STOCHASTIC FITZHUGH-NAGUMO EQUATIONS ON NETWORKS WITH IMPULSIVE NOISE

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1. INTRODUCTION

In this paper we study a system of nonlinear diffusion equations on a finite network in the presence of an impulsive noise acting on the nodes of the system. We allow a rather general nonlinear drift term, including dissipative functions of FitzHugh-Nagumo type (i.e. $f(u) = -u(u-1)(u-a)$) arising in various models of neurophysiology (see e.g. the monograph [19] for more details).

Electric signaling by neurons has been studied since the 50s, starting with the now classical Hodgkin-Huxley model [16] for the diffusion of the transmembrane electrical potential in a neuronal cell. This model consists of a system of four equations describing the diffusion of the electrical potential and the behavior of various ion channels. Successive simplifications of the model, trying to capture the key phenomena of the Hodgkin-Huxley model, lead to the reduced FitzHugh-Nagumo equation, which is a scalar equation with two stable states (see e.g. [27]).

Among other papers dealing with the case of a whole neuronal network (usually modeled as a graph with m edges and n nodes), which is intended to be a simplified model for a large region of the brain, let us mention a series of recent papers by Mugnolo et al. [21, 25], where the well-posedness of the isolated system is studied.

Note that, for a diffusion on a network, other conditions must be imposed in order to define the behavior at the nodes. We impose a continuity condition, that is, given any node in the network, the electrical potentials of all its incident edges are equal. Each edge represents an active soma, and in this part of the cell the potential evolves following a generalized Kirchhoff condition that we model with dynamical boundary conditions for the internal dynamics.

Since the classical work of Walsh [28], stochastic partial differential equations have been an important modeling tools in neurophysiology, where a random forcing is introduced to model several external perturbations acting on the system. In our neural network, we model the electrical activity of

background neurons with a stochastic input of impulsive type, to take into account the stream of excitatory and inhibitory action potentials coming from the neighbors of the network. The need to use models based on impulsive noise was already pointed out in several papers by Kallianpur and coauthors – see e.g. [17, 18].

Following the approach of [5], we use the abstract setting of stochastic PDEs by semigroup techniques (see e.g. [9, 10]) to prove existence and uniqueness of solutions to the system of stochastic equations on a network. In particular, the specific stochastic dynamics is rewritten in terms of a stochastic evolution equation driven by an additive Lévy noise on a certain class of Hilbert spaces. Even though there is a growing interest in stochastic PDEs driven by jump noise (let us just mention [20], [14], [22]), it seems like the case we are interested in, i.e. with a power-type nonlinearity, is not covered by existing results.

The rest of the paper is organized as follows: in section 2 we first introduce the problem and we motivate our assumptions in connection with the applications to neuronal networks. Then, we provide a suitable abstract setting and we prove, following [25], that the linear operator appearing as leading drift term in the stochastic PDE generates an analytic semigroup of contractions. Section 3 contains our main results. First we prove existence and uniqueness of mild solution for the problem under Lipschitz conditions on the nonlinear term (theorem 3.6). This result (essentially already known) is used to obtain existence and uniqueness in the mild sense for the SPDE with a locally Lipschitz continuous dissipative drift of FitzHugh-Nagumo type by techniques of monotone operators.

2. Setting of the problem

The network is identified with the underlying graph G, described by a set of n vertices $\mathsf{v}_1, \ldots, \mathsf{v}_n$ and m oriented edges $\mathsf{e}_1, \ldots, \mathsf{e}_m$ which we assume to be normalized, i.e., $\mathsf{e}_j = [0, 1]$. The graph is described by the *incidence matrix* $\Phi = \Phi^+ - \Phi^-$, where $\Phi^+ = (\phi_{ij}^+)_{n \times m}$ and $\Phi^- = (\phi_{ij}^-)_{n \times m}$ are given by

$$\phi_{ij}^- = \begin{cases} 1, & \mathsf{v}_i = \mathsf{e}_j(1) \\ 0, & \text{otherwise} \end{cases} \qquad \phi_{ij}^+ = \begin{cases} 1, & \mathsf{v}_i = \mathsf{e}_j(0) \\ 0, & \text{otherwise.} \end{cases}$$

The degree of a vertex is the number of edges entering or leaving the node. We denote

$$\Gamma(\mathsf{v}_i) = \{j \in \{1, \ldots, m\} \ : \ \mathsf{e}_j(0) = \mathsf{v}_i \text{ or } \mathsf{e}_j(1) = \mathsf{v}_i\}$$

hence the degree of the vertex v_i is the cardinality $|\Gamma(\mathsf{v}_i)|$.

The electrical potential in the network shall be denoted by $\bar{u}(t,x)$ where $\bar{u} \in (L^2(0,1))^m$ is the vector $(u_1(t,x), \ldots, u_m(t,x))$ and $u_j(t,\cdot)$ is the electrical potential on the edge e_j. We impose a general diffusion equation on every edge

$$\text{(1)} \qquad \frac{\partial}{\partial t} u_j(t,x) = \frac{\partial}{\partial x}\left(c_j(x)\frac{\partial}{\partial x}u_j(t,x)\right) + f_j(u_j(t,x)),$$

for all $(t,x) \in \mathbb{R}_+ \times (0,1)$ and all $j = 1, \ldots, m$. The generality of the above diffusion is motivated by the discussion in the biological literature, see for example [11], who remark, in discussing some concrete biological models, that the basic cable properties is not constant throughout the dendritic tree. The above equation shall be endowed with suitable boundary and initial conditions. Initial conditions are given for simplicity at time $t = 0$ of the form

$$\text{(2)} \qquad u_j(0,x) = u_{j0}(x) \in C([0,1]), \qquad j = 1, \ldots, m.$$

Since we are dealing with a diffusion in a network, we require first a continuity assumption on every node

$$\text{(3)} \qquad p_i(t) := u_j(t,\mathsf{v}_i) = u_k(t,\mathsf{v}_i), \qquad t > 0, \ j,k \in \Gamma(\mathsf{v}_i), \ i = 1, \ldots, n$$

and a stochastic generalized Kirchhoff law in the nodes

$$\text{(4)} \qquad \frac{\partial}{\partial t} p_i(t) = -b_i p_i(t) + \sum_{j \in \Gamma(\mathsf{v}_i)} \phi_{ij}\mu_j c_j(\mathsf{v}_i)\frac{\partial}{\partial x}u_j(t,\mathsf{v}_i) + \sigma_i \frac{\partial}{\partial t}L(t,\mathsf{v}_i),$$

for all $t > 0$ and $i = 1, \ldots, n$. Observe that the positive sign of the Kirchhoff term in the above condition is consistent with a model of purely excitatory node conditions, i.e. a model of a neuronal tissue where all synapses depolarize the postsynaptic cell. Postsynaptic potentials can have graded amplitudes modeled by the constants $\mu_j > 0$ for all $j = 1, \ldots, m$.

Finally, $L(t,\mathsf{v}_i), \ i = 1, \ldots, n$, represent the stochastic perturbation acting on each node, due to the external surrounding, and $\frac{\partial}{\partial t}L(t,\mathsf{v}_i)$ is the formal time derivative of the process L, which takes a meaning only in integral sense. Biological motivations lead us to model this term by a Lévy-type process. In fact, the evolution of the electrical potential on the molecular membrane can be perturbed by different types of random terms, each modeling the influence, at different time scale, of the surrounding medium. On a fast time scale, vesicles of neurotransmitters released by external neurons cause electrical impulses which arrive randomly at the soma causing a sudden change in the membrane voltage potential of an amount, either positive or negative, depending on the composition of the vesicle and possibly even on the state of the neuron. We model this behavior perturbing the equation

by an additive term driven by a n-dimensional impulsive noise of the form

$$(5) \qquad L(t) = \int_{\mathbb{R}^n} x \tilde{N}(t, \mathrm{d}x),$$

see Hypothesis 2.2 below for a complete description of the process. See also [18] for a related model.

Although many of the above reasonings remain true also when considering the diffusion process on the fibers, we shall not pursue such generality and assume that the random perturbation acts only on the boundary of the system, i.e. on the nodes of the network.

Let us state the main assumptions on the data of the problem.

Hypothesis 2.1.
 (1) *In (1), we assume that $c_j(\cdot)$ belongs to $C^1([0,1])$, for $j = 1, \ldots, m$ and $c_j(x) > 0$ for every $x \in [0,1]$.*
 (2) *There exists constants $\eta \in \mathbb{R}$, $c_0 > 0$ and $s \geq 1$ such that, for $j = 1, \ldots, m$, the functions $f_j(u)$ satisfy $f_j(u) + \eta u$ is continuous and decreasing, and $|f_j(u)| \leq c_0(1 + |u|^s)$.*
 (3) *In (4), we assume that $b_i \geq 0$ for every $i = 1, \ldots, n$ and at least one of the coefficients b_i is strictly positive.*
 (4) *$\{\mu_j\}_{j=1,\ldots,m}$ and $\{\sigma_i\}_{i=1,\ldots,n}$ are real positive numbers.*

Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual hypotheses and a Hilbert space $\mathcal{H}$, let us define the space $L^2_{\mathcal{F}}(\Omega \times [0,T]; \mathcal{H})$ of adapted processes $Y : [0,T] \to \mathcal{H}$ endowed with the natural norm

$$|Y|_2 = \left(\mathbb{E} \int_0^T |Y(t)|^2_{\mathcal{H}} \mathrm{d}t \right)^{1/2}.$$

We shall consider a Lévy process $\{L_t, \ t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with values in $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, i.e., a stochastically continuous, adapted process starting almost surely from 0, with stationary and independent increments and càdlàg trajectories, hence with discontinuities of jump type. By the classical Lévy-Itô decomposition theorem, the n-dimensional Lévy process $L(t)$ has a decomposition

(6)
$$L(t) = mt + qW_t + \int_{|x| \leq 1} x[N(t, \mathrm{d}x) - t\nu(\mathrm{d}x)] + \int_{|x| > 1} xN(t, \mathrm{d}x), \qquad t \geq 0$$

where $m \in \mathbb{R}^n$, $q \in M_{n \times n}(\mathbb{R})$ is a symmetric, positive defined matrix, $\{W_t, \ t \geq 0\}$ is an n-dimensional centered Brownian motion and the Lévy measure $\nu(\mathrm{d}x)$ is σ-finite on $\mathbb{R}^n \setminus \{0\}$ and such that $\int \min(1, x^2)\nu(\mathrm{d}x) < \infty$. We denote by $\tilde{N}(\mathrm{d}t, \mathrm{d}x) := N(\mathrm{d}t, \mathrm{d}x) - \mathrm{d}t\nu(\mathrm{d}x)$ the compensated Poisson measure.

Hypothesis 2.2. *We suppose that the measure ν has finite second order moment, i.e.*

$$(7) \qquad \int_{\mathbb{R}^n} |x|^2 \nu(\mathrm{d}x) < \infty.$$

Condition (7) implies that the generalized compound Poisson process $\int_{|x|>1} x\, N(t, \mathrm{d}x)$ has finite moments of first and second order. Then, with no loss of generality, we assume that

$$(8) \qquad \int_{|x|>1} x\nu(\mathrm{d}x) = 0.$$

We also assume throughout that the Lévy process is a pure jump process, i.e. $m \equiv 0$ and $q \equiv 0$, which leads to the representation (5) in view of assumptions (7) and (8).

2.1. **Well-posedness of the linear deterministic problem.** We consider the product space $\mathbb{H} = (L^2(0, 1))^m$. A general vector $\bar{u} \in \mathbb{H}$ is a collection of functions $\{u_j(x),\ x \in [0, 1],\ j = 1, \ldots, m\}$ which represents the electrical potential inside the network.

Remark 2.3. For any real number $s \geq 0$ we define the Sobolev spaces

$$\mathbb{H}^s = (H^s(0, 1))^m,$$

where $H^s(0, 1)$ is the fractional Sobolev space defined for instance in [23]. In particular we have that $\mathbb{H}^1 \subset (C[0, 1])^m$. Hence we are allowed to define the boundary evaluation operator $\Pi : \mathbb{H}^1 \to \mathbb{R}^n$ defined by

$$\Pi\bar{u} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \quad \text{where } p_i = \bar{u}(\mathsf{v}_i) = u_k(\mathsf{v}_i) \quad \text{for } k \in \Gamma(\mathsf{v}_i),\ i = 1, ..., n.$$

On the space $\mathbb{H}$ we introduce the linear operator $(A, D(A))$ defined by

$$D(A) = \{\bar{u} \in \mathbb{H}^2 \mid \exists\, p \in \mathbb{R}^n \text{ such that } \Pi\bar{u} = p\}$$

$$A\bar{u} = \left(\frac{\partial}{\partial x}\left(c_j(x)\frac{\partial}{\partial x}u_j(t, x)\right)\right)_{j=1,\ldots,m}$$

As discussed in [25], the diffusion operator A on a network, endowed with active nodes, fits the abstract mathematical theory of parabolic equations with dynamic boundary conditions and in particular it can be discussed in an efficient way by means of sesquilinear forms. Here, we shall follow the same approach.

First, notice that no other condition except continuity on the nodes is imposed on the elements of $D(A)$. This is often stated by saying that the domain is *maximal*.

The so called feedback operator, denoted by C, is a linear operator from $D(A)$ to $\mathbb{R}^n$ defined as

$$C\bar{u} = \left(\sum_{j \in \Gamma(\mathsf{v}_i)} \phi_{ij} \mu_j c_j(\mathsf{v}_i) \frac{\partial}{\partial x} u_j(t, \mathsf{v}_i) \right)_{i=1,\dots,n}.$$

On the vectorial space $\mathbb{R}^n$ we define also the diagonal matrix

$$B = \begin{pmatrix} -b_0 & & \\ & \ddots & \\ & & -b_n \end{pmatrix}.$$

With the above notation, problem (1)–(4) can be written as an abstract Cauchy problem on the product space $\mathcal{H} = \mathbb{H} \times \mathbb{R}^n$ endowed with the natural inner product

$$\langle X, Y \rangle_{\mathcal{H}} = \langle \bar{u}, \bar{v} \rangle_{\mathbb{H}} + \langle p, q \rangle_{\mathbb{R}^n}, \text{ where } X, Y \in \mathcal{H} \text{ and } X = \begin{pmatrix} \bar{u} \\ p \end{pmatrix}, \ Y = \begin{pmatrix} \bar{v} \\ q \end{pmatrix}$$

We introduce the matrix operator $\mathcal{A}$ on the space $\mathcal{H}$, given in the form

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \{X = (\bar{u}, p) \in \mathcal{H} : \bar{u} \in D(A), u_j(\mathsf{v}_i) = p_i \text{ for every } j \in \Gamma(\mathsf{v}_i)\}.$$

Then the linear deterministic part of problem (1)–(4) becomes

(9)
$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = \mathcal{A}X(t)$$
$$X(0) = x_0$$

where $x_0 = (u_j(0, x))_{j=1,\dots,m} \in C([0,1])^m$ is the m-vector of initial conditions. This problem is well posed, as the following result shows.

Proposition 2.4. *Under Hypothesis 2.1.1 and 2.1.2 the operator $(\mathcal{A}, D(\mathcal{A}))$ is self-adjoint, dissipative and has compact resolvent. In particular, it generates a C_0 analytic semigroup of contractions.*

Proof. For the sake of completeness, we provide a sketch of the proof following [25]. The idea is simply to associate the operator $(\mathcal{A}, D(\mathcal{A}))$ with a suitable form $\mathfrak{a}(X, Y)$ having dense domain $\mathcal{V} \subset \mathcal{H}$.

The space $\mathcal{V}$ is defined as

$$\mathcal{V} = \left\{ X = \begin{pmatrix} \bar{u} \\ p \end{pmatrix} \mid \bar{u} \in (H^1(0,1))^m, \ u_k(\mathsf{v}_i) = p_i \text{ for } i = 1,\dots,n, \ k \in \Gamma(\mathsf{v}_i) \right\}$$

and the form $\mathfrak{a}$ is defined as

$$\mathfrak{a}(X,Y) = \sum_{j=1}^{m} \int_{0}^{1} \mu_j c_j(x) u_j'(x) v_j'(x) \, dx + \sum_{l=1}^{n} b_l p_l q_l, \ X = \begin{pmatrix} \bar{u} \\ p \end{pmatrix}, \ Y = \begin{pmatrix} \bar{v} \\ q \end{pmatrix}.$$

The form $\mathfrak{a}$ is clearly positive and symmetric; furthermore it is closed and continuous. Then a little computation shows that the operator associated with the form $\mathfrak{a}$ is $(\mathcal{A}, D(\mathcal{A}))$ defined above. Classical results in Dirichlet forms theory, see for instance [26], lead to the desired result. $\square$

The assumption that $b_l > 0$ for some l is a dissipativity condition on $\mathcal{A}$. In particular it implies the following result (for a proof see [25]).

Proposition 2.5. *Under Hypothesis 2.1.1 and 2.1.3, the operator $\mathcal{A}$ is invertible and the semigroup $\{\mathcal{T}(t), \ t \geq 0\}$ generated by $\mathcal{A}$ is exponentially bounded, with growth bound given by the strictly negative spectral bound of the operator $\mathcal{A}$.*

3. The stochastic Cauchy problem

We can now solve the system of stochastic differential equations (1)–(4). The functions $f_j(u)$ which appear in (1) are assumed to have a polynomial growth. We remark that the classical FitzHugh-Nagumo problem requires

$$f_j(u) = u(u-1)(a_j - u) \qquad j = 1, ..., m$$

for some $a_j \in (0, 1)$, and satisfies Hypothesis 2.1.2 with

$$\eta \leq -\max_{j} \frac{(a_j^3 + 1)}{3(a_j + 1)}, \qquad s = 3.$$

We set
(10)
$$F(\bar{u}) = \left(f_j(u_j) \right)_{j=1,...,m} \quad \text{and} \quad \mathcal{F}(X) = \begin{pmatrix} -F(\bar{u}) \\ 0 \end{pmatrix} \quad \text{for } X = \begin{pmatrix} \bar{u} \\ p \end{pmatrix}$$

and we write our problem in abstract form
(11)
$$dX(t) = [\mathcal{A}X(t) - \mathcal{F}(X(t))] \, dt + \Sigma \, d\mathcal{L}(t)$$
$$X(0) = x_0$$

where Σ is the matrix defined by

$$\Sigma = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{diag}(\sigma_1, \ldots, \sigma_n) \end{pmatrix},$$

and $\mathcal{L}(t)$ is the natural embedding in $\mathcal{H}$ of the n-dimensional Lévy process $L(t)$, i.e.

$$\mathcal{L}(t) = \begin{pmatrix} 0 \\ L(t) \end{pmatrix}.$$

Remark 3.1. Note that $\mathcal{F}$ is only defined on its domain $D(\mathcal{F})$, which is strictly smaller than $\mathcal{H}$.

Let us recall the definition of mild solution for the stochastic Cauchy problem (11).

Definition 3.2. *An $\mathcal{H}$-valued predictable process $X(t)$, $t \in [0, T]$, is said to be a* mild solution *of (11) if*

$$(12) \qquad \mathbb{P}\left(\int_0^T |\mathcal{F}(X(s))|\, \mathrm{d}s < +\infty\right) = 1$$

and

$$(13) \qquad X(t) = \mathcal{T}(t)x_0 - \int_0^t \mathcal{T}(t-s)\mathcal{F}(X(s))\, \mathrm{d}s + \int_0^t \mathcal{T}(t-s)\Sigma\, \mathrm{d}\mathcal{L}(s)$$

$\mathbb{P}$-*a.s. for all $t \in [0, T]$, where $\mathcal{T}(t)$ is the semigroup generated by $\mathcal{A}$.*

Condition (12) implies that the first integral in (13) is well defined. The second integral, which we shall refer to as stochastic convolution, is well defined as will be shown in the following subsection.

3.1. **The stochastic convolution process.** In our case the stochastic convolution can be written as

$$Z(t) = \int_0^t \int_{\mathbb{R}^n} \mathcal{T}(t-s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \tilde{N}(\mathrm{d}s, \mathrm{d}x).$$

The definition of stochastic integral with respect to a compensated Poisson measure has been discussed by many authors, see for instance [1, 2, 3, 8, 13, 15]. Here we limit ourselves to briefly recall some conditions for the existence of such integrals. In particular, in this paper we only integrate deterministic functions, such as $\mathcal{T}(\cdot)\Sigma$, taking values in (a subspace of) $L(\mathcal{H})$, the space of linear operators from $\mathcal{H}$ to $\mathcal{H}$. In order to define the stochastic integral of this class of processes with respect to the Lévy martingale-valued measure

$$(14) \qquad M(t, B) = \int_B x\, \tilde{N}(t, \mathrm{d}x),$$

one requires that the mapping $\mathcal{T}(\cdot)\Sigma : [0, T] \times \mathbb{R}^n \ni (t, x) \mapsto \mathcal{T}(t)(0, \sigma x)$ belongs to the space $L^2((0, T) \times B; \langle M(\mathrm{d}t, \mathrm{d}x)\rangle)$ for every $B \in \mathcal{B}(\mathbb{R}^n)$, i.e. that

$$(15) \qquad \left(\int_0^T \int_B \left|\mathcal{T}(s)\begin{pmatrix} 0 \\ \sigma x \end{pmatrix}\right|_{\mathcal{H}}^2 \nu(\mathrm{d}x)\, \mathrm{d}s\right)^{1/2} < \infty.$$

Thanks to (7), one has

$$\int_0^T \int_B \left| \mathcal{T}(s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\mathcal{H}}^2 \nu(\mathrm{d}x)\,\mathrm{d}s$$

$$\leq |\sigma|^2 \left(\int_0^T |\mathcal{T}(s)|_{L(\mathcal{H})}^2\,\mathrm{d}s \right) \left(\int_B |x|^2\,\nu(\mathrm{d}x) \right) < \infty,$$

thus the stochastic convolution $Z(t)$ is well defined for all $t > 0$.

We shall now prove a regularity property (in space) of the stochastic convolution. In theorem 3.6 below we will prove also show that the stochastic convolution has càdlàg paths.

Let us define the product spaces $\mathcal{E} := (C[0,1])^m \times \mathbb{R}^n$ and $C_{\mathcal{F}}([0,T]; L^2(\Omega; \mathcal{E}))$, the space of $\mathcal{E}$-valued, adapted mean square continuous processes Y on the time interval $[0,T]$ such that

$$|Y|_{C_{\mathcal{F}}}^2 := \sup_{t \in [0,T]} \mathbb{E}|Y(t)|_{\mathcal{E}}^2 < \infty.$$

Lemma 3.3. *For all $t \geq 0$, the stochastic convolution $\{Z(t),\ t \in [0,T]\}$ belongs to the space $C_{\mathcal{F}}([0,T]; L^2(\Omega; \mathcal{E}))$. In particular, $Z(t)$ is predictable.*

Proof. Let us recall that the unbounded matrix operator $\mathcal{A}$ on $\mathcal{H}$ is defined by

$$\mathcal{A} = \begin{pmatrix} \partial_x^2 & 0 \\ -\partial_\nu & B \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \{X = (\bar{u}, p) \in \mathcal{H}\ :\ \bar{u} \in D(A), u_l(\mathsf{v}_i) = p_i \text{ for every } l \in \Gamma(\mathsf{v}_i)\},$$

and, by proposition 2.4, it generates a C_0-analytic semigroup of contractions on $\mathcal{H}$.

Let us introduce the interpolation spaces $\mathcal{H}_\theta = (\mathcal{H}, D(\mathcal{A}))_{\theta,2}$ for $\theta \in (0,1)$. By classical interpolation theory (see e.g. [24]) it results that, for $\theta < 1/4$, $\mathcal{H}_\theta = \mathbb{H}^{2\theta} \times \mathbb{R}^n$ while for $\theta > 1/4$ the definition of $\mathcal{H}_\theta$ involves boundary conditions, that is

$$\mathcal{H}_\theta = \left\{ \begin{pmatrix} \bar{u} \\ p \end{pmatrix} \in H^{2\theta} : \Pi\bar{u} = p \right\}.$$

Therefore, one has $(0, \sigma x) \in \mathcal{H}_\theta$ for $\theta < 1/4$. Furthermore, for $\theta > 1/2$, one also has $\mathcal{H}_\theta \subset \mathbb{H}^1 \times \mathbb{R}^n \subset (C[0,1])^m \times \mathbb{R}^n$ by Sobolev embedding theorem. Moreover, for all $x \in \mathcal{H}_\theta$ and $\theta + \gamma \in (0,1)$, it holds

$$|\mathcal{T}(t)x|_{\theta+\gamma} \leq t^{-\gamma}|x|_\theta e^{\omega_{\mathcal{A}} t},$$

where $\omega_{\mathcal{A}}$ is the spectral bound of the operator $\mathcal{A}$.

Let θ, γ be real numbers such that $\theta \in (0, 1/4)$, $\gamma \in (0, 1/2)$ and $\theta + \gamma \in (1/2, 1)$. Then for all $t \in [0, T]$

$$|Z(t)|_{\theta+\gamma} \leq \int_0^t \int_{\mathbb{R}^n} \left| \mathcal{T}(t-s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\theta+\gamma} \tilde{N}(\mathrm{d}x, \mathrm{d}s) \qquad \mathbb{P}\text{-a.s.}$$

The right hand side of the above inequality is well defined if and only if

$$\mathbb{E} \left| \int_0^T \int_{\mathbb{R}^n} \left| \mathcal{T}(t-s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\theta+\gamma} \tilde{N}(\mathrm{d}x, \mathrm{d}s) \right|^2 = \int_0^T \int_{\mathbb{R}^n} \left| \mathcal{T}(s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\theta+\gamma}^2 \nu(\mathrm{d}x)\mathrm{d}s < \infty,$$

where the identity follows by the classical isometry for Poisson integrals. On the other hand, one has

$$\int_0^T \int_{\mathbb{R}^n} \left| \mathcal{T}(s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\theta+\gamma}^2 \nu(\mathrm{d}x)\mathrm{d}s \leq$$

$$\leq \int_0^T \int_{\mathbb{R}^n} s^{-2\gamma} \left| \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_\theta^2 e^{2\omega_A s} \nu(\mathrm{d}x)\mathrm{d}s$$

$$\leq |\sigma|^2 \int_0^T s^{-2\gamma} e^{2\omega_A s}\mathrm{d}s \int_{\mathbb{R}^n} |x|^2 \nu(\mathrm{d}x) < \infty$$

using $\gamma \in (0, 1/2)$ and assumption (7). So $Z(t) \in \mathcal{H}_{\theta+\gamma}$ for $\theta + \gamma > 1/2$ and then $Z(t) \in (C[0,1])^m \times \mathbb{R}^n = \mathcal{E}$. It remains to prove that $Z(t)$ is mean square continuous as $\mathcal{E}$-valued process. For $0 \leq s < t \leq T$ we can write

$$\begin{aligned}
\mathbb{E}|Z(t) - Z(s)|_{\mathcal{E}}^2 &= \mathbb{E} \left| \int_0^t \mathcal{T}(t-r)\Sigma\,\mathrm{d}\mathcal{L}(r) - \int_0^s \mathcal{T}(s-r)\Sigma\mathrm{d}\mathcal{L}(r) \right|_{\mathcal{E}}^2 \\
&\leq 2\mathbb{E} \left| \int_0^s \int_{\mathbb{R}^n} [\mathcal{T}(t-r) - \mathcal{T}(s-r)] \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \tilde{N}(\mathrm{d}x, \mathrm{d}r) \right|_{\mathcal{E}}^2 \\
&\quad + 2\mathbb{E} \left| \int_s^t \int_{\mathbb{R}^n} \mathcal{T}(t-r) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \tilde{N}(\mathrm{d}x, \mathrm{d}r) \right|_{\mathcal{E}}^2 \\
&= 2\int_0^s \int_{\mathbb{R}^n} \left| [\mathcal{T}(t-r) - \mathcal{T}(s-r)] \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\mathcal{E}}^2 \nu(\mathrm{d}x)\mathrm{d}r \\
&\quad + 2\int_s^t \int_{\mathbb{R}^n} \left| \mathcal{T}(t-r) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\mathcal{E}}^2 \nu(\mathrm{d}x)\mathrm{d}r \longrightarrow 0
\end{aligned}$$

for the strong continuity of the semigroup $\mathcal{T}(t)$. Since the stochastic convolution $Z(t)$ is adapted and mean square continuous, it is predictable. $\square$

3.2. **Existence and uniqueness in the Lipschitz case.** We consider as a preliminary step the case of Lipschitz continuous nonlinear term and we prove existence and uniqueness of solutions in the space $C_{\mathcal{F}}$ of adapted

mean square continuous processes taking values in $\mathcal{H}$. We would like to mention that this result is included only for the sake of completeness and for the simplicity of its proof (which is essentially based only on the isometry defining the stochastic integral). In fact, a much more general existence and uniqueness result was proved by Kotelenez in [20].

Theorem 3.4. *Assume Hypothesis 2.2 and let x_0 be an $\mathcal{F}_0$-measurable $\mathcal{H}$-valued random variable such that $\mathbb{E}|x_0|^2 < \infty$. Let $G : \mathcal{H} \to \mathcal{H}$ be a function satisfying Lipschitz and linear growth conditions:*

$$(16) \quad |G(x)| \leq c_0(1 + |x|), \qquad |G(x) - G(y)| \leq c_0|x - y|, \qquad x, y \in \mathcal{H}.$$

for some constant $c_0 > 0$. Then there exists a unique mild solution $X : [0, T] \to L^2(\Omega, \mathcal{H})$ to equation (11) with $-\mathcal{F}$ replaced by G, which is continuous as $L^2(\Omega, \mathcal{H})$-valued function. Moreover, the solution map $x_0 \mapsto X(t)$ is Lipschitz continuous.

Proof. We follow the semigroup approach of [10, Theorem 7.4] where the case of Wiener noise is treated. We emphasize only the main differences in the proof.

The uniqueness of solutions reduces to a simple application of Gronwall's inequality. To prove existence we use the classical Banach's fixed point theorem in the space $C_{\mathcal{F}}([0, T]; L^2(\Omega; \mathcal{H}))$. Let $\mathcal{K}$ be the mapping

$$\mathcal{K}(Y)(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t - s)G(Y(s)) \, \mathrm{d}s + Z(t)$$

where $Y \in C_{\mathcal{F}}([0, T]; L^2(\Omega; \mathcal{H}))$ and $Z(t)$ is the stochastic convolution. $Z(\cdot)$ and $\mathcal{T}(\cdot)x_0$ belong to $C_{\mathcal{F}}([0, T]; L^2(\Omega; \mathcal{H}))$ respectively in view of Lemma 3.3 and initial assumption. Moreover, setting

$$\mathcal{K}_1(Y)(t) = \int_0^t \mathcal{T}(t - s)G(Y(s)) \, \mathrm{d}s,$$

it is sufficient to note that

$$|\mathcal{K}_1(Y)|^2_{C_{\mathcal{F}}} \leq (Tc_0)^2(1 + |Y|^2_{C_{\mathcal{F}}})$$

by the linear growth of G and the contractivity of $\mathcal{T}(t)$. Then we obtain that $\mathcal{K}$ maps the space $C_{\mathcal{F}}([0, T]; L^2(\Omega; \mathcal{H}))$ to itself. Further, using the Lipschitz continuity of G, it follows that for arbitrary processes Y_1 and Y_2 in $C_{\mathcal{F}}([0, T]; L^2(\Omega; \mathcal{H}))$ we have

$$|\mathcal{K}(Y_1) - \mathcal{K}(Y_2)|^2_{C_{\mathcal{F}}} = |\mathcal{K}_1(Y_1) - \mathcal{K}_1(Y_2)|^2_{C_{\mathcal{F}}} \leq (c_0 T)^2|Y_1 - Y_2|^2_{C_{\mathcal{F}}}.$$

If we choose an interval $[0, \tilde{T}]$ such that $\tilde{T} < c_0^{-1}$, it follows that the mapping $\mathcal{K}$ has a unique fixed point $X \in C_{\mathcal{F}}([0, \tilde{T}]; L^2(\Omega; \mathcal{H}))$. The extension to arbitrary interval $[0, T]$ follows by patching together the solutions in successive time intervals of length $\tilde{T}$.

The Lipschitz continuity of the solution map $x_0 \mapsto X$ is again a consequence of Banach's fixed point theorem, and the proof is exactly as in the case of Wiener noise.

It remains to prove the mean square continuity of X. Observe that $\mathcal{T}(\cdot)x_0$ is a deterministic continuous function and it follows, again from Lemma 3.3, that the stochastic convolution $Z(t)$ is mean square continuous. Hence it is sufficient to note that the same holds for the term $\int_0^t \mathcal{T}(t-s)G(X(s))\,\mathrm{d}s$, that is $\mathbb{P}$-a.s. a continuous Bochner integral and then continuous as the composition of continuous functions on $[0, T]$. $\square$

Remark 3.5. By standard stopping time arguments one actually show that existence and uniqueness of a mild solution holds assuming only that x_0 is $\mathcal{F}_0$-measurable.

In order to prove that the solution constructed above has càdlàg paths, unfortunately one cannot adapt the factorization technique developed for Wiener integrals (see e.g. [10]). However, the càdlàg property of the solution was proved by Kotelenez [20], under the assumption that $\mathcal{A}$ is dissipative. Therefore, thanks to proposition 2.4, the solution constructed above has càdlàg paths. One could also obtain this property proving the following a priori estimate, which might be interesting in its own right.

Theorem 3.6. *Under the assumptions of theorem 3.4 the unique mild solution of problem* (11) *verifies*

$$\mathbb{E}\sup_{t\in[0,T]}|X(t)|_{\mathcal{H}}^2 < \infty.$$

Proof. Let us consider the Itô formula for the function $|\cdot|_{\mathcal{H}}^2$, applied to the process X. Although our computations are only formal, they can be justified using a classical approximation argument. We obtain

$$\mathrm{d}|X(t)|_{\mathcal{H}}^2 = 2\langle X(t^-), \mathrm{d}X(t)\rangle_{\mathcal{H}} + \mathrm{d}[X]_t.$$

By the dissipativity of the operator $\mathcal{A}$ and the Lipschitz continuity of G, we obtain

$$\langle X(t-), \mathrm{d}X(t)\rangle_{\mathcal{H}} = \langle \mathcal{A}X(t), X(t)\rangle_{\mathcal{H}}\mathrm{d}t + \langle G(X(t)), X(t)\rangle_{\mathcal{H}}\mathrm{d}t + \langle X(t-), \Sigma\mathrm{d}\mathcal{L}(t)\rangle_{\mathcal{H}}$$
$$\leq c_0|X(t)|_{\mathcal{H}}^2 + \langle X(t-), \Sigma\mathrm{d}\mathcal{L}(t)\rangle_{\mathcal{H}}.$$

Therefore

$$|X(t)|_{\mathcal{H}}^2 \leq |x_0|_{\mathcal{H}}^2 + 2c_0\int_0^t |X(s)|_{\mathcal{H}}^2\mathrm{d}s + 2\int_0^t \langle X(s-), \Sigma\mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}} + \int_0^t |\Sigma|^2\mathrm{d}[\mathcal{L}]_s$$

and

$$\mathbb{E}\sup_{t\leq T}|X(t)|_{\mathcal{H}}^2 \leq \mathbb{E}|x_0|_{\mathcal{H}}^2 + 2c_0 T\mathbb{E}\sup_{t\leq T}|X(t)|_{\mathcal{H}}^2$$

(17)
$$+ 2\mathbb{E}\sup_{t\leq T}\Big|\int_0^t \langle X(s-), \Sigma \mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}}\Big| + T\int_{\mathbb{R}^n}|\Sigma|^2|x|^2\,\nu(dx),$$

where we have used the relation

$$\mathbb{E}\sup_{t\leq T}[X]_t \leq \mathbb{E}\int_0^T|\Sigma|^2\,d[\mathcal{L}](t) = \mathbb{E}\int_0^T|\Sigma|^2\,d\langle\mathcal{L}\rangle(t) = T\int_{\mathbb{R}^n}\Big|\Sigma\begin{pmatrix}0\\x\end{pmatrix}\Big|^2\,\nu(dx).$$

By the Burkholder-Davis-Gundy inequality applied to $M_t = \int_0^t\langle X(s-), \Sigma \mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}}$, for $p=1$, there exists a constant c_1 such that

$$\mathbb{E}\sup_{t\leq T}\Big|\int_0^t\langle X(s-), \Sigma \mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}}\Big| \leq$$

$$\leq c_1\mathbb{E}\left(\left[\int_0^{\cdot}\langle X(s-), \Sigma \mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}}\right](T)\right)^{1/2}$$

$$\leq c_1\mathbb{E}\left(\sup_{t\leq T}|X(t)|_{\mathcal{H}}^2\int_0^T|\Sigma|^2\mathrm{d}[\mathcal{L}](s)\right)^{1/2}$$

$$\leq c_1\left(\varepsilon\mathbb{E}\sup_{t\leq T}|X(t)|_{\mathcal{H}}^2 + \frac{1}{4\varepsilon}\mathbb{E}\int_0^T|\Sigma|^2\mathrm{d}[\mathcal{L}](s)\right)$$

(18)
$$= c_1\varepsilon\mathbb{E}\sup_{t\leq T}|X(t)|_{\mathcal{H}}^2 + \frac{c_1 T}{4\varepsilon}\int_{\mathbb{R}^n}|\Sigma|^2|x|^2\nu(dx),$$

where we have used Young's inequality. Then, by (17) and (18)

$$\mathbb{E}\sup_{t\leq T}|X(t)|_{\mathcal{H}}^2 \leq \mathbb{E}|x_0|_{\mathcal{H}}^2 + 2c_0 T\mathbb{E}\sup_{t\leq T}|X(t)|_{\mathcal{H}}^2 + 2c_1\varepsilon\mathbb{E}\sup_{t\leq T}|X(t)|_{\mathcal{H}}^2$$

$$+ \big(\frac{c_1}{2\varepsilon}+1\big)T\int_{\mathbb{R}^n}|\Sigma|^2|x|^2\,\nu(dx),$$

hence

$$\mathbb{E}\sup_{t\leq T}|X(t)|_{\mathcal{H}}^2 \leq N\left[\mathbb{E}|x_0|_{\mathcal{H}}^2 + T\left(1+\frac{c_1}{2\varepsilon}\right)\right] < +\infty,$$

where

$$N = N(c_0, c_1, T, \varepsilon) = \frac{1}{1-2c_0 T - 2c_1\varepsilon}.$$

Choosing $\varepsilon > 0$ and $T > 0$ such that $N < 1$, one obtains the claim for a small time interval. The extension to arbitrary time interval follows by classical extension arguments. $\square$

3.3. **FitzHugh-Nagumo type nonlinearity.** Let us now consider the general case of a nonlinear drift term $\mathcal{F}$ which is a dissipative mapping with domain $D(\mathcal{F})$ strictly contained in $\mathcal{H}$. A method to solve equations such as (11) driven by Wiener noise is given in [11]: in that approach it is necessary to find a (reflexive Banach) space $\mathcal{V}$, continuously embedded in $\mathcal{H}$, which is large enough to contain the paths of the stochastic convolution, and, on the other hand, not too large so that it is contained in the domain of the nonlinearity $\mathcal{F}$. As discussed in section 3.1, in our setting the natural candidates for this space are $\mathcal{V} = (\mathbb{H}^1(0,1))^m \times \mathbb{R}^n$ and $\mathcal{E} = (C[0,1])^m \times \mathbb{R}^n$. Unfortunately, it is not possible to give a direct application of the results in [11, Section 5.5], as we do not have continuity in time of the stochastic convolution, but only a càdlàg property. Hence, we need a different approach to the problem, based on regularizations and weak convergence techniques.

Theorem 3.7. *Equation (11) admits a unique mild solution which satisfies the estimate*

$$\mathbb{E}|X(t,x) - X(t,y)|^2 \leq e^{2\eta t}\mathbb{E}|x-y|^2.$$

Proof. As observed in section 3 above, there exists $\eta > 0$ such that $F + \eta I$ is accretive. By a standard argument one can reduce to the case of $\eta = 0$ (see e.g. [4]), which we shall assume from now on, without loss of generality. Let us set, for $\lambda > 0$, $F_\lambda(u) = F((1 + \lambda F)^{-1}(u))$ (Yosida regularization). $\mathcal{F}_\lambda$ is then defined in the obvious way.

Let $\mathcal{G}y = -\mathcal{A}y + \mathcal{F}(y)$. Then $\mathcal{G}$ is maximal monotone on $\mathcal{H}$. In fact, since $\mathcal{A}$ is self-adjoint, setting

$$\varphi(u) = \begin{cases} |\mathcal{A}^{1/2}u|^2, & u \in D(\mathcal{A}^{1/2}) \\ +\infty, & \text{otherwise,} \end{cases}$$

one has $\mathcal{A} = \partial\varphi$. Let us also set $F = \partial g$, where $g : \mathbb{R}^m \to \mathbb{R}$ is a convex function, the construction of which is straightforward. Well-known results on convex integrals (see e.g. [4, sec. 2.2]) imply that F on H is equivalently defined as $F = \partial I_g$, where

$$I_g(u) = \begin{cases} \int_{[0,1]^m} g(u(x))\,ds, & \text{if } g(u) \in L^1([0,1]^m), \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us recall that

$$\mathcal{F} = \begin{pmatrix} F \\ 0 \end{pmatrix}.$$

Since $D(\mathcal{F}) \cap D(\mathcal{A})$ is not empty, $\mathcal{G}$ is maximal monotone if $\varphi((I + \lambda\mathcal{F})^{-1}(u)) \leq \varphi(u)$ (see e.g. [6, Thm. 9]), which is verified by a direct

(but tedious) calculation using the explicit form of $\mathcal{A}$, since $(I + \lambda f_j)^{-1}$ is a contraction on $\mathbb{R}$ for each $j = 1, \ldots, m$.

Let us consider the regularized equation

$$dX_\lambda(t) + \mathcal{G}_\lambda X_\lambda(t)\, dt = \Sigma\, d\mathcal{L}(t).$$

Appealing to Itô's formula for the square of the norm one obtains

$$|X_\lambda(t)|^2 + 2\int_0^t \langle \mathcal{G}_\lambda X_\lambda(s), X_\lambda(s) \rangle\, ds = |x|^2 + 2\int_0^t \langle X_\lambda(s-), \Sigma\, d\mathcal{L}(s) \rangle + [X_\lambda](t)$$

for all $t \in [0, T]$. Taking expectation on both sides yields

$$(19)\quad \mathbb{E}|X_\lambda(t)|^2 + 2\mathbb{E}\int_0^t \langle \mathcal{G}_\lambda X_\lambda(s), X_\lambda(s) \rangle\, ds = |x|^2 + t\int_{\mathbb{R}^n} |\Sigma|^2\, |z|^2\, \nu(dz),$$

where we have used the identity

$$[X_\lambda](t) = \Big[\int_0^\cdot \Sigma\, d\mathcal{L}(s)\Big](t) = t\int_{\mathbb{R}^n} |\Sigma|^2\, |z|^2\, \nu(dz).$$

Let us define the space $\mathbb{L}^p$ as the set of $\mathcal{H}$ valued random variables with finite p-th moment. Therefore, since by (19) we have that $\{X_\lambda\}$ is a bounded subset of $L^\infty([0, T], \mathbb{L}^2)$, and $\mathbb{L}^2$ is separable, Banach-Alaoglu's theorem implies that

$$X_\lambda \overset{*}{\rightharpoonup} X \qquad \text{in } L^\infty([0, T], \mathbb{L}^2),$$

on a subsequence still denoted by λ. Thanks to the assumptions on f_j, one can easily prove that $\langle F(u), u \rangle \geq c|u|^{p+1}$ for some $c > 0$, hence (19) also gives

$$\mathbb{E}\int_0^T |X_\lambda(s)|_{p+1}^{p+1}\, ds < C,$$

which implies that

$$(20)\qquad X_\lambda \rightharpoonup X \qquad \text{in } L^{p+1}(\Omega \times [0, T] \times D, \mathbb{P} \times dt \times d\xi),$$

where $D = [0, 1]^m \times \mathbb{R}^n$. Furthermore, (19) and (20) also imply

$$\mathcal{G}_\lambda(X_\lambda) \rightharpoonup \eta \qquad \text{in } L^{\frac{p+1}{p}}(\Omega \times [0, T] \times D, \mathbb{P} \times dt \times d\xi).$$

The above convergences immediately imply that X and η are predictable, then in order to complete the proof of existence, we have to show that $\eta(\omega, t, \xi) = \mathcal{G}(X(\omega, t, \xi))$, $\mathbb{P} \times dt \times d\xi$-a.e.. For this it is enough to show that

$$\limsup_{\lambda \to 0} \mathbb{E}\int_0^T \langle \mathcal{G}_\lambda X_\lambda(s), X_\lambda(s) \rangle\, ds \leq \mathbb{E}\int_0^T \langle \eta(s), X(s) \rangle\, ds.$$

Using again Itô's formula we get

$$(21)\qquad \mathbb{E}|X(T)|^2 + 2\mathbb{E}\int_0^T \langle \eta(s), X(s) \rangle\, ds = |x|^2 + T\int_{\mathbb{R}^n} |\Sigma|^2\, |z|^2\, \nu(dz).$$

However, (20) implies that

$$\liminf_{\lambda \to 0} \mathbb{E}|X_\lambda(T)|^2 \geq \mathbb{E}|X(T)|^2$$

(see e.g. [7, Prop. 3.5]), from which the claim follows comparing (19) and (21).

The Lipschitz dependence on the initial datum as well as (as a consequence) uniqueness of the solution is proved by observing that $X(t,x) - X(t,y)$ satisfies $\mathbb{P}$-a.s. the deterministic equation

$$\frac{d}{dt}(X(t,x) - X(t,y)) = \mathcal{A}(X(t,x) - X(t,y)) - \mathcal{F}(X(t,x)) + \mathcal{F}(X(t,y)),$$

hence

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|X(t,x) - X(t,y)|^2 &= \langle \mathcal{A}(X(t,x) - X(t,y)), X(t,x) - X(t,y) \rangle \\
&\quad - \langle \mathcal{F}(X(t,x) - \mathcal{F}(X(t,y)), X(t,x) - X(t,y) \rangle \\
&\leq \eta|X(t,x) - X(t,y)|^2,
\end{aligned}
$$

where $X(\cdot, x)$ stands for the mild solution with initial datum x. By the Gronwall lemma

$$\mathbb{E}|X(t,x) - X(t,y)|^2 \leq e^{2\eta t}\mathbb{E}|x - y|^2$$

which concludes the proof of the theorem. $\qquad\square$

Remark 3.8. By arguments similar to those used in the proof of theorem 3.6 one can also obtain that

$$\mathbb{E}\sup_{t \leq T}|X_\lambda(t)|^2 < C,$$

i.e. that $\{X_\lambda\}$ is bounded in $\mathbb{L}^2(\Omega; L^\infty([0,T]; \mathcal{H}))$. By means of Banach-Alaoglu's theorem, one can only conclude that $X_\lambda \overset{*}{\rightharpoonup} X$ in $\mathbb{L}^2(\Omega; L^1([0,T]; \mathcal{H}))'$, which is larger than $\mathbb{L}^2(\Omega; L^\infty([0,T]; \mathcal{H}))$. In fact, from [12, Thm. 8.20.3], being $L^1([0,T]; \mathcal{H})$ a separable Banach space, one can only prove that if F is a continuous linear form on $\mathbb{L}^2(\Omega; L^1([0,T]; \mathcal{H}))$, then there exists a function f mapping Ω into $L^\infty([0,T]; \mathcal{H})$ that is weakly measurable and such that

$$F(g) = \mathbb{E}\langle f, g \rangle$$

for each $g \in \mathbb{L}^2(\Omega; L^1([0,T]; \mathcal{H}))$.

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