigma_2^K} \, \varphi_i^1 \circ \varphi_j^2(\mathbb{D}) \right) = D(z_{ij}, r_2)$$

$$D_{ij}' = \phi_1 \left(\varphi_i^1 \circ \varphi_j^2(\mathbb{D}) \right) = D'(z_{ij}, \sigma_2^K r_2)$$

for certain $z_{ij} \in \mathbb{D}$, where $i = 1, \ldots, m_1$ and $j = 1, \ldots, m_2$. Now Let

$$g_2(z) = \begin{cases} \sigma_2^{1-K}(z - z_{ij}) + z_{ij} & z \in D_{ij}' \\ \left| \frac{z - z_{ij}}{r_2} \right|^{\frac{1}{K}-1} (z - z_{ij}) + z_{ij} & z \in D_{ij} \setminus D_{ij}' \\ z & \text{otherwise} \end{cases}$$

Clearly, g_2 is K-quasiconformal, conformal outside of $\bigcup_{i,j}(D_{ij} \setminus D'_{ij})$, maps each D_{ij} onto itself and D'_{ij} onto $D''_{ij} = D(z_{ij}, \sigma_2 r_2)$, while the rest of the plane remains fixed. Define $\phi_2 = g_2 \circ \phi_1$.
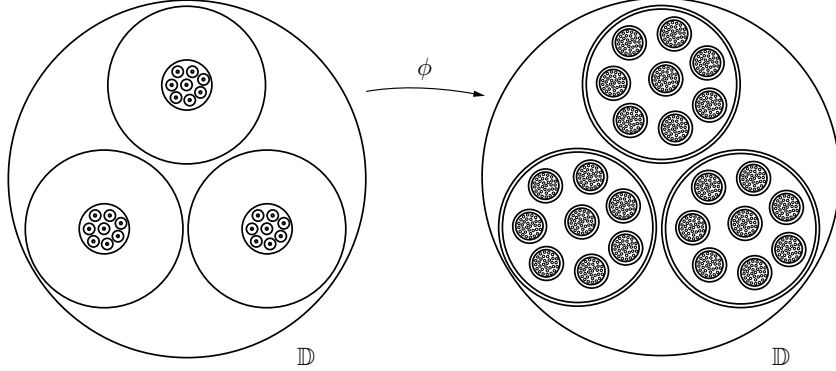
The induction step. After step $N-1$ we take m_N disjoint disks of radius R_N, with union of $D(z_l^N, R_N)$ covering at least half of the area of $\mathbb{D}$,

$$c_N = m_N R_N^2 > \frac{1}{2} \tag{5.4}$$

As before we may choose m_N so large that $m_1 \ldots m_N h(\sigma_1^K \ldots \sigma_N^K R_1 \ldots R_N) = 1$ holds for a unique σ_N, as small as we wish. Note that $\lim_{N \to \infty} \sigma_N = 0$ and

$$m_1 \ldots m_N \, \sigma_1 R_1 \ldots \sigma_N R_N \, \varepsilon(\sigma_1^K R_1 \ldots \sigma_N^K R_N)^{\frac{K+1}{2K}} (c_1 \ldots c_N)^{\frac{1-K}{2K}} = 1$$

Denote then $\varphi_j^N(z) = z_j^N + \sigma_N^K R_N z$ and $r_N = R_N \sigma_{N-1} r_{N-1}$. For any multiindex $J = (j_1, ..., j_N)$, where $1 \le j_k \le m_k$, $k = 1, ..., N$, let

$$D_J = \phi_{N-1} \left(\frac{1}{\sigma_N^K} \varphi_{j_1}^1 \circ \cdots \circ \varphi_{j_N}^N(\mathbb{D}) \right) = D(z_J, r_N)$$

$$D'_J = \phi_{N-1} \left(\varphi_{j_1}^1 \circ \cdots \circ \varphi_{j_N}^N(\mathbb{D}) \right) = D'(z_J, \sigma_N^K r_N)$$

and let

$$g_N(z) = \begin{cases} \sigma_N^{1-K}(z - z_J) + z_J & z \in D'_J \\ \left| \frac{z - z_J}{r_N} \right|^{\frac{1}{K} - 1} (z - z_J) + z_J & z \in D_J \setminus D'_J \\ z & \text{otherwise} \end{cases}$$

Clearly, g_N is K-quasiconformal, conformal outside of $\bigcup_{J=(j_1,...,j_N)}(D_J \setminus D'_J)$, maps D_J onto itself and D'_J onto $D''_J = D(z_J, \sigma_N r_N)$, while the rest

of the plane remains fixed. Now define $\phi_N = g_N \circ \phi_{N-1}$.

Since each ϕ_N is K-quasiconformal and equals the identity outside the unit disk $\mathbb{D}$, there exists a limit K-quasiconformal mapping

$$\phi = \lim_{N \to \infty} \phi_N$$

with convergence in $W^{1,p}_{loc}(\mathbb{C})$ for any $p < \frac{2K}{K-1}$. On the other hand, ϕ maps the compact set

$$E = \bigcap_{N=1}^{\infty} \left(\bigcup_{j_1,\dots,j_N} \varphi_{j_1}^1 \circ \cdots \circ \varphi_{j_N}^N (\mathbb{D}) \right)$$

to the set

$$\phi(E) = \bigcap_{N=1}^{\infty} \left(\bigcup_{j_1,\dots,j_N} \psi_{j_1}^1 \circ \cdots \circ \psi_{j_N}^N (\mathbb{D}) \right)$$

where we have written $\psi_j^i(z) = z_j^i + \sigma_i R_i z$, $j = 1, \dots, m_i$, $i \in \mathbb{N}$.

To complete the proof, write

$$s_N = (\sigma_1^K R_1) \dots (\sigma_N^K R_N) \quad \text{and} \quad t_N = (\sigma_1 R_1) \dots (\sigma_N R_N) \qquad (5.5)$$

Observe that we have chosen the parameters R_N, m_N, σ_N so that

$$m_1 \dots m_N \, h(s_N) = 1 \qquad (5.6)$$

$$m_1 \dots m_N \, t_N \, \varepsilon(s_N)^{\frac{K+1}{2K}} (c_1 \dots c_N)^{\frac{1-K}{2K}} = 1 \qquad (5.7)$$

Claim. $\mathcal{H}^h(E) \simeq 1$.

Since $\operatorname{diam}(\varphi_{j_1}^1 \circ \cdots \circ \varphi_{j_N}^N (\mathbb{D})) \leq \delta_N \to 0$ when $N \to \infty$, we have by (5.6)

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} \mathcal{H}_\delta^h(E) \leq \lim_{\delta \to 0} \sum_{j_1,\dots,j_N} h(\operatorname{diam}(\varphi_{j_1}^1 \circ \cdots \circ \varphi_{j_N}^N (\mathbb{D})))$$

$$= m_1 \dots m_N \, h(s_N) = 1$$

For the converse inequality, take a finite covering (U_j) of E by open disks of diameter $\operatorname{diam}(U_j) \leq \delta$ and let $\delta_0 = \inf_j(\operatorname{diam}(U_j)) > 0$. Denote by N_0 the minimal integer such that $s_{N_0} \leq \delta_0$. By construction, the family $(\varphi_{j_{N_0}}^{N_0} \circ \cdots \circ \varphi_{j_1}^1 (\mathbb{D}))_{j_1,\dots,j_{N_0}}$ is a covering of E with the $\mathcal{M}^h$-packing condition [18]. Thus,

$$\sum_j h(\operatorname{diam}(U_j)) \geq C \sum_{j_1,\dots,j_{N_0}} h(\operatorname{diam}(\varphi_{j_{N_0}}^{N_0} \circ \cdots \circ \varphi_{j_1}^1 (\mathbb{D}))) = C$$

Hence, $\mathcal{H}_\delta^h(E) \geq C$ and letting $\delta \to 0$, we get that

$$C \leq \mathcal{H}^h(\phi(E)) \leq 1$$

proving our first claim.

A similar argument, based this time on (5.7), gives that $\mathcal{H}^{h'}(\phi(E)) \simeq 1$ for a measure function $h'(t) = t\varepsilon'(t)$, as soon as for all indexes N

$$\varepsilon'(t_N) = \varepsilon(s_N)^{\frac{K+1}{2K}} (c_1 \dots c_N)^{\frac{1-K}{2K}} \tag{5.8}$$

Claim. One can find a continuous and nondecreasing function $\varepsilon'(t)$ satisfying (5.8) and

$$\int_0^1 \frac{\varepsilon'(t)^2}{t} dt < \infty \tag{5.9}$$

Indeed, let us first choose a continuous nondecreasing function $v(t)$ so that $v(t) \to 0$ as $t \to 0$ and so that (5.2) still holds in the form

$$\int_0 \frac{\varepsilon(t)^{1+1/K}}{t\, v(t)} dt < \infty \tag{5.10}$$

In the above inductive construction we can then choose the σ_j's so that $v(\sigma_1^K \cdots \sigma_N^K) \leq 2^{-N(1-1/K)}$ for every index N. Now (5.4) and (5.8) imply

$$\varepsilon'(t_N)^2 \leq \varepsilon(s_N)^{1+1/K}\, 2^{N(1-1/K)} \leq \frac{\varepsilon(s_N)^{1+1/K}}{v(s_N)}$$

On the other hand by (5.5) we also have $t_{N-1}/t_N \leq s_{N-1}/s_N$ and so we may extend $\varepsilon'(t)$, determined by (5.8) only at the t_N's, so that it is continuous, nondecreasing and satisfies

$$\int_0 \varepsilon'(t)^2\, \frac{dt}{t} \leq \int_0 \frac{\varepsilon(s)^{1+1/K}}{v(s)}\, \frac{ds}{s} < \infty$$

Hence the claim follows. Combining it with Mattila's theorem [19] completes the proof of the Theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lastly let us note that if we do not care for the analytic capacity of the target set, a straightforward modification of the previous Theorem, normalizing the disks of the construction so that $m_N\, t_N\, \eta(t_N) = 1$, gives

Corollary 5.2. *Let $K \geq 1$ and let $h(t) = t\,\eta(t)$ be a measure function such that*

- *η is continuous and nondecreasing, $\eta(0) = 0$ and $\eta(t) = 1$ whenever $t \geq 1$.*
- *$\lim_{t \to 0} \dfrac{t^\alpha}{\eta(t)} = 0$ for all $\alpha > 0$.*

There exists a compact set $E \subset \mathbb{D}$ and a K-quasiconformal mapping ϕ such that

$$\dim(E) = \frac{2}{K+1} \quad \text{and} \quad \mathcal{H}^h(\phi(E)) = 1 \tag{5.11}$$

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