MODEL STRUCTURES ON THE CATEGORY OF SMALL DOUBLE CATEGORIES

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Abstract. In this paper we obtain several model structures on **DblCat**, the category of small double categories. Our model structures have three sources. We first transfer across a categorification-nerve adjunction. Secondly, we view double categories as internal categories in **Cat** and take as our weak equivalences various internal equivalences defined via Grothendieck topologies. Thirdly, **DblCat** inherits a model structure as a category of algebras over a 2-monad. Some of these model structures coincide and the different points of view give us further results about cofibrant replacements and cofibrant objects. As part of this program we give explicit descriptions and discuss properties of free double categories, quotient double categories, colimits of double categories, and several nerves and categorifications.

Contents

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1. Introduction

The theory of categories enriched in **Cat**, called 2-categories, has been highly developed over the past 40 years and has found numerous applications. Beginning with Bénabou's bicategories (weak 2-categories) in [4], through Kelly's monograph [50] on enriched categories, and including the more recent article [53], as well as many others, we have seen the $n = 2$ case for higher category theory become very well understood. Limits in 2-categories [51], 2-monads on 2-categories [7], and Kan extensions for 2-functors [50] are now well known. Model structures on **2-Cat** have also been studied recently in [55], [56], and [78]. Model structures, more generally, have been used in the study of $(\infty, 1)$-categories as a means of comparison by Bergner, Joyal-Tierney, Rezk, and Toën [5], [6], [48], [71], and [77].

Recent examples, however, show that 2-categories are not enough, and that one must invoke Ehresmann's earlier notion of *double* category [31], [32]. In many mathematical situations one is interested in two types of morphisms, which may or may not interact. Between rings, for example, there are ring homomorphisms as well as bimodules. Between manifolds there are diffeomorphisms and cobordisms, which are both used in field theory. Between categories there are functors as well as adjunctions. The notion of 2-category does not capture both types of morphisms, but the notion of (pseudo) double category certainly does.

Concisely, a *small double category* is an internal category in **Cat**. A small double category consists of a set of objects, a set of horizontal morphisms, a set of vertical morphisms, and a set of squares, equipped with various associative and unital compositions satisfying the interchange law. In addition to the early work of Bastiani-Ehresmann [2], [28], [29], [30], [31], [32], and

Brown-Spencer [17], recent work on double categories has been completed by Brown and collaborators, Dawson, Fiore, Garner, Grandis, Paré, Pronk, Shulman, and others [14], [15], [16], [21], [24], [25], [26], [35], [37], [39], [40], [41], [42], [72], and [73].

Double categories are the $n = 2$ case for n-fold categories, which have been studied and applied for some time now. In the same way that *higher* categories may be defined by iterated enrichment, one may define *wider* categories or *n-fold categories* via iterated internalization. The edge symmetric[1] case has been studied by Brown under the name of cubical ω-categories. Further, n-fold categories internal to the category of groups have been used to model connected homotopy $(n+1)$-types in [59] as summarized in the survey paper [66]. Recent work includes [57] and [67]. Applications of versions of the $n = 2$ case of internalized categories include [23], [34], [35], [52], [62], [65], [72], [73]. Thus, there has been a general trend towards n-fold categories, especially the $n = \omega$ and $n = 2$ cases.

In this article we introduce model categories into the theory of double categories, anticipating a utility in the theory of *wider categories* analogous to that of model structures in the theory of *higher categories.* Already in the $n = 2$ case we see that n-fold categories and n-categories diverge: even though the homotopy theory of 2-categories resembles that of categories, the homotopy theory of double categories is quite different. This results from the numerous ways to view a double category: as an internal category in **Cat**, as a categorical structure with two directions, as certain simplicial objects in **Cat**, or as certain bisimplicial sets. Each point of view suggests different notions of weak equivalence and fibration. The new types of pasting diagrams available in a double category also create new phenomena. We take these various point of view into consideration when constructing the model structures.

Thus, our model structures have three sources. First, we transfer the categorical and Thomason diagram structures on the category of simplicial objects in **Cat** to **DblCat** via a horizontal categorification-horizontal nerve adjunction. In the Thomason structure on **Cat** in [76], a functor is a weak equivalence if and only if its nerve is a weak homotopy equivalence of simplicial sets. In the categorical structure on **Cat** of [49] and [70], a functor is a weak equivalence if and only if it is an equivalence of categories. Both the Thomason structure and the categorical structure are cofibrantly generated, and thus induce cofibrantly generated model structures on simplicial objects in **Cat** where weak equivalences and fibrations are defined levelwise. We apply Kan's Lemma on Transfer of cofibrantly generated model structures (Theorem 7.11)to transfer both of these diagram structures to **DblCat**.

[1] *Edge symmetric* means that the n-morphisms in all $n+1$ directions are the same.

However, the application is not straightforward, and we must make several double categorical preparations, including horizontal categorification and a pushout formula in **DblCat**. We also prove one negative result: it is impossible to transfer the Reedy categorical structure on $\mathbf{Cat}^{\Delta^{op}}$ to **DblCat**. The transfer from bisimplicial sets will be the subject of a later article.

We arrive at a second source for model structures on **DblCat** when we view double categories as internal categories in **Cat**. In this way we obtain double categorical versions of the categorical structure on **Cat**. Although the notion of fully faithfulness makes sense internally, essential surjectivity does not, and therefore equivalences of internal categories need further explanation. Model structures on categories internal to a good category $\mathbf{C}$ have already been developed in [33], and we apply their results to the case $\mathbf{C} = \mathbf{Cat}$. They define essential surjectivity (and hence also weak equivalences) with respect to a Grothendieck topology $\mathcal{T}$ on $\mathbf{C}$. We take simplicially surjective functors and categorically surjective functors as bases for Grothendieck topologies on **Cat**, and obtain two distinct model structures.

Third, **DblCat** inherits a model structure as a category of algebras over a 2-monad as in [54]. The underlying 1-category of a 2-category with finite limits and finite colimits always admits the so-called trivial model structure, whose weak equivalences are equivalences and fibrations are isofibrations. If $\mathcal{K}$ is a locally finitely presentable 2-category equipped with a 2-monad T with rank, then the category of T-algebras is a model category: a morphism of T-algebras is a weak equivalence or fibration if and only if its underlying morphism is a weak equivalence or fibration in the trivial model structure on $\mathcal{K}$. In our application of [54], $\mathcal{K}$ is the 2-category $\mathsf{Cat}(\mathbf{Graph})$ of internal categories in reflexive graphs, and T arises from the adjunction between reflexive graphs and **Cat**.

We prove that some of these model structures coincide. The transferred categorical diagram structure is the same as the model structure associated to the simplicially surjective topology on **Cat**, while the algebra structure is the same as the model structure associated to the categorically surjective topology on **Cat**.

These two different constructions of the same model structures yield more refined information about cofibrant replacements and cofibrant objects. For example, the cofibrant objects in the algebra structure are known to be precisely the flexible algebras, but from the second characterization we see that the flexible double categories are precisely those with object category projective with respect to functors that are surjective on objects and full. Such a description of the cofibrant objects, and explicit cofibrant replacements cannot be obtained using either model structure alone. Further, such a description allows us to conclude that the flexible 2-categories of [55] are indeed flexible algebras for a 2-monad.

In order to build our model structures we prove various general results about double categories, so far not available in the literature. These results are also of independent interest for the theory of double categories in its own right. We develop free double categories, their quotients, and colimits of double categories using a double categorical version of Street's 2-categorical notion of *derivation scheme* [75]. In particular we obtain an explicit formula for two pushouts of double categories in Theorem 10.6, which is essential for our application of Kan's Lemma on Transfer in Theorems 7.13 and 7.14.

Free double categories on reflexive double graphs have been studied in [26]. By *reflexive double graph* we mean a collection of objects, vertical edges, horizontal edges, and squares equipped with identity edges and identity squares. We factorize the adjunction of [26] between double categories and reflexive double graphs via the category of double derivation schemes. A *double derivation scheme* is a reflexive double graph in which the horizontal and vertical reflexive 1-graphs are categories. In the free double category on a double derivation scheme, the vertical and horizontal 1-categories are preserved, but squares consist of allowable compatible arrangements. Since we are considering compatible arrangements of squares in a double derivation scheme rather than in a double reflexive graph, our *allowable* compatible arrangements are different than the *composable* compatible arrangements of [25].

Free double categories on double derivation schemes and their quotients allow us to construct colimits of double categories. First one takes the colimits of the vertical and horizontal 1-categories. These together with the colimit of the *sets* of squares form a double derivation scheme. Finally, we mod out the free double category on this double derivation scheme by the smallest congruence which guarantees that the natural maps are double functors, and the result is the colimit in **DblCat**. This colimit formula is the basis of Theorem 10.6 which gives an explicit description of the pushouts of a double functor along two inclusions of external products. This theorem is crucial for our application of Kan's Lemma on Transfer. These two pushouts are special cases of a more general theorem on pushouts along inclusions of external products, which will appear in a separate paper with a comparison to [22].

Free double categories on double derivation schemes and their quotients find further application in the construction of fundamental double categories of simplicial objects in **Cat** and bisimplicial sets, *i.e.,* in our construction of left adjoints to the horizontal nerve and double nerve functors. We obtain an important example of our explicit constructions of fundamental double categories in a second way as well, namely via weighted colimits (see Example 6.5 and Proposition 6.10).

We begin in Section 1 with a review of double categories, including horizontal 2-categories, vertical 2-categories, and the external product of 2-categories. Free double categories on double derivation schemes are introduced in Section 2 and are used in Section 4 to describe colimits in **DblCat**. Horizontal and double nerves are discussed in Section 5 along with their representable definitions in terms of external products of finite ordinals. In Section 6, free double categories on double derivation schemes and their quotients are applied to construct the left adjoints to the horizontal nerve and the double nerve. Section 7 focuses on transferring model structures across the horizontal categorification-horizontal nerve adjunction, and recalls model structures on **Cat**, smallness issues, and the Kan's Lemma on Transfer. Section 8 begins with an exposition of the methods of [33], and then applies them to obtain model structures on $\mathsf{Cat}(\mathbf{Cat}) = \mathbf{DblCat}$ induced by three Grothendieck topologies on **Cat**: the simplicially surjective topology, the categorically surjective topology, and the trivial topology. In Section 9 we prove that the 2-monad structure on **DblCat** coincides with the model structure induced by the categorically surjective topology. In Section 10, the Appendix, we obtain an explicit description of certain pushouts in **DblCat**, namely Theorem 10.6. We use this to characterize the behavior of the horizontal nerve on such pushouts in Theorem 10.7. The essential application is to the generating acyclic cofibrations in the transfer in Section 7.

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2. Double Categories

We first recall the elementary notions of double category theory. In many mathematical contexts there are two interesting types of morphisms; double categories organize them into one structure. For example, between rings there are morphisms of rings as well as bimodules, between objects of any 2-category there are morphisms as well as adjunctions, and so on. Sometimes one would like to distinguish a family of squares, such as the pullback squares among the commutative squares, and double categories are also of use here. The notion of double category is not new, and goes back to Ehresmann in [31] and [32].

Definition 2.1. A *small double category* $\mathbb{D} = (\mathbb{D}_0, \mathbb{D}_1)$ is a category object in the category of small categories. This means that $\mathbb{D}_0$ and $\mathbb{D}_1$ are categories equipped with functors

$$\begin{tikzcd} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "m"] & \mathbb{D}_1 \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathbb{D}_0 \arrow[l, "u" description] \end{tikzcd}$$

that satisfy the usual axioms of a category. We call the objects and morphisms of $\mathbb{D}_0$ respectively the *objects* and *vertical morphisms* of $\mathbb{D}$, and we call the objects and morphisms of $\mathbb{D}_1$ respectively the *horizontal morphisms* and *squares* of $\mathbb{D}$.

When one expands this definition, one sees that a small double category consists of a set of *objects*, a set of *horizontal morphisms*, a set of *vertical morphisms*, and a set of *squares* equipped with various sources, targets, and associative and unital compostions. Since we only deal with small double categories, we will usually leave off the adjective small. Sources and targets are indicated as follows.

(1)
$$A \xrightarrow{f} B \qquad \begin{tikzcd} A \arrow[d, "j"'] \\ C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

We denote the set of squares with the boundary

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

by $\mathbb{D}\begin{pmatrix} & f & \\ j & & k \\ & g & \end{pmatrix}$. Then one has the categories

$$(\mathit{Obj}\,\mathbb{D}, \mathit{Hor}\,\mathbb{D}) \text{ and } (\mathit{Ver}\,\mathbb{D}, \mathit{Sq}\,\mathbb{D})$$

under horizontal composition and the categories

$$(\mathit{Obj}\,\mathbb{D}, \mathit{Ver}\,\mathbb{D}) \text{ and } (\mathit{Hor}\,\mathbb{D}, \mathit{Sq}\,\mathbb{D})$$

under vertical composition. We will write $[f\ g]$ for the horizontal composition of horizontal morphisms f and g, and similarly $[\alpha\ \beta]$ for the horizontal composition of squares α and β. We will write $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$ for the vertical composition of vertical morphisms v and w, and similarly $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$ for the vertical composition of squares α and β. Composition of squares in $\mathbb{D}$ satisfies the usual interchange law.

There are many examples of double categories. The commutative squares in a given 1-category form a double category. More generally, for a 2-category $\mathbf{C}$, Ehresmann defined the double category $\mathbb{Q}\mathbf{C}$ of *quintets of* $\mathbf{C}$. Its objects are the objects of $\mathbf{C}$, horizontal and vertical morphisms are the morphisms of $\mathbf{C}$, and the squares α as in (1) are the 2-cells $\alpha\colon\ k \circ f \Longrightarrow g \circ j$. In many examples one direction is merely a bicategory (weak 2-category), and one actually has a *pseudo double category* as defined in [41]. For example, the double category of rings, bimodules, ring homomorphism, and twisted maps of bimodules is weak in one direction. Another example is given by finite sets, Riemann surfaces with labelled analytically parametrized boundary components, bijections of finite sets, and holomorphic maps preserving the given structure. In these two examples we choose the horizontal direction to be weak, so that bimodules respectively Riemann surfaces are the horizontal morphisms. In this paper we work only with strict double categories, though pseudo double categories can also fit into our framework.

The notion of double category contains many familiar structures. If we view a category as an internal category in **Cat** with object and morphism categories discrete, it is equivalent to viewing an ordinary category as a double category with trivial vertical morphisms and trivial squares. Every 2-category $\mathbf{C}$ can be considered a double category in two ways: either as a double category $\mathbb{H}\mathbf{C}$ with trivial vertical morphisms or as a double category

$\mathbb{V}\mathbf{C}$ with trivial horizontal morphisms. Similarly, any double category $\mathbb{D}$ has an underlying horizontal 2-category $\mathbf{H}\mathbb{D}$ and an underlying vertical 2-category $\mathbf{V}\mathbb{D}$: we obtain these substructures as the full subdouble categories with only trivial vertical morphisms or trivial horizontal morphisms respectively. We denote the underlying 1-categories of $\mathbf{H}\mathbb{D}$ and $\mathbf{V}\mathbb{D}$ by $(\mathbf{H}\mathbb{D})_0$ and $(\mathbf{V}\mathbb{D})_0$ respectively.

A *double functor* $F\colon \mathbb{D}\longrightarrow\mathbb{E}$ is an internal functor in **Cat**. Such a functor consists of functions

$$\begin{aligned} Obj\ \mathbb{D} &\longrightarrow Obj\ \mathbb{E} \\ Hor\ \mathbb{D} &\longrightarrow Hor\ \mathbb{E} \\ Ver\ \mathbb{D} &\longrightarrow Ver\ \mathbb{E} \\ Sq\ \mathbb{D} &\longrightarrow Sq\ \mathbb{E} \end{aligned}$$

which preserve all sources, targets, compositions, and identities.

Internal natural transformations are also called *horizontal natural transformations.*

Definition 2.2. If $F, G\colon \mathbb{D}\longrightarrow\mathbb{E}$ are double functors, then a *horizontal natural transformation* $\theta\colon F\Longrightarrow G$ as in [41] assigns to each object A a horizontal morhism $\theta A\colon FA\longrightarrow GA$ and assigns to each vertical morphism j a square

$$\begin{tikzcd} FA \arrow[r, "\theta A"] \arrow[d, "Fj"'] \arrow[dr, phantom, "\theta j"] & GA \arrow[d, "Gj"] \\ FC \arrow[r, "\theta C"'] & GC \end{tikzcd}$$

such that:

(i) For all $A \in \mathbb{D}$, we have $\theta 1^v_A = i^v_{\theta A}$,
(ii) For composable vertical morphisms j and k,

$$\begin{tikzcd} FA \arrow[r, "\theta A"] \arrow[d, "F\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]"'] \arrow[dr, phantom, "\theta\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]"] & GA \arrow[d, "F\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]"] \\ FE \arrow[r, "\theta E"'] & GE \end{tikzcd} \quad = \quad \begin{tikzcd} FA \arrow[r, "\theta A"] \arrow[d, "Fj"'] \arrow[dr, phantom, "\theta j"] & GA \arrow[d, "Gj"] \\ FC \arrow[r, "\theta C"] \arrow[d, "Fk"'] \arrow[dr, phantom, "\theta k"] & GC \arrow[d, "Gk"] \\ FE \arrow[r, "\theta E"'] & GE, \end{tikzcd}$$

(iii) For all α as in (1),

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "Fj"'] \arrow[dr, phantom, "F\alpha"] & FB \arrow[r, "\theta B"] \arrow[d, "Fk"] \arrow[dr, phantom, "\theta k"] & GB \arrow[d, "Gk"] \\ FC \arrow[r, "Fg"'] & FC \arrow[r, "\theta C"'] & GD \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\theta A"] \arrow[d, "Fj"'] \arrow[dr, phantom, "\theta j"] & GA \arrow[r, "Gf"] \arrow[d, "Gj"] \arrow[dr, phantom, "G\alpha"] & GB \arrow[d, "Gk"] \\ FC \arrow[r, "\theta C"'] & GC \arrow[r, "Gg"'] & GD. \end{tikzcd}$$

We also need the analogous notion of vertical natural transformation.

Definition 2.3. If $F, G\colon \mathbb{D} \longrightarrow \mathbb{E}$ are double functors, then a *vertical natural transformation* $\sigma\colon F \Longrightarrow G$ as in [41] assigns to each object A a vertical morphism $\sigma A\colon FA \longrightarrow GA$ and assigns to each horizontal morphism f a square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\sigma A"'] \arrow[dr, phantom, "\sigma f"] & FB \arrow[d, "\sigma B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

such that:

(i) For all objects $A \in \mathbb{D}$, we have $\sigma 1^h_A = i^h_{\sigma A}$,
(ii) For all composable horizontal morphisms f and g,

$$\sigma[f \;\; g] = [\sigma f \;\; \sigma g],$$

(iii) For all α as in (1),

$$\begin{bmatrix} F\alpha \\ \sigma g \end{bmatrix} = \begin{bmatrix} \sigma f \\ G\alpha \end{bmatrix}.$$

Thus, double categories form a 2-category in two different ways, depending on the choice of 2-cell. Further, there is a useful adjunction with **2-Cat**.

Proposition 2.4. *Let* $\mathbf{DlbCat_h}$ *respectively* $\mathbf{DblCat_v}$ *denote the 2-categories of small double categories, double functors, and horizontal natural transformations respectively vertical natural transformations. Then the inclusion 2-functors*

$$\mathbb{H}\colon \mathbf{2\text{-}Cat} \longrightarrow \mathbf{DblCat_h}$$

$$\mathbb{V}\colon \mathbf{2\text{-}Cat} \longrightarrow \mathbf{DblCat_v}$$

have as right 2-adjoints the 2-functors

$$\mathbf{H}\colon \mathbf{DblCat_h} \longrightarrow \mathbf{2\text{-}Cat}$$

$$\mathbf{V}\colon \mathbf{DblCat_v} \longrightarrow \mathbf{2\text{-}Cat}$$

respectively.

The inclusion $\mathbb{H}$ gives us another way of constructing examples of double categories from 1- and 2-categories.

Definition 2.5. If $\mathbf{C}$ and $\mathbf{D}$ are 2-categories, then their *external product* $\mathbf{C}\boxtimes\mathbf{D}$ is the double category with objects $\mathit{Obj}\,\mathbf{C}\times \mathit{Obj}\,\mathbf{D}$, vertical morphisms

$$(f,D)\colon\ (C,D)\longrightarrow(C',D)\,,$$

horizontal morphisms

$$(C,g)\colon\ (C,D)\longrightarrow(C,D')\,,$$

and squares

$$\begin{tikzcd}
(C,D) \arrow[r, "{(C,g_1)}"] \arrow[d, "{(f_1,D)}"'] \arrow[dr, phantom, "\alpha"] & (C,D') \arrow[d, "{(f_2,D')}"] \\
(C',D) \arrow[r, "{(C',g_2)}"'] & (C',D')
\end{tikzcd}$$

given by pairs $\alpha=(\gamma,\delta)$ of 2-cells $\gamma\colon\ f_1\Longrightarrow f_2$ and $\delta\colon\ g_1\Longrightarrow g_2$ in $\mathbf{C}$ and $\mathbf{D}$ respectively. More succinctly, $\mathbf{C}\boxtimes\mathbf{D}=\mathbb{V}\mathbf{C}\times\mathbb{H}\mathbf{D}=(\mathbb{H}\mathbf{C})^t\times\mathbb{H}\mathbf{D}$ where $(\mathbb{H}\mathbf{C})^t$ denotes the transpose[2] of the double category $\mathbb{H}\mathbf{C}$. More generally, the *external product* of double categories $\mathbb{C}$ and $\mathbb{D}$ is $\mathbb{C}\boxtimes\mathbb{D}:=\mathbb{C}^t\times\mathbb{D}$.

Lemma 2.6. *The external product is a functor*

$$\boxtimes\colon\ \mathbf{2\text{-}Cat}\times\mathbf{2\text{-}Cat}\longrightarrow\mathbf{DblCat}\,.$$

Proof: Transpose is functorial. $\square$

Example 2.7. Let $[m]$ denote the partially ordered set $\{0,1,2,\ldots,m\}$. Then the double category $[m]\boxtimes[n]$ has the shape

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {}
\end{tikzcd}$$

[2]To transpose a double category, one interchanges the roles of the horizontal and vertical morphisms.

with m rows and n-columns.

Next we turn to some other adjunctions.

3. Free Double Categories and Quotients

As expected, there is a notion of free double category and quotient double category. However, the situation is richer than for ordinary categories, as there is an intermediate step between double categories and reflexive double graphs, which we call *double derivation schemes.* Double derivation schemes and quotients are crucial in the explicit description of colimits in Section 4, the construction of left adjoints to horizontal and double nerves in Section 6, and the computation of pushouts in Theorems 10.6 and 10.7.

In this section we introduce double analogues to some of the concepts in [75], with one important difference: we always require identities. We work with *reflexive* graphs, *i.e.,* graphs which are equipped with a distinguished *identity* edge $1_A\colon A \longrightarrow A$ for each vertex A. Consequently, our reflexive 2-graphs and derivation schemes always have identity edges and identity 2-edges. This is important because nontrivial squares in a double category may very well have one or more trivial edges. All graphs are directed multigraphs, also called *quivers.*

Definition 3.1. A *reflexive double graph* $\mathbb{A}$ is an internal reflexive graph in the category of reflexive graphs. This consists of a set of vertices (objects) $Obj\ \mathbb{A}$, a set of horizontal edges $Hor\ \mathbb{A}$, a set of vertical edges $Ver\ \mathbb{A}$, and a set of squares $Sq\ \mathbb{A}$ equipped with source and target maps as in (1) as well as horizontal and vertical identity edges and identity squares. A *morphism of reflexive double graphs* is a morphism of internal reflexive graphs, or equivalently, a map which preserves sources and targets as well as all identities. Reflexive double graphs form a category which we denote by **RefDblGr**. We denote the horizontal and vertical reflexive 2-graphs of a double graph $\mathbb{A}$ by $\mathbf{H}\mathbb{A}$ and $\mathbf{V}\mathbb{A}$.

A reflexive double graph is a double category without any of the compositions. The intermediate structure between reflexive double graphs and double categories is analogous to Street's notion of *derivation scheme* in [75].

Definition 3.2. A *double derivation scheme* is a reflexive double graph whose vertical reflexive 1-graph and horizontal reflexive 1-graph are categories. A *morphism of double derivation schemes* is a map which is a functor on both the horizontal and vertical 1-categories, and preserves source, target, and identity squares. Double derivation schemes form a category which we denote by **DblDerSch**. We denote the horizontal and vertical derivation schemes of a double derivation scheme $\mathbb{S}$ by $\mathbf{H}\mathbb{S}$ and $\mathbf{V}\mathbb{S}$, and their underlying categories by $(\mathbf{H}\mathbb{S})_0$ and $(\mathbf{V}\mathbb{S})_0$.

To take a free category on a reflexive graph, one merely takes paths of composable edges and identifies paths which differ only by insertion or deletion of identity edges. However, the 2-dimensional situation is more subtle, as evidenced by [46], [68], and [69]. Thus, in the construction of a free double category we need a careful definition of *allowable compatible arrangement*. We use the notion of compatible arrangement from [25], and develop it further for our purposes.

Definition 3.3. A *compatible arrangement* in a double derivation scheme $\mathbb{S}$ consists of a subdivision of a rectangle into smaller rectangles and a function which assigns to each vertex an object, to each horizontal line segment a horizontal morphism, to each vertical line segment a vertical morphism, and to each constituent rectangle a square in $\mathbb{S}$, which are compatible in the sense that

(i) for each horizontal edge in the subdivision, the domain and codomain respectively of the morphism assigned to it are the objects assigned to the left and right vertices respectively;
(ii) for each vertical edge in the subdivision, the domain and codomain respectively of the morphism assigned to it are the objects assigned to the top and bottom vertices respectively;
(iii) for each constituent rectangle the composition of the morphisms assigned to the edges on
    (a) the left side is the horizontal domain of the square assigned to it;
    (b) the right side is the horizontal codomain of the square assigned to it;
    (c) the top is the vertical domain of the square assigned to it;
    (d) the bottom is the vertical codomain of the square assigned to it.

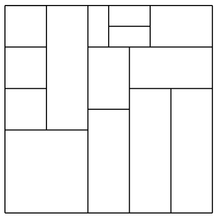
In the free double category on a double derivation scheme, the squares will be compatible arrangements for which the image under any morphism of double derivation schemes into a double category would become composable to a single square by a sequence of horizontal and vertical compositions. We will call such compatible arrangements allowable. However, such an image would just be a compatible arrangement in that double category with the same underlying subdivision of the rectangle. So whether a compatible arrangement is allowable in the free double category depends only on its shape, *i.e.*, the underlying subdivision of the rectangle. A *horizontal* (resp.

vertical) *cut* in a compatible arrangement is a horizontal (resp. vertical) line segment which consists of edges of the underlying subdivision of the rectangle. A horizontal (respectively vertical) cut is *full length* if it stretches from the left (respectively top) edge of the arrangement to the right (respectively bottom) edge of the arrangement. We can use this to characterize when a compatible arrangement is allowable.

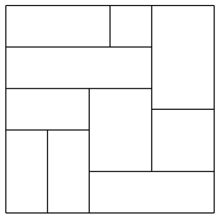
Definition 3.4. A subdivision of the rectangle is *allowable* if it is either the trivial subdivision, consisting of just the rectangle itself, or contains a full length horizontal or vertical cut which divides it into two allowable subdivisions. A compatible arrangement is *allowable* if its underlying subdivision of the square is allowable.

As an illustration, consider the following two examples of subdivisions of the rectangle.

Allowable:



Not allowable:



Note that the notion of allowable compatible arrangement differs from the notion of *composable* compatible arrangement in [25] in that Dawson and Paré call a compatible arrangement in a double category $\mathbb{D}$ composable if it is composable to a single square through the use of both factorizations and compositions in $\mathbb{D}$. So their notion does depend on the ambient double category, not only on the shape of the arrangement. Any allowable compatible arrangement in our sense is composable in the sense of Dawson and Paré.

Proposition 3.5. *A compatible arrangement in a double category is allowable if and only if it can be composed to a single square by a sequence of horizontal and vertical compositions.*

Proof: We argue by induction on the number of squares in the arrangement. The statement is trivially true for arrangements consisting of a single square. Now let C be a compatible arrangement consisting of two or more squares, with an assignment into a double category $\mathbb{D}$ which is composable by a sequence of horizontal and vertical compositions of squares. Consider the last composition used. Without loss of generality, assume that this is a horizontal composition of two squares γ_1 and γ_2 along a vertical morphism

v, as in

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\gamma_1"] & \cdot \arrow[r] \arrow[d, "v"'] \arrow[dr, phantom, "\gamma_2"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

Both γ_1 and γ_2 have been obtained by sequences of horizontal and vertical compositions of squares in C, so v is a vertical composition of vertical morphisms $v_1, \ldots, v_n$ in C. The underlying edges of these vertical morphisms form a cut in the underlying subdivision of the rectangle for C. The squares on the left side of this cut form a compatible arrangement, since they form a rectangular subset of a compatible arrangement. Call this arrangement C_1. It can be composed to γ_1 by a subsequence of the horizontal vertical compositions used for C. In the same way, the squares on the right side of this cut form a compatible arrangement C_2 which can be composed to γ_2 by a sequence of horizontal and vertical compositions. Since both C_1 and C_2 contain strictly less squares than C, the induction hypothesis gives that they are both allowable compatible arrangements.

Conversely, suppose that a compatible arrangement C of two or more squares in a double category $\mathbb{D}$ is allowable. Then it contains a horizontal (resp. vertical) cut into two allowable compatible arrangements C_1 and C_2. By induction these arrangements can be composed to single squares in $\mathbb{D}$ by sequences of horizontal and vertical compositions. Now consider the sequence of horizontal and vertical compositions used for C_1 followed by the one for C_2 and then one final vertical (resp. horizontal) composition along the cut. This shows that C is composable to a single square in $\mathbb{D}$ by a sequence of horizontal and vertical compositions of squares. □

For inductive arguments on the number of squares in an allowable compatible arrangement, we need to know that cutting an allowable arrangement along any full length cut produces two smaller compatible arrangements.

Proposition 3.6. *If CA is a compatible arrangement which is allowable, then any full length cut divides the arrangement into two allowable compatible arrangements.*

Proof: We prove this by induction on the number of squares in the arrangement. It is obviously true for compatible arrangements consisting of a single square. For an arrangement consisting of $n \geq 2$ squares, let $\mathcal{C}_1$ be the cut of this proposition and let $\mathcal{C}_2$ be the cut used to establish that CA is allowable. Assume without loss of generality that $\mathcal{C}_2$ is horizontal. Let CA_1 and CA_2 be the compatible arrangements obtained by cutting CA

along $\mathcal{C}_2$ (with CA_1 on the top and CA_2 on the bottom). Note that both of these arrangements are allowable and contain strictly less than n squares.

If $\mathcal{C}_1$ is vertical, call the compatible arrangement to its left CA_3 and the one to its right CA_4. We need to show that CA_3 and CA_4 are allowable. The cut $\mathcal{C}_1$ itself gets divided by $\mathcal{C}_2$ into two vertical full length cuts $\mathcal{C}_{1,1}$ and $\mathcal{C}_{1,2}$, for CA_1 and CA_2 respectively. The cut $\mathcal{C}_{1,1}$ divides CA_1 into compatible arrangements $CA_{1,1}$ and $CA_{1,2}$, and the cut $\mathcal{C}_{1,2}$ divides CA_2 into compatible arrangements $CA_{2,1}$ and $CA_{2,2}$. Assume that $CA_{1,1}$ and $CA_{2,1}$ lie to the left of the cuts and $\mathcal{C}_{1,2}$ and $CA_{2,2}$ lie to the right of the cuts. By the induction hypothesis, $CA_{1,1}$, $CA_{1,2}$, $CA_{2,1}$, and $CA_{2,2}$ are all allowable. It is clear CA_3 gets divided into $CA_{1,1}$ and $CA_{2,1}$ by the left side of the cut $\mathcal{C}_2$, so CA_3 is allowable. In the same way CA_4 gets cut into $CA_{1,2}$ and $CA_{2,2}$ by the right side of the cut $\mathcal{C}_2$, so CA_4 is allowable.

If $\mathcal{C}_1$ is horizontal, assume that CA_1 contains $\mathcal{C}_1$. By the induction hypothesis, $\mathcal{C}_1$ divides the allowable compatible arrangement CA_1 into two allowable compatible arrangements, say $CA_{1,a}$ and $CA_{1,b}$, where $CA_{1,a}$ is on the top. Also, $\mathcal{C}_1$ divides CA into two compatible arrangements, $CA_{1,a}$ and $CA_{1,c}$, the latter of which is divided by $\mathcal{C}_2$ into $CA_{1,b}$ and CA_2. Since both $CA_{1,b}$ and CA_2 are allowable, we conclude that $CA_{1,c}$ is allowable in addition to $CA_{1,a}$. □

Proposition 3.7. *The forgetful functors T and U admit left adjoints S and R.*

$$\begin{tikzcd} \mathbf{RefDblGr} \arrow[rr, bend left, "S"] \arrow[rr, phantom, "\perp"] & & \mathbf{DblDerSch} \arrow[ll, bend left, "T"] \arrow[rr, bend left, "R"] \arrow[rr, phantom, "\perp"] & & \mathbf{DblCat} \arrow[ll, bend left, "U"] \end{tikzcd}$$

The left adjoint S gives the free double derivation scheme on a reflexive double graph, and the left adjoint R gives the free double category on a double derivation scheme. The functor R preserves the horizontal and vertical 1-categories.

Proof: For a reflexive double graph $\mathbb{A}$, let $S\mathbb{A}$ have vertical and horizontal 1-categories the free 1-categories on the respective reflexive graphs. The set of squares remains the same. It is straightforward to verify that this defines a left adjoint to T.

For a double derivation scheme $\mathbb{S}$, let $R\mathbb{S}$ have vertical and horizontal 1-categories the vertical and horizontal 1-categories of $\mathbb{S}$ respectively. The squares of $R\mathbb{S}$ are allowable compatible arrangements of squares of $\mathbb{S}$. Such compatible arrangements are composed vertically and horizontally by concatenation. Clearly, composites of allowable compatible arrangements are allowable.

If $J\colon \mathbb{S} \longrightarrow U\mathbb{D}$ is a morphism of double derivation schemes, then it induces a double functor $J'\colon R\mathbb{S} \longrightarrow \mathbb{D}$ which is J on the horizontal and vertical 1-categories. For an allowable compatible arrangement D, $J'D$ is the composite in $\mathbb{D}$ of J applied to the constituents of D. Morphisms $R\mathbb{S} \longrightarrow \mathbb{D}$ restrict to morphisms $\mathbb{S} \longrightarrow U\mathbb{D}$, and it is not hard to check that these two operations are inverse. We conclude that $R \dashv U$. $\square$

Now that we have free notions, we also define quotients. Note that the notion of congruence for ordinary categories is an equivalence relation on the cells of highest dimension, satisfying certain compatibility properties. We imitate this in our notion of congruence for a double category.

Definition 3.8. A *congruence* on a category $\mathbf{C}$ is an equivalence relation on $\mathbf{C}(a, b)$ for each $a, b \in \mathbf{C}$, such that if $f \sim f'$ and $g \sim g'$, then $gf \sim g'f'$ whenever the composites exist.

Definition 3.9. A *congruence* on a double derivation scheme $\mathbb{S}$ consists of a congruence on the horizontal 1-category and a congruence on the vertical 1-category.

Definition 3.10. A *congruence* on a double category $\mathbb{D}$ consists of an equivalence relation on $\mathbb{D}\begin{pmatrix} & f & \\ j & & k \\ & g & \end{pmatrix}$ for each boundary $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$ such that if $\alpha \sim \alpha', \beta \sim \beta'$, and $\gamma \sim \gamma'$ then

$$\begin{bmatrix} \alpha & \beta \end{bmatrix} \sim \begin{bmatrix} \alpha' & \beta' \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ \gamma \end{bmatrix} \sim \begin{bmatrix} \alpha' \\ \gamma' \end{bmatrix}$$

whenever the composites exist. Note that the congruence does not concern the horizontal and vertical morphisms.

Proposition 3.11. *Let $\mathbb{D}$ be a double category equipped with a congruence. If two allowable compatible arrangements D_1 and D_2 with the same underlying tiling of the rectangle have congruent constituent squares, then the composites of D_1 and D_2 in $\mathbb{D}$ are congruent.*

Proof: By Theorem 1.2 of [25], any two composites of a composable compatible arrangement are equal. The compatible arrangements D_1 and D_2 are composable since they are allowable. If we compose each of D_1 and D_2 using the same sequence of pairwise compositions, then the pairwise composites in each step are congruent. An inductive argument shows that total composites are then also congruent. $\square$

Definition 3.12. The *quotient* $\mathbf{C}/\sim$ of a category $\mathbf{C}$ by a congruence $\sim$ has the same objects as $\mathbf{C}$ but has $(\mathbf{C}/\sim)(a,b) = \mathbf{C}(a,b)/\sim$.

Definition 3.13. The *quotient* $\mathbb{S}/\sim$ of a double derivation scheme $\mathbb{S}$ by a congruence $\sim$ has the same objects and squares as $\mathbb{S}$ but has horizontal and vertical 1-categories the quotient categories of $(\mathbf{H}\mathbb{S})_0$ and $(\mathbf{V}\mathbb{S})_0$.

Definition 3.14. The *quotient* $\mathbb{D}/\sim$ of a double category $\mathbb{D}$ by a congruence $\sim$ has the same objects and horizontal and vertical 1-categories as $\mathbb{D}$ but has

$$\mathbb{D}^q \left(\begin{array}{ccc} & f & \\ j & & k \\ & g & \end{array} \right) = \mathbb{D} \left(\begin{array}{ccc} & f & \\ j & & k \\ & g & \end{array} \right) / \sim .$$

These are of course not the most general notions of quotient, but more general quotients can be built from these as follows. All quotients can be characterized by the usual universal properties.

Definition 3.15. Let $\mathbf{C}$ be a category and $\mathbf{R} \subseteq \mathbf{C} \times \mathbf{C}$ a subcategory satisfying the usual axioms of an equivalence relation both on the set of objects and on the set of morphisms. Then the *quotient* $\mathbf{C}/\mathbf{R}$ of $\mathbf{C}$ by the equivalence relation $\mathbf{R}$ is defined as follows. First we obtain a graph with object set $Obj\, \mathbf{C}/Obj\, \mathbf{R}$ and morphism set $Mor\, \mathbf{C}/Mor\, \mathbf{R}$. We make this into a reflexive graph by identifying 1_A and 1_B whenever A and B are identified. Next we take the free category $\mathbf{F}$ on this reflexive graph and mod out by the smallest congruence that makes the graph morphism $\mathbf{C} \longrightarrow \mathbf{F}$ into a functor.

Such quotients of categories have been considered by [8]. However a counterexample in [8], [63], and [64] shows that the quotient functor may identify morphisms which are not equivalent. Early work on quotients is found in [45]. More recently, quotients of categories by generalized congruences have been considered in [3].

For general quotients of double categories, we need intermediate quotients of double derivation schemes.

Definition 3.16. Let $\mathbb{S}$ be a double derivation scheme and $\mathbb{R} \subseteq \mathbb{S} \times \mathbb{S}$ a sub-double derivation scheme satisfying the usual axioms of an equivalence relation on the sets of objects, vertical morphisms, horizontal morphisms, and squares. Then the quotient $\mathbb{S}/\mathbb{R}$ of $\mathbb{S}$ by the equivalence relation $\mathbb{R}$ is defined as follows. The horizontal and vertical 1-categories are the quotients of the horizontal and vertical 1-categories of $\mathbb{S}$ as in Definition 3.15. The squares are $Sq\,(\mathbb{S}/\mathbb{R}) = (Sq\,\mathbb{S})/(Sq\,\mathbb{R})$.

Definition 3.17. Let $\mathbb{D}$ be a double category and $\mathbb{R} \subseteq \mathbb{D} \times \mathbb{D}$ a sub-double category satisfying the usual axioms of an equivalence relation. Then the quotient $\mathbb{D}/\mathbb{R}$ of $\mathbb{D}$ by the equivalence relation $\mathbb{R}$ is defined as follows. First take the quotient of the underlying double derivation scheme as in Definition 3.16. Then take the free double category $\mathbb{F}$ on this double derivation scheme and mod out by the smallest congruence (Definitions 3.10 and 3.14) that makes the morphism $\mathbb{D} \longrightarrow \mathbb{F}$ of double derivation schemes into a double functor. Note that only squares get identified in this last step, since the horizontal and vertical 1-categories of the free double category on a derivation scheme are the same as the horizontal and vertical 1-categories of the derivation scheme.

We will make use of free double categories and their quotients in our discussion of categorification in Section 6 as well as in an explicit description of certain pushouts of double categories in Theorem 10.6 and Theorem 10.7. These are essential ingredients in the construction of model structures on **DblCat**. For now it is sufficient to give a colimit formula in **DblCat**.

4. Limits and Colimits of Double Categories

Model structures in general require the existence of limits and colimits. Moreover, in order to transfer model structures along certain adjunctions we will need an explicit formula for certain pushouts of double categories, as in Theorem 10.6 and Theorem 10.7. So in this section we discuss limits and colimits of double categories.

Colimits for categories were described in detail by Gabriel and Zisman. Their work was extended in [78] to a construction of colimits in 2-categories. We extend this further to a construction in double categories which goes roughly as follows. To take the colimit of a functor $F\colon I \longrightarrow \mathbf{DblCat}$, with index category I, first we take the colimit $\mathbb{S}$ of the underlying double derivation schemes, then take the free double category $\mathbb{F}$ on $\mathbb{S}$, and finally mod out by the congruence that makes the maps $Fi \longrightarrow \mathbb{F}$ into double functors. The intermediate notion of double derivation scheme allows us to deal with the quotients of morphisms and quotients of squares separately. We present the details in the following theorems.

Theorem 4.1. *The category* **DblCat** *is complete and cocomplete.*

Proof: The limits of the sets of objects, horizontal morphisms, vertical morphisms, and squares assemble to form a double category and this double category is the limit. After all, **DblCat** is a category of algebras.

The category **DblCat** is the category of models in **Cat** of a sketch with finite diagrams, and **Cat** is locally finitely presentable, so an application of Proposition 1.53 of [1] shows that **DblCat** is locally finitely presentable. Locally finitely presentable categories are cocomplete, so **DblCat** is cocomplete. □

Note that the underlying horizontal and vertical 2-categories of the limit are the limits of the underlying horizontal and vertical 2-categories, since **H** and **V** admit left adjoints by Proposition 2.4. The forgetful functor $\mathbf{2\text{-}Cat} \longrightarrow \mathbf{Cat}$ also admits a left adjoint, so similar comments hold for the horizontal and vertical 1-categories.

We work towards an explicit description of colimits in **DblCat** which mimics Gabriel and Zisman's calculation of colimits in **Cat** below.

Theorem 4.2 (Colimit Formula in **Cat** of [36])**.** *To calculate a colimit of a functor $F\colon I \longrightarrow \mathbf{Cat}$, one first calculates the colimit of the underlying reflexive graphs, then takes the free category $\mathbf{F}$ on this, and finally one mods out by the smallest congruence on $\mathbf{F}$ that makes the maps $Fi \longrightarrow \mathbf{F}$ into functors.*

Lemma 4.3. *The horizontal and vertical 1-categories of a colimit of double derivation schemes are the colimits of the underlying horizontal and vertical 1-categories. Similarly, the horizontal and vertical 1-categories of a colimit of double categories are the colimits of the underlying horizontal and vertical 1-categories.*

Proof: The right adjoint to the forgetful functor

$$\mathbf{DblDerSch} \longrightarrow \mathbf{Cat}$$

$$\mathbb{S} \longmapsto (\mathbf{H}\mathbb{S})_0$$

assigns to a category $\mathbf{E}$ the double derivation scheme $\mathbb{E}$ with horizontal 1-category $\mathbf{E}$, a unique vertical morphism between any two objects, and a unique square for each boundary. Similarly, the forgetful functor $\mathbb{S} \longmapsto (\mathbf{V}\mathbb{S})_0$ admits a right adjoint. Since left adjoints preserve colimits, the statement for double derivation schemes follows.

The same argument works for **DblCat** in place of **DblDerSch**. □

Theorem 4.4 (Colimit Formula in **DblDerSch**)**.** *The colimit $\mathbb{S}$ of a functor $F\colon I \longrightarrow \mathbf{DblDerSch}$ is obtained by first taking the colimit of the underlying reflexive double graphs and then taking the free double derivation scheme $\mathbb{F}$ on the resulting reflexive double graph, and modding out by the smallest congruence that makes the double-graph morphisms $Fi \longrightarrow \mathbb{F}$ into morphisms of double derivation schemes.*

Proof: Suppose $\mathbb{S}'$ is a double derivation scheme and $\beta_i \colon Fi \longrightarrow \mathbb{S}'$ are natural morphisms of double derivations schemes. We define a unique factorization

$$\begin{tikzcd} Fi \arrow[r, "\beta_i"] \arrow[d, "\alpha_i"'] & \mathbb{S}' \\ \mathbb{S} \arrow[ur, dotted] & \end{tikzcd}$$

on horizontal and vertical 1-categories by the universal property of Lemma 4.3, and on squares by the universal property of the colimit of the sets $Sq\, Fi$. The set of squares in the free double derivation scheme on a reflexive double graph is the same as the set of squares in the reflexive double graph by Proposition 3.7. □

Theorem 4.5 (Colimit Formula in **DblCat**)**.** *The colimit* $\mathbb{C}$ *of a functor*

$$F \colon I \longrightarrow \mathbf{DblCat}$$

is calculated as follows. Let $\mathbb{S}$ *be the colimit in* **DblDerSch** *of the underlying double derivation schemes, and* $\mathbb{F}$ *the free double category on* $\mathbb{S}$*. Then* $\mathbb{C}$ *is the quotient of* $\mathbb{F}$ *by the smallest congruence such that the natural morphisms of double derivation schemes*

$$Fi \xrightarrow{\alpha_i} \mathbb{S} \longrightarrow \mathbb{F}$$

become double functors. Note that the horizontal and vertical 1-categories of $\mathbb{S}, \mathbb{F}$*, and* $\mathbb{C}$ *are the same, in particular the horizontal and vertical 1-categories of* $\mathbb{C}$ *are the colimits of the horizontal and vertical 1-categories of the* Fi*.*

Proof: Let q denote the map of double derivation schemes from $\mathbb{S}$ to the quotient $\mathbb{C}$ of the free double category $\mathbb{F}$. Then $q \circ \alpha_i$ is a double functor for all $i \in I$. Suppose $\mathbb{C}'$ is a double category and $\beta_i \colon Fi \longrightarrow \mathbb{C}'$ are natural double functors. Then by Lemma 4.4 there exists a unique morphism J of double derivation schemes that makes the upper left triangle commute,

$$\begin{tikzcd} Fi \arrow[r, "\beta_i"] \arrow[d, "\alpha_i"'] & \mathbb{C}' \\ \mathbb{S} \arrow[ur, dotted, "\exists!\, J"] \arrow[r, "q"'] & \mathbb{C}\,. \arrow[u, dotted, "\exists!\, L"'] \end{tikzcd}$$

The morphism J induces a double functor $K \colon \mathbb{F} \longrightarrow \mathbb{C}'$ since $\mathbb{F}$ is free on $\mathbb{S}$. The commutativity of the upper left triangle says that K preserves the congruence on $\mathbb{F}$, and therefore induces a unique functor L which makes the lower right triangle commute. Therefore the square commutes, and further L is the unique double functor such that the square commutes by the uniqueness of the two fillers. $\square$

Recall that filtered colimits in **Cat** are particularly simple to calculate: the filtered colimit of the underlying reflexive graphs is already a category and this category is the filtered colimit in **Cat**. Similarly, one does not need to use free constructions and quotients to calculate filtered colimits in **DblCat**.

Theorem 4.6. *A filtered colimit of double categories is calculated by simply taking the filtered colimits of the underlying reflexive double graphs.*

Proof: The filtered colimit of the underlying reflexive double graphs admits all the associative and unital compositions necessary for a double category by the corresponding result in **Cat**. The interchange law holds because it is possible to find representatives of all four squares in a single stage, where the interchange law is known to hold. $\square$

5. Nerves of Double Categories

Grothendieck's full and faithful nerve embedding $N\colon \mathbf{Cat} \longrightarrow \mathbf{SSet}$ has been of tremendous use in higher category theory. One can expect that its n-fold version will similarly be of use. In fact, the authors of [10],[11], [12], [13] have studied edge symmetric n-fold categories from the point of view of cubical sets. A double category is a 2-truncated cubical set. We introduce in this section *simplicial* and *bisimplicial* nerves of double categories. These will be of use in Section 7 where we transfer model structures on $\mathbf{Cat}^{\Delta^{op}}$ to **DblCat** via a categorification-nerve adjunction. The first nerve we consider is the horizontal nerve, which is really an internal notion.

Definition 5.1. Let $\mathbb{D} = (\mathbb{D}_0, \mathbb{D}_1)$ be a double category. Then the *horizontal nerve* $N_h\mathbb{D}$ is the simplicial object in **Cat**

$$(N_h\mathbb{D})_0 = \mathbb{D}_0$$

$$(N_h\mathbb{D})_1 = \mathbb{D}_1$$

$$(N_h\mathbb{D})_n = \underbrace{\mathbb{D}_1 \;{}_t\times_s \mathbb{D}_1 \;{}_t\times_s \cdots {}_t\times_s \mathbb{D}_1}_{n \text{ copies of } \mathbb{D}_1}.$$

$$\begin{tikzcd} \mathit{Obj}\,(N_h\mathbb{D})_n : & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

$$\begin{tikzcd} & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \mathit{Mor}\,(N_h\mathbb{D})_n : & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

In other words,

$$\begin{aligned} Obj\,(N_h\mathbb{D})_n &= Hom_{\mathbf{Cat}}([n],(Obj\,\mathbb{D}, Hor\,\mathbb{D})) \\ Mor\,(N_h\mathbb{D})_n &= Hom_{\mathbf{Cat}}([n],(Ver\,\mathbb{D}, Sq\,\mathbb{D})). \end{aligned}$$

Composition in $(N_h\mathbb{D})_n$ is vertical.

Example 5.2. If $\mathbf{C}$ is a category, then the simplicial set $N_h(\mathbb{H}\mathbf{C})$ is the usual nerve of $\mathbf{C}$.

Like the nerve of a category, the horizontal nerve of a double category has a representable definition. Recall that $\mathbf{DblCat_v}$ denotes the 2-category of small double categories, double functors, and vertical natural transformations.

Theorem 5.3. *For every double category $\mathbb{D}$, the simplicial category*

$$[n] \mapsto \mathbf{DblCat_v}(\mathbb{H}[n], \mathbb{D})$$

is isomorphic to the horizontal nerve $N_h\mathbb{D}$. Equivalently, the object simplicial set of the horizontal nerve is

$$[n] \mapsto \mathbf{DblCat}([0] \boxtimes [n], \mathbb{D})$$

and the morphism simplicial set of the horizontal nerve is

$$[n] \mapsto \mathbf{DblCat}([1] \boxtimes [n], \mathbb{D}).$$

Proof: The double categories $\mathbb{H}[n]$ and $[0]\boxtimes[n]$ are isomorphic, and vertical natural transformations $\mathbb{H}[n] \longrightarrow \mathbb{D}$ are the same as double functors $[1] \boxtimes [n] = (\mathbb{H}[1])^t \times \mathbb{H}[n] \longrightarrow \mathbb{D}$ as pointed out in [41]. $\square$

In Section 6 we construct the left adjoint to the horizontal nerve explicitly, but for now we observe that a left adjoint exists.

Theorem 5.4. *The horizontal nerve $N_h \colon \mathbf{DblCat} \longrightarrow \mathbf{Cat}^{\Delta^{op}}$ admits a left adjoint c_h called horizontal categorification.*

Proof: This follows from Theorem 5.3, the 2-categorical cocompleteness of $\mathbf{DblCat_v}$ (a slight improvement of Theorem 4.1), and an enriched version of Theorem 5.14, which can be found in Chapter 5 of [50]. $\square$

Theorem 5.5. *The horizontal nerve N_h preserves filtered colimits.*

Proof: It follows from Theorem 4.6 that the category of horizontal morphisms and squares of a filtered colimit of double categories is the filtered colimit of the categories of horizontal morphisms and squares. Since filtered colimits commute with finite limits, in particular iterated pullbacks, N_h preserves filtered colimits. $\square$

The horizontal nerve is also well behaved with respect to external products.

Proposition 5.6. *Let* $\sigma\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}^{\Delta^{op}}$ *denote the constant functor. Let* $\nu\colon \mathbf{Set}^{\Delta^{op}} \longrightarrow \mathbf{Cat}^{\Delta^{op}}$ *be the inclusion induced by the functor* $\mathbf{Set} \longrightarrow \mathbf{Cat}$ *which takes a set to the corresponding discrete category. If* $\mathbf{A}$ *and* $\mathbf{B}$ *are categories, then* $N_h(\mathbf{A} \boxtimes \mathbf{B}) = \sigma\mathbf{A} \times \nu N\mathbf{B}$. *In other words,* $N_h(\mathbf{A} \boxtimes \mathbf{B})_k = \mathbf{A} \times N\mathbf{B}_k$ *where we view the set* $N\mathbf{B}_k$ *as a discrete category.*

The second nerve we introduce in this section is the bisimplicial nerve, which we geometrically realize to get a classifying space. Let $\mathbf{SSet^2}$ denote the category of bisimplicial sets, *i.e.,* functors from $\Delta^{op} \times \Delta^{op}$ into $\mathbf{Set}$. Since

$$\mathbf{Cat}(\Delta^{op} \times \Delta^{op}, \mathbf{Set}) \cong \mathbf{Cat}(\Delta^{op}, \mathbf{Set}^{\Delta^{op}})$$

we see that that the category of bisimplicial sets is isomorphic to the category $\mathbf{SSet}^{\Delta^{op}}$ of simplicial objects in $\mathbf{SSet}$.

Definition 5.7. The *bisimplicial nerve* or *double nerve* of a double category $\mathbb{D}$ is the bisimplicial set $N_d\mathbb{D}$ with (m, n)-bisimplices given by $m \times n$ arrays of squares of $\mathbb{D}$. In particular $(N_d\mathbb{D})_{0,0}$ is the set of objects of $\mathbb{D}$, $(N_d\mathbb{D})_{0,n}$ consists of paths of n horizontal morphisms, and $(N_d\mathbb{D})_{n,0}$ consists of paths of n vertical morphisms.

Definition 5.8. The *classifying space functor* B is the composite

$$\mathbf{DblCat} \xrightarrow{N_d} \mathbf{SSet^2} \xrightarrow{\text{diag}} \mathbf{SSet} \xrightarrow{|\cdot|} \mathbf{Top},$$

where diag is the functor induced by the diagonal and $|\cdot|$ is the geometric realization.

The traditional nerve functor $N\colon \mathbf{Cat} \longrightarrow \mathbf{SSet}$ is a fully faithfull embedding. We have a similar statement for the double nerve.

Proposition 5.9. *The double nerve* $N_d\colon \mathbf{DblCat} \longrightarrow \mathbf{SSet^2}$ *is a fully faithful embedding.*

The traditional nerve, the horizontal nerve, and the double nerve are related as follows.

Proposition 5.10. *The functor*

$$\mathbf{Dbl} \xrightarrow{N_h} \mathbf{Cat}^{\Delta^{op}} \xrightarrow{N_*} \mathbf{SSet}^{\Delta^{op}}$$

$$\mathbb{D} \longmapsto ([n] \mapsto N((N_h\mathbb{D})_n))$$

is naturally isomorphic to N_d.

Corollary 5.11. *Let* $\mathbf{C}$ *be a 2-category. Consider the bisimplicial set obtained by taking the nerves of the hom-categories of* $\mathbf{C}$*, viewing the resulting* **SSet***-category as a simplicial object in* **Cat** *with constant object set, and then composing this functor* $\Delta^{op} \longrightarrow \mathbf{Cat}$ *with the traditional nerve functor. This bisimplicial set is naturally isomorphic to* $N_d(\mathbf{C})$ *if we view* $\mathbf{C}$ *as a double category with trivial vertical morphisms.*

The traditional nerve, like its higher dimensional counterparts as in [18], [19], [27], [58], and [74], is defined via the inclusion $J\colon \Delta \longrightarrow \mathbf{Cat}$ as $(N\mathbf{C})_n = Hom_{\mathbf{Cat}}(J[n], \mathbf{C})$. We similarly have a representable definition of the double nerve using the external product of 2-categories from Definition 2.5.

Theorem 5.12. *Let* $J\colon \Delta \times \Delta \longrightarrow \mathbf{Cat}$ *be the full and faithful functor* $J([m],[n]) = [m] \boxtimes [n]$*. Then the double nerve of a double category* $\mathbb{D}$ *has* (m,n)*-bisimplices*

$$(N_d\mathbb{D})_{m,n} = Hom_{\mathbf{DblCat}}(J(m,n), \mathbb{D}). \tag{2}$$

Remark 5.13. This theorem shows how the double nerve can be extended to pseudo double categories: one takes normal homomorphisms of pseudo double categories in (2).

Theorem 5.14 (Kan's Lemma on Transfer)**.** *Let* $\mathbf{A}$ *be a small category and* $J\colon \mathbf{A} \longrightarrow \mathbf{B}$ *a functor. If* $\mathbf{B}$ *is cocomplete, then the left Kan extension of* J *along the Yoneda embedding exists and is the left adjoint of the singular functor*

$$J_*\colon \mathbf{B} \longrightarrow \mathbf{Set}^{\mathbf{A}^{op}}$$

$$B \longmapsto Hom_{\mathbf{B}}(J(-), B).$$

In Section 6 we construct the left adjoint to the double nerve explicitly, but we can already prove its existence.

Corollary 5.15. *The double nerve* $N_d\colon \mathbf{DblCat} \longrightarrow \mathbf{SSet^2}$ *admits a left adjoint* c_d *called double categorification.*

Proof: Let J be as in Theorem 5.12. Then J_* is the double nerve, and the existence of the left adjoint follows from Theorem 4.1 and Theorem 5.14. □

Example 5.16. Since N_d is a right adjoint, it preserves products. As a result, it preserves external products: if $\mathbf{C}$ and $\mathbf{D}$ are 1-categories, then

$$\begin{aligned} N\mathbf{C} \boxtimes N\mathbf{D} &= (N_d \mathbb{H}\mathbf{C})^t \times N_d \mathbb{H}\mathbf{D} \\ &= N_d(\mathbb{H}\mathbf{C})^t \times N_d \mathbb{H}\mathbf{D} \\ &= N_d((\mathbb{H}\mathbf{C})^t \times \mathbb{H}\mathbf{D}) \\ &= N_d(\mathbf{C} \boxtimes \mathbf{D}). \end{aligned}$$

Theorem 5.17. *The double nerve preserves filtered colimits.*

Proof: By Theorem 4.6 a filtered colimit of double categories is the filtered colimit of the underlying reflexive double graphs with the induced compositions and identities. For any double category $\mathbb{D}$, the set $N_d(\mathbb{D})_{m,n}$ is a finite limit, in much the same way that $N(\mathbf{C})_n$ is an n-fold pullback. Since filtered colimits commute with finite products and filtered colimits of double categories have a simple form, N_d preserves filtered colimits. □

6. Categorification

We now construct left adjoints c_h and c_d to the horizontal nerve N_h and the double nerve N_d by analogy with the usual nerve N. Both c_h and c_d are appropriately compatible with external products, as we show in Example 6.5, Proposition 6.10, and Example 6.13.

Recall the well known left adjoint $c\colon \mathbf{SSet} \longrightarrow \mathbf{Cat}$ to the nerve functor N. For a simplicial set X, the category cX is the *fundamental category of* X, or *categorification of* X. It is the free category on the reflexive graph (X_0, X_1) modulo the smallest congruence such that for every $\tau \in X_2$ with edges

$$\begin{tikzcd} & {} \arrow[dr, "g"] & \\ {} \arrow[ur, "f"] \arrow[rr, "h"'] & \tau & {} \end{tikzcd}$$

we have $g \circ f \sim h$. The following proof is our guideline for left adjoints c_h and c_d to N_h and N_d.

Proposition 6.1. *Categorification c is left adjoint to the nerve functor N.*

Proof: We need to construct a natural bijection

$$\mathbf{Cat}(cX, \mathbf{A}) \cong \mathbf{SSet}(X, N\mathbf{A}).$$

Suppose we have a map $G\colon X \longrightarrow N\mathbf{A}$ of simplicial sets. The 1-truncation is a morphism of reflexive graphs, so there is a unique functor J making the upper left triangle commute.

$$\begin{tikzcd}
(X_0, X_1) \arrow[r, "(G_0,G_1)"] \arrow[d] & \mathbf{A} \\
\mathrm{FreeCat}(X_0, X_1) \arrow[r] \arrow[ur, dotted, "\exists!\, J"] & cX \arrow[u, dotted, "\exists!\, G'"']
\end{tikzcd}$$

Since J comes from a morphism of simplicial sets, the functor J takes congruent morphisms to equal ones. Therefore there exists a unique functor G' making the lower right triangle commute.

For the converse, we first show that a morphism of simplicial sets $G\colon X \longrightarrow N\mathbf{A}$ is completely determined by its 1-truncation (G_0, G_1). If we define injective maps $e_{i,i+1}\colon \{0,1\} \longrightarrow \{0,\ldots,n\}$ by

$$e_{i,i+1}(0) = i$$
$$e_{i,i+1}(1) = i+1$$

for $0 \leq i \leq n-1$, and if σ is an n-simplex, then $G(\sigma)$ is the string of n morphisms in $\mathbf{A}$

$$\xrightarrow{G(e^*_{0,1}(\sigma))} \xrightarrow{G(e^*_{1,2}(\sigma))} \cdots \qquad \cdots \xrightarrow{G(e^*_{n-1,n}(\sigma))} .$$

Given a functor $G'\colon cX \longrightarrow \mathbf{A}$, we compose it with the morphism of reflexive graphs

$$(X_0, X_1) \longrightarrow \mathrm{FreeCat}(X_0, X_1) \longrightarrow cX$$

to obtain a morphism of reflexive graphs $(X_0, X_1) \longrightarrow \mathbf{A}$ which induces a morphism $G\colon X \longrightarrow N\mathbf{A}$ of simplicial sets. These two procedures are inverse. $\square$

6.1. **Horizontal Categorification.** We turn first to the left adjoint of the horizontal nerve. We will obtain two proofs that the horizontal categorification of the product of a category with a simplicial set is an external product of the category with the fundamental category of the simplicial set. This is done in Example 6.5 using the definition of horizontal categorification, while it is done in Proposition 6.10 using weighted colimits.

Definition 6.2. Let $X \in \mathbf{Cat}^{\Delta^{op}}$. We define a double category $c_h X$ called the *horizontal categorification* or *fundamental double category of* X as follows. First we define a double derivation scheme $\mathbb{S}$ with vertical 1-category X_0 and with horizontal 1-category the fundamental category of the simplicial set *Obj* X. The squares of $\mathbb{S}$ are the morphisms of X_1. We equip the free double category $\mathbb{F}$ on the double derivation scheme $\mathbb{S}$ with the smallest congruence $\sim$ such that

(i) If $\alpha, \beta \in Mor\, X_1$ are vertically composable in $\mathbb{F}$ then $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$ is congruent to the composite of β and α in X_1,

(ii) For all $\tau \in Mor\, X_2$ with boundary

$$\begin{tikzcd} & \cdot \arrow[dr, "\beta"] & \\ \cdot \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] \arrow[r, phantom, shift left=3, "\tau"] & & \cdot \end{tikzcd}$$

we have

$$[\alpha\beta] \sim \gamma,$$

(iii) For any vertical morphism j, the horizontal identity i^h_j is congruent to the degeneracy of j in $Mor\, X_1$.

We define $c_h X$ as the quotient of $\mathbb{F}$ by the congruence $\sim$. The horizontal and vertical 1-categories of $c_h X$ are the horizontal and vertical 1-categories of $\mathbb{S}$.

Remark 6.3. In the definition of horizontal categorification it is not necessary to mod out by additional relations to make the identity squares functorial. If $g \circ f \sim h$ because of $\tau \in Obj\, X_2$, then the identity morphism on τ in the category X_2 implies we have $i^v_h \sim [i^v_f\, i^v_g]$ (the face maps are functors). For vertically composable morphisms j and k, we have

$$i^h_{\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]} \sim \begin{bmatrix} i^h_j \\ i^v_k \end{bmatrix}$$

because degeneracy is a functor and by (i) and (ii).

Example 6.4. If X is a simplicial set, then $c_h \nu X = \mathbb{H}cX$. By definition, the horizontal 1-category is cX, and the vertical 1-category is the discrete category X_0. Since X_1 is also discrete, there are no nontrivial squares.

Example 6.5. Recall from Proposition 5.6 that $\sigma\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}^{\Delta^{op}}$ denotes the constant functor and $\nu\colon \mathbf{Set}^{\Delta^{op}} \longrightarrow \mathbf{Cat}^{\Delta^{op}}$ denotes the inclusion. If $\mathbf{A}$ is a category and Y is a simplicial set, then the horizontal categorification of the simplicial category $\sigma\mathbf{A} \times \nu Y$ is $\mathbf{A} \boxtimes cY$. In fact, the horizontal 1-category of the double derivation scheme $\mathbb{S}$ is

$$c(Obj\,(\sigma\mathbf{A} \times \nu Y)) = c(Obj\,\mathbf{A} \times Y) = (\mathbf{H}(\mathbf{A} \boxtimes cY))_0.$$

The vertical 1-category of $\mathbb{S}$ is

$$(\sigma\mathbf{A} \times \nu Y)_0 = \mathbf{A} \times Y_0 = (\mathbf{V}(\mathbf{A} \boxtimes cY))_0.$$

The squares of $\mathbb{S}$ are

$$Mor\,(\sigma\mathbf{A} \times \nu Y)_1 = (Mor\,\mathbf{A}) \times Y_1.$$

The congruence on $\mathbb{F}$ corresponds precisely to the relations in $\mathbf{A} \boxtimes cY$. We present an alternative conceptual proof of this example in Proposition 6.10.

Proposition 6.6. *Horizontal categorification c_h is left adjoint to the horizontal nerve N_h.*

Proof: We use the notation of Definition 6.2 and construct a natural bijection

$$\mathbf{DblCat}(c_h X, \mathbb{D}) \cong [\Delta^{op}, \mathbf{Cat}](X, N_h\mathbb{D}).$$

Suppose $G\colon X \longrightarrow N_h\mathbb{D}$ is a morphism of simplicial objects in **Cat**. This induces a morphism of double deriviation schemes $\mathbb{S} \longrightarrow \mathbb{D}$ and a unique double functor J making the upper left triangle commute,

$$\begin{tikzcd} \mathbb{S} \arrow[r] \arrow[d] & \mathbb{D} \\ \mathbb{F} \arrow[r] \arrow[ur, dotted, "\exists!\, J"] & c_h X \arrow[u, dotted, "\exists!\, G'"'] \end{tikzcd}$$

Since G is a morphism of simplicial objects in **Cat**, J takes congruent squares to equal squares, and there exists a unique double functor G' making the lower right triangle commute.

On the other hand, given a double functor $G'\colon c_h X \longrightarrow \mathbb{D}$ we compose it with the morphism of 1-truncated simplicial objects in **Cat**

$$(X_0, X_1) \longrightarrow \mathbb{F} \longrightarrow c_h X\ .$$

The resulting morphism of 1-truncated simplicial objects determines a morphism $G\colon X \longrightarrow N_h\mathbb{D}$ of simplicial objects in **Cat**: such a morphism is determined by its 1-truncation since

$$G^{Obj} :\ Obj\,X \longrightarrow Obj\,N_h\mathbb{D}$$

$$G^{Mor} :\ Mor\,X \longrightarrow Mor\,N_h\mathbb{D}$$

are determined by their 1-truncations as in the proof of Proposition 6.1.

These two procedures are inverse to one another. □

We now move towards a conceptual proof of Example 6.5 in Proposition 6.10.

Remark 6.7. Recall that if S is a set and A is an object of a category, then the *copower* $S \cdot A$ is the coproduct of A with itself S-times. In some categories, the copower has a simple description. For example, if $\mathbf{C}$ is a category, then the copower in $\mathbf{Cat}$ is

$$S \cdot \mathbf{C} = \coprod_S \mathbf{C} = S \times \mathbf{C}.$$

If X is a simplicial set and $Y \in [\Delta^{op}, \mathbf{Cat}]$, then $X \cdot Y$ is the simplicial object in $\mathbf{Cat}$

$$[n] \longmapsto X_n \cdot Y_n = \textstyle\coprod_{X_n} Y_n = X_n \times Y_n \ ,$$

which is the same as $\nu X \times Y$.

Lemma 6.8. *For categories* $\mathbf{C}$ *and* $\mathbf{D}$ *and a double category* $\mathbb{E}$ *we have a natural isomorphism of categories*

$$\mathbf{DblCat_v}(\mathbb{V}\mathbf{C} \times \mathbb{H}\mathbf{D}, \mathbb{E}) \cong \mathbf{Cat}(\mathbf{C}, \mathbf{DblCat_v}(\mathbb{H}\mathbf{D}, \mathbb{E})).$$

Proof: Recall that $\mathbb{V}\mathbf{C} \times \mathbb{H}\mathbf{D} = \mathbf{C} \boxtimes \mathbf{D}$ and that $N_h(\mathbf{C} \boxtimes \mathbf{D}) = \sigma\mathbf{C} \times \nu N\mathbf{D}$ by Proposition 5.6. Since N_h is 2-fully faithful, we have

$$\mathbf{DblCat_v}(\mathbb{V}\mathbf{C} \times \mathbb{H}\mathbf{D}, \mathbb{E}) \cong [\Delta^{op}, \mathbf{Cat}](\sigma\mathbf{C} \times \nu N\mathbf{D}, N_h\mathbb{E}). \tag{3}$$

Since $(\mathbf{Cat}, \times)$ is closed symmetric monoidal, it follows from a general fact that $[\Delta^{op}, \mathbf{Cat}]$ has a tensor product

$$(Y \otimes Z)_n := Y_n \times Z_n = (Y \times Z)_n$$

and an internal hom

$$\begin{aligned} [Y, Z]_n &:= [\Delta^{op}, \mathbf{Cat}](\Delta[n] \cdot Y, Z) \\ &\cong [\Delta^{op}, \mathbf{Cat}](Y \times \nu\Delta[n], Z) \end{aligned}$$

for all $Y, Z \in [\Delta^{op}, \mathbf{Cat}]$. Applying this to the right hand side of equation (3) we obtain

$$\mathbf{DblCat_v}(\mathbb{V}\mathbf{C} \times \mathbb{H}\mathbf{D}, \mathbb{E}) \cong [\Delta^{op}, \mathbf{Cat}](\sigma\mathbf{C}, [\nu N\mathbf{D}, N_h\mathbb{E}]). \tag{4}$$

But recall that $\sigma\mathbf{C} = sk_0\,\mathbf{C}$ where sk_0 is the left adjoint to the 0-truncation tr_0 of simplicial objects in $\mathbf{Cat}$. Thus

$$\begin{aligned} \mathbf{DblCat_v}(\mathbb{V}\mathbf{C} \times \mathbb{H}\mathbf{D}, \mathbb{E}) &\cong [\Delta^{op}, \mathbf{Cat}](\sigma\mathbf{C}, [\nu N\mathbf{D}, N_h\mathbb{E}]) \\ &\cong \mathbf{Cat}(\mathbf{C}, [\nu N\mathbf{D}, N_h\mathbb{E}]_0) \\ &\cong \mathbf{Cat}(\mathbf{C}, [\Delta^{op}, \mathbf{Cat}](\nu N\mathbf{D}, N_h\mathbb{E})) \\ &\cong \mathbf{Cat}(\mathbf{C}, [\Delta^{op}, \mathbf{Cat}](N_h\mathbb{H}\mathbf{D}, N_h\mathbb{E})) \\ &\cong \mathbf{Cat}(\mathbf{C}, \mathbf{DblCat_v}(\mathbb{H}\mathbf{D}, \mathbb{E})) \end{aligned}$$

by the 2-fully faithfulness of N_h. $\square$

Lemma 6.9. *If X and Y are simplicial objects in* **Cat**, *then $X \times Y$ is the weighted colimit $X * G$ of the* **Cat**-*functor*

$$G\colon \Delta \longrightarrow [\Delta^{op}, \mathbf{Cat}]$$

$$[n] \longmapsto Y \times \nu\Delta[n]$$

with weighting $X\colon \Delta^{op} \longrightarrow \mathbf{Cat}$.

Proof: For any $Z \in [\Delta^{op}, \mathbf{Cat}]$,

$$[\Delta^{op}, \mathbf{Cat}](G([n]) \times Z)$$

is the n-th category of the internal hom $[Y, Z]$ as in the proof of Lemma 6.8. Thus we have a natural isomorphism

$$[\Delta^{op}, \mathbf{Cat}](X, [\Delta^{op}, \mathbf{Cat}](G(-), Z)) \cong [\Delta^{op}, \mathbf{Cat}](X \times Y, Z)$$

and $X \times Y$ satisfies the universal property of the weighted colimit $X * G$. □

We finish the conceptual proof of Example 6.5.

Proposition 6.10. *If* $\mathbf{A}$ *is a category and Y is a simplicial set, then the horizontal categorification of the simplicial category $\sigma\mathbf{A} \times \nu Y$ is $\mathbf{A} \boxtimes cY$ where cY is the traditional categorification of Y.*

Proof: By Lemma 6.9, $\sigma\mathbf{A} \times \nu Y$ is the weighted colimit $\sigma\mathbf{A} * G$ of

$$G\colon \Delta \longrightarrow [\Delta^{op}, \mathbf{Cat}]$$

$$[n] \longmapsto \nu Y \times \nu\Delta[n]$$

with weighting $\sigma\mathbf{A}\colon \Delta^{op} \longrightarrow \mathbf{Cat}$.

Let $J\colon \Delta \longrightarrow \mathbf{DblCat}$ be the horizontal embedding. Then from the enriched version of Theorem 5.14 (Kan's Lemma), found in Chapter 5 of [50], for each $Z \in [\Delta^{op}, \mathbf{Cat}]$, $c_h(Z) \cong Z * J$. Hence

$$\begin{aligned} c_h(\sigma\mathbf{A} \times \nu Y) &= c_h(\sigma\mathbf{A} * G) \\ &\cong (\sigma\mathbf{A} * G) * J \\ &\cong \sigma\mathbf{A} * (G * J) \end{aligned} \tag{5}$$

by Fubini's Theorem, also in [50]. The functor $G * J\colon \Delta \longrightarrow \mathbf{DblCat}$ in the last line takes $[n]$ to

$$G([n]) * J \cong c_h(G([n])) = c_h(\nu Y \times \nu\Delta[n]).$$

From Example 6.4 and the fact that c preserves finite products, we have

$$c_h(\nu Y \times \nu\Delta[n]) = \mathbb{H}c(Y \times \Delta[n]) \cong \mathbb{H}cY \times \mathbb{H}[n].$$

We conclude that (5) has the form

$$c_h(\sigma\mathbf{A} \times \nu Y) \cong \sigma\mathbf{A} * (\mathbb{H}cY \times \mathbb{H}[-]). \tag{6}$$

We claim that the right hand side of (6) is isomorphic to $\mathbb{V}\mathbf{A} \times \mathbb{H}cY$. In fact, Lemma 6.8 and the adjunction $sk_0 \dashv tr_0$ give, for all $\mathbb{E} \in \mathbf{DblCat_v}$

$$\begin{aligned}
\mathbf{DblCat_v}(\mathbb{V}\mathbf{A} \times \mathbb{H}cY, \mathbb{E})) &\cong \mathbf{Cat}(\mathbf{A}, \mathbf{DblCat_v}(\mathbb{H}cY, \mathbb{E})) \\
&\cong \mathbf{Cat}(\mathbf{A}, tr_0\,\mathbf{DblCat_v}(\mathbb{H}cY \times \mathbb{H}[-], \mathbb{E})) \\
&\cong [\Delta^{op}, \mathbf{Cat}](\sigma\mathbf{A}, \mathbf{DblCat_v}(\mathbb{H}cY \times \mathbb{H}[-], \mathbb{E})).
\end{aligned}$$

The claim follows now from the definition of weighted colimit. Hence, (6) implies that

$$c_h(\sigma\mathbf{A} \times \nu Y) \cong \mathbb{V}\mathbf{A} \times \mathbb{H}cY = \mathbf{A} \boxtimes cY.$$

$\square$

The vertical categorification of a simplicial object X in $\mathbf{Cat}$ is the transpose of c_hX.

6.2. **Double Categorification.** Finally, we turn to the left adjoint c_d to the double nerve N_d, and show that c_d preserves external products. This section is not needed for the rest of the paper, so it may be skipped on a first reading. This section will be of use when we construct a model structure on $\mathbf{DblCat}$ from bisimplicial sets in future work.

Definition 6.11. Let $X \in \mathbf{SSet^2}$ be a bisimplicial set. We define a double category c_dX called the *double categorification* or *fundamental double category of* X as follows. First define a double derivation scheme $\mathbb{S}$ with horizontal 1-category cX_{0*} and vertical 1-category cX_{*0}, and with squares X_{11}. We equip the free double category $\mathbb{F}$ on the double derivation scheme $\mathbb{S}$ with the smallest congruence $\sim$ such that

(i) For all $\tau \in X_{12}$ with boundary

$$\begin{tikzcd} & {} \arrow[d, phantom, "\tau"] \arrow[dr, "\beta"] & \\ {} \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & {} & {} \end{tikzcd}$$

we have

$$[\alpha\beta] \sim \gamma,$$

(ii) For all $\tau \in X_{21}$ with boundary

$$\begin{tikzcd} & {} \arrow[d, phantom, "\tau"] \arrow[dr, "\beta"] & \\ {} \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & {} & {} \end{tikzcd}$$

we have
$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \sim \gamma,$$

(iii) For all $f \in X_{01}$, the vertical identity i^v_f is congruent to the degeneracy of f in X_{11},

(iv) For all $j \in X_{10}$, the horizontal identity i^h_j is congruent to the degeneracy of j in X_{11}.

We define $c_d X$ as the quotient of $\mathbb{F}$ by the congruence $\sim$. The horizontal and vertical 1-categories of $c_d X$ are cX_{0*} and cX_{*0} respectively.

Remark 6.12. In the definition of double categorification it is not necessary to mod out by additional relations to make the identity squares functorial. They already are functorial in the free double category $\mathbb{F}$ on the double derivation scheme $\mathbb{S}$.

Example 6.13. If X and Y are simplicial sets, and $X \boxtimes Y$ is the bisimplicial set with (m,n)-bisimplices $X_m \times X_n$, then $c_d(X \boxtimes Y) = cX \boxtimes cY$.

Proposition 6.14. *Double categorification c_d is left adjoint to the double nerve N_d.*

Proof: We use the notation of Definition 6.11 and prove a natural bijection
$$\mathbf{DblCat}(c_d X, \mathbb{D}) \cong \mathbf{SSet^2}(X, N_d\mathbb{D}).$$
Suppose $G\colon X \longrightarrow N_d\mathbb{D}$ is a morphism of bisimplicial sets. Then the restriction to the map of double graphs $(X_{ij})_{0\leq i,j\leq 1} \longrightarrow \mathbb{D}$ induces a map of double derivation schemes $\mathbb{S} \longrightarrow \mathbb{D}$ as in the 1-category case, which induces a unique double functor J making the upper left triangle commute,
$$\begin{tikzcd}
\mathbb{S} \arrow[r] \arrow[d] & \mathbb{D} \\
\mathbb{F} \arrow[r] \arrow[ur, dotted, "\exists!\, J"] & c_d X \arrow[u, dotted, "\exists!\, G'"']
\end{tikzcd}$$
However, J takes congruent squares to equal squares since G is a map of bisimplicial sets, thus J induces a unique double functor G' making the lower right triangle commute.

If we are given a double functor $G'\colon c_d X \longrightarrow \mathbb{D}$, then we compose it with the morphism of double graphs
$$(X_{ij})_{0\leq i,j\leq 1} \longrightarrow \mathbb{F} \longrightarrow c_d X$$
to induce a morphism of bisimplicial sets $G\colon X \longrightarrow N_d\mathbb{D}$. Such a morphism G is determined by its restriction to $(X_{ij})_{0\leq i,j\leq 1}$.

These two procedures are inverse to one another. □

7. Model Structures Arising from $\mathbf{Cat}^{\Delta^{op}}$

Now that we have the adjunction $\mathbf{Cat}^{\Delta^{op}} : c_h \dashv N_h : \mathbf{DblCat}$ in place we can use it to transfer model structures from $\mathbf{Cat}^{\Delta^{op}}$ to $\mathbf{DblCat}$ using Kan's Lemma 7.11 on Transfer. This theorem says that one can lift a model structure across an adjunction under certain smallness conditions, which guarantee functorial factorizations. This is our first method for constructing model structures on $\mathbf{DblCat}$. In Section 8 we will adopt the point of view of double categories as internal categories and apply the results of [33]. In Section 9 we will consider $\mathbf{DblCat}$ as a category of algebras for a 2-monad and use [54].

The category $\mathbf{Cat}^{\Delta^{op}}$ has four model structures of interest to us. These arise as diagram structures and Reedy structures associated to two cofibrantly generated model structures on $\mathbf{Cat}$: the categorical structure and the Thomason structure. In this section we first describe the structures on $\mathbf{Cat}$ and their associated diagram structures and then recall some preliminaries about model categories, such as smallness arguments and the Kan's Lemma on Transfer. We then transfer the diagram structures to $\mathbf{DblCat}$ across the horizontal categorification-horizontal nerve adjunction. In the proofs of our transfer results we crucially need to know the behavior of certain pushouts, and these are treated in Theorems 10.6 and 10.7 of the Appendix. We also show that the Reedy categorical structure cannot transfer.

Recall the notion of cofibrantly generated model category.

Definition 7.1. A model category $\mathbf{C}$ is *cofibrantly generated* if there exist sets of morphism I and J in $\mathbf{C}$ such that

(i) The domains of I are small with respect to I-cell as defined in Definitions 7.5 and 7.8,
(ii) The domains of J are small with respect to J-cell,
(iii) The class of fibrations is precisely the class of morphisms with the right lifting property with respect J,
(iv) The class of acyclic fibrations is precisely the class of morphisms with the right lifting property with respect to I.

In this case, I is the set of *generating cofibrations* and J is the set of *generating acyclic cofibrations.*

7.1. **Model Structures on Cat.** In the *Thomason structure* on $\mathbf{Cat}$ in [76] a functor F is a weak equivalence (respectively fibration) if and only if $\mathrm{Ex}^2 NF$ is a weak equivalence (respectively fibration) of simplicial sets. The functor Ex is superfluous for weak equivalences, as Thomason proved that F is a weak equivalence if and only if NF is. The functor $\mathrm{Ex}\colon \mathbf{SSet} \longrightarrow \mathbf{SSet}$ is the left adjoint to barycentric subdivision $\mathrm{Sd}\colon \mathbf{SSet} \longrightarrow \mathbf{SSet}$, which we recall below. The Thomason structure is cofibrantly generated. The generating cofibrations are the inclusions of categorical boundaries

$$c\mathrm{Sd}^2\partial\Delta[m] \longrightarrow c\mathrm{Sd}^2\Delta[m]$$

while the generating acyclic cofibrations are the inclusions of categorical horns

$$c\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c\mathrm{Sd}^2\Delta[m]\,.$$

We now recall the definition of barycentric subdivision Sd. The simplicial sets $\mathrm{Sd}\Delta[m]$ and $\mathrm{Sd}\Lambda^k[m]$ are respectively the nerves of the posets of nondegenerate simplices of $\Delta[m]$ and $\Lambda^k[m]$. The ordering is the face relation. Thus a q-simplex of $\mathrm{Sd}\Delta[m]$ is a tuple $(v_0, \ldots, v_q)$ of nondegenerate simplices (faces) of $\Delta[m]$ such that v_i is a face of v_{i+1} for all $0 \leq i \leq q-1$. Such a tuple is a q-simplex of $\mathrm{Sd}\Lambda^k[m]$ if and only if all $v_0, \ldots, v_q$ are in $\Lambda^k[m]$. A p-simplex u is a face of a q-simplex v in $\mathrm{Sd}\Delta[m]$ if and only if

$$\{u_0, \ldots, u_p\} \subseteq \{v_0, \ldots, v_q\}.$$

A p-simplex u of $\mathrm{Sd}\Delta[m]$ is nondegenerate if and only if all u_i are distinct.

The subdivision of a simplicial set Y is by definition

$$\operatorname*{colim}_{\Delta[n]\to Y} \mathrm{Sd}\Delta[n]$$

where the colimit is indexed over the category of simplices of Y. It follows from Page 311 of [76] that $c\mathrm{Sd}^2\Delta[m]$ and $c\mathrm{Sd}^2\Lambda^k[m]$ are respectively the posets of nondegenerate simplices of $\mathrm{Sd}\Delta[m]$ and $\mathrm{Sd}\Lambda^k[m]$ and the generating acyclic cofibration $c\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c\mathrm{Sd}^2\Delta[m]$ is the inclusion of these posets.

The other model structure on **Cat** is the *categorical structure* of [49]. In the categorical structure a functor is a weak equivalence if and only if it is an equivalence of categories. A functor $F\colon \mathbf{A} \longrightarrow \mathbf{B}$ is a fibration if for each isomorphism $g\colon b \cong Fa$ in $\mathbf{B}$ there is an isomorphism $f\colon a' \cong a$ in $\mathbf{A}$ such that $Fa' = b$ and $Ff = g$. These fibrations of categories are also called *isofibrations*. A cofibration is a functor that is injective on objects. The categorical structure on **Cat** is also cofibrantly generated. There are three generating cofibrations:

$$\emptyset \hookrightarrow \{1\}$$

$$\{0,1\} \hookrightarrow \{0 \to 1\}$$

$$\{0 \rightrightarrows 1\} \longrightarrow \{0 \to 1\}$$

and one generating acyclic cofibration:

$$\{1\} \hookrightarrow \{0 \cong 1\} = \mathbf{I}\,.$$

7.2. **Diagram Model Structures on $\mathbf{Cat}^{\Delta^{op}}$.** Given a model category $\mathbf{M}$ and a small category $\mathbf{C}$, one might hope that the category $\mathbf{M}^{\mathbf{C}}$ of functors $\mathbf{C} \longrightarrow \mathbf{M}$ is also a model category with levelwise weak equivalences and levelwise fibrations. By this we mean that a natural transformation is a weak equivalence (respectively fibration) if and only if each of its components is. Unfortunately, this definition does not always give rise to a model structure on $\mathbf{M}^{\mathbf{C}}$. However, if $\mathbf{M}$ is a *cofibrantly generated* model category, Theorem 7.2 guarantees that this definition does indeed give rise to a model structure on $\mathbf{M}^{\mathbf{C}}$, which is even cofibrantly generated.

Theorem 7.2 (11.6.1 in [43])**.** *Let $\mathbf{C}$ be a small category and $\mathbf{M}$ a cofibrantly generated model category with I the set of generating cofibrations and J the set of generating acyclic cofibrations. Then $\mathbf{M}^{\mathbf{C}}$ is a cofibrantly generated model category with levelwise weak equivalences and levelwise fibrations. The generating cofibrations are natural transformations of the form*

$$\coprod_{\mathbf{C}(C,-)} A \xrightarrow{\coprod_{\mathbf{C}(C,-)} f} \coprod_{\mathbf{C}(C,-)} B$$

for $f\colon A \longrightarrow B$ in I. The generating acyclic cofibrations are defined similarly with f in J. A morphism in $\mathbf{M}^{\mathbf{C}}$ is a cofibration if it is a retract of a transfinite composition of pushouts of generating cofibrations. The components of a cofibration are also cofibrations.

Thus, the category $\mathbf{Cat}^{\Delta^{op}}$ inherits two model structures from Section 7.1. In the *diagram Thomason structure* on $\mathbf{Cat}^{\Delta^{op}}$, a natural transformation α is a weak equivalence (respectively fibration) if and only if $N\alpha_i$ is a weak equivalence (respectively fibration) of simplicial sets for each $i \geq 0$. In the *diagram categorical structure* on $\mathbf{Cat}^{\Delta^{op}}$, a natural transformation α is a weak equivalence (respectively fibration) if and only if α_i is an equivalence of categories (respectively isofibration) for all $i \geq 0$.

If $\mathbf{C}$ is a *Reedy category*, then a model structure on $\mathbf{M}$ also induces a *Reedy model structure* on $\mathbf{M}^{\mathbf{C}}$ (see for example [43] or [44]). The category Δ is a Reedy category, so the Thomason and categorical structures on $\mathbf{Cat}$ also give rise to two more model structures on $\mathbf{Cat}^{\Delta^{op}}$. However, we do not study these in more detail because of the following Remark and also because of Theorem 7.16.

Remark 7.3. If $\mathbf{C}$ is a Reedy category and $\mathbf{M}$ is a cofibrantly generated model category, then the Reedy structure on $\mathbf{M}^{\mathbf{C}}$ is Quillen equivalent to the diagram structure.

7.3. **Smallness.** We will need some knowledge about smallness to use Kan's Lemma on Transfer. We recall some of the relevant notions from [44]. Appropriate smallness conditions also allow us to conclude that a transfinite composition of weak equivalences is a weak equivalence.

Definition 7.4. Let κ be a cardinal. An ordinal λ is *κ-filtered* if it is a limit ordinal and, if $A \subseteq \lambda$ and $|A| \leq \kappa$, then $\sup A < \lambda$.

Definition 7.5. Let $\mathbf{C}$ be a category with all small colimits and κ a cardinal. An object A of $\mathbf{C}$ is called *κ-small* if for all κ-filtered ordinals λ and all colimit-preserving functors $X\colon \lambda \longrightarrow \mathbf{C}$ the map of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathbf{C}(A, X_\beta) \longrightarrow \mathbf{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta) \tag{7}$$

is a bijection. An object A is said to be *small* if it is κ-small for some cardinal κ. An object A is said to be *finite* if it is κ-small for a finite cardinal κ, *i.e.,* for *any* limit ordinal λ and colimit-preserving functor X, the map (7) is a bijection. We say the concepts hold *relative to a class of morphisms* $\mathbf{D}$ *in* $\mathbf{C}$ if they hold true for all X with $X_\beta \longrightarrow X_{\beta+1}$ in $\mathbf{D}$ for all $\beta + 1 < \lambda$.

For example, categories are small as follows, and we conclude also that double categories are small.

Proposition 7.6. *Any category* $\mathbf{A}$ *is κ-small where*

$$\kappa = |\mathit{Obj}\,\mathbf{A}| + |\mathit{Mor}\,\mathbf{A}| + |\mathit{Mor}\,\mathbf{A}_s \times_t \mathit{Mor}\,\mathbf{A}|.$$

In particular, if Mor $\mathbf{A}$ *is a finite set, then* $\mathbf{A}$ *is finite as an object of* $\mathbf{Cat}$.

Proof: Let $\mathbf{X}\colon \lambda \longrightarrow \mathbf{Cat}$ be a colimit-preserving functor from a κ-filtered ordinal λ. Recall that ordinals are filtered categories and filtered colimits of categories are formed by simply taking the filtered colimits of the object set and the morphism set.

Suppose $F\colon \mathbf{A} \longrightarrow \operatorname{colim} \mathbf{X}$ is a functor. For each $A \in \mathit{Obj}\ \mathbf{A}$ and $f \in \mathit{Mor}\,\mathbf{A}$ there are $\alpha_1(A)$ and $\alpha_2(f)$ such that $F(A)$ and $F(f)$ are in the image of $\mathbf{X}_{\alpha_1(A)}$ and $\mathbf{X}_{\alpha_2(A)}$. Let β be the suprema of all the $\alpha_1(a)$ and $\alpha_2(f)$. Then $\beta < \lambda$ and we obtain maps of sets

$$G^{Obj} : \mathit{Obj}\,\mathbf{A} \longrightarrow \mathit{Obj}\,\mathbf{X}_\beta$$
$$G^{Mor} : \mathit{Mor}\,\mathbf{A} \longrightarrow \mathit{Mor}\,\mathbf{X}_\beta$$

which factor the functor F. There exists for each $f \in \mathit{Mor}\ \mathbf{A}$ an index $\gamma(f)$ such that $s(G(f)) = G(s(f))$ and $t(G(f)) = G(t(f))$ in $\mathbf{X}_{\gamma(f)}$. For each $A \in \mathit{Obj}\,\mathbf{A}$ there is an index $\delta(A)$ such that $G(1_A) = 1_{G(A)}$ in $\mathbf{X}_{\delta(A)}$.

For each $(\ell, k) \in \mathit{Mor}\, \mathbf{A}_s \times_t \mathit{Mor}\, \mathbf{A}$ there exists an index $\epsilon(\ell, k)$ such that $G(\ell \circ k) = G(\ell) \circ G(k)$ in $\mathbf{X}_{\epsilon(\ell,k)}$. Let ζ be the supremum of all these indices γ, δ, ϵ. Then $\zeta < \lambda$ and G induces a functor $\mathbf{A} \longrightarrow \mathbf{X}_\zeta$ which factors F. Hence (7) is onto.

Suppose $M\colon \mathbf{A} \longrightarrow \mathbf{X}_\alpha$ and $N\colon \mathbf{A} \longrightarrow \mathbf{X}_\beta$ are functors that become equal in the colimit. Then for each $a \in \mathit{Obj}\, \mathbf{A}$ and each $f \in \mathit{Mor}\, \mathbf{A}$ there are indices $\gamma(a)$ and $\delta(f)$ such that $M(a) = N(a)$ and $M(f) = N(f)$ in $\mathbf{X}_{\gamma(a)}$ and $\mathbf{X}_{\delta(f)}$ respectively. Let $\zeta < \lambda$ be the supremum of all these indices $\gamma(a)$ and $\delta(f)$. Then M and N become equal at the stage ζ and the map (7) is injective. $\square$

Corollary 7.7. *Let $\mathbb{D}$ be a double category and s^h, s^v, t^h, t^v the horizontal and vertical source and target maps. Then $\mathbb{D}$ is κ-small where*

$$\begin{aligned}\kappa =& |\mathit{Obj}\, \mathbb{D}| + |\mathit{Hor}\, \mathbb{D}| + |\mathit{Hor}\, \mathbb{D}_{s^h} \times_{t^h} \mathit{Hor}\, \mathbb{D}| \\ &+ |\mathit{Ver}\, \mathbb{D}| + |\mathit{Ver}\, \mathbb{D}_{s^v} \times_{t^v} \mathit{Ver}\, \mathbb{D}| \\ &+ |\mathit{Sq}\, \mathbb{D}| + |\mathit{Sq}\, \mathbb{D}_{s^v} \times_{t^v} \mathit{Sq}\, \mathbb{D}| \\ &+ |\mathit{Sq}\, \mathbb{D}_{s^h} \times_{t^h} \mathit{Sq}\, \mathbb{D}|.\end{aligned}$$

In particular, if $\mathit{Sq}\, \mathbb{D}$ is a finite set, then $\mathbb{D}$ is finite as an object of **DblCat**.

Proof: We first obtain a map of the underlying quadruple of sets, and then we go out far enough to make it into a double functor by considering the various compositions and identities as in Proposition 7.6. $\square$

Note that this corollary easily generalizes to n-fold categories.

One useful application of finiteness is to transfinite compositions of weak equivalences.

Definition 7.8. If $\mathbf{C}$ is a category with all small colimits, λ is an ordinal, $\mathbf{D}$ is a class of morphisms in $\mathbf{C}$, and $X\colon \lambda \longrightarrow \mathbf{C}$ is a colimit preserving functor such that $X_\beta \longrightarrow X_{\beta+1}$ is in $\mathbf{D}$ for all $\beta+1 < \lambda$, then the morphism

$$X_0 \longrightarrow \operatorname{colim} X$$

is called a *transfinite composition of morphisms in* $\mathbf{D}$. If I is a class of morphisms in $\mathbf{C}$, then a transfinite composition of pushouts of elements of I is called a *relative I-cell complex.* The class of relative I-cell complexes is denoted *I-cell.*

Proposition 7.9 (7.4.2 of [44])**.** *Suppose* $\mathbf{C}$ *is a cofibrantly generated model category in which the domains and codomains of the generating cofibrations and generating acyclic cofibrations are finite. Then every transfinite composition of weak equivalences is a weak equivalence.*

Example 7.10. In both the Thomason structure and the categorical structure on **Cat**, every transfinite composition of weak equivalences is a weak equivalence, as the domains and codomains of the generating cofibrations and generating acyclic cofibrations only have finitely many morphisms. Since weak equivalences and colimits in $\mathbf{Cat}^{\Delta^{op}}$ are levelwise, every transfinite composition of weak equivalences in the diagram structures is also a weak equivalence.

7.4. **Kan's Lemma on Transfer.** Our first main tool for constructing model structures on **DblCat** is Kan's Lemma on Transfer. The form we will use is Corollary 7.12.

Theorem 7.11 (Kan's Lemma on Transfer, 11.3.2 in [43])**.** *Let* $\mathbf{C}$ *be a cofibrantly generated model category with generating cofibrations* I *and generating acyclic cofibrations* J*. Suppose* $\mathbf{D}$ *is complete and cocomplete, and that*

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd} \tag{8}$$

is an adjunction. Assume the following.

(i) *For every* $i \in I$*, dom* Fi *is small with respect to* FI*-cell. For every* $j \in J$*, dom* Fj *is small with respect to* FJ*-cell.*
(ii) *The functor* G *maps every relative* FJ*-complex to a weak equivalence in* $\mathbf{C}$*.*

Then there exists a cofibrantly generated model structure on $\mathbf{D}$ *with generating cofibrations* FI *and generating acyclic cofibrations* FJ*. Further,* f *is a weak equivalence in* $\mathbf{D}$ *if and only* $G(f)$ *is a weak equivalence in* $\mathbf{C}$*, and* f *is a fibration in* $\mathbf{D}$ *if and only* $G(f)$ *is a fibration in* $\mathbf{C}$*.*

Along the lines of [78], we have the following corollary.

Corollary 7.12. *Let* $\mathbf{C}$ *be a cofibrantly generated model category with generating cofibrations* I *and generating acyclic cofibrations* J*. Suppose* $\mathbf{D}$ *is complete and cocomplete, and that* $F \dashv G$ *is an adjunction as in (8). Assume the following.*

(i) *For every* $i \in I$ *and* $j \in J$*, the objects dom* Fi *and dom* Fj *are small with respect to the entire category* $\mathbf{D}$*.*
(ii) *For any ordinal* λ *and any colimit preserving functor* $X\colon \lambda \longrightarrow \mathbf{C}$ *such that* $X_\beta \longrightarrow X_{\beta+1}$ *is a weak equivalence, the transfinite composition*
$$X_0 \longrightarrow \mathrm{colim}\, X$$
is a weak equivalence.

(iii) *G preserves filtered colimits.*

(iv) *If j' is a pushout of $F(j)$ in $\mathbf{D}$ for $j \in J$, then $G(j')$ is a weak equivalence in $\mathbf{C}$.*

Then there exists a cofibrantly generated model structure on $\mathbf{D}$ with generating cofibrations FI and generating acyclic cofibrations FJ. Further, f is a weak equivalence in $\mathbf{D}$ if and only $G(f)$ is a weak equivalence in $\mathbf{C}$, and f is a fibration in $\mathbf{D}$ if and only $G(f)$ is a fibration in $\mathbf{C}$.

Proof: Clearly, (i) of Theorem 7.11 follows from the hypotheses. To see (ii), we recall that a relative FJ-complex is a transfinite composition of pushouts of morphisms Fj where $j \in J$. If a relative FJ-complex f is a transfinite composition of $Y\colon \lambda \longrightarrow \mathbf{D}$, then Gf is the transfinite composition of $X = G \circ Y$. Since Gf is a transfinite composition of weak equivalences, Gf is also a weak equivalence. Hence G takes relative FJ-complexes to weak equivalences in $\mathbf{C}$. $\square$

7.5. **Transfer of the Diagram Thomason Structure on $\mathbf{Cat}^{\Delta^{op}}$.** With these preliminaries and our free constructions on double categories, we can transfer the diagram Thomason structure to **DblCat**. Recall the diagram Thomason structure on $\mathbf{Cat}^{\Delta^{op}}$ from Section 7.2.

Theorem 7.13. *There is a cofibrantly generated model structure on* **DblCat** *such that a double functor K is a weak equivalence (respectively fibration) if and only if $N_h K$ is levelwise a weak equivalence (respectively fibration) in the Thomason structure on* **Cat**.

Proof: We apply 7.12 to the adjunction $F = c_h \dashv N_h = G$. First we point out that

$$\begin{aligned} c_h\Big(\coprod_{\Delta^{op}([n],-)} c\mathrm{Sd}^2\Lambda^k[m]\Big) &= c_h(c\mathrm{Sd}^2\Lambda^k[m] \times \Delta[n]) \\ &= (c\mathrm{Sd}^2\Lambda^k[m]) \boxtimes c\Delta[n] \\ &= (c\mathrm{Sd}^2\Lambda^k[m]) \boxtimes [n] \end{aligned}$$

by Example 6.5 or Proposition 6.10 (for simplicity we suppress σ and ν). Similarly,

$$c_h\Big(\coprod_{\Delta^{op}([n],-)} c\mathrm{Sd}^2\Delta[m]\Big) = (c\mathrm{Sd}^2\Delta[m]) \boxtimes [n]$$

and the horizontal categorification of the generating acyclic cofibrations j in Theorem 7.2 are the inclusions $i \boxtimes 1_{[n]}$ for the inclusions

$$i\colon\, c\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c\mathrm{Sd}^2\Delta[m]$$

and $[n] \in \Delta$.

(i) The double categories $(c\mathrm{Sd}^2\Delta[m]) \boxtimes [n]$ and $(c\mathrm{Sd}^2\Lambda^k[m]) \boxtimes [n]$ have a finite number of squares, hence they are finite by Corollary 7.7.
(ii) A transfinite composition of weak equivalences in $\mathbf{Cat}^{\Delta^{op}}$ is a weak equivalence by Example 7.10.
(iii) The horizontal nerve N_h preserves filtered colimits by Theorem 5.5.
(iv) Consider the pushout in **DblCat**,

$$\begin{tikzcd}
(c\mathrm{Sd}^2\Lambda^k[m]) \boxtimes [n] \arrow[r] \arrow[d, "c_h(j)=i\boxtimes 1_{[n]}"'] & \mathbb{D} \arrow[d, "j'"] \\
(c\mathrm{Sd}^2\Delta[m]) \boxtimes [n] \arrow[r] & \mathbb{P}\,.
\end{tikzcd}$$

Then by Proposition 5.6 $N_h c_h(j) = j$ and by Theorem 10.7 the induced map in the pushout in $\mathbf{Cat}^{\Delta^{op}}$

$$\begin{tikzcd}
(c\mathrm{Sd}^2\Lambda^k[m]) \times \Delta[n] \arrow[r] \arrow[d, "N_h c_h(j)=j"'] & N_h\mathbb{D} \arrow[d, "\overline{j}"] \arrow[ddr, bend left, "N_h(j')"] & \\
(c\mathrm{Sd}^2\Delta[m]) \times \Delta[n] \arrow[r] \arrow[drr, bend right] & P \arrow[dr, dotted] & \\
& & N_h\mathbb{P}
\end{tikzcd}$$

is an isomorphism. But $\overline{j}$ is an acyclic cofibration, as it is a pushout of an acyclic cofibration. Hence $N_h(j')$ is a weak equivalence in $\mathbf{Cat}^{\Delta^{op}}$ by the 2-out-of-3 property.

□

7.6. **Transfer of the Diagram Categorical Structure on $\mathbf{Cat}^{\Delta^{op}}$.** Our preparations allow us to also quickly transfer the diagram categorical structure. Recall the diagram categorical structure on $\mathbf{Cat}^{\Delta^{op}}$ from Section 7.2. In Section 8.2 we will show that this model structure on **DblCat** coincides with the model structure induced by the simplicially surjective topology τ on **Cat** using the methods of [33]. An important reason for interest in the equality of these two structures lies in the fact that the second construction yields an explicit form for the cofibrant replacement, which is not all all transparent using only the transferred structure.

Theorem 7.14. *There is a cofibrantly generated model structure on* **DblCat** *such that a double functor K is a weak equivalence (respectively* *fibration) if and only if $N_h K$ is levelwise a weak equivalence (respectively fibration) in the categorical structure on* **Cat**.

Proof: We apply 7.12 to the adjunction $F = c_h \dashv N_h = G$. All generating acyclic cofibrations j for the categorical diagram structure on $\mathbf{Cat}^{\Delta^{op}}$ are natural transformations of the form

$$\coprod_{\Delta^{op}([n],-)} \{1\} \longrightarrow \coprod_{\Delta^{op}([n],-)} \mathbf{I}$$

and have horizontal categorification

$$\{1\} \boxtimes [n] \longrightarrow \mathbf{I} \boxtimes [n]$$

by Example 6.5 or Proposition 6.10 (for simplicity we suppress σ and ν).

(i) The double categories $\{1\} \boxtimes [n]$ and $\mathbf{I} \boxtimes [n]$ have a finite number of squares, hence they are finite by Corollary 7.7.

(ii) A transfinite composition of weak equivalences in $\mathbf{Cat}^{\Delta^{op}}$ is a weak equivalence by Example 7.10.

(iii) The horizontal nerve N_h preserves filtered colimits by Theorem 5.5.

(iv) Consider the pushout in **DblCat**,

$$\begin{tikzcd} \{1\} \boxtimes [n] \arrow[r] \arrow[d, "c_h(j)"'] & \mathbb{D} \arrow[d, "j'"] \\ \mathbf{I} \boxtimes [n] \arrow[r] & \mathbb{P}. \end{tikzcd}$$

Then by Proposition 5.6 $N_h c_h(j) = j$ and by Theorem 10.7 the induced map in the pushout in $\mathbf{Cat}^{\Delta^{op}}$

$$\begin{tikzcd} \{1\} \times \Delta[n] \arrow[r] \arrow[d, "N_h c_h(j)=j"'] & N_h\mathbb{D} \arrow[d, "\overline{j}"] \arrow[ddr, bend left, "N_h(j')"] & \\ \mathbf{I} \times \Delta[n] \arrow[r] \arrow[drr, bend right] & P \arrow[dr, dotted] & \\ & & N_h\mathbb{P} \end{tikzcd}$$

is an isomorphism. But $\overline{j}$ is an acyclic cofibration, as it is a pushout of an acyclic cofibration. Hence $N_h(j')$ is a weak equivalence in $\mathbf{Cat}^{\Delta^{op}}$ by the 2-out-of-3 property.

$\square$

7.7. **No Transfer of the Reedy Categorical Structure on $\mathbf{Cat}^{\Delta^{op}}$.** In this subsection we consider the category $\mathbf{Cat}^{\Delta^{op}}$ of simplicial objects in **Cat** equipped with the Reedy model structure associated with the categorical model structure on **Cat**. The weak equivalences in this Reedy model structure are the levelwise equivalences of categories and the fibrations are the Reedy fibrations. (For further details, see [43].) In this section we will show that it is impossible to transfer this model structure to **DblCat** via the adjunction $c_h \dashv N_h$, where N_h is the horizontal nerve and c_h is the horizontal categorification. We will need the following theorem.

Theorem 7.15 (Theorem 1 in [47])**.** *For a given functor $G\colon \mathbf{B} \longrightarrow \mathbf{C}$, the canonical comparison functor from the pullback of F along G to the pseudo pullback of F along G is an equivalence of categories for all functors $F\colon \mathbf{A} \longrightarrow \mathbf{C}$ if and only if G is an isofibration.*

Now we turn to the objective of this subsection.

Theorem 7.16. *There does not exist a model structure on* **DblCat** *such that a double functor K is a weak equivalence (respectively fibration) if and only if N_hK is a weak equivalence (respectively fibration) in the Reedy model structure on $\mathbf{Cat}^{\Delta^{op}}$ associated to the categorical structure on* **Cat***.*

Proof: Suppose that such a transferred model structure on **DblCat** does exist. Then (c_h, N_h) is a Quillen pair. Let $\mathbb{D}$ be a double category and consider a Reedy fibrant replacement $r\colon N_h\mathbb{D} \longrightarrow V_\bullet$ in $\mathbf{Cat}^{\Delta^{op}}$, that is, $V_\bullet$ is a Reedy fibrant object and r is an acyclic cofibration in the Reedy structure. Our strategy is to prove that at least one of the source and the target functors $(N_h\mathbb{D})_1 \longrightarrow (N_h\mathbb{D})_0$ is an isofibration, which will lead to a contradiction.

The morphism c_hr is an acyclic cofibration since c_h is a left Quillen functor. Moreover, since $V_\bullet$ is Reedy fibrant, the map

$$V_1 \xrightarrow{(d_0,d_1)} V_0 \times V_0$$

is an isofibration in **Cat**. Also, since every object in **Cat** is fibrant in the categorical structure, the two projections $\pi_1, \pi_2\colon V_0 \times V_0 \longrightarrow V_0$ are isofibrations. It follows that the maps $d_0, d_1\colon V_1 \longrightarrow V_0$ are themselves isofibrations. By Theorem 7.15 this implies that the canonical functor

$$V_1 \times_{V_0} V_1 \longrightarrow V_1 \overset{\mathrm{ps}}{\times}_{V_0} V_1 \tag{9}$$

from the pullback to the pseudo pullback is an equivalence of categories.

Next we similarly show that (12) is an equivalence of categories. By definition, $c_h(V)_0 = V_0$. We claim that the functor

$$(c_hV)_1 \xrightarrow{(d_0,d_1)} (c_hV)_0 \times (c_hV)_0 \tag{10}$$

is also a fibration. First note that $V_1 \longrightarrow V_0 \times V_0$ is an isofibration precisely when any diagram

$$\begin{tikzcd} \cdot \arrow[d, "v"', "\wr\|"] & \cdot \arrow[d, "w", "\|\wr"'] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

with vertical isomorphisms can be filled with a vertically invertible "square" α (morphism of V_1),

$$\begin{tikzcd} \cdot \arrow[r, "g'"] \arrow[d, "v"', "\wr\|"] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "w", "\|\wr"'] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd} .$$

Now consider a similar diagram in $c_h(V)$. This has the form

$$\begin{tikzcd} \cdot \arrow[d, "v"', "\wr\|"] & & & & \cdot \arrow[d, "w", "\|\wr"'] \\ \cdot \arrow[r, "g_1"'] & \cdot \arrow[r, "g_2"'] & \cdots & \cdot \arrow[r, "g_n"'] & \cdot \end{tikzcd} , \tag{11}$$

where the bottom edge is the equivalence class of a path of composable horizontal morphisms by Definition 6.2. We next insert vertical identity morphisms and fill in the individual squares ($V_1 \longrightarrow V_0 \times V_0$ is an isofibration) to obtain the following compatible arrangement,

$$\begin{tikzcd} \cdot \arrow[r, "g_1'"] \arrow[d, "v"', "\wr\|"] \arrow[dr, phantom, "\alpha_1"] & \cdot \arrow[r, "g_2'"] \arrow[d, equal] \arrow[dr, phantom, "\alpha_2"] & \cdot \arrow[r, dotted] \arrow[d, equal] & \cdot \arrow[r, "g_n'"] \arrow[d, equal] \arrow[dr, phantom, "\alpha_n"] & \cdot \arrow[d, "w", "\|\wr"'] \\ \cdot \arrow[r, "g_1"'] & \cdot \arrow[r, "g_2"'] & \cdot \arrow[r, dotted] & \cdot \arrow[r, "g_n"'] & \cdot \end{tikzcd} .$$

The equivalence class in $c_h(V)$ of this compatible arrangement gives the required filling for (11). So (10) is indeed a fibration. Reasoning as for (9), this implies that the following functor is an equivalence of categories.

$$(c_hV)_1 \times_{(c_hV)_0} (c_hV)_1 \longrightarrow (c_hV)_1 \overset{\text{ps}}{\times}_{(c_hV)_0} (c_hV)_1 \tag{12}$$

Next we show that at least one of the source and the target functors $(N_h\mathbb{D})_1 \longrightarrow (N_h\mathbb{D})_0$ is an isofibration, which we will see is a contradiction. We claim that the unit η_V is a weak equivalence. Since the nerve functor

N_h is fully faithful, the counit $\varepsilon\colon c_hN_h \Longrightarrow \mathrm{Id}_{\mathbf{DblCat}}$ is an isomorphism. One of the triangle identities states that $N_h\varepsilon_{\mathbb{D}} \cdot \eta_{N_h\mathbb{D}} = \mathrm{id}$, so $\eta_{N_h\mathbb{D}}$ is an isomorphism. The naturality of the unit η therefore gives us a commutative diagram

$$\begin{tikzcd}
N_h\mathbb{D} = N_hc_hN_h\mathbb{D} \arrow[d, "N_hc_hr"'] & N_h\mathbb{D} \arrow[l, "\eta_{N_h\mathbb{D}}"'] \arrow[d, "r"] \\
N_hc_hV & V \arrow[l, "\eta_V"]
\end{tikzcd}$$

in which the morphism $\eta_{N_h\mathbb{D}}$ is a levelwise equivalence (since it is an isomorphism), and the morphism r is a levelwise equivalence (by hypothesis). The morphism N_hc_hr is one as well, since c_hr is a weak equivalence (c_hr is even an acyclic cofibration). By the 2-out-of-3 property, it follows that η_V is a levelwise equivalence of categories, as claimed.

Consider the following commutative diagram in **Cat**.

$$\begin{tikzcd}
(N_h\mathbb{D})_2 \arrow[r, "\mathrm{Segal}", "\cong"'] \arrow[d, "r_2"', "\wr"] \arrow[dr, phantom, "(D)"] & (N_h\mathbb{D})_1 \times_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \arrow[r] \arrow[d, "(r_1,r_1)"] \arrow[dr, phantom, "(C)"] & (N_h\mathbb{D})_1 \overset{\mathrm{ps}}{\times}_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \arrow[d, "(r_1,r_1)"] \\
V_2 \arrow[r] \arrow[d, "(\eta_V)_2"', "\wr"] \arrow[dr, phantom, "(B)"] & V_1 \times_{V_0} V_1 \arrow[r, "\sim"] \arrow[d, "((\eta_V)_1,(\eta_V)_1)"] \arrow[dr, phantom, "(A)"] & V_1 \overset{\mathrm{ps}}{\times}_{V_0} V_1 \arrow[d, "((\eta_V)_1,(\eta_V)_1)"] \\
(N_hc_hV)_2 \arrow[r, "\cong", "\mathrm{Segal}"'] & (N_hc_hV)_1 \times_{(N_hc_hV)_0} (N_hc_hV)_1 \arrow[r, "\sim"] & (N_hc_hV)_1 \overset{\mathrm{ps}}{\times}_{(N_hc_hV)_0} (N_hc_hV)_1
\end{tikzcd}$$

Note that (B) and (D) commute by the definition of Segal maps, while the commutativity of (A) and (C) follows from the universal property of the pseudo pullbacks. The vertical functors r_2 and $(\eta_V)_2$ are equivalences of categories, since r and η_V are weak equivalences from above. The bottom edge of (C) is an equivalence, since it is (9). The bottom edge of (A) is an equivalence as it is (12) (recall that $(N_h\mathbb{E})_0 = \mathbb{E}_0$ and $(N_h\mathbb{E})_1 = \mathbb{E}_1$ for any double category $\mathbb{E}$).

We claim that the top edge of (C) is an equivalence of categories. Since r_0 and r_1 are equivalences, the vertical functor

$$(r_1, r_1)\colon (N_h\mathbb{D})_1 \overset{\mathrm{ps}}{\times}_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \longrightarrow V_1 \overset{\mathrm{ps}}{\times}_{V_0} V_1$$

is an equivalence. Moreover, since η_V is a levelwise equivalence, the 2-out-of-3 property and the commutativity of (A) imply that the functor

$$V_1 \times_{V_0} V_1 \longrightarrow (N_hc_hV)_1 \times_{V_0} (N_hc_hV)_1$$

is an equivalence. Also, the commutativity of (B) and the 2-out-of-3 property imply that $V_2 \longrightarrow V_1 \times_{V_0} V_1$ is an equivalence. The commutativity of (D) then implies that

$$(N_h\mathbb{D})_1 \times_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \longrightarrow V_1 \times_{V_0} V_1$$

is an equivalence. Finally, the commutativity of (C) implies that the canonical map

$$(N_h\mathbb{D})_1 \times_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \longrightarrow (N_h\mathbb{D})_1 \overset{\text{ps}}{\times}_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \tag{13}$$

is an equivalence of categories, as claimed.

The map (13) is nothing but

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1 \overset{\text{ps}}{\times}_{\mathbb{D}_0} \mathbb{D}_1. \tag{14}$$

The objects of $\mathbb{D}_1 \overset{\text{ps}}{\times}_{\mathbb{D}_0} \mathbb{D}_1$ are diagrams of the form

$$\begin{tikzcd} & \cdot \arrow[d, "\cong"] \arrow[r, "g"] & \cdot \\ \cdot \arrow[r, "f"'] & \cdot & \end{tikzcd} \tag{15}$$

and morphisms of $\mathbb{D}_1 \overset{\text{ps}}{\times}_{\mathbb{D}_0} \mathbb{D}_1$ are diagrams of the form

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d] & \cdot \arrow[l, "\cong"'] \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\beta"] & \cdot \arrow[d] \\ \cdot \arrow[r, "f'"'] & \cdot & \cdot \arrow[l, "\cong"] \arrow[r, "g'"'] & \cdot \end{tikzcd} \tag{16}$$

where the middle square is a *commutative square of vertical morphisms.* The objects and morphisms of $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ are those of (15) and (16) where the isomorphisms are identities. The canonical functor (14) is given by this inclusion.

We now describe a double category $\mathbb{D}$ where the canonical functor (14) is not an equivalence, which then implies that our original assumption on the existence of a transferred Reedy structure on **DblCat** is false. Let $\mathbb{D}$ be the double category with four distinct objects A, B, C, D and only the following nontrivial arrows.

(17)
$$\begin{tikzcd} & C \arrow[r, "g"] \arrow[d, "\cong"] & D \\ A \arrow[r, "f"'] & B & \end{tikzcd}$$

There are no nontrivial squares. Suppose that the canonical functor (14) is essentially surjective. Then there exist objects X, Y, Z and morphisms as in the following diagram

(18)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\alpha"] & B \arrow[d] & C \arrow[l, "\cong"'] \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\beta"] & D \arrow[d] \\ X \arrow[r, "f'"'] & Y \arrow[r, equal] & Y \arrow[r, "g'"'] & Z \end{tikzcd}$$

with α and β vertically invertible squares. However, since all squares in $\mathbb{D}$ are trivial, we conclude that $B = Y = C$, a contradiction. Hence, the canonical functor (14) is not essentially surjective and is not an equivalence.

We conclude that it is impossible to transfer the categorical Reedy model structure on $\mathbf{Cat}^{\Delta^{op}}$ to $\mathbf{DblCat}$. $\square$

8. Model Structures Arising from Grothendieck Topologies

Until now we have considered model structures transferred from $\mathbf{Cat}^{\Delta^{op}}$. But one can also view double categories as internal categories, and for these homotopy theory has already been developed. Model structures on internal categories in a category $\mathbf{C}$ satisfying certain hypotheses have been studied by Everaert, Kieboom, and Van der Linden in [33]. As they point out, there are various notions of internal equivalence of internal categories. The notions full and faithful representably make sense for internal functors as in 8.7, but notions of essential surjectivity depend on a class of morphisms $\mathcal{E}$ in $\mathbf{C}$. If this class of morphisms is the class $\mathcal{E}_\mathcal{T}$ of $\mathcal{T}$-epimorphisms for a Grothendieck topology $\mathcal{T}$ on $\mathbf{C}$, then the internal equivalences are the weak equivalences for a model structure on $\mathsf{Cat}(\mathbf{C})$ in good cases. The classes $\mathsf{fib}(\mathcal{T}), \mathsf{cof}(\mathcal{T})$, and $\mathsf{we}(\mathcal{T})$ are defined in [33] so that the following theorem holds.

Theorem 8.1 (5.5 of [33])**.** *Let $\mathbf{C}$ be a finitely complete category such that $\mathsf{Cat}(\mathbf{C})$ is finitely complete and finitely cocomplete and $\mathcal{T}$ is a Grothendieck topology on $\mathbf{C}$. If the class $\mathsf{we}(\mathcal{T})$ of $\mathcal{T}$-equivalences has the 2-out-of-3 property and $\mathbf{C}$ has enough $\mathcal{E}_\mathcal{T}$-projectives, then*

$$(\mathsf{Cat}(\mathbf{C}), \mathsf{fib}(\mathcal{T}), \mathsf{cof}(\mathcal{T}), \mathsf{we}(\mathcal{T}))$$

is a model category.

We apply this theorem to $\mathbf{C} = \mathbf{Cat}$ for various Grothendieck topologies in this section. In Section 8.2 we show that the model structure associated to the simplicially surjective topology τ is the same as the transferred diagram categorical structure. This second construction using [33] is advantageous, as it gives us more information about the model structures, such as simple descriptions of cofibrations and cofibrant replacements. We will show in Section 9 that the model structure associated to the categorically surjective basis τ' in Section 8.3 turns out to be the same as the model structure on **DblCat** viewed as a category of algebras over a 2-monad.

8.1. **Homotopy Theory of Internal Categories as in** [33]**.** First we recall the notions and results of [33] for the special case of internal categories in $\mathbf{C} = \mathbf{Cat}$.

Definition 8.2. Let $\mathsf{iso}\colon \mathsf{Cat}(\mathbf{Cat}) \longrightarrow \mathsf{Grpd}(\mathbf{Cat})$ be the right adjoint to the inclusion $\mathsf{Grpd}(\mathbf{Cat}) \longrightarrow \mathsf{Cat}(\mathbf{Cat})$. For $\mathbb{B} \in \mathsf{Cat}(\mathbf{Cat})$, this means that $\mathsf{iso}(\mathbb{B})_1$ has objects the invertible horizontal morphisms of $\mathbb{B}$ and morphisms the horizontally invertible squares. This is a category under vertical composition of squares.

Definition 8.3. If $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a double functor, then the *mapping path object* is the category $(\mathbb{P}_F)_0$ defined as the pullback below,

$$\begin{tikzcd}
(\mathbb{P}_F)_0 \arrow[r, "\overline{F}_0"] \arrow[d, "\overline{\delta}_1"'] & \mathsf{iso}(\mathbb{B})_1 \arrow[d, "\delta_1"] \\
\mathbb{A}_0 \arrow[r, "F_0"'] & \mathbb{B}_0 .
\end{tikzcd}$$

The objects of $(\mathbb{P}_F)_0$ are $(a, f\colon b \xrightarrow{\cong} F_0 a)$ for a an object of $\mathbb{A}$ and f a horizontal isomorphism of $\mathbb{B}$. The morphisms are pairs

$$\left(\begin{tikzcd} a \arrow[d, "k"'] \\ a' \end{tikzcd} , \begin{tikzcd} b \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & F_0 a \arrow[d, "F_0 k"] \\ b' \arrow[r, "\cong"'] & F_0 a' \end{tikzcd} \right) \tag{19}$$

where k is a vertical morphism in $\mathbb{A}$ and α is a horizontally invertible square in $\mathbb{B}$. Composition in $(\mathbb{P}_F)_0$ comes from the vertical composition in $\mathbb{A}$ and $\mathbb{B}$. The functor $\delta_0\colon \mathsf{iso}(\mathbb{B})_1 \longrightarrow \mathbb{B}_0$ is the source for horizontal composition.

Definition 8.4. Let $\mathcal{T}$ be a topology on **Cat**. We denote the composition of the Yoneda embedding with the sheafification functor

$$\mathbf{Cat} \xrightarrow{Y} \mathbf{Set}^{\mathbf{Cat}^{op}} \longrightarrow Sh(\mathbf{Cat}, \mathcal{T})$$

by $Y_{\mathcal{T}}$. A functor $p\colon \mathbf{E} \longrightarrow \mathbf{B}$ is $\mathcal{T}$*-epi* if $Y_{\mathcal{T}}(p)$ is epi. We denote the class of $\mathcal{T}$-epimorphisms by $\mathcal{E}_{\mathcal{T}}$.

To show that a functor is $\mathcal{T}$-epi, we will use the following characterization of $\mathcal{T}$-epimorphisms.

Proposition 8.5 (Corollary III.7.5 and III.7.6 in [61], Proposition 2.12 in [33])**.** *Let $\mathcal{T}$ be a topology on a small category. A morphism $p\colon E \longrightarrow B$ is $\mathcal{T}$-epi if and only if for every morphism $g\colon X \longrightarrow B$ there exists a covering sieve $\{ f_i\colon U_i \longrightarrow X \}_i$ and a family of morphisms $\{ u_i\colon U_i \longrightarrow E \}_i$ such that for every $i \in I$ the diagram*

$$\begin{tikzcd} U_i \arrow[r, "f_i"] \arrow[d, "u_i"'] & X \arrow[d, "g"] \\ E \arrow[r, "p"'] & B \end{tikzcd} \tag{20}$$

commutes.

Remark 8.6. Suppose K is a basis for the topology $\mathcal{T}$ in Proposition 8.5 and such g and p are given. Then there exists a covering *sieve* $\{ f_i\colon U_i \longrightarrow X \}_i$ in $\mathcal{T}$ and a family of morphisms $\{ u_i\colon U_i \longrightarrow E \}_i$ making (20) commute if and only if there exists a covering *family* $\{ g_j\colon V_j \longrightarrow X \}_j$ in K and a family of morphisms $\{ v_j\colon V_j \longrightarrow E \}_j$ making (20) commute. Thus, in Proposition 8.5 one could equivalently replace the phrase "covering sieve" by the phrase "covering family in a given basis".

Proof: A sieve S is a covering sieve in the topology $\mathcal{T}$ generated by the basis K if and only if it contains a covering family R from the basis K. Suppose such a covering sieve $\{f_i\}_i$ with morphisms $\{u_i\}_i$ is given. Then this covering sieve contains a covering family in L for which (20) commutes. Conversely, given such a covering family $\{g_j\}_j$ with morphisms $\{v_j\}_j$, we may take the sieve

$$\{g_j \circ w | w \text{ a morphism such that } g_j \circ w \text{ exists}\}_j$$

generated by the family $\{v_j\}_j$. Then the family $\{v_j \circ w\}$ makes (20) commute. $\square$

Definition 8.7. Let $\mathcal{E}_\mathcal{T}$ be the class of $\mathcal{T}$-epimorphisms for a Grothendieck topology on **Cat** and let $\delta_0\colon \mathsf{iso}(\mathbb{B})_1 \longrightarrow \mathbb{B}_0$ be the source map for horizontal composition. A double functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is *essentially $\mathcal{T}$-surjective* if the functor

$$(\mathbb{P}_F)_0 \xrightarrow{\overline{F}_0} \mathsf{iso}(\mathbb{B})_1 \xrightarrow{\delta_0} \mathbb{B}_0$$

given by

$$\delta_0 \circ \overline{F}_0(a, b \xrightarrow{\cong} F_0 a) = b,$$

and

$$\delta_0 \circ \overline{F}_0 \left(\begin{tikzcd} a \arrow[d, "k"'] \\ a' \end{tikzcd} , \begin{tikzcd} b \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & F_0 a \arrow[d, "F_0 k"] \\ b' \arrow[r, "\cong"'] & F_0 a' \end{tikzcd} \right) = \begin{tikzcd} b \arrow[d, "j"'] \\ b' \end{tikzcd}$$

is in $\mathcal{E}_\mathcal{T}$. If F is additionally *fully faithful* in the sense of [20], *i.e.,* if

$$\begin{tikzcd} \mathbb{A}_1 \arrow[r, "F_1"] \arrow[d, "(s,t)"'] & \mathbb{B}_1 \arrow[d, "(s,t)"] \\ \mathbb{A}_0 \times \mathbb{A}_0 \arrow[r, "F_0 \times F_0"'] & \mathbb{B}_0 \times \mathbb{B}_0 \end{tikzcd}$$

is a pullback square in **Cat**, then F is called a *$\mathcal{T}$-equivalence.* We denote the class of $\mathcal{T}$-equivalences by $\mathsf{we}(\mathcal{T})$. Note that a double functor F is fully faithful if and only if the restricted functors

$$(\mathit{Obj}\,\mathbb{A}, \mathit{Hor}\,\mathbb{A}) \longrightarrow (\mathit{Obj}\,\mathbb{B}, \mathit{Hor}\,\mathbb{B})$$
$$(\mathit{Ver}\,\mathbb{A}, \mathit{Sq}\,\mathbb{A}) \longrightarrow (\mathit{Ver}\,\mathbb{B}, \mathit{Sq}\,\mathbb{B})$$

are both fully faithful.

Remark 8.8. If **A** and **B** are 1-categories, then a functor $F\colon \mathbf{A} \longrightarrow \mathbf{B}$ is essentially surjective in the usual sense if and only if $\delta_0 \circ \overline{F}_0$ is surjective. The functor F is fully faithful in the sense of 8.7 if and only if it is fully faithful in the usual sense. The notions of essential surjectivity and fully faithfulness can be found in any standard reference on category theory, such as Pages 19 and 115 of [9] or Pages 14, 15, and 93 of [60].

Remark 8.9. We conclude from Proposition 8.5 that if $\mathcal{T}' \subseteq \mathcal{T}$ are Grothendieck topologies, then every $\mathcal{T}'$-epimorphism is also a $\mathcal{T}$-epimorphism. Thus, every $\mathcal{T}'$-equivalence is a $\mathcal{T}$-equivalence. Thus, finer topologies give rise to model categories with more weak equivalences. [3]

[3]We thank Joachim Kock for posing this question.

Definition 8.10. A category $\mathbf{P}$ is *$\mathcal{E}_{\mathcal{T}}$-projective*, or simply *$\mathcal{T}$-projective*, if for every $\mathcal{T}$-epi functor $G\colon \mathbf{Q} \twoheadrightarrow \mathbf{R}$ and every functor $H\colon \mathbf{P} \longrightarrow \mathbf{R}$ there exists a factorization $F\colon \mathbf{P} \longrightarrow \mathbf{Q}$ such that $GF = H$,

$$\begin{tikzcd} & \mathbf{P} \arrow[dl, dashed, "\exists F"'] \arrow[d, "H"] \\ \mathbf{Q} \arrow[r, two heads, "G"'] & \mathbf{R}\,. \end{tikzcd}$$

Definition 8.11. We say that **Cat** *has enough $\mathcal{E}_{\mathcal{T}}$-projectives* if for every category $\mathbf{C}$ there exists an $\mathcal{E}_{\mathcal{T}}$-projective category $\mathbf{P}$ and a $\mathcal{T}$-epi functor $\mathbf{P} \twoheadrightarrow \mathbf{C}$.

Definition 8.12. A double functor $F\colon \mathbb{E} \longrightarrow \mathbb{B}$ is a *$\mathcal{T}$-fibration* if the induced morphism $(r_F)_0$ in the diagram below is $\mathcal{T}$-epi,

(21)
$$\begin{tikzcd} \mathsf{iso}(\mathbb{E})_1 \arrow[dr, dashed, "(r_F)_0"] \arrow[drr, bend left, "\mathsf{iso}\,(F)_1"] \arrow[ddr, bend right, "\delta_1"'] & & \\ & (\mathbb{P}_F)_0 \arrow[r, "\overline{F}_0"] \arrow[d, "\overline{\delta}_1"'] & \mathsf{iso}(\mathbb{B})_1 \arrow[d, "\delta_1"] \\ & \mathbb{E}_0 \arrow[r, "F_0"'] & \mathbb{B}_0\,. \end{tikzcd}$$

Remark 8.13. If $\mathbf{E}$ and $\mathbf{B}$ are 1-categories, then $(r_F)_0$ is surjective if and only if F is an isofibration. Recall from Section 7.1 that a functor $F\colon \mathbf{E} \longrightarrow \mathbf{B}$ is said to be an *isofibration* if for any object e of $\mathbf{E}$ and any isomorphism $b \cong Fe$ in $\mathbf{B}$, there exists a lift to an isomorphism $b' \cong e$ in $\mathbf{E}$.

Proposition 8.14 (Proposition 5.6 of [33]). *Under the assumptions of Theorem 8.1, a double functor $F\colon \mathbb{E} \longrightarrow \mathbb{B}$ is an acyclic $\mathcal{T}$-fibration if and only if it is fully faithful and F_0 is a $\mathcal{T}$-epi functor.*

Definition 8.15. A double functor is a *$\mathcal{T}$-cofibration* if it has the left lifting property with respect to all acyclic $\mathcal{T}$-fibrations.

Proposition 8.16 (Proposition 5.9 of [33]). *Under the assumptions of Theorem 8.1, a double functor $J\colon \mathbb{A} \longrightarrow \mathbb{X}$ is a $\mathcal{T}$-cofibration if and only if J_0 has the left lifting property with respect to all $\mathcal{T}$-epi functors.*

Corollary 8.17. *A double category $\mathbb{X}$ is cofibrant in the $\mathcal{T}$-model structure if and only if $\mathbb{X}_0$ is $\mathcal{E}_{\mathcal{T}}$-projective.*

Proof: By Proposition 8.16, $\mathbb{X}$ is cofibrant if and only if for any $\mathcal{T}$-epi functor G and any functor H, a lift ℓ

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathbf{Q} \arrow[d, two heads, "G"] \\ \mathbb{X}_0 \arrow[r, "H"'] \arrow[ur, dashed, "\ell"] & \mathbf{R} \end{tikzcd}$$

exists, or equivalently, if $\mathbb{X}_0$ is $\mathcal{E}_{\mathcal{T}}$-projective. □

8.2. **Model Structure from the Simplicially Surjective Basis.** For a category $\mathbf{C}$, we write $\mathbf{C}_k$ for the k-th set of the nerve $N\mathbf{C}$. Similarly for a functor F we write $(NF)_k = F_k$. We say that a functor F is *simplicially surjective* if F_k is surjective for all $k \geq 0$. We prove that the associated topology on **Cat** induces a model structure on **DblCat** which coincides with the transferred diagram categorical structure of Section 7.6. This second construction gives additional information about the transferred diagram categorical structure, including an explicit form for the cofibrant replacement functor.

Lemma 8.18. *For a category* $\mathbf{C}$ *define*

$$K(\mathbf{C}) := \{\{F\colon \mathbf{D} \longrightarrow \mathbf{C}\} \mid F \text{ a simplicially surjective functor }\}.$$

Then K is a basis for a Grothendieck topology τ on **Cat**.

Proof:

(i) If F is an isomorphism, then NF is an isomorphism and each F_k is bijective.
(ii) If $\{F\} \in K(\mathbf{C})$ and $G\colon \mathbf{C}' \longrightarrow \mathbf{C}$ is any functor, consider the pullback $\pi_2\colon \mathbf{D} \times_{\mathbf{C}} \mathbf{C}' \longrightarrow \mathbf{C}'$ in **Cat** of F along G. Since the nerve functor preserves limits, $N\pi_2$ is the pullback of NF along NG. Then $N\pi_2$ is simplicially surjective, since limits of simplicial sets are formed pointwise.
(iii) If $G \circ F$ exists and F_k and G_k are surjective for all $k \geq 0$, then clearly $G_k \circ F_k$ is surjective for all $k \geq 0$, and $\{G \circ F\}$ is a covering.

□

Lemma 8.19. *A functor $p\colon \mathbf{E} \longrightarrow \mathbf{B}$ is τ-epi for the Grothendieck topology τ if and only if p is simplicially surjective.*

Proof: If p is τ-epi, then take $g = 1_{\mathbf{B}}$ in Proposition 8.5 with Remark 8.6 to obtain $pu_i = f$ for some covering family $\{f\}$ in K. Then f_k is surjective for all $k \geq 0$. Hence p is simplicially surjective.

If p is simplicially surjective, then $\{p\}$ is a covering family in K, and so is the pullback π_2 of p along g. Applying Proposition 8.5 with Remark 8.6 again, we see that p is τ-epi. $\square$

Recall that the objects of the k-th category $((\mathbb{A}_0)_k, (\mathbb{A}_1)_k) = (N_v\mathbb{A})_k$ of the vertical nerve are composable strings of k vertical morphisms, and the morphisms are vertically composable strings of k squares. The composition is horizontal composition of vertical strings of squares. Fully faithful double functors and τ-equivalences have a useful characterization in terms of the vertical nerve.

Proposition 8.20. *A double functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is fully faithful if and only for every $k \geq 0$ the functor*

$$((F_0)_k, (F_1)_k)\colon\ ((\mathbb{A}_0)_k, (\mathbb{A}_1)_k) \longrightarrow ((\mathbb{B}_0)_k, (\mathbb{B}_1)_k)$$

is fully faithful.

Proof: Since the nerve functor preserves pullbacks, and pullbacks of simplicial sets are formed pointwise, it follows from Definition 8.7 that F is fully faithful if and only if each $((F_0)_k, (F_1)_k)$ is fully faithful. $\square$

Proposition 8.21. *A double functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a τ-equivalence if and only if for every $k \geq 0$ the functor*

$$((F_0)_k, (F_1)_k)\colon\ ((\mathbb{A}_0)_k, (\mathbb{A}_1)_k) \longrightarrow ((\mathbb{B}_0)_k, (\mathbb{B}_1)_k)$$

is an equivalence of categories.

Proof: The double functor F is essentially τ-surjective if and only if $\delta_0 \circ \overline{F}_0$ is τ-epi. But this occurs if and only if $(\delta_0 \circ \overline{F}_0)_k$ is surjective for each k, which is equivalent to the essential surjectivity of $((F_0)_k, (F_1)_k)$ by Remark 8.8. Fully faithfullness follows from Proposition 8.20. $\square$

Corollary 8.22. *The class $\mathsf{we}(\tau)$ of τ-equivalences has the 2-out-of-3 property.*

Proposition 8.23. Cat *has enough $\mathcal{E}_\tau$-projectives.*

Proof: We first construct an $\mathcal{E}_\tau$-projective category $\mathbf{P}$ from a category $\mathbf{C}$. Let

$$\mathbf{P} := \coprod_{n\geq 1} \mathbf{C}_n \cdot [n] = \coprod_{n\geq 1} \coprod_{(f_1,\dots,f_n)\in \mathbf{C}_n} [n]$$

where $[n] = \{0, 1, \dots, n\}$ is the $(n+1)$-element ordinal viewed as a category and $\mathbf{C}_n \cdot [n]$ denotes the copower of the category $[n]$ with the set $\mathbf{C}_n$, as recalled in Remark 6.7. Suppose we have functors

$$\begin{tikzcd} & \mathbf{P} \arrow[d, "H"] \\ \mathbf{Q} \arrow[r, "G"'] & \mathbf{R}, \end{tikzcd}$$

and G is τ-epi. We denote H on the $(f_1, \ldots, f_n)$-summand of $\mathbf{P}$ by

$$H_{(f_1,\ldots,f_n)} \colon [n] \longrightarrow \mathbf{R}.$$

If $H_{(f_1,\ldots,f_n)}(j-1 \leq j) = r_j$ for $1 \leq j \leq n$, then there exists $(q_1, \ldots, q_n) \in \mathbf{Q}_n$ such that $G_n(q_1, \ldots, q_n) = (r_1, \ldots, r_n)$ since G is τ-epi. We define a functor

$$F_{(f_1,\ldots,f_n)} \colon [n] \longrightarrow \mathbf{Q}$$
$$F_{(f_1,\ldots,f_n)}(j-1 \leq j) := q_j$$

for $1 \leq j \leq n$. Putting these together, we obtain a functor $F \colon \mathbf{P} \longrightarrow \mathbf{Q}$ such that $GF = H$, and we conclude that $\mathbf{P}$ is $\mathcal{E}_\tau$-projective.

Next we construct a τ-epi functor $L \colon \mathbf{P} \longrightarrow \mathbf{C}$. On the $(f_1, \ldots, f_n)$-summand of $\mathbf{P}$ define L as

$$L_{(f_1,\ldots,f_n)}(j-1 \leq j) := f_j$$

for $1 \leq j \leq n$. We claim that for each $k \geq 1$, $L_k \colon \mathbf{P}_k \longrightarrow \mathbf{C}_k$ is surjective. Note that

$$\mathbf{P}_k = \coprod_{n \geq 1} \coprod_{(f_1,\ldots,f_n) \in \mathbf{C}_n} [n]_k.$$

If $(f_1, \ldots, f_k) \in \mathbf{C}_k$, then L_k maps $(0 \leq 1, 1 \leq 2, \ldots, k-1 \leq k)$ in the $(f_1, \ldots, f_k)$-component of $\mathbf{P}_k$ to $(f_1, \ldots, f_k)$. From the surjectivity of L_1 it follows that L_0 is surjective: if $x \in \mathbf{C}_0$, then 1_x lies in the image of L_1. Hence L_k is surjective for all $k \geq 0$, L is τ-epi, and **Cat** has enough $\mathcal{E}_\tau$-projectives □

Theorem 8.24. *The simplicially surjective topology τ on* **Cat** *determines a model structure*

$$(\mathsf{Cat}(\mathbf{Cat}), \mathsf{fib}(\tau), \mathsf{cof}(\tau), \mathsf{we}(\tau)).$$

Proof: The category $\mathsf{Cat}(\mathbf{Cat})$ is complete and cocomplete by Theorem 4.1. The class of τ-equivalences has the 2-out-of-3 property by Corollary 8.22 and **Cat** has enough $\mathcal{E}_\tau$-projectives by Proposition 8.23, so we can apply Theorem 8.1. □

We now give a more explicit description of the fibrations, acyclic fibrations, cofibrant objects, and fibrant objects.

Proposition 8.25. *Let $F\colon \mathbb{E} \longrightarrow \mathbb{B}$ be a double functor.*

(i) *F is a τ-fibration if and only if for each $k \geq 0$ the functor*

$$((F_0)_k, (F_1)_k)\colon ((\mathbb{E}_0)_k, (\mathbb{E}_1)_k) \longrightarrow ((\mathbb{B}_0)_k, (\mathbb{B}_1)_k)$$

is an isofibration.

(ii) *F is an acyclic τ-fibration if and only if for each $k \geq 0$ the functor $((F_0)_k, (F_1)_k)$ is fully faithful and surjective on objects.*

Proof:

(i) Applying the nerve to Diagram (21), we see that F is a τ-fibration if and only if $(r_F)_{0k}$ is surjective for all $k \geq 0$. By Remark 8.13, this is the case if and only if for each $k \geq 0$ the functor $((F_0)_k, (F_1)_k)$ is an isofibration. Here $(\mathsf{iso}(\mathbb{B})_1)_k = \mathsf{iso}((\mathbb{B}_0)_k, (\mathbb{B}_1)_k)$ is the category with objects composable strings of k vertical morphisms and with morphisms vertical strings of vertically composable squares that are each horizontally invertible.

(ii) From the proof of Theorem 8.21, F is fully faithful if and only if each $((F_0)_k, (F_1)_k)$ is fully faithful. Since F_0 is τ-epi if and only if $(F_0)_k$ is surjective for each $k \geq 0$, the statement follows from Proposition 8.14.

□

Corollary 8.26. *The model structure on* **DblCat** *induced by the simplicially surjective topology τ on* **Cat** *coincides with the transferred diagram categorical structure in Section 7.6.*

Proof: From Propositions 8.21 and 8.25 we see that the weak equivalences and fibrations of the two model structures coincide. □

Remark 8.27. These results allow us to construct a cofibrant replacement $\mathbb{E}$ for a double category $\mathbb{B}$. Let $\mathbb{E}_0$ be the $\mathcal{E}_\tau$-projective category associated to $\mathbb{B}_0$ with projection $L_0 := L$ as in the proof of Proposition 8.23. Let $\mathbb{E}_1$ be the following pullback in **Cat**,

$$\begin{tikzcd}
\mathbb{E}_1 \arrow[r, "L_1"] \arrow[d, "(s,t)"'] & \mathbb{B}_1 \arrow[d, "(s,t)"] \\
\mathbb{E}_0 \times \mathbb{E}_0 \arrow[r, "L_0 \times L_0"'] & \mathbb{B}_0 \times \mathbb{B}_0 .
\end{tikzcd}$$

Then the double graph $\mathbb{E}$ carries a unique double category structure such that (L_0, L_1) is a double functor by Lemma 5.14 of [33]. Since L is fully faithful and L_0 is τ-epi, $L\colon \mathbb{E} \longrightarrow \mathbb{B}$ is an acyclic fibration by Proposition

8.14. By Corollary 8.17, $\mathbb{E}$ is a cofibrant double category, and hence a cofibrant replacement for $\mathbb{B}$.

Proposition 8.28. *Let F be a τ-equivalence. Then BF, as in Definition 5.8, is a weak homotopy equivalence of topological spaces.*

Proof: By Proposition 8.21, F is a τ-equivalence if and only if $(N_vF)_k$ is an equivalence of categories for each $k \geq 0$. Since $((N_vF)_k)_\ell = (N_dF)_{k\ell}$, we see that $(N_dF)_{k*}$ is a weak equivalence of simplicial sets for each $k \geq 0$. Hence $\mathrm{diag}(N_dF)$ is a weak equivalence of simplicial sets, and $BF = |\mathrm{diag}(N_dF)|$ is a weak homotopy equivalence. $\square$

Remark 8.29. In the τ model structure we have chosen one direction to take the nerve. We obtain a completely analogous model structure by choosing the other direction.

Remark 8.30. For each $m \in \mathbb{N}$, the assignment

$$\mathbf{C} \mapsto K_m(\mathbf{C}) := \{\{F\colon \mathbf{D} \longrightarrow \mathbf{C}\} | F_k \text{ surjective for all } 0 \leq k \leq m\}$$

is a basis for a Grothendieck topology τ_m on **Cat**. We obtain a τ_m-model structure as above, though τ_m-equivalences will not necessarily be weak homotopy equivalences of classifying spaces.

8.3. **Model Structure from the Categorically Surjective Basis.** A functor is said to be *categorically surjective* if it is surjective on objects and full. It is straightforward to check that a basis for a Grothendieck topology on **Cat** is given by declaring a covering family to be a single categorically surjective functor. We call this topology τ'. In this section we study the model structure on **DblCat** induced by τ'. In Section 9 we show that this model structure is the model structure on **DblCat** viewed as a category of algebras over a 2-monad.

As before we start with a characterization of the τ'-epi functors. We will use this to prove a 2-out-of-3 property for the τ'-equivalences.

Proposition 8.31. *A functor $p\colon \mathbf{E} \longrightarrow \mathbf{B}$ is τ'-epi if and only if there is a subcategory $\mathbf{H} \hookrightarrow \mathbf{E}$ such that $p|_{\mathbf{H}}\colon \mathbf{H} \longrightarrow \mathbf{B}$ is surjective on objects and full. Thus, a τ'-epi functor is not necessarily categorically surjective.*

Proof: Suppose that p is τ'-epi. Then by Proposition 8.5 and Remark 8.6 there is a commutative square

$$\begin{tikzcd} \mathbf{U} \arrow[r, "f"] \arrow[d, "u"'] & \mathbf{B} \arrow[d, "1_{\mathbf{B}}"] \\ \mathbf{E} \arrow[r, "p"'] & \mathbf{B}, \end{tikzcd}$$

where f is surjective on objects and full. For each pair of objects x, y in $\mathbf{U}$, we have a commutative triangle

$$\begin{tikzcd}
\mathbf{U}(x,y) \arrow[rr, "f(x,y)"] \arrow[dr, "u(x,y)"'] & & \mathbf{B}(fx, fy) \\
& \mathbf{E}(ux, uy) \arrow[ur, "p(ux,uy)"'] &
\end{tikzcd}.$$

Since $f(x, y)$ is surjective, so is $p(ux, uy)$. Let $\mathbf{H} = Im(u)$. Thus, $p|_{\mathbf{H}}$ is surjective on objects and full.

Conversely, let $\mathbf{H} \overset{\ell}{\hookrightarrow} \mathbf{E}$ be a subcategory such that $p|_{\mathbf{H}}$ is surjective on objects and full, and let $g\colon \mathbf{X} \longrightarrow \mathbf{B}$ be any functor. Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{H} \times_{\mathbf{B}} \mathbf{X} \arrow[r, "p'"] \arrow[d, "s"'] \arrow[dr, phantom, "\text{pb}"] & \mathbf{X} \arrow[dd, "g"] \\
\mathbf{H} \arrow[d, "\ell"'] \arrow[dr, "p|_{\mathbf{H}}"] & \\
\mathbf{E} \arrow[r, "p"'] & \mathbf{B}\,.
\end{tikzcd}$$

Then p' is surjective on objects and full since $p|_{\mathbf{H}}$ is. Further, $gp' = p\ell s$. By Proposition 8.5 and Remark 8.6, it follows that p is τ'-epi. $\square$

Even though the τ'-epi functors do not coincide with the categorically surjective functors, they do give rise to the same projective objects.

Corollary 8.32. *A category $\mathbf{P}$ is τ'-projective if and only if it is projective with respect to categorically surjective functors.*

Proof: We use the same notation as in Definition 8.10. If $\mathbf{P}$ is τ'-projective, then $\mathbf{P}$ is projective with respect to categorically surjective functors because every categorically surjective functor is τ'-projective by Proposition 8.31. For the converse, suppose $\mathbf{P}$ is projective with respect to categorically surjective functors, and suppose G is τ'-epi. Then by Proposition 8.31 again, there exists an inclusion $\ell\colon \mathbf{Q}' \longrightarrow \mathbf{Q}$ such that $G\ell$ is a categorically surjective functor. Thus there exists an F' such that $G\ell F' = H$. If we let $F = \ell F'$ then we see that $\mathbf{P}$ is surjective with respect to categorically surjective functors. $\square$

Proposition 8.33. *If a functor $p\colon \mathbf{A} \longrightarrow \mathbf{B}$ is τ'-epi, then for all $k \geq 0$, $p_k\colon \mathbf{A}_k \longrightarrow \mathbf{B}_k$ is surjective.*

Proof: By Proposition 8.5 and Remark 8.6 there exists a functor f surjective on objects and full such that $1_{\mathbf{B}} \circ f = p \circ u$ for some u. Since $f_k = p_k \circ u_k$ is surjective, so is p_k. $\square$

Proposition 8.34. *If a double functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a τ'-equivalence, then for every $k \geq 0$ the functor*

$$((F_0)_k, (F_1)_k)\colon ((\mathbb{A}_0)_k, (\mathbb{A}_1)_k) \longrightarrow ((\mathbb{B}_0)_k, (\mathbb{B}_1)_k)$$

is an equivalence of categories.

Proof: Since F is fully faithful, $((F_0)_k, (F_1)_k)$ is fully faithful for all $k \geq 0$ by Proposition 8.20. Since F is essentially τ'-surjective, $\delta_0 \circ \overline{F}_0$ is τ'-epi and hence $(\delta_0 \circ \overline{F}_0)_k = (\delta_0)_k \circ (\overline{F}_0)_k$ is surjective for all $k \geq 0$ by Proposition 8.33. Remark 8.8 then implies that $((F_0)_k, (F_1)_k)$ is essentially surjective for all $k \geq 0$. $\square$

Lemma 8.35. *Suppose $\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}$ are double functors and two of GF, G, or F are τ'-equivalences. Then the third double functor is fully faithful.*

Proof: By Proposition 8.34 the vertical nerves of the two τ'-equivalences are levelwise equivalences of categories. Hence the vertical nerve of the third double functor is also levelwise an equivalence of categories, and in particular levelwise fully faithful. By Proposition 8.20, this implies that the third functor is fully faithful. $\square$

Lemma 8.36. *Suppose $\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}$ are double functors and GF and F are τ'-equivalences. Then G is essentially τ'-surjective.*

Proof: We need to show that $\delta_0 \circ \overline{G}_0$ is τ'-epi. Let $\mathbf{H}_F \subseteq (\mathbb{P}_F)_0$ and $\mathbf{H}_{GF} \subseteq (\mathbb{P}_{GF})_0$ be subcategories such that $\delta_0 \circ \overline{F}_0|_{\mathbf{H}_F}$ and $\delta_0 \circ \overline{(GF)}_0|_{\mathbf{H}_{GF}}$ are surjective on objects and full. Define a full subcategory $\mathbf{H}_G$ of $(\mathbb{P}_G)_0 = \mathbb{B}_0 \times_{\mathbb{C}_0} \mathsf{iso}(\mathbb{C})_1$ by applying F_0 to the first coordinate of $\mathbf{H}_{GF}$ as follows. For any object $(a, c \overset{\cong}{\to} G_0F_0a)$ in $\mathbf{H}_{GF}$, we have an object $(F_0a, c \overset{\cong}{\to} G_0(F_0a))$ in $\mathbf{H}_G$. For any morphism

$$\left(\begin{tikzcd} a \arrow[d, "k"'] \\ a' \end{tikzcd}, \begin{tikzcd} c \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & G_0F_0a \arrow[d, "G_0F_0k"] \\ c' \arrow[r, "\cong"'] & G_0F_0a' \end{tikzcd}\right)$$

in $\mathbf{H}_{GF}$ we have a morphism

$$\left(\begin{array}{c} F_0a \\ {\scriptstyle F_0k}\Big\downarrow \\ F_0a' \end{array}, \begin{array}{ccc} c & \xrightarrow{\cong} & G_0(F_0a) \\ {\scriptstyle j}\Big\downarrow & \alpha & \Big\downarrow{\scriptstyle G_0(F_0k)} \\ c' & \xrightarrow[\cong]{} & G_0(F_0a') \end{array} \right)$$

in $\mathbf{H}_G$. Then we see as follows that $\delta_0 \circ \overline{G}_0|_{\mathbf{H}_G} \colon \mathbf{H}_G \to \mathbb{C}_0$ is surjective on objects and full. If $c \in \mathbb{C}_0$, there exists an object $(a, c \overset{\cong}{\to} G_0F_0a) \in \mathbf{H}_{GF}$, with $(F_0a, c \overset{\cong}{\to} G_0(F_0a)) \in \mathbf{H}_G$ and $\delta_0\overline{G}_0((F_0a, c \overset{\cong}{\to} G_0F_0a)) = c$. So $\delta_0 \circ \overline{G}_0|_{\mathbf{H}_G}$ is surjective on objects. If $c \overset{j}{\to} c'$ is a morphism in $\mathbb{C}_0$ and $(F_0a, c \overset{\cong}{\to} G_0(F_0a))$ and $(F_0a', c' \overset{\cong}{\to} G_0(F_0a'))$ are objects of $\mathbf{H}_G$, then there exists a morphism

$$\left(\begin{array}{c} a \\ {\scriptstyle k}\Big\downarrow \\ a' \end{array}, \begin{array}{ccc} c & \xrightarrow{\cong} & G_0F_0a \\ {\scriptstyle j}\Big\downarrow & \alpha & \Big\downarrow{\scriptstyle G_0F_0k} \\ c' & \xrightarrow[\cong]{} & G_0F_0a' \end{array} \right)$$

in $\mathbf{H}_{GF}$ which gives rise to a morphism

$$\left(\begin{array}{c} a \\ {\scriptstyle F_0k}\Big\downarrow \\ a' \end{array}, \begin{array}{ccc} c & \xrightarrow{\cong} & G_0(F_0a) \\ {\scriptstyle j}\Big\downarrow & \alpha & \Big\downarrow{\scriptstyle G_0(F_0k)} \\ c' & \xrightarrow[\cong]{} & G_0(F_0a') \end{array} \right)$$

in $\mathbf{H}_G$ that maps to j under $\delta_0 \circ \overline{G}_0|_{\mathbf{H}_G}$. We conclude that $\delta_0 \circ \overline{G}_0|_{\mathbf{H}_G}$ is surjective on objects and full and therefore $\delta_0 \circ \overline{G}_0$ is τ'-epi. This implies that G is essentially τ'-surjective. $\square$

Lemma 8.37. *Suppose* $\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}$ *are double functors and* GF *and* G *are* τ'*-equivalences. Then* F *is essentially* τ'*-surjective.*

Proof: We need to show that $\delta_0 \circ \overline{F}_0$ is τ'-epi. Let $\mathbf{H}_G \subseteq (\mathbb{P}_G)_0$ and $\mathbf{H}_{GH} \subseteq (\mathbb{P}_{GF})_0$ be subcategories such that $\delta_0 \circ \overline{G}_0|_{\mathbf{H}_G}$ and $\delta_0 \circ \overline{GF}_0|_{\mathbf{H}_{GF}}$ are surjective on objects and full. Define a full subcategory $\mathbf{H}_F$ of $(\mathbb{P}_F)_0 = \mathbb{A}_0 \times_{\mathbb{B}_0} \mathsf{iso}(\mathbb{B})_1$ with object set

$$Obj\, \mathbf{H}_F := \{(a, b \overset{\cong}{\to} F_0a) | (a, G_0(b \overset{\cong}{\to} F_0a)) \in Obj\, \mathbf{H}_{GF}\}.$$

Then we can see as follows that $\delta_0 \circ \overline{F}_0|_{\mathbf{H}_F} \colon \mathbf{H}_F \to \mathbb{B}_0$ is surjective on objects and full. If $b \in \mathbb{B}_0$, then $G_0 b \in \mathbb{C}_0$, and there is an object $(a, G_0 b \overset{\cong}{\to} G_0 F_0 a) \in \mathbf{H}_{GF}$, because $\delta_0 \circ (\overline{GF})_0|_{\mathbf{H}_{GF}}$ is surjective on objects. However, $((G_0)_0, (G_1)_0)$ is fully faithful, *i.e.,* G restricted to the objects of $\mathbb{B}$ and the horizontal morphisms of $\mathbb{B}$ is a fully faithful functor of categories. So there is a unique isomorphism $b \overset{\cong}{\to} F_0 a$ whose image under G is $G_0 b \overset{\cong}{\to} G_0 F_0 a$. Hence, $(a, b \overset{\cong}{\to} F_0 a) \in \mathbf{H}_F$ and this object maps to b under $\delta_0 \circ \overline{F}_0|_{\mathbf{H}_F}$. We conclude that $\delta_0 \circ \overline{F}_0|_{\mathbf{H}_F}$ is surjective on objects.

Moreover, if $b \overset{j}{\to} b'$ is a morphism in $\mathbb{B}_0$ and $(a, b \overset{\cong}{\to} F_0 a)$ and $(a', b' \overset{\cong}{\to} F_0 a')$ are objects of $\mathbf{H}_F$, then $G_0 j$ is a morphism of $\mathbb{C}_0$, and since $\delta_0 \circ (\overline{GF})_0|_{\mathbf{H}_{GF}}$ is full, there is a morphism of the form

$$\left(\begin{tikzcd} a \arrow[d, "k"'] \\ a' \end{tikzcd} , \begin{tikzcd} G_0 b \arrow[r, "G(\cong)"] \arrow[d, "G_0 j"'] \arrow[dr, phantom, "\alpha"] & G_0 F_0 a \arrow[d, "G_0 F_0 k"] \\ G_0 b' \arrow[r, "G(\cong)"'] & G_0 F_0 a' \end{tikzcd} \right)$$

in $\mathbf{H}_{GF}$. However, the functor $((G_0)_1, (G_1)_1)$ is fully faithful, so there is a unique square β, such that

$$G \left(\begin{tikzcd} b \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\beta"] & F_0 a \arrow[d, "F_0 k"] \\ b' \arrow[r, "\cong"'] & F_0 a' \end{tikzcd} \right) = \alpha.$$

Moreover, β is also horizontally invertible. Hence

$$\left(\begin{tikzcd} a \arrow[d, "k"'] \\ a' \end{tikzcd} , \begin{tikzcd} b \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\beta"] & F_0 a \arrow[d, "F_0 k"] \\ b' \arrow[r, "\cong"'] & F_0 a' \end{tikzcd} \right)$$

is a morphism in $\mathbf{H}_F$ which maps to j under $\delta_0 \circ \overline{F}_0$. We conclude that $\delta_0 \circ \overline{F}_0|_{\mathbf{H}_F}$ is surjective on objects and full, so $\delta_0 \circ \overline{F}_0$ is τ'-epi and F is essentially τ'-surjective. $\square$

Lemma 8.38. *Suppose that* $\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}$ *are double functors, and* F *and* G *are* τ'*-equivalences. Then* $G \circ F$ *is essentially* τ'*-surjective.*

Proof: We need to show that $\delta_0 \circ (\overline{GF})_0$ is τ'-epi. Let $\mathbf{H}_F \subseteq (\mathbb{P}_F)_0$ and $\mathbf{H}_G \subseteq (\mathbb{P}_G)_0$ be subcategories such that $\delta_0 \circ \overline{F}_0|_{\mathbf{H}_F}$ and $\delta_0 \circ \overline{G}_0|_{\mathbf{H}_G}$ are surjective on objects and full. Let $\mathbf{H}_{GF}$ be the full subcategory of $(\mathbb{P}_{GF})_0 = \mathbb{A}_0 \times_{\mathbb{C}_0} \mathsf{iso}(\mathbb{C})_1$, with objects

$$\begin{aligned} \mathit{Obj}\, \mathbf{H}_{GF} :=& \{(a, c \overset{\cong}{\to} G_0 b \overset{\cong}{\to} G_0 F_0 a) \mid \\ & \quad (b, c \overset{\cong}{\to} G_0 b) \in \mathbf{H}_G, (a, b \overset{\cong}{\to} F_0 a) \in \mathbf{H}_F\}. \end{aligned}$$

Suppose that $c \in \mathbb{C}_0$, then there are objects $(b, c \overset{\cong}{\to} G_0 b) \in \mathbf{H}_G$ and $(a, b \overset{\cong}{\to} F_0 a) \in \mathbf{H}_F$, because $\delta_0 \circ \overline{G}_0|_{\mathbf{H}_G}$ and $\delta_0 \circ \overline{F}_0|_{\mathbf{H}_F}$ are surjective on objects. Thus $(a, c \overset{\cong}{\to} G_0 b \overset{\cong}{\to} G_0 F_0 a) \in \mathbf{H}_{GF}$, and this object maps to c under $\delta_0 \circ \overline{GF}_0|_{\mathbf{H}_{GF}}$.

Next, suppose that $c \overset{j}{\to} c'$ is a morphism of $\mathbb{C}_0$ and $(a, c \overset{\cong}{\to} G_0 b \overset{\cong}{\to} G_0 F_0 a)$ and $(a', c' \overset{\cong}{\to} G_0 b' \overset{\cong}{\to} G_0 F_0 a')$ are objects of $\mathbf{H}_{GF}$. Then there exist morphisms

$$\left(\begin{tikzcd} b \arrow[d, "k_b"'] & , & c \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & G_0 b \arrow[d, "G_0 kb"] \\ b' & & c' \arrow[r, "\cong"'] & G_0 b' \end{tikzcd} \right)$$

in $\mathbf{H}_G$, and

$$\left(\begin{tikzcd} a \arrow[d, "k_a"'] & , & b \arrow[r, "\cong"] \arrow[d, "k_b"'] \arrow[dr, phantom, "\beta"] & F_0 a \arrow[d, "F_0 k_a"] \\ a' & & b' \arrow[r, "\cong"'] & F_0 a' \end{tikzcd} \right)$$

in $\mathbf{H}_F$, and therefore

$$\left(\begin{tikzcd} a \arrow[d, "k_a"'] & , & c \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & G_0 b \arrow[r, "\cong"] \arrow[d, "G_0 k_b"] \arrow[dr, phantom, "G\beta"] & G_0 F_0 a \arrow[d, "G_0 F_0 k_a"] \\ a' & & c' \arrow[r, "\cong"'] & G_0 b' \arrow[r, "\cong"'] & G_0 F_0 a' \end{tikzcd} \right)$$

is a morphism of $\mathbf{H}_{GF}$ that maps to $c \overset{j}{\to} c'$ under $\delta_0 \circ (\overline{GF})_0|_{\mathbf{H}_{GF}}$. So we have proved that $\delta_0 \circ (\overline{GF})_0|_{\mathbf{H}_{GF}}$ is surjective on objects and full. We conclude that GF is τ'-epi and essentially τ'-surjective. $\square$

The previous four lemmas are summarized in the following theorem.

Theorem 8.39. *The class* $\mathsf{we}(\tau')$ *of* τ'*-equivalences has the 2-out-of-3 property.*

Lemma 8.40. *Let* $\mathcal{E}_{\tau'}$ *denote the class of* τ'*-epis in* **Cat**. *Then* **Cat** *has enough* $\mathcal{E}_{\tau'}$*-projectives.*

Proof: For a category $\mathbf{C}$, we will denote the free category on its underlying graph by $\mathbf{P_C}$. The functor $\mathbf{P_C} \to \mathbf{C}$, which is the identity on objects and defined by composition on paths of morphisms, is surjective on objects and full, so it is τ'-epi.

Suppose $G\colon \mathbf{Q} \twoheadrightarrow \mathbf{R}$ is a τ'-epi and $H\colon \mathbf{P_C} \longrightarrow \mathbf{R}$ is a functor. Let $\mathbf{Q}' \subseteq \mathbf{Q}$ be a subcategory such that $G|_{\mathbf{Q}'}\colon \mathbf{Q}' \twoheadrightarrow \mathbf{R}$ is surjective on objects and full. Let $U\colon \mathbf{Cat} \to \mathbf{Graph}$ denote the forgetful functor. Then there is a map of graphs which makes the following diagram commute,

$$\begin{tikzcd} & U\mathbf{C} \arrow[dl, dashed] \arrow[d, "U(H|_{\mathbf{C}})"] \\ U\mathbf{Q}' \arrow[r, twoheadrightarrow, "U(G|_{\mathbf{Q}'})"'] & U\mathbf{R} \end{tikzcd}$$

and induces a functor F such that

$$\begin{tikzcd} & \mathbf{P_C} \arrow[dl, dashed, "F"'] \arrow[d, "H"] \\ \mathbf{Q}' \arrow[r, "G|_{\mathbf{Q}'}"'] & \mathbf{R} \end{tikzcd}$$

commutes. Hence the diagram

$$\begin{tikzcd} & & \mathbf{P_C} \arrow[dl] \arrow[dd, "H"] \\ & \mathbf{Q}' \arrow[dl, hookrightarrow] & \\ \mathbf{Q} \arrow[rr, twoheadrightarrow, "G"'] & & \mathbf{R} \end{tikzcd}$$

commutes, and **Cat** has enough $\mathcal{E}_{\tau'}$-projectives. $\square$

Theorem 8.41. *The categorically surjective topology* τ' *determines a model structure*

$$(\mathsf{Cat}(\mathbf{Cat}), \mathsf{fib}(\tau'), \mathsf{cof}(\tau'), \mathsf{we}(\tau')).$$

Proof: The category $\mathsf{Cat}(\mathbf{Cat})$ is complete and cocomplete by Lemmas 4.1 and 4.1. The class of τ'-equivalences has the 2-out-of-3 property by Corollary 8.39 and **Cat** has enough $\mathcal{E}_{\tau'}$-projectives by Proposition 8.40, so we can apply Theorem 8.1. $\square$

Proposition 8.42. *A double category $\mathbb{X}$ is cofibrant in the τ'-model structure if and only if $\mathbb{X}_0$ is projective with respect to functors that are surjective on objects and full.*

Proof: By Corollary 8.17 a double category is cofibrant if and only if $\mathbb{X}_0$ is projective with respect to τ'-epi functors. But by Corollary 8.32 $\mathbb{X}_0$ is projective with respect to τ'-epis if and only if it is projective with respect to functors that are surjective on objects and full. $\square$

As an immediate consequence of Propositions 8.21 and 8.34, we see that every τ'-equivalence is a τ-equivalence. This also follows from Remark 8.9, since the categorically surjective τ'-topology is contained in the simplicially surjective τ-topology. An interesting question is whether or not a condition slightly stronger than simplicial surjectivity but also slightly weaker than categorical surjectivity would give rise to a model structure with weak equivalences between those of the τ'-structure and the τ-structure. For example, such a condition on a functor is to be *U-split*. However, this condition recovers the τ'-topology instead of something new. In fact, this condition only gives a different basis for the τ'-topology which will be of use in Section 9.

Definition 8.43. Let $U\colon \mathbf{Cat} \longrightarrow \mathbf{Graph}$ be the forgetful functor from categories to directed graphs. We say that a functor p is *U-split* if there exists a morphism q of directed graphs such that $(Up) \circ q = id$.

Lemma 8.44. *A functor $p\colon \mathbf{E} \longrightarrow \mathbf{B}$ is U-split if and only if there is a subcategory $\mathbf{H} \hookrightarrow \mathbf{E}$ such that $p|_{\mathbf{H}}\colon \mathbf{H} \longrightarrow \mathbf{B}$ is surjective on objects and full.*

Proof: Suppose p is U-split. Then there exists a morphism of directed graphs q such that $Up \circ q = id$. Let $\mathbf{H}$ be the full subcategory $\mathbf{E}$ whose objects are in the image of q. Then $p|_{\mathbf{H}}$ is surjective on objects and full, as one sees using the directed graph section q.

Conversely, suppose there exists a subcategory $\mathbf{H}$ of $\mathbf{E}$ such that $p|_{\mathbf{H}}$ is categorically surjective. Then $p|_{\mathbf{H}}$ is U-split, and $id = U(p|\mathbf{H}) \circ q = Up \circ q$ so that p is also U-split. $\square$

Proposition 8.45. *The assignment*

$$\mathbf{C} \mapsto L(\mathbf{C}) := \{\{F\colon \mathbf{D} \longrightarrow \mathbf{C}\} |\ F \text{ is } U\text{-split}\ \}$$

is a basis for the τ'-topology on $\mathbf{Cat}$.

Proof: We omit the proof that this is a basis.

Recall that a sieve is a covering sieve in the topology induced by a basis if and only if it contains a covering family from the basis. If S is a τ'-covering sieve, it contains a categorically surjective functor, and hence a

U-split functor by Lemma 8.44, so that S is also a covering sieve in the topology induced by L.

Conversely, suppose S is a sieve on $\mathbf{B} \in \mathbf{Cat}$ that contains a U-split functor $p\colon \mathbf{E} \longrightarrow \mathbf{B}$ and $p|_{\mathbf{H}}$ is categorically surjective. Then $p \circ i \in S$ for the inclusion $i\colon \mathbf{H} \longrightarrow \mathbf{E}$, and S is a sieve in the τ'-topology. □

8.4. **Model Structure from the Trivial Topology.** On any 2-category $\mathcal{K}$ with finite limits and finite colimits there is the **Cat**-enriched *trivial model structure* as proved in [54] using pseudo limits. A weak equivalence (fibration) in this model structure is a morphism $f\colon A \longrightarrow B$ such that $\mathcal{K}(E, f)\colon \mathcal{K}(E, A) \longrightarrow \mathcal{K}(E, B)$ is a weak equivalence (fibration) for all E in the categorical model structure on **Cat**. Thus f is a weak equivalence if and only if there is a morphism $g\colon B \longrightarrow A$ such that gf and fg are isomorphic via 2-cells to the respective identities. A morphism f is a fibration, or *isofibration*, if for all morphisms $a\colon X \longrightarrow A$ and $b\colon X \longrightarrow B$ and any invertible 2-cell $\beta\colon b \cong fa$, there exists a morphism $a'\colon X \longrightarrow A$ and an invertible 2-cell $\alpha\colon a' \cong a$ with $fa' = b$ and $f\alpha = \beta$. If the 2-category $\mathcal{K}$ is merely a 1-category, then the trivial model structure agrees with the usual trivial model structure: weak equivalences are isomorphisms and all morphisms are fibrations and cofibrations.

Thus **DblCat** admits three trivial model structures, depending on whether we take as 2-cells the horizontal natural transformations, the vertical natural transformations, or only trivial 2-cells. When we say *trivial model structure on* **DblCat** we mean the one arising from the 2-category with horizontal natural transformations as 2-cells.

Remark 8.46 ([33])**.** If τ_{tr} is the trivial topology[4] on a finitely complete and finitely cocomplete category $\mathbf{C}$ with enough $\mathcal{E}_{\tau_{tr}}$-projectives and if the τ_{tr}-equivalences have the 2-out-of-3 property, then the τ_{tr}-model structure is the trivial model structure on the 2-category $\mathsf{Cat}(\mathbf{C})$. Note that Proposition 8.5 implies that a morphism p in $\mathbf{C}$ is τ_{tr}-epi if and and only if there exists a morphism q such that $pq = \mathrm{id}$.

Proof: A morphism f is essentially τ_{tr}-surjective (fully faithful) if and only if $\mathsf{Cat}(\mathbf{C})(E, f)\colon \mathsf{Cat}(\mathbf{C})(E, A) \longrightarrow \mathsf{Cat}(\mathbf{C})(E, B)$ is essentially surjective (fully faithful) for all E. Hence the weak equivalences in the τ_{tr}-model structure on $\mathsf{Cat}(\mathbf{C})$ are the weak equivalences in the trivial model structure.

Fibrations are seen to be isofibrations using the diagram of Definition 8.12 adapted to the general case. □

[4] In the trivial topology the only covering sieve on an object is the maximal sieve.

The trivial model structure on a category of internal categories is much like the Strøm structure, as studied in [33].

After the discussion of model structures on **DblCat** as a category of internal categories in Section 8, we now turn to a model structure on **DblCat** as a category of algebras and show that this model structure is the same as the categorically surjective model structure. We will make use of the trivial model structure.

9. A Model Structure for **DblCat** as the 2-Category of Algebras for a 2-Monad

Every 2-category of algebras over a 2-monad T with rank (*i.e.,* which preserves α-filtered colimits for some α) on a locally finitely presentable 2-category $\mathcal{K}$ admits a canonical cofibrantly generated **Cat**-enriched model structure as in [54]. It is obtained by transferring from the trivial model structure on the 2-category $\mathcal{K}$ described in Section 8.4. A strict morphism of strict T-algebras is a weak equivalence (fibration) if and only if its underlying morphism is an equivalence (isofibration). We prove that the model structure induced by the categorically surjective topology τ' can be recovered in this way. The interest in having these two different descriptions lies in the fact that they allow a characterization of the flexible double categories (Corollary 9.4) that cannot be obtained using only the description of [54]. Further, **DblCat** provides a good setting for comparing the categorical model structure on **2-Cat** in [55] and [56] to a model structure induced by a 2-monad.

Recall that the adjunction $F : \mathbf{Graph} \dashv \mathbf{Cat} : U$ induces a 2-adjunction

$$\begin{tikzcd} \mathsf{Cat}(\mathbf{Graph}) \arrow[rr, bend left, "\overline{F}"] \arrow[rr, phantom, "\perp"] & & \mathsf{Cat}(\mathbf{Cat}) \arrow[ll, bend left, "\overline{U}"] \end{tikzcd}$$

which is 2-monadic. The category of algebras for $\overline{UF}$ is **DblCat**. An internal category in **Graph** is a double graph with a category structure on $(\mathit{Obj}, \mathit{Hor})$ and on $(\mathit{Ver}, \mathit{Sq})$, in other words horizontal compositions are defined but vertical compositions are not.

Theorem 9.1. *The model structure induced by the 2-monad $\overline{UF}$ is the τ'-model structure.*

Proof: First we prove that the weak equivalences are the same. Note that a double functor G is fully faithful if and only if $\overline{U}G$ is fully faithful as in Definition 8.7. A double functor G is a weak equivalence as a morphism of algebras if and only if $\overline{U}G$ is a weak equivalence in the trivial model structure on $\mathsf{Cat}(\mathbf{Graph})$, which is the case if and only if $\overline{U}G$ is fully faithful and there exists a morphism q of directed graphs such that $U(\delta_0 \circ \overline{G}_0) \circ q = \mathrm{id}_{\mathbb{B}_0}$ by Remark 8.46. That is equivalent to G being fully faithful and $\delta_0 \circ \overline{G}_0$ being U-split, which is precisely the definition of weak equivalence in the τ'-model structure using Proposition 8.31 and Lemma 8.44. Hence the weak equivalences coincide.

Similarly, a double functor G is a fibration as a morphism of algebras if and only if $\overline{U}G$ is a fibration in the trivial model structure on $\mathsf{Cat}(\mathbf{Graph})$, which is the case if and only if there exists a morphism q of direct graphs such that $(U(r_G)_0) \circ q = id$ in Diagram (21), which is the case if and only if $(r_G)_0$ is U-split. This is equivalent to G being a fibration in the τ'-model structure. Hence the fibrations coincide. $\square$

Corollary 9.2. *The categorically surjective τ'-model structure is* **Cat**-*enriched and cofibrantly generated.*

For a 2-monad T on $\mathcal{K}$ as above, let $T\text{-Alg}_s$ denote the 2-category of strict T-algebras, strict morphisms, and 2-cells. We denote, as usual, by T-Alg the 2-category of strict T-algebras, pseudo morphisms, and 2-cells. As shown in [7], the inclusion $T\text{-Alg}_s \longrightarrow T\text{-Alg}$ admits a left 2-adjoint denoted $A \mapsto A'$. The counit $q\colon A' \longrightarrow A$ is a strict morphism, and if q admits a section in $T\text{-Alg}_s$, then A is called *flexible*. The flexible algebras are the closure under flexible colimits of the free algebras. Pseudo morphisms from A to B are in bijective correspondence with strict morphisms from A' to B.

Theorem 9.3 (Theorem 4.12 in [54])**.** *The cofibrant objects of T-Alg$_s$ are precisely the flexible algebras; in particular, any algebra of the form A' is cofibrant, and is thus a cofibrant replacement for A. Every free algebra is flexible.*

Corollary 9.4. *The cofibrant objects in the τ'-model structure are precisely the flexible double categories. In particular, a double category $\mathbb{X}$ is flexible if and only if $\mathbb{X}_0$ is projective with respect to functors that are surjective on objects and full.*

Proof: This follows from Proposition 8.42 and Theorem 9.3. $\square$

The categorical model structure on **2-Cat** of [55] and [56] has weak equivalences the strict 2-functors that are biequivalences, fibrations the equivfibrations, and cofibrations those 2-functors whose underlying functor has the left lifting property with respect to functors that are surjective on objects and full. We can compare this with the τ'-model structure as follows.

Proposition 9.5. *Consider* **2-Cat** *vertically embedded in* **DblCat**. *If a 2-functor is a cofibration in* **DblCat**, *then it is a cofibration in* **2-Cat**. *A 2-category is cofibrant in* **2-Cat** *if and only if it is cofibrant in* **DblCat**. *Thus a 2-category is flexible as in* [55] *if and only if it is flexible as an algebra over the 2-monad* $\overline{UF}$.

Proof: If the underlying functor of a 2-functor has the left lifting property with respect to all τ'-epis, then it has the left lifting property with respect to all functors that are surjective on objects and full by Proposition 8.31. The underlying functor of a vertically embedded 2-functor is the functor on object categories.

A 2-category is cofibrant in **2-Cat** if and only if its underlying category is projective with respect to all functors that are surjective on objects and full. But this coincides with cofibrant 2-categories in **DblCat** by Proposition 8.42. □

The sets of weak equivalences with source and target 2-categories in the two model structures have nontrivial intersection, but neither set of weak equivalences is contained in the other. Biequivalences are not in general fully faithful in the sense of internal categories.

It is interesting to note that the **Cat**-analogue of Theorem 9.1 does not hold. In other words, if we view **Cat** as the category of algebras over the 2-monad UF on **Graph**, then the associated model structure on **Cat** is not the model structure associated to the topology of surjective functions on **Set**. A covering family in a basis for this topology is a single surjective function, so that the epis for this topology are the same as the epis for the trivial topology by Proposition 8.5 and Remark 8.6, namely the surjective maps themselves. In fact, the trivial topology, simplicially surjective topology, and categorically surjective topology on **Set** all give rise to the categorical model structure on **Cat**, while the 2-monad structure on **Cat** has weak equivalences the isomorphisms of categories. When we pass to **DblCat** on the other hand, the three model structures associated to these three topologies become distinct, and one of them agrees with the 2-monad structure.

10. Appendix: Horizontal Nerves and Pushouts

Though the horizontal nerve and bisimplicial nerve preserve filtered colimits, they certainly do not preserve general colimits, not even pushouts. The purpose of this appendix is to explicitly describe the behavior of the horizontal nerve on pushouts in **DblCat** along

$$i \boxtimes 1_{\mathbf{C}}\colon \mathbf{A} \boxtimes \mathbf{C} \longrightarrow \mathbf{B} \boxtimes \mathbf{C}$$

where $i\colon \mathbf{A} \longrightarrow \mathbf{B}$ is either of the following full inclusions from Section 7.1.

$$c\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c\mathrm{Sd}^2\Delta[m]$$

$$\{1\} \longrightarrow \mathbf{I}$$

Theorem 10.7 is the main technical result needed for an application of Kan's Lemma on Transfer 7.11 to transfer model structures across the adjunction $c_h \dashv N_h$ in Theorems 7.13 and 7.14. In the following, we use "$\backslash$" to denote set-theoretic complement. We begin with some pushouts in **Cat** which will aid us in our description of the horizontal and vertical 1-categories of the pushouts in Theorem 10.6. The squares will require an induction argument.

Lemma 10.1. *If $A \subseteq B$ and D are sets, then the pushout in* **Set**

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & D \arrow[d] \\ B \arrow[r] & P \end{tikzcd}$$

is $P = D \coprod (B \backslash A)$.

Lemma 10.2. *Suppose $\mathbf{A}$ is a full subcategory of $\mathbf{B}$ and*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "F"] \arrow[d, hook] & \mathbf{D} \arrow[d] \\ \mathbf{B} \arrow[r] & \mathbf{P} \end{tikzcd}$$

is a pushout in **Cat**. *Then the objects of $\mathbf{P}$ are*

$$Obj\,\mathbf{P} = Obj\,\mathbf{D} \coprod (Obj\,\mathbf{B} \backslash Obj\,\mathbf{A})$$

and morphisms of $\mathbf{P}$ have two forms:

(i) *A morphism $B_0 \xrightarrow{f} B_1$ with $f \in (Mor\,\mathbf{B} \backslash Mor\,\mathbf{A})$.*

(ii) *A path $X_1 \xrightarrow{f_1} D_1 \xrightarrow{d} D_2 \xrightarrow{f_2} X_2$ where d is a morphism in $\mathbf{D}$, and $f_1, f_2 \in (Mor\,\mathbf{B} \backslash Mor\,\mathbf{A}) \cup \{identities\ on\ Obj\,\mathbf{P}\}$. If f_1 is nontrivial, then $D_1 \in \mathbf{A}$. If f_2 is nontrivial, then $D_2 \in \mathbf{A}$.*

Proof: To calculate a pushout of categories, one takes the free category on the pushout of the underlying graphs, and then mods out by the relations necessary to make the natural maps from $\mathbf{A}, \mathbf{B}, \mathbf{D}$ to the free category into functors as in Theorem 4.2. Thus the objects of $\mathbf{P}$ are

$$\mathit{Obj}\,\mathbf{D} \coprod (\mathit{Obj}\,\mathbf{B} \backslash \mathit{Obj}\,\mathbf{A})$$

by Lemma 10.1. The edges of the pushout graph are

$$\mathit{Mor}\,\mathbf{D} \coprod (\mathit{Mor}\,\mathbf{B} \backslash \mathit{Mor}\,\mathbf{A}),$$

again by Lemma 10.1. The free category on this consists of finite composable paths of these edges.

Suppose

$$P_0 \xrightarrow{f_1} P_1 \xrightarrow{f_2} P_2 \cdots\cdots\cdots P_{k-1} \xrightarrow{f_k} P_k$$

is a morphism in the pushout $\mathbf{P}$. Then we can reduce it to the form (i) or (ii) using the relations induced by $\mathbf{A}, \mathbf{B}$, and $\mathbf{D}$ as follows. Suppose f_{i-1} and f_{i+1} are in $\mathit{Mor}\,\mathbf{D}$, while f_i is in $(\mathit{Mor}\,\mathbf{B} \backslash \mathit{Mor}\,\mathbf{A})$. Then P_{i-1} and P_i must be objects of $\mathbf{A}$. But by the fullness of $\mathbf{A}$, f_i must be in $\mathit{Mor}\,\mathbf{A}$, and we have arrived at a contradiction. Thus no morphism of $(\mathit{Mor}\,\mathbf{B} \backslash \mathit{Mor}\,\mathbf{A})$ can be surrounded by morphisms of $\mathbf{D}$: there exist $0 \leq m \leq n \leq k+1$ such that for all $0 \leq i \leq m$ and all $n \leq i \leq k$ we have $f_i \in (\mathit{Mor}\,\mathbf{B} \backslash \mathit{Mor}\,\mathbf{A})$, and for all $m < i < n$ we have $f_i \in \mathit{Mor}\,\mathbf{D}$. Next we compose the f_i in each range, and we obtain a path of the form (i) or (ii). □

Remark 10.3. A morphism j of $\mathbf{B}$ is in $\mathit{Mor}\,\mathbf{B} \backslash \mathit{Mor}\,\mathbf{A}$ if and only if its source or target is in $\mathit{Obj}\,\mathbf{B} \backslash \mathit{Obj}\,\mathbf{A}$ by the fullness of $\mathbf{A}$ in $\mathbf{B}$.

Lemma 10.4. *If $A \subseteq B$ are sets and $\mathbf{C}$ and $\mathbf{D}$ are categories, then the pushout in* **Cat**

$$\begin{tikzcd}
A_{\text{disc}} \times \mathbf{C} \arrow[r] \arrow[d, hook, "i \times 1_{\mathbf{C}}"'] & \mathbf{D} \arrow[d] \\
B_{\text{disc}} \times \mathbf{C} \arrow[r] & \mathbf{P}
\end{tikzcd}$$

is $\mathbf{P} = \mathbf{D} \coprod ((B \backslash A)_{\text{disc}} \times \mathbf{C})$. *(The subscript* ‘disc’ *means discrete category on a given set.)*

Proof: Since $B_{\text{disc}} \times \mathbf{C} = A_{\text{disc}} \times \mathbf{C} \coprod ((B \backslash A)_{\text{disc}} \times \mathbf{C})$, the pushout of the underlying graphs is

$$\mathbf{D} \coprod ((B \backslash A)_{\text{disc}} \times \mathbf{C}) \tag{22}$$

by Lemma 10.1. The free category on this graph, modulo the appropriate relations as in Theorem 4.2, is once again (22). □

Lemma 10.5. *Suppose* $\mathbf{A}$ *is a full subcategory of* $\mathbf{B}$, C *is a set, and*

$$\begin{tikzcd} \mathbf{A} \times C_{\mathrm{disc}} \arrow[r, "F"] \arrow[d, hook] & \mathbf{D} \arrow[d] \\ \mathbf{B} \times C_{\mathrm{disc}} \arrow[r] & \mathbf{P} \end{tikzcd}$$

is a pushout in $\mathbf{Cat}$*. Then the objects of* $\mathbf{P}$ *are*

$$Obj\,\mathbf{P} = Obj\,\mathbf{D} \coprod ((Obj\,\mathbf{B} \backslash Obj\,\mathbf{A}) \times C)$$

and the morphisms of $\mathbf{P}$ *have two forms:*

(i) *A morphism* $(B_0, c) \xrightarrow{f} (B_1, c)$ *with* $c \in C$ *and* $f = (f', c) \in (Mor\,\mathbf{B} \backslash Mor\,\mathbf{A}) \times C$.

(ii) *A path* $X_1 \xrightarrow{f_1} D_1 \xrightarrow{d} D_2 \xrightarrow{f_2} X_2$ *where* d *is a morphism in* $\mathbf{D}$*, and each of* f_1 *and* f_2 *is either in* $Mor\,\mathbf{B} \backslash Mor\,\mathbf{A} \times C$ *or an identity morphism.*

Moreover, if f_1 *or* f_2 *is not an identity morphism in (ii), then the path has one of the two respective forms*

$$(B_1, c_1) \xrightarrow{(f_1', c)} (A_1, c_1) \xrightarrow{d} D_2 \xrightarrow{f_2} X_2$$

$$X_1 \xrightarrow{f_1} D_1 \xrightarrow{d} (A_2, c_2) \xrightarrow{(f_2', c_2)} (B_2, c_2)$$

where $c_1, c_2 \in C$, $B_1, B_2 \in Obj\ \mathbf{B} \backslash Obj\ \mathbf{A}$, $A_1, A_2 \in Obj\ \mathbf{A}$, $f_1', f_2' \in Mor\,\mathbf{B} \backslash Mor\,\mathbf{A}$*, and* $d \in Mor\,\mathbf{D}$.

Proof: This follows from Lemma 10.2. □

Let us recall the two full inclusions $i \colon \mathbf{A} \longrightarrow \mathbf{B}$ under consideration. The first case in which we are interested is the full inclusion of posets $c\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c\mathrm{Sd}^2\Delta[m]$. Here $c \colon \mathbf{SSet} \longrightarrow \mathbf{Cat}$ denotes the fundamental category functor as described in Section 6 and $\mathrm{Sd} \colon \mathbf{SSet} \longrightarrow \mathbf{SSet}$ is the subdivision functor defined in [38] and recalled on Page 35.

The second full inclusion $i \colon \mathbf{A} \longrightarrow \mathbf{B}$ of interest is $\{1\} \longrightarrow \mathbf{I}$. The category $\mathbf{I}$ consists of two objects 0 and 1 and four morphisms: an isomorphism between 0 and 1, and the identity maps. The discrete subcategory $\{1\}$ is clearly full.

We can now give an explicit description of pushouts in $\mathbf{DblCat}$ along $i \boxtimes 1_{\mathbf{C}} \colon \mathbf{A} \boxtimes \mathbf{C} \longrightarrow \mathbf{B} \boxtimes \mathbf{C}$ which we use immediately in Theorem 10.7 for the transfer.

Theorem 10.6. *Let* $i\colon \mathbf{A} \longrightarrow \mathbf{B}$ *be either of the following full inclusions.*

$$c\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c\mathrm{Sd}^2\Delta[m]$$

$$\{1\} \longrightarrow \mathbf{I}$$

Let $\mathbf{C}$ *be a category (e.g.,the finite ordinal* $[n]$*), and* $\mathbb{D}$ *a double category. Then the pushout*

$$\begin{tikzcd} \mathbf{A} \boxtimes \mathbf{C} \arrow[r, "F"] \arrow[d, "i \boxtimes 1_{\mathbf{C}}"'] & \mathbb{D} \arrow[d] \\ \mathbf{B} \boxtimes \mathbf{C} \arrow[r] & \mathbb{P} \end{tikzcd}$$

in **DblCat** *has the following explicit description.*

$$Obj\, \mathbb{P} = Obj\, \mathbb{D} \coprod ((Obj\, \mathbf{B} \backslash Obj\, \mathbf{A}) \times Obj\, \mathbf{C}) \tag{23}$$

$$(\mathbf{H}\mathbb{P})_0 = (\mathbf{H}\mathbb{D})_0 \coprod ((Obj\, \mathbf{B} \backslash Obj\, \mathbf{A})_{disc} \times \mathbf{C}) \tag{24}$$

$$\begin{aligned} Mor\, (\mathbf{V}\mathbb{P})_0 = \{ & \textit{paths of the form (i) and (ii)} \\ & \textit{in Lemma 10.5 with } C = Obj\, \mathbf{C} \\ & \textit{and } \mathbf{D} = (\mathbf{V}\mathbb{D})_0 \} \end{aligned} \tag{25}$$

Squares of $\mathbb{P}$ *have two forms:*

(i) *A square* $\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ *in* $Sq\,(\mathbf{B} \boxtimes \mathbf{C}) \backslash Sq\,(\mathbf{A} \boxtimes \mathbf{C})$.

(ii) *A vertical path of squares* $\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_1"] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\delta"] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_2"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ *where* δ *is a square in* $\mathbb{D}$ *and each of* β_1 *and* β_2 *is either a vertical identity square (on a horizontal morphism) in* $\mathbb{P}$ *or is in* $Sq\,(\mathbf{B} \boxtimes \mathbf{C}) \backslash Sq\,(\mathbf{A} \boxtimes \mathbf{C})$*. Moreover, in the case of* $c\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c\mathrm{Sd}^2\Delta[m]$*, the square* β_1 *is always a vertical identity square.*

Note that $Sq\,(\mathbf{B}\boxtimes\mathbf{C})\backslash Sq\,(\mathbf{A}\boxtimes\mathbf{C}) =$

$$\left\{\begin{array}{c|c} \begin{tikzcd} (B,C) \arrow[r, "(1_B,g)"] \arrow[d, "(f,1_C)"'] & (B,C') \arrow[d, "(f,1_{C'})"] \\ (B',C) \arrow[r, "(1_{B'},g)"'] & (B',C') \end{tikzcd} & \begin{array}{c} g \in Mor\,\mathbf{C}, \\ f \in Mor\,\mathbf{B}\backslash Mor\,\mathbf{A} \end{array} \end{array}\right\}.$$

Proof: We use Theorem 4.5. First we calculate the pushout $\mathbb{S}$ of the underlying double derivation schemes. The object set $Obj\,\mathbb{S} = Obj\,\mathbb{P}$ is the pushout of the object sets, so (23) follows from Lemma 10.1. The horizontal and vertical 1-categories of $\mathbb{S}$ (and $\mathbb{P}$) are the pushouts of the horizontal and vertical 1-categories, so (24) follows from Lemma 10.4 and (25) follows from Lemma 10.5. By Lemma 10.1 again, the pushout of the *sets* of squares is

$$Sq\,\mathbb{S} = Sq\,\mathbb{D}\coprod(Sq\,(\mathbf{B}\boxtimes\mathbf{C})\backslash Sq\,(\mathbf{A}\boxtimes\mathbf{C})). \tag{26}$$

Thus we have calculated the pushout $\mathbb{S}$ of the underlying double derivation schemes, its horizontal and vertical 1-categories coincide with those of $\mathbb{P}$, and they have the form claimed in the theorem. It only remains to show that the squares of $\mathbb{P}$ have the form claimed in the theorem.

The double category $\mathbb{P}$ is the free double category on the double derivation scheme $\mathbb{S}$ modulo the smallest congruence making the natural morphisms of double derivation schemes from $\mathbf{A}\boxtimes\mathbf{C}, \mathbf{B}\boxtimes\mathbf{C}$, and $\mathbb{D}$ to $\mathbb{P}$ into double functors. Squares of $\mathbb{P}$ are represented by allowable compatible arrangements in $\mathbb{S}$. To prove that squares of $\mathbb{P}$ have the form (i) or (ii), it suffices to show that any allowable compatible arrangement of squares in $\mathbb{S}$ can be transformed into (i) or (ii) using the relations of the congruence and the double category associativity, identity, and interchange axioms. The congruence allows us to compose squares according to the relations in the double categories $\mathbf{A}\boxtimes\mathbf{C}, \mathbf{B}\boxtimes\mathbf{C}$, and $\mathbb{D}$.

We must treat the two inclusions i separately.

Let $i\colon\, \mathbf{A}\longrightarrow\mathbf{B}$ be the full inclusion $c\mathrm{Sd}^2\Lambda^k[m]\longrightarrow c\mathrm{Sd}^2\Delta[m]$. Recall from Page 35 that $c\mathrm{Sd}^2\Lambda^k[m]$ and $c\mathrm{Sd}^2\Delta[m]$ are respectively the posets of nondegenerate simplices of $\mathrm{Sd}\Lambda^k[m]$ and $\mathrm{Sd}\Delta[m]$, and that there is a morphism $(u_0,\ldots,u_p)\longrightarrow(v_0,\ldots,v_q)$ in $\mathbf{B}$ if and only if

$$\{u_0,\ldots,u_p\}\subseteq\{v_0,\ldots,v_q\}.$$

Also, an object $(v_0,\ldots,v_q)$ of $\mathbf{B}$ is in $\mathbf{A}$ if and only *all* v_i are faces of $\Lambda^k[m]$. Thus, we see for any path of composable morphisms in $\mathbf{B}$

$$B_0\xrightarrow{f_1}B_1\xrightarrow{f_2}B_2\cdots\cdots B_{n-1}\xrightarrow{f_n}B_n$$

with B_i not in $\mathbf{A}$, all f_j and B_j with $j \geq i$ are also not in $\mathbf{A}$. Thus, once a path leaves $\mathbf{A}$, it cannot return to $\mathbf{A}$. In particular, if $B \longrightarrow B'$ is a morphism in $\mathbf{B}$ and B is not in $\mathbf{A}$, then B' is also not in $\mathbf{A}$. Another useful property of $\mathbf{B}$ is that every morphism has a unique decomposition into irreducibles. These special features of the posets $\mathbf{A}$ and $\mathbf{B}$ allow us to put the squares of $\mathbb{P}$ into the desired form (i) or (ii), as we do now.

Suppose R is an allowable compatible arrangement of squares in $\mathbb{S}$, *i.e.,* a representative of a square in $\mathbb{P}$. If R consists entirely of squares in $\mathbb{D}$, then it is equivalent to its composition in $\mathbb{D}$, so it has the form (ii) and we are finished.

So suppose that R contains at least one square in $Sq\,\mathbf{B} \boxtimes \mathbf{C} \backslash Sq\,\mathbf{A} \boxtimes \mathbf{C}$. Then R has at least one vertex (B, C) in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$, *i.e.,* B is in $\mathbf{B}$ but not in $\mathbf{A}$. Any horizontal morphism in R with source (respectively target) (B, C) is in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$, as (B, C) is not in $\mathbf{A} \boxtimes \mathbf{C}$. Thus the target (respectively source) of such a morphism has the form (B, C') and is also in $\mathbf{B} \boxtimes \mathbf{C}$ but not $\mathbf{A} \boxtimes \mathbf{C}$. Any vertical morphism in R with *source* (B, C) is in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$, as (B, C) is not in $\mathbf{A} \boxtimes \mathbf{C}$. Thus the target of such a vertical morphism is of the form (B', C) with B' not in $\mathbf{A}$ by the special feature of the posets $\mathbf{A}$ and $\mathbf{B}$ described in the preceding paragraph. From the original vertex (B, C) we traverse down a vertical morphism with source (B, C) if there is one, otherwise we traverse to the right along a horizontal morphism with source (B, C). In either case, we arrive at another vertex (B_1, C_1) which is in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$. From this vertex we repeat the procedure, moving either to the right or down. We continue in this way until we reach the bottom edge of the allowable compatible arrangement R. We conclude that the entire bottom edge of the diagram consists of objects and horizontal morphisms in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$, and hence not in $\mathbb{D}$.

Each of these horizontal morphisms on the bottom edge is the bottom edge of a square in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$, since squares of $\mathbb{D}$ only have vertices in $\mathbb{D}$ (some objects of $\mathbb{D}$ are identified with objects of $\mathbf{A} \boxtimes \mathbf{C}$). Thus, the bottom portion of R looks like Figure 1 with all squares in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$.

Next we factor the vertical morphisms of Figure 1 into irreducibles, which we can do since these vertical morphisms are of the form (f, C) where f is a morphism in $\mathbf{B}$ and C is an object of $\mathbf{C}$. By the uniqueness of the factorization and the form of squares in $\mathbf{B} \boxtimes \mathbf{C}$, we can factor these squares at the height of the shortest one as illustrated in Figure 2. We include these new horizontal morphisms into the allowable compatible arrangement R, and obtain a new compatible arrangement R_1. The compatible arrangement

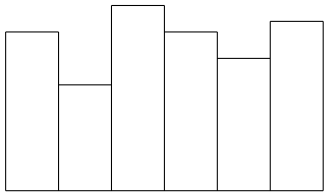


Figure 1. The bottom portion of R.

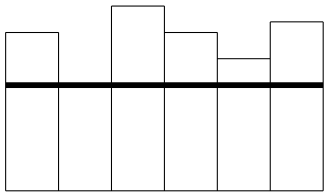


Figure 2. A factorization of the squares in Figure 1.

R_1 is also allowable, since the same cuts that make R allowable also make R_1 allowable.

The bold horizontal line in Figure 2 is a full length cut on an allowable compatible arrangement R_1, hence it divides R_1 into two allowable compatible arrangements by Proposition 3.6. We denote the upper allowable compatible arrangement by $R_{1,1}$ and the lower allowable compatible arrangement by $R_{1,2}$. Then $R_{1,1}$ has at least one square less than R, since we cut off at the height of the shortest square whose bottom edge is on the bottom edge of R. If we argue by induction on the number of squares in an allowable compatible arrangement, we may assume that $R_{1,1}$ is equivalent to a square of the form (i) or (ii). The allowable compatible arrangement $R_{1,2}$ is equivalent to a square of the form (i), as it can be composed horizontally. Finally, we compose $R_{1,1}$ with $R_{1,2}$ to conclude that R is also equivalent to a compatible arrangement of the form (i) or (ii).

We only need an argument for the triviality of β_1 whenever a compatible arrangement is equivalent to one of the form (ii). Suppose β_1 is in $Sq\,(\mathbf{B} \boxtimes \mathbf{C})\backslash Sq\,(\mathbf{A} \boxtimes \mathbf{C})$. Then its lower two vertices cannot be in $\mathbf{A} \boxtimes \mathbf{C}$ (for if they were, the upper two vertices must also be in $\mathbf{A} \boxtimes \mathbf{C}$, and the square β_1 would be in $\mathbf{A} \boxtimes \mathbf{C}$). Thus, the upper two vertices of the square δ are not in $\mathbf{A} \boxtimes \mathbf{C}$, a contradiction. Thus β_1 must be trivial. This completes the proof of Theorem 10.6 for the case $c\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c\mathrm{Sd}^2\Delta[m]$.

Now we turn to the squares in the second case. Let $i\colon \mathbf{A} \longrightarrow \mathbf{B}$ be the full inclusion $\{1\} \longrightarrow \mathbf{I}$ where $\mathbf{I}$ is the category with two objects 0 and 1

$$
\begin{tikzcd}
(0,C_1) \arrow[r] \arrow[d, equal] & (0,C_2) \arrow[d, equal] \\
(0,C_1) \arrow[r] & (0,C_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
(1,C_1) \arrow[r] \arrow[d, equal] & (1,C_2) \arrow[d, equal] \\
(1,C_1) \arrow[r] & (1,C_2)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(0,C_1) \arrow[r] \arrow[d] & (0,C_2) \arrow[d] \\
(1,C_1) \arrow[r] & (1,C_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
(1,C_1) \arrow[r] \arrow[d] & (1,C_2) \arrow[d] \\
(0,C_1) \arrow[r] & (0,C_2)
\end{tikzcd}
$$

FIGURE 3. The Four Types of Squares in $\mathbf{B} \boxtimes \mathbf{C}$.

and an isomorphism between them. We will again argue by induction on the number of squares in the allowable compatible arrangement, but the special features of the inclusion $\{1\} \longrightarrow \mathbf{I}$ are different from those of the previous case. Note that $\mathbf{B} \boxtimes \mathbf{C}$ only has the four types of squares listed in Figure 3. The only vertical morphisms in $\mathbf{B} \boxtimes \mathbf{C}$ that are identified with a morphism in $\mathbb{D}$ are the trivial vertical morphisms $\mathrm{id}^v_{(1,C)} \colon (1,C) \longrightarrow (1,C)$.

Suppose that any allowable compatible arrangement of squares in $\mathbb{S}$ with fewer than n squares is equivalent in $\mathbb{P}$ to one of the form (i) or (ii). Let R be an allowable compatible arrangement of n squares in $\mathbb{S}$. Since R is allowable, it admits a full length cut $\mathcal{C}$ which divides R into two allowable compatible arrangements each with fewer than n squares. We now recombine these two smaller allowable compatible arrangements to show that R is equivalent to a compatible arrangement of the form (i) or (ii), but the argument is slightly different depending on whether $\mathcal{C}$ is horizontal or vertical.

Suppose the full length cut $\mathcal{C}$ is horizontal. Let R_1 and R_2 be the allowable compatible arrangements above and below $\mathcal{C}$ respectively. Since R_1 and R_2 have fewer than n squares, they must be equivalent to compatible arrangements of the form (i) or (ii). If R_1 and R_2 both are equivalent to compatible arrangements of the form (ii), then by the fullness of $\mathbf{A}$ in $\mathbf{B}$ their vertical composite is also of the form (ii), and hence R is equivalent to a compatible arrangement of the form (ii). If one or both of R_1 and R_2 has the form (i), then one can similarly conclude that R is equivalent to a compatible arrangement of form (i) or (ii).

Suppose the full length cut $\mathcal{C}$ is vertical. Let Q^ℓ and Q^r be the allowable compatible arrangements to the left and to the right of $\mathcal{C}$ respectively. Since Q^ℓ and Q^r have fewer than n squares, they must be equivalent to compatible arrangements of the form (i) or (ii). There are several cases to consider.

$$
Q^\ell \qquad
\begin{tikzcd}
(B_1, C_1^\ell) \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta^\ell"] & (B_1, C) \arrow[d, "p^\ell"] \\
(B_4, C_1^\ell) \arrow[r] & (B_4, C)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(B_1, C) \arrow[r] \arrow[d, "j^r"'] \arrow[dr, phantom, "\beta_1^r"] & {} \arrow[d] \\
(B_2^r, C) \arrow[r] \arrow[d, "k^r"'] \arrow[dr, phantom, "\delta^r"] & {} \arrow[d] \\
(B_3^r, C) \arrow[r] \arrow[d, "m^r"'] \arrow[dr, phantom, "\beta_2^r"] & {} \arrow[d] \\
(B_4, C) \arrow[r] & {}
\end{tikzcd}
\qquad Q^r
$$

FIGURE 4.

If both Q^ℓ and Q^r are equivalent to compatible arrangements of the form (i), then their horizontal composite R is clearly in $\mathbf{B} \boxtimes \mathbf{C}$, and hence also equivalent to a compatible arrangement of the form (i) or (ii).

If Q^ℓ is equivalent to a compatible arrangement of the form (i) and Q^r is equivalent to a compatible arrangement of the form (ii), then β_1^r and β_2^r must be in $\mathbf{B} \boxtimes \mathbf{C}$ as in Figure 4. Further, the vertical morphism $k^r \colon (B_2^r, C) \longrightarrow (B_3^r, C)$ must be the identity

$$\mathrm{id}^v_{(1,C)} \colon (1, C) \longrightarrow (1, C),$$

since $k^r = (m^r)^{-1} p^\ell (j^r)^{-1}$ lies in both $\mathbf{B} \boxtimes \mathbf{C}$ and $\mathbb{D}$, and the only vertical morphisms in both $\mathbf{B} \boxtimes \mathbf{C}$ and $\mathbb{D}$ are such vertical identities. Then we can subdivide β^ℓ in $\mathbf{B} \boxtimes \mathbf{C}$ as in Figure 5. The middle square of Q^ℓ is now an identity square on a horizontal morphism in $\mathbb{D}$, and hence is also a square in $\mathbb{D}$. Finally, we horizontally compose Q^ℓ and Q^r and use the interchange law to obtain a compatible arrangement of the form (i) or (ii). Hence R is equivalent to a compatible arrangement of the form (i) or (ii).

Next we consider the case where Q^ℓ and Q^r are both equivalent to compatible arrangements of the form (ii) *and* the squares $\beta_1^\ell, \beta_2^\ell, \beta_1^r, \beta_2^r$ are in $\mathbf{B} \boxtimes \mathbf{C}$ as in Figure 6. Then $B_2^\ell = 1 = B_2^r$ and $B_3^\ell = 1 = B_3^r$, since the only objects of $\mathbf{B} \boxtimes \mathbf{C}$ that are identified with an object of $\mathbb{D}$ are of the form $(1, C)$. Thus $j^\ell = j^r$ and $m^\ell = m^r$, as there is a unique vertical morphism from any object of $\mathbf{B} \boxtimes \mathbf{C}$ to another. Since j^ℓ and m^ℓ are invertible and $m^\ell k^\ell j^\ell = m^\ell k^r j^\ell$, we see also that $k^\ell = k^r$. Hence Q^ℓ and Q^r can be horizontally composed to obtain a compatible arrangement equivalent to (i) or (ii).

$$\begin{tikzcd}
 & (B_1, C_1^\ell) \arrow[r] \arrow[d] & (B_1, C) \arrow[d] & & (B_1, C) \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4ex, "\beta_1^r"] & {} \arrow[d] & \\
Q^\ell & (1, C_1^\ell) \arrow[r] \arrow[d, equal] & (1, C) \arrow[d, equal] & & (1, C) \arrow[r] \arrow[d, equal] \arrow[r, phantom, shift right=4ex, "\delta^r"] & {} \arrow[d] & Q^r \\
 & (1, C_1^\ell) \arrow[r] \arrow[d] & (1, C) \arrow[d] & & (1, C) \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4ex, "\beta_2^r"] & {} \arrow[d] & \\
 & (B_4, C_1^\ell) \arrow[r] & (B_4, C) & & (B_4, C) \arrow[r] & {} &
\end{tikzcd}$$

FIGURE 5.

$$\begin{tikzcd}
 & {} \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4ex, "\beta_1^\ell"] & (B_1, C_1) \arrow[d, "j^\ell"] & & (B_1, C_1) \arrow[r] \arrow[d, "j^r"'] \arrow[r, phantom, shift right=4ex, "\beta_1^r"] & {} \arrow[d] & \\
Q^\ell & {} \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4ex, "\delta^\ell"] & (B_2^\ell, C_1) \arrow[d, "k^\ell"] & & (B_2^r, C_1) \arrow[r] \arrow[d, "k^r"'] \arrow[r, phantom, shift right=4ex, "\delta^r"] & {} \arrow[d] & Q^r \\
 & {} \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4ex, "\beta_2^\ell"] & (B_3^\ell, C_2) \arrow[d, "m^\ell"] & & (B_3^r, C_2) \arrow[r] \arrow[d, "m^r"'] \arrow[r, phantom, shift right=4ex, "\beta_2^r"] & {} \arrow[d] & \\
 & {} \arrow[r] & (B_4, C_2) & & (B_4, C_2) \arrow[r] & {} &
\end{tikzcd}$$

FIGURE 6.

Next we consider the case where Q^ℓ and Q^r are both equivalent to compatible arrangements of the form (ii), but the squares $\beta_1^\ell, \beta_2^\ell, \beta_1^r, \beta_2^r$ may be vertical identity squares in $\mathbb{P}$, *i.e.,* not necessarily in $\mathbf{B} \boxtimes \mathbf{C}$, as in Figure 7. Suppose β_1^ℓ is a vertical identity square. Then $P_1 = D_1^\ell$. We claim that β_1^r is also a vertical identity square; there are two cases to prove. If D_1^ℓ is an object of $\mathbb{D}$ that is not of the form $(1, C)$, then f^r cannot be in $\mathbf{B} \boxtimes \mathbf{C}$ (as its source is not in $\mathbf{B} \boxtimes \mathbf{C}$). Hence β_1^r is a vertical identity square. For the second case, if D_1^ℓ is of the form $(1, C)$, then j^r is a vertical arrow in $\mathbf{B} \boxtimes \mathbf{C}$ with source and target $(1, C)$. By the special form of squares in $\mathbf{B} \boxtimes \mathbf{C}$ in Figure 3, we see that β_1^r is also a vertical identity square. Thus, we have proved, if β_1^ℓ is a vertical identity square, then β_1^r is also a vertical identity

$$\begin{tikzcd}
 & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_1^\ell"] & P_1 \arrow[d, "j^\ell"] & & P_1 \arrow[r, "f^r"] \arrow[d, "j^r"'] \arrow[dr, phantom, "\beta_1^r"] & {} \arrow[d] & \\
Q^\ell & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\delta^\ell"] & D_1^\ell \arrow[d, "k^\ell"] & & D_1^r \arrow[r] \arrow[d, "k^r"'] \arrow[dr, phantom, "\delta^r"] & {} \arrow[d] & Q^r \\
 & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_2^\ell"] & D_2^\ell \arrow[d, "m^\ell"] & & D^r \arrow[r] \arrow[d, "m^r"'] \arrow[dr, phantom, "\beta_2^r"] & {} \arrow[d] & \\
 & {} \arrow[r] & P_2 & & P_2 \arrow[r] & {} &
\end{tikzcd}$$

Figure 7.

square. One can similarly show that if any one of $\beta_1^\ell, \beta_2^\ell, \beta_1^r, \beta_2^r$ is a vertical identity square, then the square next to it is also.

Let us continue the case where Q^ℓ and Q^r are both equivalent to compatible arrangements of the form (ii) as in Figure 7, and suppose again that β_1^ℓ is a vertical identity square. Then β_1^r is also a vertical identity square. If either of β_2^ℓ or β_2^r is a trivial identity square, then so is the other, in which case Q^ℓ and Q^r can be horizontally composed to give a compatible arrangement equivalent to one of the form (i) or (ii). If neither β_2^ℓ nor β_2^r is a vertical identity square, then they are both in $\mathbf{B} \boxtimes \mathbf{C}$, and we can argue as in Figure 6 to conclude $(B_4, C_2) = P_2 = (B_4, C_2)$, $m^\ell = m^r$, and $k^\ell = k^r$, in which case Q^ℓ and Q^r can be horizontally composed to give a compatible arrangement equivalent to one of the form (i) or (ii).

The other cases of Figure 7, where one or more of $\beta_1^\ell, \beta_2^\ell, \beta_1^r, \beta_2^r$ is a vertical identity square in $\mathbb{P}$, are similar.

Thus every every square of $\mathbb{P}$ is equivalent to a compatible arrangement of the form (i) or (ii), for both inclusions i under consideration. This completes the proof of Theorem 10.6. $\square$

The two inclusions of Theorem 10.6 have some features in common, and the theorem holds for an entire class of inclusions $i\colon \mathbf{A} \longrightarrow \mathbf{B}$. We will return to the this topic and its interaction with [22] in the future. Theorem 10.6 allows us to characterize the behavior of the horizontal nerve on such pushouts in Theorem 10.7, which we need to transfer the model structures from $\mathbf{Cat}^{\Delta^{op}}$ in Section 7.

Theorem 10.7. *Let* $i\colon \mathbf{A} \longrightarrow \mathbf{B}$ *be either of the following full inclusions.*

$$c\mathrm{Sd}^2\Lambda^k[m] \longrightarrow c\mathrm{Sd}^2\Delta[m]$$

$$\{1\} \longrightarrow \mathbf{I}$$

Let $\mathbf{C}$ *be a finite ordinal* $[n]$ *viewed as a category,* $\mathbb{D}$ *a double category, and* $\mathbb{P}$ *the pushout*

$$\begin{tikzcd} \mathbf{A} \boxtimes \mathbf{C} \arrow[r] \arrow[d, "i\boxtimes 1_{\mathbf{C}}"'] & \mathbb{D} \arrow[d] \\ \mathbf{B} \boxtimes \mathbf{C} \arrow[r] & \mathbb{P} \end{tikzcd}$$

in **DblCat**. *Then the induced map*

$$N_h(\mathbb{D}) \coprod_{N_h(\mathbf{A}\boxtimes\mathbf{C})} N_h(\mathbf{B} \boxtimes \mathbf{C}) \longrightarrow N_h(\mathbb{P})$$

is an isomorphism of simplicial objects in **Cat**.

Proof: We calculate the pushout

$$N_h(\mathbb{D}) \coprod_{N_h(\mathbf{A}\boxtimes\mathbf{C})} N_h(\mathbf{B} \boxtimes \mathbf{C}) \tag{27}$$

levelwise and compare it with $N_h(\mathbb{P})$, which was described in Theorem 10.6. The horizontal nerve of an external product of categories is known from Proposition 5.6.

In level 0, the pushout (27) is

$$\mathbb{D}_0 \coprod_{\mathbf{A}\times(Obj\,\mathbf{C})_{disc}} \mathbf{B} \times (Obj\,\mathbf{C})_{\mathrm{disc}}$$

which is the same as the vertical 1-category of $\mathbb{P}$ and thus $(N_h\mathbb{P})_0$.

In level $k \geq 1$ the pushout (27) is

$$N_h(\mathbb{D})_k \coprod_{\mathbf{A}\times N\mathbf{C}_k} \mathbf{B} \times N\mathbf{C}_k$$

by Proposition 5.6. An application of Lemma 10.5 to level k shows that it is equal to $N_h(\mathbb{P})_k$ by Theorem 10.6. □

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