THE EQUATION $x^p y^q = z^r$ AND GROUPS THAT ACT FREELY ON Λ-TREES

N. BRADY, L. CIOBANU, A. MARTINO, S. O ROURKE

Abstract. Let G be a group that acts freely on a Λ-tree, where Λ is an ordered abelian group, and let x, y, z be elements in G. We show that if $x^p y^q = z^r$ with integers p, q, $r \geq 4$, then x, y and z commute. As a result, the one-relator groups with $x^p y^q = z^r$ as relator, are examples of hyperbolic and CAT(-1) groups which do not act freely on any Λ-tree.

1. Introduction

There has recently been a great deal of interest in tree-free groups, that is, groups which act freely and without inversions by isometries on some Λ-tree. The principal source of this interest has been related to the solution of the Tarski problem, where one of the main objects of study, limit groups, have been shown to act freely on $\mathbb{Z}^n$-trees for some n. Groups that act freely on Λ-trees — so-called Λ-*free groups* — generalise free groups in the sense that $\mathbb{Z}$-free groups are precisely free groups. Moreover, for general Λ, these groups satisfy properties reminiscent of free groups. For example, they are torsion-free, closed under free products, and commutativity is a transitive relation on non-identity elements. In addition, all known examples of finitely generated Λ-free groups that contain no copy of $\mathbb{Z} \times \mathbb{Z}$ are hyperbolic.

The purpose of this article is to generalise a classical theorem in free groups to the broad class of tree-free groups. The classical result of Lyndon and Schützenberger ([10]) states that any elements x, y, and z of F, a free group, that satisfy the relation $x^p y^q = z^r$ for p, q, $r \geq 2$ commute. (See also [9], [2], [15], [14], [4].) Therefore all solutions to this equation are contained in a cyclic subgroup of F. Here we show,

Theorem 3.2 *Let G be a group that acts freely, and without inversions, by isometries on a Λ-tree, where Λ is an ordered abelian group, and let x, y, z be elements in G. If $x^p y^q = z^r$ with p, q, $r \geq 4$, then x, y and z commute.*

While the argument of Lyndon and Schützenberger relies on combinatorics of words in the free group, our argument relies on the information provided by the action via isometries of the group on the Λ-tree.

A Λ-metric space can be defined in the same way as a conventional metric space with $\mathbb{R}$ replaced by Λ. A Λ-tree can be characterised as a geodesically convex Λ-metric space (X, d) which is 0-hyperbolic and which satisfies $d(x, v) + d(y, v) - d(x, y) \in 2\Lambda$ for all $x, y, v \in X$ (see [3]). When the group Λ is archimedean the free actions on Λ-trees are well understood. In particular, the finitely generated groups that act freely on $\mathbb{R}$-trees have been completely classified by Rips. They are the groups that can be written as a free product $G_1 \star G_2 \star \dots G_n$ for some integer $n \geq 1$, where each G_i is either a finitely generated free abelian group or a non-exceptional surface group. In the non-archimedean case, Martino and O Rourke (see [12]) have provided examples of $\mathbb{Z}^n$-free groups. Also, it is known that among the groups that act freely on $\mathbb{R}^n$-trees are all the fully residually free groups, or *limit groups* ([8], [16], [7]). The fact that limit groups are exactly the groups with the same universal theory as free groups (see [13]) immediately implies that solutions of $x^p y^q = z^r$ commute in limit groups. We show that in addition to limit groups, the commutativity of solutions to $x^p y^q = z^r$ holds in all the groups that act freely on Λ-trees, with some restriction on the exponents.

We would like to point out one intriguing difference in the behaviour of the equation $x^2 y^2 = z^2$ in free groups versus groups that act freely on Λ-trees. In free groups the satisfiability of the equation $x^2 y^2 = z^2$ implies that x, y and z commute, while in Λ-free groups this is not true, since the exceptional surface group $\langle x, y, z, \mid x^2 y^2 z^2 = 1 \rangle$ acts freely on a $\mathbb{Z}^2$-tree ([5]). By the Base-Change Functor Theorem (see Section 2) it follows that this group acts freely on any Λ-tree, where Λ is non-archimedean.

One interesting question to ask is where the arguments for free groups and tree-free groups must diverge when considering equations. In fact, most of our techniques work in the various cases we consider for all equations of the type $x^p y^q = z^r$, with $p, q, r \geq 2$. The real difference seems to be that in free groups for the cases of small exponents one has to use inductive arguments on length which cannot work for a general Λ-tree, since there are generally infinitely many lengths less than any given one. For example, if $p = q = r = 3$ we can successfully employ the same techniques we used for larger values of p, q and r. However, we encountered difficulties in part (3.) of our proof, when the intersection Δ of the axes A_x and A_y has exactly the same length as the shortest of the translations, y^q.

Nevertheless, one immediate consequence of our result is that one can construct many groups which cannot act freely on any Λ-tree. In particular, if we look at the one-relator groups, defined as follows,

$$G_{pqr} = \langle x, y, z \mid x^p y^q = z^r \rangle,$$

we get a sequence of groups which do not act freely on any Λ-tree, for $p, q, r \geq 4$. Moreover, these groups are all small-cancellation groups; they are $C(6) - T(4)$ for all $p, q, r \geq 2$, and so are word hyperbolic (see [6]). Therefore we obtain

Corollary 1.1. *The groups G_{pqr} form a sequence of word hyperbolic groups which cannot act freely, and without inversions, by isometries on any Λ-tree.*

While the result above is not surprising it is, as far as we are aware, new. Note that by contrast the groups

$$\langle x_1, x_2, \ldots, x_n \mid x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} = 1 \rangle$$

are expressible as amalgamated free products of free groups over maximal cyclic subgroups for $n \geq 4$ provided at least four α_i are non-zero. It follows that these groups are $\mathbb{Z}^2$-free (see [1],[12]).

In the final section of the paper we show that these groups G_{pqr} have CAT(-1) structures. This question arises naturally since, intuitively, a result true for tree-like structures often has a weaker analogue for hyperbolic structures. Our aim, initially, was to try to provide examples of word hyperbolic groups which do not have CAT(-1) structures, using the sorts of length arguments we employ for the case of Λ-trees. While this naive approach doesn't seem to work, there is still some hope that the arguments may provide some restrictions to the possible CAT(-1) structures, in particular translation lengths. The aim would then be to construct multiple HNN-extensions from the groups G_{pqr} which are word hyperbolic via the Bestvina and Feighn Combination Theorem on the one hand, and which violate the translation length restrictions for CAT(-1) structures on the other hand.

2. Background

A complete account of Λ-trees is given in [3]. Here we recall the basic relevant definitions and results. An *ordered abelian group* is an abelian group Λ, together with a total ordering $\leq$ on Λ, such that for all a, b and $c \in \Lambda$, $a \leq b$ implies $a + c \leq b + c$. For a and b as before, we define $[a, b]_\Lambda = \{x \in \Lambda \mid a \leq x \leq b\}$. A Λ-*metric space* (X, d) can be defined in the same way as a conventional metric space. That is, $d : X \times X \to \Lambda$ is symmetric, satisfies the triangle inequality and satisfies $d(x, y) = 0$ if and only if $x = y$. A *segment* in X is the image of an isometry $\alpha : [a, b]_\Lambda \to X$

for some a, b in Λ, with $\alpha(a), \alpha(b)$ the endpoints of the segment. A Λ-metric space is *geodesic* if for all x and y in X there is a segment $[x, y]$ with endpoints x and y.

Definition. A Λ-tree is a geodesic Λ-metric space (X, d) such that:

: (a) if two segments of (X, d) intersect in a single point, which is an endpoint of both, then their union is a segment;

: (b) the intersection of two segments with a common endpoint is also a segment.

It follows that there is a unique segment having x and y as endpoints. We denote this segment by $[x, y]$.

Let X be a Λ-tree, where Λ is an arbitrary ordered abelian group. A subtree of X is a subset $A \subseteq X$ such that $x, y \in A$ implies $[x, y] \subseteq A$.

Let G be a group that acts on X via isometries. Isometries of Λ-trees are analogous to those of ordinary trees in that we can classify them as *inversions, elliptic* and *hyperbolic* isometries. In this paper we consider only *free* actions, that is, actions without inversions in which no non-trivial element of the group fixes a point in the tree. Thus all non-trivial isometries are hyperbolic. Let $g \in G$ be an isometry of X. Then we can define A_g, the *characteristic set* or *axis* of g as

$$A_g = \{p \in X \mid [pg^{-1}, p] \cap [p, pg] = \{p\}\}.$$

If g is hyperbolic, A_g is the maximal g-invariant linear subtree of X on which g acts by translation. For every element g in G that is not an inversion one can then define the *translation length function* given by $\|g\| = \min\{d(x, xg) \mid x \in X\}$. It can be shown that this minimum is always realised, and that it is different from 0 for g hyperbolic. To better visualize axes of translation in Λ-trees you can consult the figure on page 83 of [3].

In a Λ-tree, X, every triple of points, p_1,p_2, p_3 has a Y-point, $Y(p_1, p_2, p_3)$, which uniquely lies on all segments $[p_i, p_j]$, for $i \neq j$. The axis of a hyperbolic element g is then equal to $\{Y(pg^{-1}, p, pg) \mid p \in X\}$.

If g and h are hyperbolic isometries of X such that $A_g \cap A_h \neq \emptyset$, and g and h translate in the same direction along $A_g \cap A_h$, then we say that g and h meet *coherently.* If $A_g \cap A_h \neq \emptyset$ and g and h translate in different directions along $A_g \cap A_h$, then g and h meet *incoherently.*

The following is a basic lemma about translation lengths.

Lemma 2.1 ([3], Lemma 3.1.7, page 86)**.** *Let g and h be hyperbolic isometries of a Λ-tree X.*

: *(a) If n is a non-zero integer then $\|g^n\| = |n|\|g\|$ and $A_g = A_{g^n}$.*

: *(b) If g and h meet coherently then $\|gh\| = \|g\| + \|h\|$.*

One of the characteristics of free actions on Λ-trees is that for $gh \neq hg$ we have $A_g \cap A_h$ is a segment of length not exceeding $\|g\| + \|h\|$, since otherwise the commutator of g and h would be an elliptic element, contradicting the freeness of the action. We state this formally, because of its importance, even though it amounts to a fairly trivial observation.

Lemma 2.2 ([3], Remark, page 111)**.** *Let G be a group acting freely without inversions on a Λ-tree, and let $g, h \in G$. Then if g and h do not commute, $A_g \cap A_h$ cannot contain a segment of length greater than or equal to $\|g\| + \|h\|$. Conversely, if g and h commute, they share an axis and hence $A_g \cap A_h$ will contain a segment of length greater than or equal to $\|g\| + \|h\|$.*

One property of Λ-free groups that we will use in this paper is that of *commutative-transitivity* of non-identity elements, which is equivalent to saying that centralisers of non-identity elements are abelian and follows from the fact that two non-identity elements commute if and only if they have the same axis.

Another useful fact about actions on Λ-trees is the device that relates actions on Λ_1-trees to actions on Λ_2-trees, as in the following theorem.

Theorem 2.3 ([11], [3], Corollary 2.4.9, page 76)**.** *(Base-Change Functor) Let $h : \Lambda_1 \to \Lambda_2$ be an order preserving homomorphism between ordered abelian groups and let G be a group acting by isometries on a Λ_1-tree, (X_1, d_1). Then there is a Λ_2-tree, (X_2, d_2) on which G acts by isometries and a mapping $\phi : X_1 \to X_2$ such that*

: *(i) $d_2(\phi(x), \phi(y)) = h(d_1(x, y))$, for all $x, y \in X_1$,*
: *(ii) $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X_1$,*
: *(iii) $\|g\|_{X_2} = h(\|g\|_{X_1})$ for all $g \in G$.*

For the X_2 constructed in the proof of Theorem 2.3 we have that if the action of G on X_1 is free and h is injective, then the action of G on X_2 is also free. In particular we can construct the barycentric subdivision X' of X_1 by taking the endomorphism h to be $\lambda \to 2\lambda$; the resulting action is then without inversions.

Note also that since $\mathbb{Z}^n$ embeds in $\mathbb{R}^n$ every free action on a $\mathbb{Z}^n$-tree gives rise to a free action on an $\mathbb{R}^n$-tree.

3. The main theorem

The strategy of the proof that follows is to consider the equation $x^p y^q = z^r$ and simply look at the various configurations of axes of x, y, z and find a contradiction based on length. The source of the contradiction will primarily be the assumption no pair from x, y, z commute (since if any two commute, all three must commute), which will be expressed via Lemma 2.2.

The proof itself is elementary from the point of view of Λ-tree theory, though the justification of the figures we provide would be rather technical from first principles. For this reason we shall refer where we can, to [3], where much of the basic work is already done. However, we should stress that the technical proofs we refer to are largely formal demonstrations that one's geometric intuition works perfectly well in the context of Λ-trees. With that in mind, we mention the following Lemma, which provides a useful tool for determining the position of axes.

Lemma 3.1. *Let u, v be distinct points in a Λ-tree, X, and g a hyperbolic isometry of X. Then if u, v, ug, vg are collinear in the order given, both u and v must lie on the axis of g.*

Informally, one chooses a point u which one wants to show is on the axis of g, and a point v, 'close' to u in the positive g direction and checks collinearity of the images in the given order.

Theorem 3.2. *Let G be a group that acts freely, and without inversions, by isometries on a Λ-tree, where Λ is an ordered abelian group, and let x, y, z be elements in G. If $x^p y^q = z^r$ with $p,\ q,\ r \geq 4$, then x, y and z commute.*

Proof. Let us assume that x, y and z do not commute and let A_x, A_y and A_z be the axes of translation of x, y and z, respectively. By Lemma 2.1 we know that $A_x = A_{x^n}$ and $\|x^n\| = |n|\|x\|$ (the same is clearly true for y and z) for every non-zero integer n. We can assume without loss of generality that p, q, r are positive and,

$$r\|z\| \leq q\|y\| \leq p\|x\|. \tag{1}$$

If $A_x \cap A_y = \emptyset$, then $\|xy\| = \|x\| + \|y\| + 2d(A_x, A_y)$ (Lemma 3.2.2, [3]) and we have

$$r\|z\| = \|z^r\| = \|x^p y^q\| > \|x^p\| + \|y^q\| = p\|x\| + q\|y\|,$$

which contradicts assumption (1).

Now let us assume that $A_x \cap A_y \neq \emptyset$. Let $\Delta(x, y)$ be the intersection of the two axes, and let $\Delta = |\Delta(x, y)| \in \Lambda$, the length of this segment. Since we assume that x and y do not commute and the action is free, by Lemma 2.2 we have

$$\Delta < \|x\| + \|y\|. \tag{2}$$

If A_x and A_y meet coherently, then we have Figure 1.

In this case, by Lemma 2.1 (b)

$$r\|z\| = \|z^r\| = \|x^p y^q\| = \|x^p\| + \|y^q\| = p\|x\| + q\|y\|,$$

which also contradicts assumption (1).

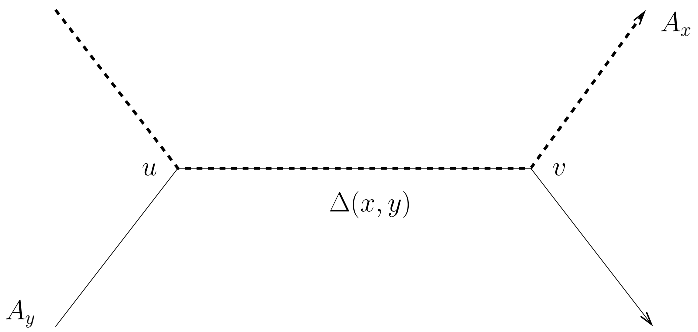

FIGURE 1. Coherent axes

Now let us assume that A_x and A_y meet incoherently. Let $\Delta(x,z)$ and $\Delta(y,z)$ be the intersection of A_x and A_z, and A_y and A_z, respectively. Then we have three cases to consider, depending on the relative length of Δ with respect to $\|y^q\|$.

(1) Let us assume that

$$\Delta > \|y^q\|, \tag{3}$$

as in Figure 2.

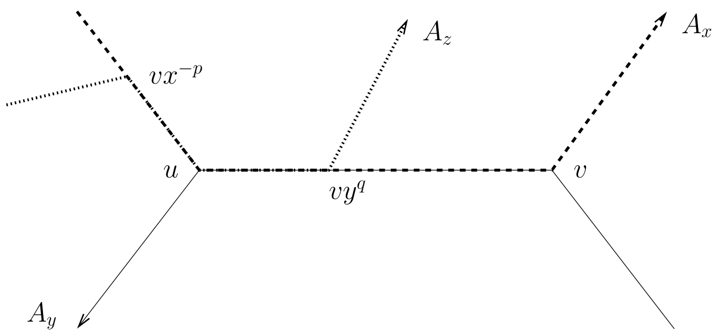

FIGURE 2. A_x and A_y have large intersection

By Lemma 3.3.4 of [3], A_z meets A_x coherently and A_y incoherently, and

$$\|z^r\| = \|x^p\| - \|y^q\|. \tag{4}$$

Since x, y and z do not commute, we get

$$\|x\| + \|z\| > |\Delta(x,z)| = \|z^r\| \implies \|x\| > (r-1)\|z\| \tag{5}$$

and (2) together with (3) give

$$\|x\| > (q-1)\|y\|. \tag{6}$$

Since $p \geq 4$ we have

$$\|x^p\| = \|x^{p-4}\| + \|x^2\| + \|x^2\| > \|x^{p-4}\| + q\|y\| + r\|z\|$$

as $q \geq 3$ and $r \geq 3$ implies $2\|x\| > q\|y\|$ and $2\|x\| > r\|z\|$ by (5) and (6). This contradicts (4).

(2) Now let us assume that the intersection of A_x and A_y is relatively small:

$$\Delta < \|y^q\|. \tag{7}$$

Then by Lemma 3.3.3 in [3] A_z meets both A_x and A_y coherently, and we have the configuration in Figure 3.

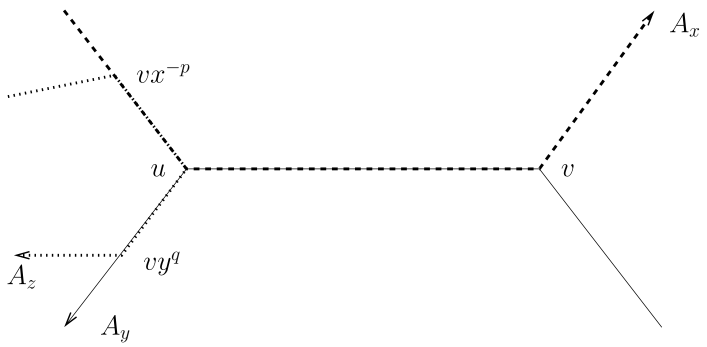

Figure 3. A_x and A_y have small intersection

Since $\|z^r\| \leq \|y^q\|$ and $\|z^r\| = \|x^p\| + \|y^q\| - 2\Delta$ we have

$$\Delta \geq \frac{\|x^p\|}{2} \tag{8}$$

Inequalities (2) and (8) give $(p-2)\|x\| < 2\|y\|$ and by using the assumption (1) we also get $(q-2)\|y\| < 2\|x\|$.

By putting these inequalities together we get that, if $p \geq 4$,

$$(q-2)\|y\| < 2\|x\| \leq (p-2)\|x\| < 2\|y\|. \tag{9}$$

This implies $q < 4$, which is not in our range, and so this configuration cannot happen.

(3) The last case to consider is $\Delta = \|y^q\|$, which corresponds to the configuration described in [3], Lemma 3.3.5.

By (2) the above equality gives $(q-1)\|y\| < \|x\|$, which implies $\Delta = \|y^q\| < 2\|x\|$ since $q > 2$. Let w be $Y(vx^{-p}y^{-q}, v, vy^qx^p)$,

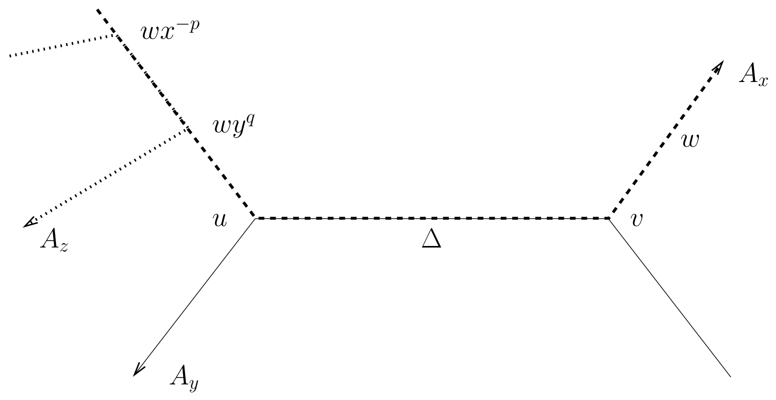

FIGURE 4. A_x and A_y have intersection of length $\|y^q\|$

that is, $[v, w] = [v, vx^{-p}y^{-q}] \cap [v, vy^q x^p]$ (see also Lemma 1.1.2 [3]). Then w is clearly on the axis of x, since $[v, vy^q x^p] = [v, ux^p] \subseteq A_x$; it is also a point on the axis of $y^q x^p$. To see this, note that $wy^q x^p$ must belong to the segment $[v, vy^q x^p]$. If $wy^q x^p$ belongs to $[v, w]$ then, passing to the barycentric subdivision if necessary, it can be shown using arguments similar to those used below that $y^q x^p$ fixes the midpoint of $[w, wy^q x^p]$, forcing $y^q = x^{-p}$, a contradiction. Therefore $wy^q x^p \in [w, wy^q x^p]$, and one can easily apply Lemma 3.1 to conclude that w lies on the axis of $y^q x^p$. Moreover, wy^q is a point on the axis of x, with $wy^q \in [v, vx^{-p}y^{-q}]y^q = [u, vx^{-p}]$, and wy^q lies on the axis of $x^p y^q = z^r$.

To justify the picture the reader should also note that if one considers any point, t, on the axis of x in the positive direction from w, then ty^q does not lie on the axis of x. This is because $vx^{-p}y^{-q}, w, t$ are collinear with w being the nearest point on the segment to the axis of y. Hence vx^{-p}, wy^q, ty^q are also collinear, with wy^q again being the closest point to the axis of y. This immediately implies that ty^q is not on the axis of x.

To completely justify the figure above, all that remains is to argue why wx^{-p} is in the negative A_x direction from wy^q. This is equivalent to saying that $\|x^p\| - \|y^q\| - 2l > 0$, where $l = d(v, w)$. This follows since z^r is not elliptic; one can read this off from Lemma 3.3.5. of [3] which shows that the translation length of z^r is $\min(\|x^p\| - \|y^q\| - 2l, 0)$, or by directly finding a fixed point for $x^p y^q$ using arguments similar to the one we use below.

Having these facts at our disposal, we can easily see that $[wx^{-p}, wy^q]$ is the intersection of A_x and A_z and,

$$|\Delta(x, z)| = \|z^r\| = \|x^p\| - \|y^q\| - 2l \tag{10}$$

We will show that

$$\|x\| < \Delta < 2\|x\| - 2l. \tag{11}$$

Suppose that the second inequality does not hold. Then, passing to the barycentric subdivision if necessary, there exist points $t \in [v, w]$ and $t' \in [wy^q, u]$ such that $d(t, t') = 2\|x\|$ and $t' = ty^q$ (see Theorem 2.3). This implies $ty^qx^2 = t$ and so y^qx^2 is an elliptic element, a situation possible only if $y^q = x^{-2}$. However, by commutative-transitivity this implies that x and y commute, which contradicts our initial assumption.

Now let us assume the first inequality of (11) does not hold, that is, $\Delta \leq \|x\|$. If $\Delta + 2l > \|x\|$ then we can repeat the previous argument and get that y^qx is an elliptic element, which implies that x and y commute, and we obtain a contradiction. So let us suppose $\Delta + 2l < \|x\|$. Since x and z do not commute, we have $|\Delta(x, z)| < \|x\| + \|z\|$, which implies $\|z^r\| < \|x\| + \|z\|$. But then $\Delta + 2l < \|x\|$ and (10) imply $\|z\| > (p - 2)\|x\|$, which is false by (1). This concludes the proof of (11).

From (10) and (11) we get $\|z^r\| > (p - 2)\|x\|$. Since we assume that x and z do not commute we get $|\Delta(x, z)| = \|z^r\| < \|x\| + \|z\|$, so in conclusion $(p - 3)\|x\| < \|z\|$, which contradicts (1) if $p, q \geq 4$.

□

4. CAT(-1) Structures

In this section we show that the groups

$$G_{pqr} = \langle\, x, y, z \,|\, x^p y^q = z^r \,\rangle$$

are all CAT(-1) groups. As mentioned in the introduction, the original motivation for studying these CAT(-1) structures, was to see if certain multiple HNN extensions (described in Remark 4.3 below) of G_{pqr} gave examples of hyperbolic groups which are not CAT(-1). We do not know whether or not these HNN groups are CAT(-1). However they should admit high dimensional CAT(0) structures by the techniques of Tim Hsu and Dani Wise.

Proposition 4.1. *Let $p, q, r \geq 2$ be integers. The groups*

$$G_{pqr} = \langle\, x, y, z \,|\, x^p y^q = z^r \,\rangle$$

admit CAT(-1) structures corresponding to each isometry class of triangles in the hyperbolic plane.

Proof. In order to see that the groups G_{pqr} are all CAT(-1), take a triangle in the hyperbolic plane with positive angles α, β and γ. Subdivide the side opposite the angle α (respectively β, γ) into p (respectively q, r) subsegments of equal length, and label the subsegments by x (respectively y, z^{-1}) as shown on the left side of Figure 5.

The quotient space of this triangle obtained by isometrically identifying all the x edges (respectively y-edges, z-edges) is a cell complex, with one vertex, three 1-cells (labelled x, y and z respectively) and a single 2-cell corresponding to the triangle. This cell complex is a presentation 2-complex for the group G_{pqr}. It is a piecewise hyperbolic 2-complex.

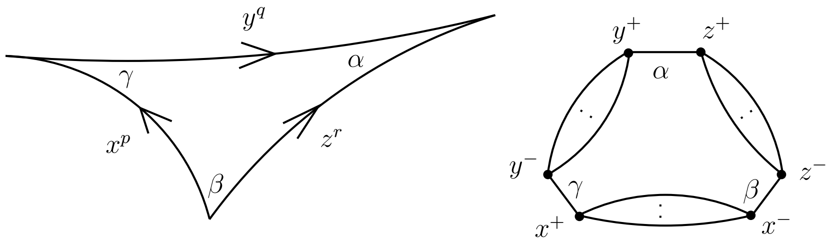

FIGURE 5. 2-cell of G_{pqr} presentation 2-complex and vertex link.

The link of the single vertex is the metric graph shown on the right side of Figure 5. There are $p-1$ edges from x^+ to x^-, $q-1$ edges from y^+ to y^-, and $r-1$ edges from z^+ to z^-. These all have length π. The remaining three edges have lengths α, β and γ as indicated in the figure. There are no nontrivial loops in the link of length less than 2π. Thus the link is a CAT(1) metric graph, and the 2-complex is a locally CAT(-1) presentation complex for G_{pqr}.

There are no restrictions on the original triangle. Thus we have exhibited CAT(-1) structures for G_{pqr} corresponding to each isometry class of triangles in the hyperbolic plane. □

The characterizations of the G_{pqr} groups in Remarks 4.1 and 4.2 are essentially equivalent, 4.1 being of a topological nature, and 4.2 adopting the combinatorial point of view.

Remark 4.1. There are 3-dimensional CAT(-1) structures for G_{pqr} too. One way to see this is to note that G_{pqr} is the fundamental group of the 2-complex obtained from a "thrice punctured sphere" (compact, orientable surface with three circle boundary components and with Euler characteristic

-2) by wrapping one boundary circle p times around a target circle, wrapping another boundary circle q times around a second target circle, and wrapping the third boundary circle r times around a third target circle. This thickens up to give a compact, hyperbolic 3-manifold with boundary an orientable surface of genus 2.

Remark 4.2. Here is another way to look at the G_{pqr}. They are the fundamental groups of graphs of groups with underlying graph a tripod (tree with 1 valence 3 vertex and 3 valence 1 vertices), edge groups all infinite cyclic, valence 1 vertex groups all infinite cyclic, and valence 3 vertex group being free of rank 2. The inclusions from the edge groups to the valence 3 vertex group map to generators a, b and their product ab. The inclusion maps from the edge groups to the valence 1 vertex groups are just multiplication by p, q and r.

Remark 4.3. It is easy to produce hyperbolic groups from the G_{pqr} via multiple HNN extensions over infinite cyclic subgroups. For instance, one can add a stable letter which conjugates one generator to another, add another stable letter which conjugates a generator to a commutator of two generators, and so on. One uses the Bestvina-Feighn combination theorem after each HNN extension to ensure that the resulting groups are hyperbolic. The group G_{pqr} is a subgroup of these multiple HNN groups. Therefore, if these HNN groups are CAT(-1), one gets an action of G_{pqr} by semi-simple isometries on a CAT(-1) space with various restrictions on translation lengths. For example, it seems hard to find an action by semi-simple isometries of the G_{pqr} on a CAT(-1) space in which the translation lengths of the 3 generators and their 3 pairwise commutators are all equal.

The graph of free groups over infinite cyclic edge groups viewpoint of the G_{pqr} in Remark 4.2 leads to a large class of CAT(0) structures. The Sageev construction techniques being developed by Hsu and Wise will give lots of new CAT(0) cubical structures for the G_{pqr}. Their techniques should also apply to give CAT(0) cubical structures for the hyperbolic multiple HNN extensions of the G_{pqr}. But these appear to be very far from CAT(-1) structures.

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Noel Brady, Dept. of Mathematics, University of Oklahoma, Norman, OK 73019, USA, *E-mail*: nbrady@math.ou.edu

Laura Ciobanu, Centre de Recerca Matemàtica, 08193 Bellaterra, SPAIN, *E-mails*: ciobanu@math.auckland.ac.nz, LCiobanu@crm.es

Armando Martino, Universitat Politècnica de Catalunya, Dept. of Mathematics, 08860 Castelldefels, SPAIN, *E-mail*: Armando.Martino@upc.edu

Shane O Rourke, Dept. of Mathematics, Cork Institute of Technology, Cork, IRELAND, *E-mail*: Shane.ORourke@cit.ie