CONTRACTIVE PROBABILITY METRICS AND ASYMPTOTIC BEHAVIOR OF DISSIPATIVE KINETIC EQUATIONS

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Abstract. The present notes are intended to present a detailed review of the existing results in dissipative kinetic theory which make use of the contraction properties of two main families of probability metrics: optimal mass transport and Fourier-based metrics. The first part of the notes is devoted to a self-consistent summary and presentation of the properties of both probability metrics, including new aspects on the relationships between them and other metrics of wide use in probability theory. These results are of independent interest with potential use in other contexts in Partial Differential Equations and Probability Theory. The second part of the notes makes a different presentation of the asymptotic behavior of Inelastic Maxwell Models than the one presented in the literature and it shows a new example of application: particle's bath heating. We show how starting from the contraction properties in probability metrics, one can deduce the existence, uniqueness and asymptotic stability in classical spaces. A global strategy with this aim is set up and applied in two dissipative models.

Contents

1. Introduction

In kinetic theory of rarefied gases, the spatially homogenous Boltzmann equation for elastic Maxwell molecules [50] is one of the most intensively studied models, in reason of the simplification consequent to the property that the collision rate is independent of the relative velocity of the colliding pair. The investigation both of the spatially homogenous Boltzmann equation and of its simplified models made possible to achieve essential progresses and to verify or discard conjectures. The first of these theoretical studies is due to McKean Jr. [81], who was able to find explicit rates of convergence towards the Maxwellian equilibrium for the Kac caricature of a Maxwell gas, a one-dimensional model introduced in the fifties by Mark Kac [75]. The pioneering paper by McKean contains a lot of enlightening remarks, and introduces into the matter the role of the entropy production and of the Fisher information [78], fruitfully used later on in different contexts [42]. At the beginning of the seventies, always motivated by the problem of the convergence to equilibrium for Kac equation, Tanaka [100] introduced into kinetic theory the concept of a metric nowadays known with the name of the Russian mathematician L.N. Vasershtein, who introduced it independently in a different field [109]. Tanaka's work, however, first contains the noticeable idea to obtain results for the large-time behavior of a nonlinear equation in consequence of the contractivity of the metric. The properties of Wasserstein metric where subsequently dealt with in a subsequent paper by Tanaka [101], who added to the previous ideas the interesting connection between the problem of convergence to equilibrium for the Boltzmann equation for Maxwell molecules and the central limit problem of probability

theory [92]. While innovative in the methodology introduced into kinetic theory, these papers do not contain results about the rate of convergence to equilibrium. This represents a point of weakness, in reason of the fact that McKean [81] proved that, at least for a certain class of initial data, the solution to Kac equation converges to equilibrium exponentially in time. The same result was conjectured to hold by Cercignani [48] for the spatially homogeneous Boltzmann equation in any dimension and with any kernel, provided a suitable lower bound on the entropy production holds true; for an exhaustive discussion on this conjecture see [105]. It is now clear that, while powerful in getting convergence to equilibrium for the Boltzmann equation, the Wasserstein metric is not suitable to obtain precise rates of convergence to equilibrium [29].

The exponential convergence towards equilibrium for both Kac equation and the Boltzmann equation for Maxwell molecules was obtained by Gabetta, Toscani and Wennberg in [62]. The result takes advantage of the possibility, discovered by Bobylev [17], to write the Boltzmann equation for Maxwell molecules by passing to Fourier transform, and makes use of a new metric for probability measures which results particularly flexible to obtain precise rates of exponential convergence towards the Maxwellian equilibrium. In the same paper, various relationships of this metric with other known metrics, including the Wasserstein one, allowed to obtain rates of convergence in the physical space. The same metric was subsequently used in [103] to prove uniqueness of the solution to the Boltzmann equation for Maxwell molecules without cut-off, as well as in [37], always in connection with the spatially homogeneous Boltzmann equation for Maxwell molecules and its representation in Wild sums.

Hence, the study of the Boltzmann equation for elastic Maxwell molecules was responsible of the introduction of new mathematical tools, and among them, two metrics for probability measures, which are at the basis of most of the results concerned with the large-time behavior of kinetic equations of Maxwell type were emerging.

In the last years, the interest in kinetic theory of dissipative systems, such as granular gases and fluids, has caused a great revival in the study of the Boltzmann equation. Not surprisingly, the work of Bobylev, Carrillo and Gamba [20], who introduced a dissipative Maxwell model with its energy independent collision rate which simplifies the nonlinear collision term to a convolution product, has had a great impact in that revival. A second main fact which was responsible of a noticeable increasing of interest was the discovery of an exact scaling solution for a freely cooling one-dimensional Maxwell model [5].

The application of the techniques based on the use of the probability metrics, both of Wasserstein and Fourier based, allowed to generalize the results of convergence to the equilibrium in classical elastic kinetic theory to the new problem of convergence to the self-similar solution (homogeneous cooling state) both in simpler one-dimensional models [91, 88] and in Maxwell models [22, 25, 13, 29, 23, 24]. For the driven case in which one introduces a source of energy to avoid total cooling of the system, the convergence towards stationary states have been analysed in [51, 21, 12, 29].

The present notes are intended to present a detailed review of the existing results in dissipative kinetic theory which make use of the contraction properties of these metrics. The first part of the notes, however, will be devoted to an almost complete presentation of the properties of both Wasserstein and Fourier based metrics, including new aspects on the relationships between them and other metrics of wide use in probability theory. These results are of independent interest, and can be used in other contexts to obtain both regularity and convergence to equilibrium for solutions to nonlinear friction equations [43, 70, 44] and nonlinear diffusion equations [40, 41].

The second part focuses on the analysis of the asymptotic behavior for Inelastic Maxwell Models. We start by doing a self-consistent introduction to the subject where most of the material has appeared in the existing literature. We concentrate later on the application of the probability metrics to these models in three situations: the stochastic heating, the particle's bath heating and the free cooling of the gas. The stochastic heating case has been here rephrased and re-addressed with respect to the existing literature in the subject. We show for instance, how to start from the contractivity of the optimal transport metric to deduce existence of steady states and apply a general strategy drawn at the beginning of Section 7.

We then attack the case of the particle's bath heating in which the source of energy is introduced by a particle's bath modelled by a linear inelastic Boltzmann operator. We show that the contraction of probability metrics is kept and from this we deduce the existence, uniqueness and global asymptotic stability of equilibria in this case. This provides another example not covered in the literature in which the strategy devised in these notes applies. Finally, we also revise the convergence towards self-similar solutions in the free cooling of a gas. Here, we show the limitations of the contractions of optimal transport metrics compared to Fourier-based metrics since in this case we cannot obtain the existence of homogeneous cooling states from the contraction of the former. We elaborate trying to do summaries of the main proofs that are found in the literature on the subject while keeping the details in the most original parts.

We believe this set of notes may be found useful for newcomers to the subject and young researchers in kinetic theory that want to get a quick overview on the main properties of probability metrics and their applications to the particular case of dissipative kinetic equations without going to the scattered information in different sources.

2. A Review on Probability metrics

This section is devoted to give a self-consistent review of the main properties and relations between two of the most useful probability metrics for long time asymptotics analysis of kinetic and diffusion models: the optimal mass transport metric W_2 and Fourier-based metrics d_s.

2.1. **Optimal Mass Transportation Metrics.** Given two probability measures $f, g \in \mathcal{P}(\mathbb{R}^N)$, the Euclidean Wasserstein Distance is defined as

$$W_2(f,g) = \inf_{\Pi\in\Gamma} \left\{ \iint_{\mathbb{R}^N\times\mathbb{R}^N} |v-x|^2 \, d\Pi(v,x) \right\}^{1/2} \tag{2.1}$$

where Π runs over the set of transference plans Γ, that is, the set of joint probability measures on $\mathbb{R}^N \times \mathbb{R}^N$ with marginals f and $g \in \mathcal{P}(\mathbb{R}^N)$, i.e.,

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \varphi(v)\, d\Pi(v,x) = \int_{\mathbb{R}^N} \varphi(v)\, f(v)\, dv$$

and

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \varphi(x)\, d\Pi(v,x) = \int_{\mathbb{R}^N} \varphi(x)\, g(x)\, dx$$

for all $\varphi \in C_b(\mathbb{R}^N)$, the set of continuous and bounded functions on $\mathbb{R}^N$. From a probabilistic point of view, the Wasserstein distance can be alternatively defined as

$$W_2(f,g) = \inf_{(V,X)\in\tilde{\Gamma}} \left\{ \mathbb{E}\left[|V-X|^2 \right] \right\}^{1/2} \tag{2.2}$$

where $\tilde{\Gamma}$ is the set of all possible couples of random variables (V,X) with f and g as respective laws, i.e., $V, X : (\mathcal{S}, \mathcal{A}, P) \longrightarrow (\mathbb{R}^N, \mathcal{B}_d)$ measurable maps from a probability space of reference $(\mathcal{S}, \mathcal{A}, P)$ onto the Lebesgue space $(\mathbb{R}^N, \mathcal{B}_d)$ such that the laws or image measures are $V\#P = f$ and $X\#P = g$. Let us remind that the law or the image measure by the measurable map $V : (\mathcal{S}, \mathcal{A}, P) \longrightarrow (\mathbb{R}^N, \mathcal{B}_d)$ or the push-forward of P through the map V is defined as

$$V_\# P[K] := P[V^{-1}(K)]$$

for each Borel set $K \subset \mathbb{R}^N$, or equivalently, by duality as the measure $V_\# P$ satisfying

$$\int_{\mathbb{R}^N} \varphi \, d(V_\# P) = \int_{\mathcal{S}} (\varphi \circ V) \, dP$$

for all $\varphi \in C_b(\mathbb{R}^N)$. We will also make use of the expression V transports P onto f whenever $V \# P = f$.

Let us point out that the Euclidean Wasserstein distance W_2 is finite for any two probability measures with finite second moments $f, g \in \mathcal{P}_2(\mathbb{R}^N)$. Also, let us remark that in the sequel we will denote by $f(v)\, dv$ or $df(v)$ the integration with respect to the measure $f(v)$ independently of being absolutely continuous with respect to Lebesgue measure or not. If there is any need of such a distinction we will explicitly mention it.

Finally, let us remark that this distance is related to the classical Monge's optimal mass transport problem, namely to the problem of finding a map $T : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ such that

$$I := \inf_{T \, with \, g = T \# f} \left\{ \int_{\mathbb{R}^N} |v - T(v)|^2 \, df(v) \right\}^{1/2}.$$

In fact, the definition of the Euclidean Wasserstein distance is a relaxed variational problem of the previous question since by taking $\Pi_T = (1_{\mathbb{R}^N} \times T)\#\mu$ as candidate transference plan Π, one can see the previous set of maps as a subset of all possible transference plans.

In what follows, we summarize the main properties of the Euclidean Wasserstein distance W_2 that will be used in the rest, referring to [28, 106, 108] for the proofs. Further information on the connections to optimal mass transport theory can be found in [64, 92, 106, 108].

Proposition 2.1 (W_2-properties)**.** *The space $(\mathcal{P}_2(\mathbb{R}^N), W_2)$ is a complete metric space. Moreover, the following properties of the distance W_2 hold:*

i) **Optimal transference plan:** *The infimum in the definition of the distance W_2 is achieved at a joint probability measure Π_o called an optimal transference plan satisfying:*

$$W_2^2(f, g) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} |v - x|^2 \, d\Pi_o(v, x).$$

ii) **Convergence of measures:** *Given $\{f_n\}_{n \geq 1}$ and f in $\mathcal{P}_2(\mathbb{R}^N)$, the following three assertions are equivalent:*
   a) *$W_2(f_n, f)$ tends to 0 as n goes to infinity.*
   b) *f_n tends to f weakly-* as measures as n goes to infinity and*

$$\sup_{n \geq 1} \int_{|v| > R} |v|^2 \, f_n(v) \, dv \to 0 \;\; as \;\; R \to +\infty.$$

c) *f_n tends to f weakly-* as measures and*

$$\int_{\mathbb{R}^N} |v|^2 f_n(v)\, dv \to \int_{\mathbb{R}^N} |v|^2 f(v)\, dv \ \text{ as } \ n \to +\infty.$$

iii) **Lower semicontinuity:** *W_2 is weakly-* lower semicontinuous in each argument.*

iv) **Relation to Temperature:** *If f belongs to $\mathcal{P}_2(\mathbb{R}^N)$ and δ_a is the Dirac mass at a in $\mathbb{R}^N$, then*

$$W_2^2(f, \delta_a) = \int_{\mathbb{R}^N} |v - a|^2 df(v).$$

v) **Scaling:** *Given f in $\mathcal{P}_2(\mathbb{R}^N)$ and $\theta > 0$, let us define*

$$\mathcal{S}_\theta[f] = \theta^{N/2} f(\theta^{1/2} v) \tag{2.3}$$

for absolutely continuous measures with respect to Lebesgue measure or its corresponding definition by duality for general measures; then for any f and g in $\mathcal{P}_2(\mathbb{R}^N)$, we have

$$W_2^2(\mathcal{S}_\theta[f], \mathcal{S}_\theta[g]) = \frac{1}{\theta} W_2^2(f, g).$$

vi) **Convexity:** *Given f_1, f_2, g_1 and g_2 in $\mathcal{P}_2(\mathbb{R}^N)$ and α in $[0, 1]$, then*

$$W_2^2(\alpha f_1 + (1-\alpha) f_2, \alpha g_1 + (1-\alpha) g_2) \le \alpha W_2^2(f_1, g_1) + (1-\alpha) W_2^2(f_2, g_2).$$

As a simple consequence, given f, g and h in $\mathcal{P}_2(\mathbb{R}^N)$, then

$$W_2(h * f, h * g) \le W_2(f, g)$$

where $*$ stands for the convolution in $\mathbb{R}^N$.

vii) **Additivity with respect to convolution:** *Given f_1, f_2, g_1 and g_2 in $\mathcal{P}_2(\mathbb{R}^N)$ with with equal mean values, then*

$$W_2^2(f_1 * f_2, g_1 * g_2) \le W_2^2(f_1, g_1) + W_2^2(f_2, g_2).$$

Remark 2.2 (Superadditivity with respect to convolution)**.** *Coupling Property vii) with the Scaling property v), shows that, for any constant λ such that $0 < \lambda < 1$*

$$\begin{aligned} W_2^2(\mathcal{S}_{1/\lambda}[f_1] * \mathcal{S}_{1/(1-\lambda)}[f_2], \mathcal{S}_{1/\lambda}[g_1] * \mathcal{S}_{1/(1-\lambda)}[g_2]) \le \\ \le \quad \lambda W_2^2(f_1, g_1) + (1-\lambda) W_2^2(f_2, g_2). \end{aligned}$$

This property is usually referred as superadditivity with respect to convolutions*. To our knowledge, this property has been first derived by Tanaka in* [101]*, and it is at the basis of most of the applications of Wasserstein metric to kinetic theory. Property vii) of Proposition* 2.1 *is a direct consequence of its definition in terms of random variables. Let (X_1, Y_1), (X_2, Y_2) be two*

independent pairs of random variables, and let f_i (resp. g_i) be the laws of X_i (resp. Y_i) $i = 1, 2$. Suppose moreover that X_i and Y_i have the same mean value, namely $\mathbb{E}[X_i] = \mathbb{E}[Y_i]$ $i = 1, 2$. If the pairs (X_1, Y_1), (X_2, Y_2) realize the optimal transference plans, then for $i = 1, 2$

$$W_2^2(f_i, g_i) = \mathbb{E}\left[|X_i - Y_i|^2\right].$$

In this case

$$\begin{aligned} W_2^2(f_1 * f_2, g_1 * g_2) &\leq \mathbb{E}\left[|(X_1 + X_2) - (Y_1 + Y_2)|^2\right] \\ &= \mathbb{E}\left[|X_1 - Y_1|^2\right] + \mathbb{E}\left[|X_2 - Y_2|^2\right] + 2\,\mathbb{E}\left[(X_1 - Y_1) \cdot (X_2 - Y_2)\right] \\ &= W_2^2(f_1, g_1) + W_2^2(f_2, g_2) \end{aligned}$$

In fact, the term $\mathbb{E}\left[(X_1 - Y_1) \cdot (X_2 - Y_2)\right]$ is equal to zero due to the independence of the pairs, and to the equality of the mean values. This property will be quite useful in Section 6.

Remark 2.3 (Completeness of Spheres in W_2)**.** *A simple consequence of the previous Proposition is that the set*

$$\mathcal{M}_\theta = \left\{ \mu \in \mathcal{P}_2(\mathbb{R}^N) \text{ such that } \int_{\mathbb{R}^3} |v|^2 df(v) = 3\theta \right\},$$

i.e., the "sphere" of radius $\sqrt{3\theta}$ in $\mathcal{P}_2(\mathbb{R}^N)$ centered at δ_0, endowed with the distance W_2 is a complete metric space.

We also remind the reader that convergence in W_2-sense implies the convergence of averages or observables in physical space. We will denote by $\mathrm{Lip}(\mathbb{R}^N)$ the set of Lipschitz functions on $\mathbb{R}^N$ and by $W^{1,\infty}(\mathbb{R}^N)$ the set of bounded and Lipschitz functions on $\mathbb{R}^N$.

Corollary 2.4 (Convergence of averages with W_2)**.** *Given $\varphi \in \mathrm{Lip}(\mathbb{R}^N)$ with Lipschitz constant L, then we have*

$$\left| \int_{\mathbb{R}^N} \varphi(v)(f(v) - g(v))\, dv \right| \leq L\, W_2(f, g).$$

Proof.- Let $\Pi_o(v, w)$ the optimal plan between f and $g \in \mathcal{P}_2(\mathbb{R}^N)$ for W_2. Then

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} |v - w|^2 d\Pi_o(v, w) = W_2^2(f, g),$$

and we can write

$$\int_{\mathbb{R}^N} \varphi(v)(f(v) - g(v))\, dv = \int_{\mathbb{R}^N \times \mathbb{R}^N} (\varphi(v) - \varphi(w))\, d\Pi_o(v, w).$$

Using that φ is Lipschitz with constant L and estimating by Hölder's inequality, we get

$$\left|\int_{\mathbb{R}^N}\varphi(v)(f(v)-g(v))\,dv\right| \le \int_{\mathbb{R}^N\times\mathbb{R}^N}|\varphi(v)-\varphi(w)|\,d\Pi_o(v,w)$$
$$\le L\int_{\mathbb{R}^N\times\mathbb{R}^N}|v-w|\,d\Pi_o(v,w)\le L\,W_2(f,g)$$

giving the assertion. □

Let us point out that the previous corollary is based on the fact that the distance W_2 controls the distance associated with the cost $c(v,x)=|v-x|$. In fact, the optimal mass transportation cost distances can be generalized in the following way:

$$W_p(f,g)=\inf_{\pi\in\Gamma}\left\{\iint_{\mathbb{R}^N\times\mathbb{R}^N}|v-x|^p\,d\pi(v,x)\right\}^{1/p}=\inf_{(V,X)\in\tilde{\Gamma}}\left\{\mathbb{E}\left[|V-X|^p\right]\right\}^{1/p} \tag{2.4}$$

for any given $1\le p<\infty$ and $f,g\in\mathcal{P}(\mathbb{R}^N)$. Denoting by $\mathcal{P}_p(\mathbb{R}^N)$ the set of $f,g\in\mathcal{P}(\mathbb{R}^N)$ with moments up to order p bounded, the distance W_p is well-defined and finite on $\mathcal{P}_p(\mathbb{R}^N)$ and properties similar to those of Proposition 2.1 hold. Again, we refer to [106, 108] for further details. Finally, let us remark that by Hölder's inequality it follows that the sequence $W_p(f,g)$ is nondecreasing as a function of p, and thus, the distance $W_\infty(f,g)$ can be defined as

$$W_\infty(f,g):=\lim_{p\nearrow\infty}W_p(f,g).$$

In the noticeable case $p=1$, the distance W_1 is also called the Kantorovich-Rubinstein distance or the dual-Lipschitz norm. In fact, as a consequence of Fenchel-Rockafellar's duality principle one has [106, Theorem 1.14] that the W_1 distance can be characterized as

$$W_1(f,g)=\sup\left\{\left|\int_{\mathbb{R}^N}\varphi(v)(f(v)-g(v))\,dv\right|,\varphi\in\mathrm{Lip}(\mathbb{R}^N),\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^N)}\le 1\right\} \tag{2.5}$$
$$=\sup\left\{\left|\int_{\mathbb{R}^N}\varphi(v)(f(v)-g(v))\,dv\right|,\varphi\in W^{1,\infty}(\mathbb{R}^N),\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^N)}\le 1\right\}.$$

As it was already observed above, Corollary 2.4 can be seen as a simple consequence of the W_1-characterization together with $W_1(f,g)\le W_2(f,g)$ for any $f,g\in\mathcal{P}_2(\mathbb{R}^N)$.

These metrics has been considered in the PDE's analytic community quite recently in connection to gradient flows and steepest descent schemes

of linear and nonlinear diffusions [74, 87, 1, 47, 3] and homogeneous kinetic models [43, 3, 44] as well as for describing their asymptotic behavior.

It is maybe not so well known that Wasserstein metrics have a very rich history, with a number of historical sources. Apparently, the denomination *Vasershtein distance* appeared for the first time in [55]. For any pair of probability measures (f, g) on a metric space $(\mathcal{M}, d)$, L.N. Vasershtein [109] indeed introduced the metric

$$\nu(f,g) = \inf_{(V,X)\in\tilde{\Gamma}} \{\, \mathbb{E}\,[d(X,V)]\,\}\,.$$

His work had a great impact especially in ergodic theory in connection with generalizations of the Ornstein isomorphism theorem [69]. In subsequent times it became common both to use *Wasserstein* as the English version of the Russian name and the notation $W(f, g)$ for $\nu(f, g)$. However, the minimal L_1-metric ν was introduced and investigated already in 1940 by L.V. Kantorovich for compact metric spaces [76]. His work was motivated by the classical Monge transportation problem. Subsequently, the transportation distance was generalized to general cost functionals. The famous Kantorovich-Rubinshtein theorem [77] gave a dual representation of the minimal L_1-metric ν in terms of a Lipschitz metric. From this point of view, the notion of a Kantorovich metric or minimal L_1-metric also would be historically appropriate. Related works, however, were already present -and presumably unknown to the Russian school- in the probabilistic literature. In fact, in 1914, C. Gini, while introducing a *simple index of dissimilarity*, first defined the metric in a discrete setting on the real line and T. Salvemini (the discrete case, [95]) and G. Dall'Aglio (the general case, [53]) proved the basic one-dimensional representation

$$W_p^p(F,G) = \int_0^1 \left|F^{-1}(\eta) - G^{-1}(\eta)\right|^p \, d\eta,$$

where F, G are the distribution functions of f, g. Gini had given this formula for empirical distributions and $p = 1, 2$. This influential work initiated a lot of research on measures with given marginals in the Italian School of probability, while M. Fréchet [61] explicitly dealt with metric properties of these distances.

Almost at the same time of the work of Wasserstein, C.L. Mallows [80] introduced independently the ν-metric in a statistical context. He used its properties for proving a central limit theorem and reobtained the representation above. Based on Mallows work, P.J. Bickel and D.A. Freedman [11] described topological properties and investigated applications to statistical problems such as the bootstrap. They introduced the notion of a Mallows

metric for the nowadays W_2-distance. This notion is used mainly in the statistics literature and in some literature on algorithms.

Amazingly the W_2-metric was introduced at the same time into kinetic theory by H. Tanaka in [100] to recover the long time asymptotics of the Kac caricature of a Maxwell gas [75]. In this case also, convergence to the Maxwellian equilibrium corresponds to prove a generalized central limit theorem. The importance of this metric in connection with the large-time behavior of more realistic Boltzmann-type equations was subsequently dealt with by Tanaka in 1978 [101]. This seminal work had a noticeable impact in the kinetic community, where the W_2-metric has been known for many years under the denomination of Tanaka functional [27, 90].

The preceding historical discussion enlightens at least two facts. First, Wasserstein-like metrics are quite useful into several different fields of applications. Second, taking into account the various historical sources, maybe the unbearable name $GDKRVMT$-metric, (Gini-Dall'Aglio-Kantorovich-Rubinstein-Vasershtein-Mallows-Tanaka)-metric!, would be more correct for this class of distances.

2.2. **One-dimensional Wasserstein metric.** As briefly discussed in Subsection 2.1, the one-dimensional case is of independent interest, due to its colourful history, that goes back to the works of [95, 53]. In this case, in fact one can resort to a basic representation, which allows in general for almost explicit computations. In the sequel we will present a simple way to derive this one-dimensional representation, by resorting to a key result of Höffding [72]. This method of proof was suggested to Tanaka [100] as an alternative to its proof, and it is reported at the end of his paper.

Let Γ denote as in (2.1) the set of transference plans, that is, the set of joint probability measures on $\mathbb{R} \times \mathbb{R}$ with marginals f and $g \in \mathcal{P}(\mathbb{R})$. Denoting by $F(v)$ the distribution function of f,

$$F(v) = \int_{-\infty}^{v} df,$$

and $G(x)$ the distribution function of G, then the set of transference plans is equivalent to the set $\Gamma(F, G)$ of cumulative probability distributions functions in $(v, x) \in \mathbb{R}^2$ for which the corresponding measure in $\mathbb{R} \times \mathbb{R}$ has marginals f and g.

Within $\Gamma(F, G)$ there are cumulative probability distribution functions H^* and H_* discovered by Hoeffding [72] and Fréchet [61] which have maximum and minimum correlation. Let $x^+ = \max\{0, x\}$ and $x \wedge y = \min\{x, y\}$. Then, owing to the properties of the probability distributions, it is a simple exercise to conclude that in $\Gamma(F, G)$ for all $(v, x) \in \mathbb{R}^2$,

$$H^*(v, x) = F(v) \wedge G(x) \quad \text{and} \quad H_*(v, x) = [F(v) + G(x) - 1]^+.$$

The extremal distributions can also be characterized in another way, based on certain familiar properties of uniform distributions. Given any $\eta \in (0,1)$, let

$$F^{-1}(\eta) = \inf\{v : F(v) > \eta\}$$

denote the pseudo inverse function of the distribution function $F(v)$. If X is a real–valued random variable with distribution function F, and U is a random variable uniformly distributed on $[0,1]$, it follows that $F^{-1}(U)$ has distribution function F, or equivalently, $f = F^{-1}\#d\eta$ where $d\eta$ is the Lebesgue measure in the interval $[0,1]$.

Moreover, for any F, G with finite variances the pair of random variables $[F^{-1}(U), G^{-1}(U)]$ has cumulative distribution function H^*, or equivalently, $(F^{-1}\times G^{-1})\#d\eta$ has joint distribution function $H^*(v,x) = \min(F(v), G(x))$ [110, 106, 108]. Consequently,

$$W_2(f,g)^2 = \inf_{\Pi\in\Gamma} \iint_{\mathbb{R}\times\mathbb{R}} |v-x|^2 \, d\Pi(v,x) = \iint_{\mathbb{R}\times\mathbb{R}} |v-x|^2 \, d\Pi_o(v,x) \quad (2.6)$$

where in the last integral Π_o denotes the measure in the product space $\mathbb{R}\times\mathbb{R}$ induced by the joint distribution function H^*.

In fact, given an arbitrary random vector (V,X) with cumulative distribution function H with marginals f and g, thanks to a result by Höffding [72]

$$\begin{aligned}\mathbb{E}(VX) - \mathbb{E}(V)\mathbb{E}(X) &= \iint_{\mathbb{R}\times\mathbb{R}} [H(v,x) - F(v)G(x)] \, dv\, dx \\ &\le \iint_{\mathbb{R}\times\mathbb{R}} [H^*(v,x) - F(v)G(x)] \, dv\, dx,\end{aligned}$$

and this implies (2.6). Recalling now that $[F^{-1}(U), G^{-1}(U)]$ has cumulative joint distribution function H^* [110], or equivalently, that the measure $(F^{-1}\times G^{-1})\#d\eta$ has joint distribution function $H^*(v,x)$, one can conclude that the Wasserstein distance between F and G can be rewritten as the L^2-distance of the pseudo inverse functions

$$W_2(f,g) = \left(\int_0^1 [F^{-1}(\eta) - G^{-1}(\eta)]^2 \, d\eta\right)^{1/2}. \quad (2.7)$$

Hence, in the one-dimensional case, one has the explicit expression of the optimal transference plan, $\Pi_o = (F^{-1}\times G^{-1})\#d\eta$ with joint distribution H^*.

This easy expression of the optimal plan is not only for the euclidean cost but for all convex costs in one dimension [106, Theorem 2.18]. In fact, for all $1 \le p < \infty$, we obtain that the optimal plan for the variational problem

(2.4) coincides with $\Pi_o = (F^{-1} \times G^{-1})\#d\eta$ and that

$$W_p(f,g) = \left(\int_0^1 |F^{-1}(\eta) - G^{-1}(\eta)|^p \, d\eta \right)^{1/p}. \tag{2.8}$$

This also defines the ∞-Wasserstein distance in one dimension as

$$W_\infty(f,g) := \lim_{p \nearrow \infty} W_p(f,g) = \|F^{-1} - G^{-1}\|_{L^\infty(0,1)}. \tag{2.9}$$

2.3. **Fourier-based metrics.** Given $f \in \mathcal{P}(\mathbb{R}^N)$, its Fourier transform or characteristic function is defined as

$$\hat{f}(k) = \int_{\mathbb{R}^N} e^{-iv\cdot k} \, df(v).$$

Given a smooth function $\Psi(k)$, we will denote by $D^\beta \Psi(k)$ its derivative of order $|\beta|$ given by the multi-index $\beta \in \mathbb{N}^r$, $r \in \mathbb{N}$, and by $D^m \Psi$, for all $m \in \mathbb{N}$, its differential of order m verifying for all $k, a \in \mathbb{R}^N$

$$D^m \Psi(a)(k, \dots, k) = \sum_{|\beta|=m} \frac{k^\beta}{\beta!} D^\beta \Psi(a).$$

With this notation, Taylor's formula up to order m centered at 0 can be written as

$$\Psi(k) = \sum_{l=0}^{m-1} D^l \Psi(0)(k, \dots, k) + \int_0^1 D^m \Psi(tk)(k, \dots, k) \, dt, \tag{2.10}$$

for all $k \in \mathbb{R}^N$, and $\widehat{(-iv)^\beta} f = D^\beta \hat{f}$.

Given any $s > 0$, the Fourier based metric d_s is defined as

$$d_s(f,g) = \sup_{k \in \mathbb{R}^N_o} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^s} \tag{2.11}$$

where $\mathbb{R}^N_o = \mathbb{R}^N - \{0\}$, for any pair of probability measures $f, g \in \mathcal{P}(\mathbb{R}^N)$.

Despite the coloured history of the Wasserstein metric, the Fourier based metric has been introduced only recently in connection with the study of the large-time asymptotics of the Boltzmann equation for Maxwell molecules in [62]. There, the case $s = 2+\alpha$, $\alpha > 0$, was considered. Further applications of d_s, with $s = 4$, were studied in [38], while the cases $s = 2$ and $s = 2 + \alpha$, $\alpha > 0$, have been considered in [37] in connection with the so-called Mc Kean graphs. The case $s = 2$ was subsequently used in [103], in connection with the uniqueness of the non cut-off Boltzmann equation for Maxwell molecules. A further application of the general case $s > 0$ to the finding of Berry–Essen type bounds in the central limit theorem for a stable law has been given in [68]. Only recently, various applications to the large-time

behavior of the dissipative Boltzmann equation [91, 12, 13] enlightened the importance of this distance even in this case.

In order to check under which conditions the metric d_s is well-defined and finite, we need the following result showing us how to trade integrability estimates for f for regularity of the Fourier transform $\hat{f}$:

Lemma 2.5 (Uniform Modulus of Continuity for Derivatives)**.** [62, Lemma 3.1] *Given an strictly increasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\phi(r)/r$ non increasing and let $\psi(y) := [\phi(y^{-1})]^{-1}$. Given $m \in \mathbb{N}$, if*

$$M_\phi := \int_{\mathbb{R}^N} (1 + |v|^m)\phi(|v|)\, df(v) < \infty,$$

then

$$|D^\beta \hat{f}(k) - D^\beta \hat{f}(\tilde{k})| \leq 2\, M_\phi\, \psi(|k - \tilde{k}|)$$

for all β multi-index of order m.

Proof.- Since f has moments of order m bounded and $\widehat{(-iv)^\beta f} = D^\beta \hat{f}$, we deduce that

$$\begin{aligned} |D^\beta \hat{f}(k) - D^\beta \hat{f}(\tilde{k})| &\leq 2 \int_{\mathbb{R}^N} \left| \sin\left(\frac{(k - \tilde{k}) \cdot v}{2} \right) \right| |v|^\beta\, df(v) \\ &\leq 2 \left(\int_{\mathbb{R}^N} (1 + |v|^m)\phi(|v|) df(v) \right) \left(\sup_{v \in \mathbb{R}^N_o} \frac{\left| \sin\left(\frac{(k-\tilde{k})v}{2} \right) \right|}{\phi(|v|)} \right). \end{aligned}$$

Now, using the elementary inequality $|\sin(z)| \leq \max(|z|, 1)$ for all $z \in \mathbb{R}$, we get

$$\sup_{v \in \mathbb{R}^N_o} \frac{\left| \sin\left(\frac{(k-\tilde{k}) \cdot v}{2} \right) \right|}{\phi(|v|)} \leq \sup_{v \in \mathbb{R}^N_o} \frac{\max\left(\frac{|k-\tilde{k}||v|}{2}, 1 \right)}{\phi(|v|)} \leq \left[\phi\left(\frac{2}{|k - \tilde{k}|} \right) \right]^{-1} \leq \psi(|k - \tilde{k}|)$$

due to the assumptions on ϕ since

$$\frac{\max(xy, 1)}{\phi(x)} \leq \left[\phi\left(\frac{1}{y} \right) \right]^{-1}$$

for all $x, y \in \mathbb{R}^+$. □

In order to precise some statements below, we will say that two probability measures $f, g \in \mathcal{P}(\mathbb{R}^N)$ have equal moments up to $m \in \mathbb{N}$ if

$$\int_{\mathbb{R}^N} v^\beta\, df(v) = \int_{\mathbb{R}^N} v^\beta\, dg(v) < \infty$$

for all multi-indices $|\beta| \leq m$.

Proposition 2.6 (Finiteness of d_s)**.** *Given any two probability measures $f, g \in \mathcal{P}_s(\mathbb{R}^N)$ with $s > 0$ with equal moments up to $[s]$ if $s \notin \mathbb{N}$, or equal moments up to $s-1$ if $s \in \mathbb{N}$, then $d_s(f,g) < +\infty$.*

Proof.- Assume $s \notin \mathbb{N}$, equality of moments up to order $[s]-1$ and Taylor expansion (2.10) up to order $m = [s]$ for any $f \in \mathcal{P}_s(\mathbb{R}^N)$ imply that

$$\begin{aligned} |\hat{f}(k) - \hat{g}(k)| &\leq \int_0^1 \left| (D^m \hat{f}(tk) - D^m \hat{g}(tk)) \left(\frac{k}{|k|}, \dots, \frac{k}{|k|} \right) \right| \, dt \, |k|^m \\ &\leq C \sum_{|\beta|=m} \int_0^1 \left| (D^\beta \hat{f}(tk) - D^\beta \hat{g}(tk)) \right| \, dt \, |k|^m. \end{aligned} \tag{2.12}$$

Lemma 2.5 with $\phi(|v|) = |v|^{s-m}$ asserts that

$$\max\{|D^\beta \hat{f}(k) - D^\beta \hat{f}(0)|, |D^\beta \hat{g}(k) - D^\beta \hat{g}(0)|\} \leq C|k|^\alpha. \tag{2.13}$$

with $s = m + \alpha$ since $f, g \in \mathcal{P}_s(\mathbb{R}^N)$. Since moments of order m are equal, then $D^\beta \hat{f}(0) = D^\beta \hat{g}(0)$ for $|\beta| = m$ and

$$|\hat{f}(k) - \hat{g}(k)| \leq C \sum_{|\beta|=m} \int_0^1 \left| (D^\beta \hat{f}(tk) - D^\beta \hat{f}(0)) - (D^\beta \hat{g}(tk) - D^\beta \hat{g}(0)) \right| \, dt \, |k|^m.$$

We conclude that $d_s(f,g) < \infty$ by triangular inequality using (2.13). Finally, the case $s \in \mathbb{N}$ follows directly from Taylor expansion (2.12) up to order $m = s$ since all moments up to order $s-1$ are equal and trivially

$$\max\{|D^\beta \hat{f}(k)|, |D^\beta \hat{g}(k)|\} \leq C \, \max\{\int_{\mathbb{R}^N} |v|^m \, df(v), \int_{\mathbb{R}^N} |v|^m \, dg(v)\},$$

for all multi-indices β of order m and all $k \in \mathbb{R}^N$. □

We now show that these distances endow certain probability sets with a complete metric. Given $s, \alpha > 0$, let us denote by $X_{s,\alpha,M}$ the set of probability measures $f \in \mathcal{P}_{s+\alpha}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} v^\beta \, df(v) = M_\beta \in \mathbb{R}^+$$

for all multi-indices $|\beta| \leq [s]$ with M_β fixed numbers and

$$\int_{\mathbb{R}^N} |v|^{s+\alpha} \, df(v) \leq M_{s+\alpha} \in \mathbb{R}^+$$

being the set of all M_β and $M_{s+\alpha}$ denoted simply by M.

Proposition 2.7 (Completeness in d_s)**.** *The set $X_{s,\alpha,M}$ endowed with the distance d_s is a complete metric space.*

Proof.- Let us consider $\{f_n\}$ a Cauchy sequence in $X_{s,\alpha,M}$ for the distance d_s. The definition of d_s implies obviously that for all $k \in \mathbb{R}^N_o$, the sequence $\{\hat{f}_n(k)\}$ is Cauchy and thus convergent towards a limit defined as $g(k)$; on the other hand, $\hat{f}_n(0) = 1 := g(0)$. Moreover, a direct use of Lemma 2.5, implies that all derivatives of the characteristic functions $\hat{f}_n$ are equicontinuous, precisely,

$$|D^\beta \hat{f}_n(k) - D^\beta \hat{f}_n(\tilde{k})| \leq \psi_{|\beta|}(|k - \tilde{k}|)$$

for all β multi-index of order $|\beta| \leq [s]$ and suitable modulus of continuity functions $\psi_{|\beta|}$ given in Lemma 2.5. Thus, Ascolí-Arzelá theorem implies that the convergence of the characteristic functions $\{\hat{f}_n\}$ is not only pointwise convergence but also uniformly in $C^{|s|}(\mathbb{R}^N)$. In particular, the limit of the characteristics functions verify $g \in C_b(\mathbb{R}^N)$.

Levy's continuity theorem [56] implies that g is a characteristic function, i.e., $g = \hat{f}$ with $f \in \mathcal{P}(\mathbb{R}^N)$ and that $f_n \rightharpoonup f$ weakly-* as measures; it remains to prove that $f \in X_{s,\alpha,M}$. This is a simple consequence of the tightness estimate

$$\int_{\mathbb{R}^N} |v|^{s+\alpha}\, df_n(v) \leq M_{s+\alpha}$$

for all $n \in \mathbb{N}$, that together with the weak-* convergence $f_n \rightharpoonup f$ implies that

$$\int_{\mathbb{R}^N} |v|^{s+\alpha}\, df(v) \leq M_{s+\alpha} \qquad \text{and} \qquad \int_{\mathbb{R}^N} v^\beta\, df(v) = M_\beta$$

for all multi-indices $|\beta| \leq [s]$.

Now, let us show the convergence of $\{f_n\}$ towards f in d_s. Proceeding as in the proof of Proposition 2.6, equality of moments up to order $[s]$ and Taylor expansion (2.10) up to order $m = [s]$ for any $f \in \mathcal{P}_s(\mathbb{R}^N)$ imply that

$$\begin{aligned}
\frac{|\hat{f}_n(k) - \hat{f}(k)|}{|k|^m} &\leq \int_0^1 \left| (D^m \hat{f}_n(tk) - D^m \hat{f}(tk)) \left(\frac{k}{|k|}, \dots, \frac{k}{|k|} \right) \right| dt \\
&\leq C \sum_{|\beta|=m} \int_0^1 \left| (D^\beta \hat{f}_n(tk) - D^\beta \hat{f}(tk)) \right| dt \\
&\leq C \sum_{|\beta|=m} \int_0^1 \left| (D^\beta \hat{f}_n(tk) - D^\beta \hat{f}_n(0)) - (D^\beta \hat{f}(tk) - D^\beta \hat{f}(0)) \right| dt
\end{aligned}$$

for all $k \in \mathbb{R}^N_o$. Lemma 2.5 with $\phi(|v|) = |v|^{s+\gamma-m}$ and $\gamma > 0$ such that $0 < s + \gamma - m < \min(1, s - m + \alpha)$ asserts that

$$\max\{|D^\beta \hat{f}_n(k) - D^\beta \hat{f}_n(0)|, |D^\beta \hat{f}(k) - D^\beta \hat{f}(0)|\} \leq C|k|^{s+\gamma-m}$$

for all multi-indices $|\beta| = m$, uniformly in n, since

$$\int_{\mathbb{R}^N} |v|^{s+\alpha}\, df_n(v) \le M_{s+\alpha} \qquad \text{and} \qquad \int_{\mathbb{R}^N} |v|^{s+\alpha}\, df(v) \le M_{s+\alpha}.$$

Summarizing we deduce that, for all $k \in \mathbb{R}^N_o$

$$\frac{|\hat{f}_n(k) - \hat{f}(k)|}{|k|^s} \le C\, |k|^\gamma$$

uniformly in n, and thus, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{0<|k|<\delta} \frac{|\hat{f}_n(k) - \hat{f}(k)|}{|k|^s} \le \epsilon$$

uniformly in n. On the other hand, since we are dealing with probability measures, it is immediate to conclude that there exists $R > 0$ such that

$$\sup_{|k|>R} \frac{|\hat{f}_n(k) - \hat{f}(k)|}{|k|^s} \le \epsilon$$

uniformly in n. Collecting the last estimates, we deduce

$$\begin{aligned} d_s(f_n, f) &\le \max\left\{\epsilon, \sup_{\delta<|k|<R} \frac{|\hat{f}_n(k) - \hat{f}(k)|}{|k|^s}\right\} \\ &\le \max\left\{\epsilon, \frac{1}{\delta^s} \sup_{\delta<|k|<R} |\hat{f}_n(k) - \hat{f}(k)|\right\} \end{aligned}$$

Finally, since $\{\hat{f}_n\} \to \hat{f}$ uniformly in $C_b(\mathbb{R}^N)$, there exists $n_o \in \mathbb{N}$ such that for $n \ge n_0$, we get

$$\sup_{\delta<|k|<R} |\hat{f}_n(k) - \hat{f}(k)| \le \epsilon$$

and thus, $d_s(f_n, f) \le \epsilon$ finishing the proof. □

Remark 2.8 (Open problem about Completeness)**.** *Previous proof does not assert the completeness of the set of probability measures $f \in \mathcal{P}_m(\mathbb{R}^N)$, with $m \in \mathbb{N}$, such that*

$$\int_{\mathbb{R}^N} v^\beta\, df(v) = M_\beta \in \mathbb{R}^+$$

for all multi-indices $|\beta| \le m$ with M_β given, endowed with the distance d_m. In fact, we need to control a $m + \alpha$-moment, with α arbitrarily small. It would be nice to prove or rather disprove such statement at least for the d_2 distance, cf. [103, Theorem 1]. *This makes an important difference between W_2 and d_2 in view of Remark* 2.3.

Let us now review the main properties of the d_s metrics.

Proposition 2.9 (d_s-properties)**.** *The distances d_s with $s > 0$ verify the following properties:*

i) **Interpolation of metrics:** *Given any two probability measures $f, g \in \mathcal{P}_s(\mathbb{R}^N)$ with $q > 0$ with equal moments up to $[s]$ if $s \notin \mathbb{N}$, or equal moments up to $s-1$ if $s \in \mathbb{N}$, then*

$$d_p(f,g) \leq 2\left(\frac{s-p}{2\,p}\right)^{p/s} \frac{s}{s-p}\,[d_s(f,g)]^{p/s} = C_{p,s}\,[d_s(f,g)]^{p/s}\,.$$

for any $0 < p < s$.

ii) **Control of moments:** *Given any two probability measures $f, g \in \mathcal{P}_s(\mathbb{R}^N)$ with $s \in \mathbb{N}$, $s > 0$, with equal moments up to $s-1$, then*

$$\left|\int_{\mathbb{R}^N} v^\beta\, df(v) - \int_{\mathbb{R}^N} v^\beta\, dg(v)\right| \leq C\, d_s(f,g),$$

for all multi-indices β with $|\beta| = s$.

iii) **Scaling:** *Given any two probability measures $f, g \in \mathcal{P}_s(\mathbb{R}^N)$ with $s > 0$ with equal moments up to $[s]$ if $s \notin \mathbb{N}$, or equal moments up to $s-1$ if $s \in \mathbb{N}$, then*

$$d_s(\mathcal{S}_\theta[f], \mathcal{S}_\theta[g]) = \theta^{-s/2}\, d_s(f,g),$$

where $\mathcal{S}_\theta[f]$ is given by (2.3)*.*

iv) **Convexity:** *Given f_1, f_2, g_1 and g_2 in $\mathcal{P}_s(\mathbb{R}^N)$ with $s > 0$ with equal moments up to $[s]$ if $s \notin \mathbb{N}$, or equal moments up to $s-1$ if $s \in \mathbb{N}$ and α in $[0,1]$, then*

$$d_s(\alpha f_1 + (1-\alpha) f_2, \alpha g_1 + (1-\alpha) g_2) \leq \alpha d_s(f_1, g_1) + (1-\alpha) d_s(f_2, g_2).$$

v) **Superadditivity with respect to convolution:** *Given f_1, f_2, g_1 and g_2 in $\mathcal{P}_s(\mathbb{R}^N)$ with $s > 0$ with equal moments up to $[s]$ if $s \notin \mathbb{N}$, or equal moments up to $s-1$ if $s \in \mathbb{N}$, then*

$$d_s(f_1 * f_2, g_1 * g_2) \leq d_s(f_1, g_1) + d_s(f_2, g_2).$$

Proof.- The first statement *i)* comes from

$$d_p(f,g) = \sup_{k \in \mathbb{R}^N_o} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^p} \leq \sup_{0<|k|\leq R} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^p} + \sup_{|k|>R} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^p}$$

$$\leq \sup_{0<|k|\leq R} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^s}\, R^{s-p} + \frac{2}{R^p} \leq d_s(f,g)\, R^{s-p} + \frac{2}{R^p}\,. \quad (2.14)$$

for any $R > 0$. Optimizing the function in the right-hand side of (2.14) over R, we obtain the desired result.

Let us now focus on statement *ii)*. Given $f, g \in \mathcal{P}_s(\mathbb{R}^N)$ their Fourier transforms belong to $C^{[s]}(\mathbb{R}^N)$ and Taylor's formula (2.10) of order $m = [s]$ implies

$$D^m(\hat{g}_1 - \hat{g}_2)(0)(\eta, \cdots, \eta) = \lim_{\lambda \to 0^+} \frac{\hat{g}_1(\lambda\eta) - \hat{g}_2(\lambda\eta)}{\lambda^m},$$

for all $\eta \in \mathbb{R}^N$ with $|\eta| = 1$. Now, putting this together with the definition of d_m, we get

$$|D^m(\hat{g}_1 - \hat{g}_2)(0)(\eta, \cdots, \eta)| \leq d_m(\hat{g}_1, \hat{g}_2)$$

for all $\eta \in \mathbb{R}^N$ with $|\eta| = 1$. It is not difficult to see -but cumbersome to write- that an induction argument, based on taking particular choices of the vector η as the canonical vectors in $\mathbb{R}^N$ and normalized sums of them, show that

$$\left| \int_{\mathbb{R}^N} v^\beta \, df(v) - \int_{\mathbb{R}^N} v^\beta \, dg(v) \right| \leq C \, d_m(f, g),$$

for all multi-indices β with $|\beta| = m$.

The third statement *iii)* is an easy consequence of the scaling property of the Fourier transform $\widehat{\mathcal{S}_\theta[f]}(k) = \hat{f}(\theta^{-1/2}k)$ and the definition of d_s.

The fourth statement *iv)* follows trivially from triangular inequality and the definition of d_s.

Finally, the convolution property *v)* is straightforward due to $\widehat{f * g} = \hat{f}\hat{g}$, the triangular inequality and the definition of d_s. □

Remark 2.10 (Comparison of Convolution Properties of W_4 and d_4)**.** *While properties from i) to iv) of Proposition* 2.9 *are analogous to thats of Proposition* 2.1*, the convolution property v) of the Fourier metric is noticeably different, in that it holds independently of the value of the index s. Unlikely, this is not the case for the Wasserstein distance W_s, with $s > 2$, as it can be easily checked by using the same arguments of Remark* 2.2*. Let (X_1, Y_1), (X_2, Y_2) be two independent pairs of random variables in $\mathcal{P}_4(\mathbb{R}^N)$, and let f_i (resp. g_i) be the laws of X_i (resp. Y_i) $i = 1, 2$. Suppose moreover that X_i and Y_i have the same mean value, namely $\mathbb{E}[X_i] = \mathbb{E}[Y_i]$ $i = 1, 2$. If the pairs (X_1, Y_1), (X_2, Y_2) realize the optimal transference plans for W_4, then for $i = 1, 2$*

$$W_4^4(f_i, g_i) = \mathbb{E}\left[|X_i - Y_i|^4\right].$$

Therefore, developing the fourth power, one obtains

$$\begin{aligned} W_4^4(f_1 * f_2, g_1 * g_2) &\leq \mathbb{E}\left[|(X_1 + X_2) - (Y_1 + Y_2)|^4\right] \\ &\leq \mathbb{E}\left[|X_1 - Y_1|^4\right] + \mathbb{E}\left[|X_2 - Y_2|^4\right] \\ &\quad + 6\,\mathbb{E}\left[|X_1 - Y_1|^2\right]\mathbb{E}\left[|X_2 - Y_2|^2\right] \end{aligned}$$

In fact, all the terms containing an odd power of the difference are equal to zero due to the independence of the pairs, and to the equality of their mean values. On the other hand, the last term on the right-hand side of the inequality is not negligible, since

$$\mathbb{E}\left[|X_1 - Y_1|^2\right] \mathbb{E}\left[|X_2 - Y_2|^2\right] \geq W_2^2(f_1, g_1) W_2^2(f_2, g_2).$$

This does not make possible to obtain a strict contraction with high-order W_s distances as it will be noted in Section 6.

Let us now come back to two of the metrics which will become important later on. On the basis of their definition, it is evident that both Euclidean Wasserstein distance W_2 and the Fourier based metric d_2 enjoy various common properties. In particular, W_2^2 and d_2 scale in the same way, and have the convexity and convolution properties as seen in Propositions 2.1 and 2.9.

Remark 2.11 (Superadditivity of metrics)**.** *In their applications to the central limit theorem, however, the importance of Wasserstein metric W_2 and d_2 mainly relies on their* superadditivity *with respect to rescaled convolutions. Let (X_0, Y_0), (X_1, Y_1) be two independent pairs of random variables, and let f_i (resp. g_i) be the laws of X_i (resp. Y_i), $i = 0, 1$. For $0 < \lambda < 1$, let f_λ (resp. g_λ) be the law of $\sqrt{\lambda} X_0 + \sqrt{1-\lambda} X_1$ (resp. $\sqrt{\lambda} Y_0 + \sqrt{1-\lambda} Y_1$), i.e.*

$$f_\lambda = \frac{1}{\lambda^{N/2}} f_0\left(\frac{\cdot}{\sqrt{\lambda}}\right) * \frac{1}{(1-\lambda)^{N/2}} f_1\left(\frac{\cdot}{\sqrt{1-\lambda}}\right).$$

Then, Propositions 2.1 *and* 2.9 *imply*

$$W_2^2(f_\lambda, g_\lambda) \leq \lambda W_2^2(f_0, g_0) + (1-\lambda) W_2^2(f_1, g_1) \tag{2.15}$$

and

$$d_2(f_\lambda, g_\lambda) \leq \lambda d_2(f_0, g_0) + (1-\lambda) d_2(f_1, g_1). \tag{2.16}$$

Superadditivity is also known for convex functionals (relative entropies), like Boltzmann's relative entropy

$$H(f|M^f) = \int_{\mathbb{R}^N} f(v) \log \frac{f(v)}{M^f(v)}\, dv,$$

where f is a probability density and M^f is the Gaussian density with the same mean vector and variance as those of f. This means that the property (2.11) *holds with W_2^2 or d_2 replaced by H and $\{g_0, g_1\}$ replaced by $\{M^{f_0}, M^{f_1}\}$. This is a consequence of Shannon's entropy power inequality* [99, 16]*. The same property holds for the relative Fisher information*

[99, 16],

$$I(f|M^f) = \int_{\mathbb{R}^N} \left|\nabla \log f(v) - \nabla \log M^f(v)\right|^2 f(v)\, dv.$$

As discussed by Csiszar [52], *by means of the relative entropy H, one can define the so-called H-neighborhoods. Even if those do not define a topological space, in the usual sense, their topological structure is finer than the metric topology defined by the L^1-distance,*

$$\int_{\mathbb{R}^N} |f(v) - g(v)|\, dv \leq \sqrt{2H(f|g)},$$

which is the so-called Csiszar-Kullback inequality.

2.4. **Equivalence between probability metrics.** Thanks to the Remark 2.11, it is natural to ask wether or not the topology induced by the metrics W_2 and d_2 is equivalent in the sense of giving the same weak-* uniformity in a set of probability measures. In the rest of this section we will answer positively to this question, by establishing various relations between W_2 and d_2. In addition, this equivalence will help us to establish some properties of the Fourier-based metric d_2. Connections to other metrics for probability densities [111] will be established as well.

Proposition 2.12 (From W_2 to d_2)**.** *Given $f, g \in \mathcal{P}_2(\mathbb{R}^N)$ with equal mean value, then*

$$d_2(f,g) \leq \frac{1}{2} W_2^2(f,g) + \min\left(\int_{\mathbb{R}^N} |v|^2\, df(v), \int_{\mathbb{R}^N} |v|^2\, dg(v)\right)^{1/2} W_2(f,g)$$

or in probabilistic terms,

$$d_2(f,g) \leq \frac{1}{2} W_2^2(f,g) + \min\left(Var[X] + \mathbb{E}[X]^2,\, Var[Y] + \mathbb{E}[Y]^2\right)^{1/2} W_2(f,g)$$

for X and Y with laws f and g respectively such that $\mathbb{E}[X] = \mathbb{E}[Y]$.

Proof.- Let $\Pi_o(v,w)$ the optimal plan between f and $g \in \mathcal{P}_2(\mathbb{R}^N)$ for the Euclidean Wasserstein distance W_2. Since they have equal mean velocity, then

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} (v - w) d\Pi_o(v,w) = 0,$$

and we can write

$$\hat{f}(k) - \hat{g}(k) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \left(e^{-iv\cdot k} - e^{-iw\cdot k} + ik\cdot(v-w)\right) d\Pi_o(v,w).$$

Now, we can estimate the integrand as

$$\begin{aligned}|e^{-iv\cdot k}-e^{-iw\cdot k}+ik\cdot(v-w)| &\le |e^{-iw\cdot k}\left(e^{-i(v-w)\cdot k}-1+ik\cdot(v-w)\right)| \\ &\quad + |(e^{-iw\cdot k}-1)ik\cdot(v-w)| \\ &\le \frac{1}{2}|k\cdot(v-w)|^2+|k||w||k\cdot(v-w)| \\ &\le \frac{1}{2}|k|^2\,|v-w|^2+|k|^2|w||v-w|\end{aligned}$$

by applying Taylor's formula (2.10) to the function $e^{-i(v-w)\cdot k}$ up to order 2 and mean value theorem for the function $e^{-iw\cdot k}$. Finally, integrating we get

$$\begin{aligned}\frac{|\hat{f}(k)-\hat{g}(k)|}{|k|^2} &\le \int_{\mathbb{R}^N\times\mathbb{R}^N}\left(\frac{1}{2}|v-w|^2+|w||v-w|\right)d\Pi_o(v,w) \\ &\le \frac{1}{2}W_2^2(f,g)+\left(\int_{\mathbb{R}^N}|v|^2df(v)\right)^{1/2}W_2(f,g)\end{aligned}$$

for all $k\in\mathbb{R}^N_o$. Symmetrizing the inequality we conclude the assertion. □

Let us connect now the Fourier-based distance with dual norms of smooth functional spaces. Given the set $C_b^m(\mathbb{R}^N)$ with $m\ge 1$ of bounded and differentiable functions up to order m with bounded derivatives up to order m, we define its dual metric $\|f-g\|_m^*$ as

$$\|f-g\|_m^*=\sup\left\{\left|\int_{\mathbb{R}^N}\varphi(v)(f(v)-g(v))\,dv\right|;\varphi\in C_b^m(\mathbb{R}^N),\ \|\varphi\|_m\le 1\right\}, \tag{2.17}$$

where $\|\cdot\|_m$ is the classical norm on $C_b^m(\mathbb{R}^N)$. Although this result was presented in [103, Theorem 2], we include it here in a somewhat different simplified proof.

Proposition 2.13 (From d_2 to $\|\cdot\|_m^*$)**.** [103, Theorem 2] *Given $f,g\in\mathcal{P}_2(\mathbb{R}^N)$ with equal mean value, let $m=N+3$ if N is odd or $m=N+4$ if N is even, then there exists a constant $C=C(N,m)$ such that*

$$\|f-g\|_m^*\le C\left[d_2(f,g)\,+M_2^{\frac{2}{N+2}}\,d_2(f,g)^{\frac{N}{N+2}}\right]$$

with

$$M_2=\max\left\{\int_{\mathbb{R}^N}|v|^2f\,dv,\int_{\mathbb{R}^N}|v|^2g\,dv\right\}.$$

Proof.- Given $\varphi \in C_b^m(\mathbb{R}^N)$, $\|\varphi\|_m \leq 1$ and let $R \geq 1$. Let us consider χ_R a smooth function such that $0 \leq \chi_R \leq 1$, $\chi_R = 1$ for $|v| \leq R$, $\chi_R(v) = 0$ for $|v| \geq R+1$ and $\|\chi_R\|_m \leq C_1 = C_1(N,m)$ for all $R \geq 1$.

The mass outside a large ball can be estimated as usual by

$$\left| \int_{\mathbb{R}^N} (1-\chi_R(v))\varphi(v)\,(f(v)-g(v))\,dv \right| \leq \int_{|v|\geq R} df(v) + \int_{|v|\geq R} dg(v) \leq \frac{2M_2}{R^2}.$$

Now, Parseval's identity -approximating by convolution- implies

$$\left| \int_{\mathbb{R}^N} (\chi_R\varphi)(v)\,(f(v)-g(v))\,dv \right| = \frac{1}{2\pi} \left| \int_{\mathbb{R}^N} \widehat{\chi_R\varphi}(k)\overline{[\widehat{f}(k)-\widehat{g}(k)]}\,dk \right|$$

$$\leq d_2(f,g)\,I \sup_{k\in\mathbb{R}^N} [|\widehat{\chi_R\varphi}(k)|\,(1+|k|^m)]$$

with

$$I = \frac{1}{2\pi} \int_{\mathbb{R}^N} \frac{|k|^2}{1+|k|^m}\,dk < \infty$$

and $m = N+3$ if N is odd or $m = N+4$ if N is even.

Using that $\widehat{D^\beta f} = (ik)^\beta \hat{f}$ and taking into account that $m = N+3$ if N is odd or $m = N+4$ if N is even, there exists a constant $C_2 = C_2(N,m)$ such that

$$\sup_{k\in\mathbb{R}^N} \{(1+|k|^m)|\widehat{\chi_R\varphi}(k)|\} \leq C_2 \sup_{k\in\mathbb{R}^N} \left\{ |\widehat{\chi_R\varphi}(k)| + \sum_{|\beta|=m} |\widehat{D^\beta\chi_R\varphi}(k)| \right\}.$$

Since $\chi_R\varphi$ has support in the ball of radius $1+R$ and taking into account Leibnitz's rule and the bounds on the derivatives of χ_R and φ, we deduce

$$\sup_k \{(1+|k|^m)|\widehat{\chi_R\varphi}(k)|\} \leq C_2 \int_{\mathbb{R}^N} \left\{ |\chi_R\varphi(v)| + \sum_{|\beta|=m} |D^\beta\chi_R\varphi(v)| \right\} dv$$

$$\leq C_2\|\chi_R\varphi\|_m(1+R)^N \leq C_2\,C_3\,C_1\,(1+R)^N$$

with $C_3 = C_3(N,m)$, for all $R \geq 1$. We conclude that

$$\left| \int_{\mathbb{R}^N} \varphi(v)\,(f(v)-g(v))\,dv \right| \leq \frac{2M_2}{R^2} + C_4\,R^N\,d_2(f,g) \tag{2.18}$$

for all $R \geq 1$ with

$$C_4 = C_1\,C_2\,C_3\,I\,2^N.$$

Now, if $M_2 \leq \frac{N}{4}\,C_4\,d_2(f,g)$ taking $R = 1$, we have

$$\left| \int_{\mathbb{R}^N} \varphi(v)\,(f(v)-g(v))\,dv \right| \leq \frac{N+2}{2}\,C_4\,d_2(f,g). \tag{2.19}$$

On the contrary, the minimum over $R > 1$ of the right-hand side of (2.18) is achieved at

$$R = \left(\frac{4M_2}{N\, C_4\, d_2(f,g)} \right)^{\frac{1}{N+2}}$$

giving

$$\left| \int_{\mathbb{R}^N} \varphi(v)\, (f(v) - g(v))\, dv \right| \leq C_5\, d_2(f,g)^{\frac{2}{N+2}}\, M_2^{\frac{N}{N+2}} \tag{2.20}$$

with

$$C_5 = 2 \left(\frac{4}{N\, C_4} \right)^{-\frac{2}{N+2}} + C_4 \left(\frac{4}{N\, C_4} \right)^{\frac{N}{N+2}}$$

Defining

$$C = \max \left\{ \frac{N+2}{2}\, C_4, C_5 \right\}$$

and adding (2.19) and (2.20), we deduce the desired result. □

As a simple corollary, we deduce the following control of averages for the Fourier-distance d_2.

Corollary 2.14 (Convergence of averages with d_2)**.** *Given a $\varphi \in C_b^m(\mathbb{R}^N)$, with $m = N+3$ if N is odd or $m = N+4$ if N is even, and $f, g \in \mathcal{P}_2(\mathbb{R}^N)$ with equal mean value, then*

$$\left| \int_{\mathbb{R}^N} \varphi(v)(f(v) - g(v))\, dv \right| \leq C \left[d_2(f,g) \, + M_2^{\frac{2}{N+2}}\, d_2(f,g)^{\frac{N}{N+2}} \right] \, \|\varphi\|_m.$$

Now, we will be able to connect the Fourier-based distance d_2 with the Kantorovich-Rubinstein distance due to the dual characterization (2.5). With this purpose we start relating the dual-C^m norm with W_1.

Proposition 2.15 (From $\|\cdot\|_m^*$ to W_1)**.** *Given $f, g \in \mathcal{P}_1(\mathbb{R}^N)$ and any $m \geq 2$, then there exists a constant C depending on N and m such that*

$$W_1(f,g) \leq C \left[\|f-g\|_m^* \, + (\|f-g\|_m^*)^{\frac{1}{m}} \right]$$

Proof.- Since $f, g \in \mathcal{P}_1(\mathbb{R}^N)$ we use the W_1 characterization (2.5) to reduce ourselves to functions $\varphi \in W^{1,\infty}(\mathbb{R}^N)$ with $\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^N)} \leq 1$. Now, we will regularize them by convolution. More precisely, we fixed $\omega \in C^\infty(\mathbb{R}^N)$ a positive function with support in $B(0,1)$ and unit mass. Given any $\epsilon > 0$, we consider as usual

$$\omega_\epsilon(x) = \epsilon^{-N} \omega \left(\frac{x}{\epsilon} \right)$$

and then $\varphi_\epsilon = \omega_\epsilon * \varphi \in W^{m,\infty}(\mathbb{R}^N)$ for any $m \geq 1$ with

$$\|\varphi_\epsilon\|_{\mathrm{Lip}(\mathbb{R}^N)} = \|\omega_\epsilon * D\varphi\|_{L^\infty(\mathbb{R}^N)} \leq 1$$
$$\|D^\beta \varphi_\epsilon\|_{L^\infty(\mathbb{R}^N)} = \|D^{\tilde\beta}\omega_\epsilon * D_i\varphi\|_{L^\infty(\mathbb{R}^N)} \leq C\,\epsilon^{1-m}$$

where β is any multi-index of order $|\beta| = m$ admitting a decomposition $\beta = \tilde\beta + e_i$ where e_i is the i-th canonical vector in $\mathbb{R}^N$ and $\tilde\beta$ is any multi-index of order $m-1$. Here, C will denote several constants depending only on ω, N and m but not on ϵ or φ. Moreover,

$$\|\varphi_\epsilon - \varphi\|_{L^\infty(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} \omega_\epsilon(y)|y|\,dy = C\,\epsilon \tag{2.21}$$

and we have uniform convergence in $\mathbb{R}^N$ of φ_ϵ towards φ as $\epsilon \to 0$. We refer for instance to [60] for a review of convolution and Lipschitz functions.

Now, given any two probability metrics $f, g \in \mathcal{P}_2(\mathbb{R}^N)$, we can estimate the difference on test functions $\varphi \in W^{1,\infty}(\mathbb{R}^N)$ with $\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^N)} \leq 1$ as

$$\left|\int_{\mathbb{R}^N} \varphi(v)(f(v)-g(v))\,dv\right| \leq \left|\int_{\mathbb{R}^N} (\varphi-\varphi_\epsilon)(v)\,df(v)\right| + \left|\int_{\mathbb{R}^N} (\varphi-\varphi_\epsilon)(v)\,dg(v)\right|$$
$$+ \left|\int_{\mathbb{R}^N} \varphi_\epsilon(v)(f(v)-g(v))\,dv\right|.$$

The first two terms are bounded due to (2.21) by $C\,\epsilon$ while the last term can be controlled by the definition of the dual-C^m norm giving

$$\left|\int_{\mathbb{R}^N} \varphi_\epsilon(v)(f(v)-g(v))\,dv\right| \leq \|\varphi_\epsilon\|_m\,\|f-g\|_m^* \leq C(1+\epsilon^{1-m})\,\|f-g\|_m^*$$

with C a constant independent of ϵ and φ. As a summary, we deduce

$$\left|\int_{\mathbb{R}^N} \varphi(v)(f(v)-g(v))\,dv\right| \leq C\,\epsilon + C(1+\epsilon^{1-m})\,\|f-g\|_m^*$$

for any $\epsilon > 0$. Optimizing the inequality over ϵ, we finally conclude

$$\left|\int_{\mathbb{R}^N} \varphi(v)(f(v)-g(v))\,dv\right| \leq C\left[\|f-g\|_m^* + (\|f-g\|_m^*)^{\frac{1}{m}}\right]$$

for a suitable C and for all $\varphi \in W^{1,\infty}(\mathbb{R}^N)$ with $\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^N)} \leq 1$. □

Again as a simple corollary, we deduce the following improved control of averages for the Fourier-distance d_2.

Corollary 2.16 (Convergence of averages with d_2)**.** *Given a $\varphi \in Lip(\mathbb{R}^N)$ and $f, g \in \mathcal{P}_2(\mathbb{R}^N)$ with equal mean value, then there exist positive constants*

C and exponents γ_1, γ_2, β_1, β_2, β_3 depending on N

$$\left|\int_{\mathbb{R}^N} \varphi(v)(f(v)-g(v))\,dv\right| \le C\,\max\left\{d_2(f,g), M_2^{\gamma_1} d_2(f,g)^{\beta_1}, M_2^{\gamma_2} d_2(f,g)^{\beta_2}, d_2(f,g)^{\beta_3}\right\}\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^N)}.$$

Putting together the last two results, we deduce our final relation between d_2 and W_2.

Corollary 2.17 (From d_2 to W_2)**.** *Given $f, g \in \mathcal{P}_2(\mathbb{R}^N)$ with equal mean value such that*

$$M_{2+\alpha} = \max\left\{\int_{\mathbb{R}^N} |v|^{2+\alpha} f\,dv, \int_{\mathbb{R}^N} |v|^{2+\alpha} g\,dv\right\} < \infty$$

with $\alpha > 0$, then there exist positive constants C and exponents γ_1, γ_2, β_1, β_2, β_3 and $0 < \gamma_3 < 1$ depending on N and α such that

$$W_2(f,g) \le \\ \le C\left(\max\left\{d_2(f,g), M_2^{\gamma_1} d_2(f,g)^{\beta_1}, M_2^{\gamma_2} d_2(f,g)^{\beta_2}, d_2(f,g)^{\beta_3}\right\}\right)^{\gamma_3} M_{2+\alpha}^{1-\gamma_3}$$

with

$$M_2 = \max\left\{\int_{\mathbb{R}^N} |v|^2 f\,dv, \int_{\mathbb{R}^N} |v|^2 g\,dv\right\}.$$

Proof.- In order to estimate $W_2(f,g)$ we just use a simple interpolation inequality in terms of $W_1(f,g)$ and $W_{2+\alpha}(f,g)$ as follows. Take $\Pi_o^1(v,w)$ the optimal plan for the W_1 metric, then

$$W_2(f,g) \le \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} |v-w|^2\,d\Pi_o^1(v,w)\right)^{1/2}.$$

Interpolation in the last integral between exponents 1 and $2+\alpha$ gives

$$W_2(f,g) \le \\ \le \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} |v-w|\,d\Pi_o^1(v,w)\right)^{\gamma_3}\left(\int_{\mathbb{R}^N\times\mathbb{R}^N} |v-w|^{2+\alpha}\,d\Pi_o^1(v,w)\right)^{1-\gamma_3}$$

with $\gamma_3 = \frac{\alpha}{2(1+\alpha)}$. By definition of the optimal plan and using that its marginals are f and g, we deduce

$$W_2(f,g) \le C\,W_1(f,g)^{\gamma_3} M_{2+\alpha}^{1-\gamma_3}$$

with C depending on α. Now, just by collecting the inequalities shown in Propositions 2.13 and 2.15, we deduce that

$$W_1(f,g) \le C\max\left\{d_2(f,g), M_2^{\gamma_1} d_2(f,g)^{\beta_1}, M_2^{\gamma_2} d_2(f,g)^{\beta_2}, d_2(f,g)^{\beta_3}\right\}$$

for certain constants and exponents depending on N explicitly computable from previous propositions, if needed. We conclude by putting together the last two inequalities. □

Remark 2.18 (Completeness). *The set $\mathcal{X}_{\mu,\theta,\alpha,M}$ of $f \in \mathcal{P}_{2+\alpha}(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N} v\, df(v) = \mu,\ \int_{\mathbb{R}^N} |v|^2 df(v) = N\theta \text{ and } \int_{\mathbb{R}^N} |v|^{2+\alpha} df(v) \leq M$$

is a complete metric space endowed with d_2, for any $\alpha > 0$ as a particular case of Proposition 2.7. The proof of this statement follows easily from Corollary 2.17, Proposition 2.1 and Proposition 2.12. Corollary 2.17 ensures that Cauchy sequences for d_2 inside the set $\mathcal{X}_{\mu,\theta,\alpha,M}$ are Cauchy sequences for W_2. These Cauchy sequences are convergent in W_2 to a limit f inside $\mathcal{X}_{\mu,\theta,\alpha,M}$ due to Proposition 2.1. Finally, Proposition 2.12 shows that convergent sequences in W_2 inside $\mathcal{X}_{\mu,\theta,\alpha,M}$ are also convergent in the distance d_2.

Remark 2.19 (Prokhorov's distance). *In the set $\mathcal{X}_{\mu,\theta,\alpha,M}$, we can consider another related metric. For any $\delta \geq 0$ and $U \subset \mathbb{R}^N$, we define*

$$U^\delta = \{v \in \mathbb{R}^N; D(v,U) < \delta\} \qquad \text{and} \qquad U^{\delta]} = \{v \in \mathbb{R}^N; D(v,U) \leq \delta\}$$

where $D(v,U) = \inf\{|v-w|, w \in U\}$. Let

$$\nu^*(f,g) = \inf\{\epsilon > 0 \text{ such that } f(A) \leq g(A^\epsilon) + \epsilon \text{ for all closed } A \subset \mathbb{R}^N\};$$

we set $\nu(f,g) = \max\{\nu^*(f,g), \nu^*(g,f)\}$. Here, we have denoted, abusing on the notation, the measure and its corresponding density, if any, by the same symbol. This distance is called the Prokhorov's distance introduced in [89]. *This distance verifies the following with respect to the above introduced distances (see* [62, 103]*): Given $f, g \in \mathcal{X}_{\mu,\theta,\alpha,M}$, then*

$$W_2^2(f,g) \leq (2M+8)\, \nu(f,g)^{\frac{\alpha}{\alpha+2}} + 4\, \nu(f,g)^2$$

and

$$\nu(f,g) \leq C \max\left\{ (\|f-g\|_m^*)^{\frac{1}{m+1}}, \|f-g\|_m^* \right\}$$

with $C = C(N,m)$ for all $m \in \mathbb{N}$, $m \geq 1$.

Remark 2.20 (General Statement on weak uniformity). *We shall say that two metrics m_1 and m_2 define the same weak-* uniformity on a set $S \subset \mathcal{P}(\mathbb{R}^N)$ if for all $\varepsilon > 0$ there exists $\eta > 0$ such that for all $f, g \in S$,*

$$m_1(f,g) \leq \eta \Longrightarrow m_2(f,g) \leq \varepsilon,$$

$$m_2(f,g) \leq \eta \Longrightarrow m_1(f,g) \leq \varepsilon.$$

We can summarize all the results in this section by stating that the metrics W_2, W_1, d_2, ν and $\|\cdot\|_m^*$ define the same weak- uniformity on the set $\mathcal{X}_{\mu,\theta,\alpha,M}$.*

2.5. **Equivalence in the one-dimensional case.** In this section we will show that in the one-dimensional case the estimates on the equivalence between the Wasserstein and the Fourier based metrics are much easier with easily computable constants. The analysis of the previous Section indicates that the difficult part is to obtain bounds on the Wasserstein metric in terms of d_2-metric, since Proposition 2.12 (with $N = 1$) allows to conclude that the counterpart holds with explicit constants. The one-dimensional analysis takes advantage of the explicit expression (2.8). This is particularly evident in case $p = 1$, where

$$W_1(f,g) = \int_0^1 |F^{-1}(\eta) - G^{-1}(\eta)|\, d\eta. \tag{2.22}$$

In this case, the value of the integral in (2.22) is given by the measure of the area between the two distribution functions F and G, so that

$$W_1(f,g) = \int_{\mathbb{R}} |F(v) - G(v)|\, dv.$$

Suppose for the moment that, as in Corollary 2.17, $f, g \in \mathcal{P}_2(\mathbb{R})$ with equal mean value are such that

$$M_{2+\alpha} = \max\left\{\int_{\mathbb{R}} |v|^{2+\alpha} f(v)\, dv, \int_{\mathbb{R}} |v|^{2+\alpha} g(v)\, dv\right\} < \infty$$

with $\alpha > 0$. Then, by virtue of Chebyshev's inequality, if X is a random variable with law f,

$$P(|X| > \epsilon) \le \frac{\mathbb{E}[X^{2+\alpha}]}{\epsilon^{2+\alpha}},$$

and this implies

$$\lim_{R\to\infty} R^2\left(F(-R) + 1 - F(R)\right) = \lim_{R\to\infty} R^2 P(|X| > R) = 0. \tag{2.23}$$

Thus, we can integrate by parts to get the inequality

$$\int_{|v|\ge R} |v||F(v) - G(v)|\, dv \le \int_{|v|\ge R} \frac{v^2}{2}\left(f(v) + g(v)\right)\, dv.$$

For this, observe that

$$\int_{-\infty}^{-R} |v||F(v) - G(v)|\, dv \le -\int_{-\infty}^{-R} v(F(v) + G(v))\, dv$$

and

$$\int_R^{\infty} |v||F(v) - G(v)|\, dv \le \int_R^{\infty} v((1 - F(v)) + (1 - G(v))\, dv.$$

If $f, g \in \mathcal{P}_2(\mathbb{R})$, with no extra moments bounded, we can easily arrive to prove inequality (2.23) by standard approximation arguments. Using (2.23) we obtain

$$\begin{aligned}\int_{\mathbb{R}} |F(v) - G(v)|\, dv &\leq \int_{-R}^{R} |F(v) - G(v)|\, dv + \frac{1}{R}\int_{|v|\geq R} |v||F(v) - G(v)|\, dv \\ &\leq (2R)^{1/2}\left(\int_{\mathbb{R}} |F(v) - G(v)|^2\, dv\right)^{1/2} + \frac{M_2}{R}\end{aligned}$$

where

$$M_2 = \max\left\{\int_{\mathbb{R}} |v|^2 f(v)\, dv, \int_{\mathbb{R}} |v|^2 g(v)\, dv\right\}.$$

Optimizing over R we obtain the bound

$$\int_{\mathbb{R}} |F(v) - G(v)|\, dv \leq \left(2^{1/3} + 2^{-2/3}\right) M^{1/3}\left(\int_{\mathbb{R}} |F(v) - G(v)|^2\, dv\right)^{1/3}. \tag{2.24}$$

By Parseval's formula

$$\int_{\mathbb{R}} |F(v) - G(v)|^2\, dv = \frac{1}{2\pi}\int_{\mathbb{R}} |(\widehat{F - G})(k)|^2\, dk.$$

Consider now that

$$(\widehat{F - G})(k) = \frac{\hat{f}(k) - \hat{g}(k)}{ik}$$

and thus,

$$\begin{aligned}\int_{\mathbb{R}} |F(v) - G(v)|^2\, dv &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{|\hat{f}(k) - \hat{g}(k)|^2}{k^2}\, dk \\ &\leq \frac{1}{2\pi}\left(\int_{-R}^{R} \frac{|\hat{f}(k) - \hat{g}(k)|^2}{k^2}\, dk + 2\int_{|k|\geq R} \frac{1}{k^2}\, dk\right) \\ &\leq \frac{1}{2\pi}\left(\int_{-R}^{R} k^2 d_2(f,g)^2\, dk + \frac{2}{R}\right) \\ &= \frac{1}{\pi}\left(\frac{R^3}{3}\, d_2(f,g)^2 + \frac{1}{R}\right).\end{aligned}$$

Once again, optimizing over R we obtain the bound

$$\int_{\mathbb{R}} |F(v) - G(v)|^2\, dv \leq \frac{4}{3\pi}\sqrt{d_2(f,g)}. \tag{2.25}$$

Putting together bounds (2.24) and (2.25) we finally obtain

Theorem 2.21 (From d_2 to W_1)**.** *Given $f, g \in \mathcal{P}_2(\mathbb{R})$ such that $d_2(f,g)$ is bounded, then*

$$W_1(f,g) \leq \left(\frac{18 M_2}{\pi}\right)^{1/3} d_2(f,g)^{1/6} \tag{2.26}$$

with

$$M_2 = \max\left\{\int_{\mathbb{R}} v^2\, df(v), \int_{\mathbb{R}} v^2\, dg(v)\right\}.$$

Thanks to Theorem 2.21 we can easily obtain the corresponding of Corollary 2.17, with explicitly computable constants. Let in fact $f, g \in \mathcal{P}_2(\mathbb{R})$ such that

$$M_{2+\alpha} = \max\left\{\int_{\mathbb{R}} |v|^{2+\alpha}\, df(v), \int_{\mathbb{R}} |v|^{2+\alpha}\, df(v)\right\} < \infty$$

with $\alpha > 0$. Given Π_o the optimal transference plan for all costs in one dimension obtained in Subsection 2.2, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |v-w|^2\, d\Pi_o(v,w) \leq \\
&\qquad \leq \int_{|v-w|<R} |v-w|^2 d\Pi_o(v,w) + \frac{1}{R^\alpha}\int_{|v-w|>R} |v-w|^{2+\alpha} d\Pi_o(v,w) \\
&\qquad \leq R\int_{|v-w|<R} |v-w| d\Pi_o(v,w) + \frac{2^{2+\alpha} M_{2+\alpha}}{R^\alpha} \\
&\qquad \leq R\, W_1(f,g) + \frac{2^{2+\alpha} M_{2+\alpha}}{R^\alpha}.
\end{aligned}$$

Optimizing over R we get

$$W_2(f,g) \leq 2^{(2+\alpha)/(1+\alpha)} \left(\alpha^{1/(1+\alpha)} + \alpha^{-\alpha/(1+\alpha)}\right) W_1(f,g)^{\alpha/(1+\alpha)}\, M_{2+\alpha}^{1/(1+\alpha)}. \tag{2.27}$$

We can now use the bound of Theorem 2.21 to conclude

Corollary 2.22 (From d_2 to W_2 in one-d)**.** *Given $f, g \in \mathcal{P}_2(\mathbb{R})$ such that*

$$M_{2+\alpha} = \max\left\{\int_{\mathbb{R}} |v|^{2+\alpha} f\, dv, \int_{\mathbb{R}} |v|^{2+\alpha} g\, dv\right\} < \infty$$

with $\alpha > 0$, then

$$W_2(f,g) \leq C_\alpha\, d_2(f,g)^{\,\alpha/[6(1+\alpha)]}\, M_2^{\alpha/[3(1+\alpha)]}\, M_{2+\alpha}^{1/(1+\alpha)}$$

with

$$C_\alpha = 2^{(2+\alpha)/(1+\alpha)} \left(\alpha^{1/(1+\alpha)} + \alpha^{-\alpha/(1+\alpha)}\right) \left(\frac{18}{\pi}\right)^{\alpha/[3(1+\alpha)]}.$$

2.6. **Relation to other more classical functional spaces.** Sobolev spaces $H^r(\mathbb{R}^N)$, with $r \geq 0$ are defined as usual in terms of the Fourier transform as those functions in $L^2(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (1+|k|^2)^r |\hat{f}(k)|^2 \, dk < \infty.$$

and its norm is defined as

$$\|f\|^2_{H^r(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (1+|k|^2)^r \, |\hat{f}(k)|^2 \, dk.$$

Since moments in Fourier space will have simpler relations and inductive formulas for many of the applications, we shall use in the sequel, due to notational convenience, the homogeneous Sobolev norms, with $r \geq 0$, defined as

$$\|f\|^2_{\dot{H}^r(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |k|^{2r} |\hat{f}(k)|^2 \, dk. \tag{2.28}$$

We will show by some interpolation inequalities that the control of Fourier-based distances and the control of arbitrary "large" Sobolev norms implies the control of distances in arbitrary Sobolev norms and in L^2. Moreover, a control of arbitrary "large" moments for the probability measures yields a control of the distance in L^1.

Proposition 2.23 (From d_2 to Sobolev norms)**.** [38, Theorem 4.1] *Let $r \geq 0$, and $\beta_1, \beta_2 > 0$, $0 < \beta_2 < 1$ be given. Then*

$$\begin{aligned} &\|f-g\|_{\dot{H}^r(\mathbb{R}^N)} \leq \\ &\quad \leq \quad C(\beta_1,\beta_2)\, d_2(f,g)^{(1-\beta_2)} \min\Big(\|f-g\|_{\dot{H}^{r_1}(\mathbb{R}^N)}, \|f-g\|_{\dot{H}^{r_2}(\mathbb{R}^N)} \Big)^{\beta_2}, \end{aligned}$$

with

$$r_1 = \frac{r+2(1-\beta_2)}{\beta_2} \qquad , \qquad r_2 = \frac{2r+(4+\beta_1+N)(1-\beta_2)}{2\beta_2},$$
$$C(\beta_1,\beta_2) = \big(|B^N|(1+N/\beta_1)\big)^{1-\beta_2},$$

and where $|B^N|$ denotes the volume of the unit ball in $\mathbb{R}^N$.

Proof.- Given any $p > 0$ and $0 < \beta_2 < 1$, we start by writing

$$\begin{aligned}
&\|f-g\|^2_{\dot{H}^r(\mathbb{R}^N)} = \\
&\quad = \int_{\mathbb{R}^N} |k|^{2r}\, |\hat{f}(k) - \hat{g}(k)|^2\, dk \\
&\quad \le d_2(f,g)^{2-2\beta_2} \int_{\mathbb{R}^N} |k|^{2r+2(2-2\beta_2)} |\hat{f}(k) - \hat{g}(k)|^{2\beta_2}\, dk \\
&\quad \le d_2(f,g)^{2-2\beta_2} I_{\beta_2,p} \left(\int_{\mathbb{R}^N} |k|^{2r_1} (1+|k|^p) |\hat{f}(k) - \hat{g}(k)|^2\, dk \right)^{\beta_2}
\end{aligned}$$

where we have multiplied and divided by $(1+|k|^p)^{\beta_2}$ and applied Hölder's inequality with exponent $1/\beta_2$ and

$$I_{\beta_2,p} = \left(\int_{\mathbb{R}^N} (1+|k|^p)^{-\frac{\beta_2}{1-\beta_2}}\, dk \right)^{1-\beta_2} < \infty$$

for which $p\beta_2 > (1-\beta_2)N$. Choosing now p such that $p\beta_2 = (1-\beta_2)(N+\beta_1)$ we obtain the statement. □

Lemma 2.24 (From L^2+moments to L^1 norm)**.** [38, Theorem 4.2] *Let $f \in L^1 \cap L^2(\mathbb{R}^N)$ with $|v|^{2r} f \in L^1(\mathbb{R}^N)$, then, for all $r > 0$,*

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(v)|\, dv \le \\
&\quad \le \quad C(N,r) \left(\int_{\mathbb{R}^N} |f(v)|^2\, dv \right)^{2r/(N+4r)} \left(\int_{\mathbb{R}^N} |v|^{2r} |f(v)|\, dv \right)^{N/(N+4r)}
\end{aligned}$$

with

$$C(N,r) = \left[\left(\frac{N}{4r} \right)^{4r/(N+4r)} + \left(\frac{4r}{N} \right)^{N/(N+4r)} \right] |B^N|^{2r/(N+4r)}.$$

Proof.- Given any $R > 0$, we estimate as

$$\begin{aligned}
\int_{\mathbb{R}^N} |f(v)|\, dv &\le \int_{|v|\le R} |f(v)|\, dv + \int_{|v|\ge R} |f(v)|\, dv \\
&\le (|B^N| R^N)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |f(v)|^2\, dv \right)^{\frac{1}{2}} + \frac{1}{R^{2r}} \int_{\mathbb{R}^N} |v|^{2r} |f(v)|\, dv
\end{aligned}$$

from which the statement follows by optimizing over $R > 0$. □

3. Kinetic Equations for Inelastic Interactions

The Granular Kinetic Theory has been proposed for modelling the collective behavior of a huge system of particles performing inelastic collisions. We refer to the humongous literature included in the surveys and books

[73, 67, 49, 32, 65, 33] for physical discussions on the justifications and limitations of this theory, experimental issues, computational methods and hydrodynamical approximations.

Granular Kinetic Theory has been used by fluid mechanics experts as a toolbox to produce relevant hydrodynamic systems of equations capable of reproducing rapid granular flows as pattern formation in vertically oscillated granular layers [83, 104, 15, 31, 96, 46] and shock waves in granular flows [94, 45]. Clustering and transient structures have also been investigated by means of molecular dynamical models and hydrodynamic approaches [71, 34]. A good account on the mathematics for granular materials can be found in the survey [107].

3.1. **Inelastic Boltzmann Equation.** As in the perfect elastic case [50], let us consider a system of perfect homogeneous spheres of diameter $d > 0$ and assume their positions and velocities before a binary inelastic collision are given by (x, v) and $(x - dn, w)$, where $n \in S^2$ is the unit vector along the impact direction. The post-collisional velocities are found assuming that the relative velocity in the impact direction after the collision is reduced by a factor, i.e.,

$$(v' - w') \cdot n = -e((v - w) \cdot n) \tag{3.1}$$

where $0 < e \leq 1$ is called the restitution coefficient considered constant in the sequel. The component in the orthogonal direction to n is kept, i.e.,

$$v' - w' - ((v' - w') \cdot n)n = v - w - ((v - w) \cdot n)n$$

and thus, the post-collisional velocities are given by:

$$\begin{aligned} v' &= \frac{1}{2}(v + w) + \frac{u'}{2} \\ w' &= \frac{1}{2}(v + w) - \frac{u'}{2} \end{aligned} \tag{3.2}$$

where $u' = u - (1 + e)(u \cdot n)n$, $u = v - w$ and $u' = v' - w'$.

Remark 3.1 (Variable Restitution Coefficient)**.** *From the modelling point of view a constant restitution coefficient is a quite simplified model for real materials. A dependence of the restitution coefficient upon the magnitude of the relative velocity in the impact direction is more realistic, getting the collisions more and more elastic as the modulus of this relative velocity gets smaller and smaller. This assumption avoids the inelastic collapse for particle models as reported in* [66, 82] *and it has been incorporated in the kinetic models by a number of authors* [93, 97]*. This high-nonlinear dependency is considered by various authors upon average quantities of the flow as local granular temperature* [20, 85, 102]*.*

Based on this microscopic collision mechanism and analogous arguments as in the elastic case [50], i.e., assuming that collisions are binary, localized in time and space, Enskog correction is negligible, the diameter $\pi d^2 \simeq 1$ is normalized and that correlations can be neglected then one arrives formally to the inelastic Boltzmann equation

$$\begin{aligned}\frac{\partial f}{\partial t}+(v\cdot\nabla_x)f &= Q_e(f,f) \\ &= \frac{1}{\pi}\int_{\mathbb{R}^3}\int_{S^2_+}((v-w)\cdot n)\left[\frac{1}{e^2}f(v^*)f(w^*)-f(v)f(w)\right]dn\,dw.\end{aligned} \tag{3.3}$$

where v^* and w^* are the pre-collisional velocities associated to (3.2) and $S^2_+ = \{n \in S^2$ such that $((v-w)\cdot n) > 0\}$. The right-hand side of the equation takes into account the change on velocity of the particles due to inelastic collisions while the left-hand side accounts for the free transport of particles between collisions. The factor e^{-2} in front of the gain part of the operator $Q_e(f,f)$ comes from the relation between pre and post-collisional relative velocities (3.1) and the jacobian of the linear transformation $(v',w') \longrightarrow (v,w)$ defined by (3.2) whose value is $\frac{1}{e}$.

The weak or Maxwell formulation of the nonlinear inelastic collision operator $Q_e(f,f)$ is given by

$$\begin{aligned}< \varphi, Q_e(f,f) > &= \frac{1}{\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2_+}((v-w)\cdot n)f(v)f(w)\Big[\varphi(v')-\varphi(v)\Big]dndv\,dw \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2_+}((v-w)\cdot n)f(v)f(w)\,\Delta_c\varphi\,dn\,dv\,dw\end{aligned} \tag{3.4}$$

on test functions $\varphi \in C(\mathbb{R}^3)$ where the variation of φ through a collision $\Delta_c\varphi$ is defined by

$$\Delta_c\varphi = \varphi(v') + \varphi(w') - \varphi(v) - \varphi(w).$$

As it has also become standard in elastic kinetic theory, it is quite useful to work with a different parameterization of the set of post-collisional velocities (see for instance [30, 20]). This change of variables relies on the fact that the set of all possible post-collisional velocities v' lies in a sphere of center $\frac{1+e}{4}(v+w)+\frac{1-e}{2}v$ and radius $\frac{1+e}{4}|u|$ as sketched in Figure 1. We refer to [63] for a detailed explanation of the inelastic collision mechanism.

In fact, one obtains

$$\int_{S^2}(u\cdot n)_+\,\varphi(n(u\cdot n))\,dn = \frac{|u|}{4}\int_{S^2}\varphi\left(\frac{u-|u|\sigma}{2}\right)d\sigma \tag{3.5}$$

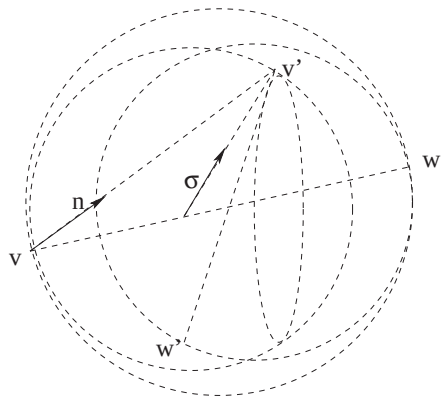

FIGURE 1. Sketch of the geometry of inelastic collisions.

for any function $\varphi \in C(\mathbb{R}^3)$. In this way, the weak formulation of the Boltzmann operator becomes

$$< \varphi, Q_e(f,f) > = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} |v-w| f(v) f(w) \Big[\varphi(v') - \varphi(v) \Big] d\sigma \, dv \, dw \tag{3.6}$$

$$= \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} |v-w| f(v) f(w) \Delta_c \varphi \, d\sigma \, dv \, dw$$

where the post-collisional velocities are determined in terms of a unit vector $\sigma \in S^2$ pointing from the center of the sphere in which post-collisional velocities lie in. More precisely, the collision mechanism is now written as

$$\begin{aligned} v' &= \frac{1}{2}(v+w) + \frac{1-e}{4} u + \frac{1+e}{4} |u| \sigma \\ w' &= \frac{1}{2}(v+w) - \frac{1-e}{4} u - \frac{1+e}{4} |u| \sigma \end{aligned} \tag{3.7}$$

with $u = v - w$. In the sequel, we will work with this form of the nonlinear collision operator unless explicitly stated.

3.2. **Basic Properties of the collision operator.** The basic conservation identities of the inelastic Boltzmann equation are obtained by taking $\varphi = 1, v$ in the weak formulation (3.6) resulting in

$$< \begin{pmatrix} 1 \\ v \end{pmatrix}, Q_e(f,f) > = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \end{pmatrix} Q_e(f,f)(v) \, dv = 0,$$

that is, conservation of mass or number of particles and mean velocity. As expected, conservation of energy does not hold in contrast with the elastic

case. In fact, we obtain

$$\Delta_c|v|^2 = -\frac{1-e^2}{4}|v-w|^2\left(1-\frac{v-w}{|v-w|}\cdot\sigma\right),$$

and thus from (3.6), we deduce

$$\begin{aligned} <|v|^2, Q_e(f,f)> &= \int_{\mathbb{R}^3}|v|^2\,Q_e(f,f)(v)\,dv \\ &= -\frac{1-e^2}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-w|^3 f(v)f(w)\,dv\,dw, \end{aligned} \tag{3.8}$$

where we observe the dissipation of kinetic energy.

In the elastic case, there exists another important quantity which is central for both the existence theory and the asymptotic properties, the entropy of the system. In the inelastic case, the entropy is not decreasing anymore, the change of entropy is given by

$$<\log f, Q_e(f,f)>= \tag{3.9}$$

$$\begin{aligned} &= \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}|u|f_vf_w\left(\log\frac{f'_vf'_w}{f_vf_w}-\frac{f'_vf'_w}{f_vf_w}+1\right)d\sigma\,dv\,dw \\ &\quad+\frac{1-e^2}{2e^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|u|f_vf_w\,dv\,dw \end{aligned} \tag{3.10}$$

where the last term gives a positive contribution, see [63]. Here, we have used the shortcuts $f'_v = f(v')$ and so on, for the sake of simplicity.

Let us analyse the main consequences over the homogeneous solutions to the inelastic Boltzmann equation (IBE) (3.3), i.e., for the solutions $f(t,v)$ of

$$\frac{\partial f}{\partial t} = \frac{1}{\pi}\int_{\mathbb{R}^3}\int_{S^2_+}((v-w)\cdot n)\left[\frac{1}{e^2}f(v^*)f(w^*)-f(v)f(w)\right]dn\,dw. \tag{3.11}$$

Conservation of mass and mean velocity allows to assume without loss of generality that the solutions are probability densities with zero mean velocity, i.e.,

$$\int_{\mathbb{R}^3}\begin{pmatrix}1\\ v\end{pmatrix}f(t,v)\,dv = \begin{pmatrix}1\\ 0\end{pmatrix}$$

while the kinetic energy is dissipated following the law:

$$\frac{d}{dt}\int_{\mathbb{R}^3}|v|^2\,f(t,v)\,dv = -\frac{1-e^2}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-w|^3 f(v)f(w)\,dv\,dw \tag{3.12}$$

Jensen's inequality applied to the right-hand side implies

$$\int_{\mathbb{R}^3}|v-w|^3 f(w)\,dw \geq \left|v-\int_{\mathbb{R}^3}wf(w)\,dw\right|^3 = |v|^3$$

and thus,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3}|v|^2\,f(t,v)\,dv &\leq -\frac{1-e^2}{8}\int_{\mathbb{R}^3}|v|^3 f(v)\,dv \leq \\ &\leq -\frac{1-e^2}{8}\left(\int_{\mathbb{R}^3}|v|^2 f(v)\,dv\right)^{3/2}\end{aligned}$$

which is known as the Haff's law of cooling. In fact, denoting the temperature of the granular gas by

$$\theta(t)=\frac{1}{3}\int_{\mathbb{R}^3}|v|^2\,f(t,v)\,dv$$

then, $\theta(t)$ satisfies the differential inequality

$$\theta'(t)\leq -\sqrt{3}\frac{1-e^2}{8}\,\theta(t)^{3/2} \tag{3.13}$$

from which, the temperature decays towards zero more rapidly than t^{-2}. These simple observations based on the main properties of the collision operator allow us to obtain the first important remark on the asymptotic behavior of solutions to (3.11).

Proposition 3.2 (Convergence towards Mono-kinetic)**.** *Assume $f(t,v)$ is a fast-decaying smooth solution to* (3.11) *such that*

$$\int_{\mathbb{R}^3}\begin{pmatrix}1\\ v\end{pmatrix}f(t,v)\,dv=\begin{pmatrix}1\\ 0\end{pmatrix},$$

then

$$W_2(f(t),\delta_0)\leq C(1+t)^{-1}$$

with $C=C(\theta(0))$.

Proof.- This proposition is just a simple consequence of the convergence towards zero of the second moment due to (3.13) and the relation to temperature of the W_2 distance given in Proposition 2.1. □

Let us mention that the Cauchy problem for the inelastic Boltzmann equation together with the proof of the Haff's law for its weak solutions has recently been tackled in [85, 84]. They are able to show a bound from below of the temperature decaying as t^{-2} at infinity under suitable assumptions on the solutions. In fact, this result is generalized [85, 84] to the case of velocity or energy dependent restitution coefficients where the cooling may happen even in finite time due to this dependence.

3.3. **Inelastic Maxwell Models.** A nice simplification of the IBE (3.3) based on the Maxwellian molecules case for the elastic Boltzmann equation was introduced in [20]. The main assumption relies on assuming that the typical collision frequency in the weak formulation of the inelastic collision operator (3.6) is of the order of the thermal speed, i.e.,

$$|v-w| \simeq B\sqrt{\theta(t)}$$

with $B>0$ chosen such that the temperature dissipation coincides with the one for the hard-spheres case (3.13).

More precisely, the weak formulation of the simplified Maxwellian-type collision operator $\tilde{Q}_e(f,f)$ becomes

$$\begin{aligned} <\varphi, \tilde{Q}_e(f,f)> &= \frac{B}{4\pi}\sqrt{\theta(t)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)f(w)\Big[\varphi(v')-\varphi(v)\Big]d\sigma\, dv\, dw \\ &= \frac{B}{8\pi}\sqrt{\theta(t)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)f(w)\Delta_c\varphi\, d\sigma\, dv\, dw. \end{aligned} \tag{3.14}$$

Although a formulation of the collision operator in strong form can be done [20], we will not write it here since it will be never used.

It is quite straightforward to check that conservation of mass and momentum remains and that the temperature dissipation becomes now an identity, that is,

$$\frac{d}{dt}\int_{\mathbb{R}^3}|v|^2 f(t,v)\, dv = -\frac{1-e^2}{8}B\sqrt{\theta(t)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-w|^2 f(v)f(w)\, dv\, dw.$$

Expanding the square, we deduce that any smooth solution of the Inelastic Maxwell Equation (IME)

$$\frac{\partial f}{\partial t} = \tilde{Q}_e(f,f) \tag{3.15}$$

such that

$$\int_{\mathbb{R}^3}\begin{pmatrix} 1 \\ v \end{pmatrix} f(t,v)\, dv = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

verifies the cooling Haff's law

$$\theta'(t) = -\frac{1-e^2}{4}B\,\theta(t)^{3/2}. \tag{3.16}$$

It is obvious that the constant B can be chosen to match the dissipation of temperature for the hard-spheres case (3.13). In this case, the granular case cools down in infinite time and as above we deduce:

Corollary 3.3 (Convergence towards Mono-kinetic)**.** *Assume $f(t,v)$ is a fast-decaying smooth solution to* (3.15) *such that*

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \end{pmatrix} f(t,v)\, dv = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then

$$W_2^2(f(t),\delta_0) = \frac{4\theta(0)}{(\frac{1-e^2}{4} B\sqrt{\theta(0)}t+2)^2}.$$

In the sequel, we will always work with normalized solutions to (3.15) with unit mass and zero mean velocity unless explicitly stated. The collision operator can be split as $\tilde{Q}_e(f,f) = B\sqrt{\theta(t)}(\tilde{Q}_e^+(f,f) - \tilde{Q}_e^-(f,f))$ with

$$< \varphi, \tilde{Q}_e^+(f,f) >= \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)f(w)\varphi(v')d\sigma\, dv\, dw$$

and $\tilde{Q}_e^-(f,f) = f$. One of the main simplifications over the theory of Boltzmann-type equations that a collision frequency not depending on $|u|$ induces, is the fact that the nonlinear operator expresses in Fourier variables in a simple closed form reducing drastically the dimensionality of the integral operator. The basic idea is a smart change of variables introduced by A. V. Bobylev [17, 18, 19] and reviewed in a recent paper of the Porto Ercole summer school lectures [54, Theorem 12].

Lemma 3.4 ($\tilde{Q}_e^+(f,f)$ in Fourier: Bobylev's identity)**.** [17, 18, 19, 20] *The following formula holds:*

$$\widehat{\tilde{Q}_e^+(f,f)} = \frac{1}{4\pi}\int_{S^2} \hat{f}(t,k_-)\hat{f}(t,k_+)\, d\sigma$$

with

$$\begin{aligned} k_- &= \frac{1+e}{4}(k-|k|\sigma), \\ k_+ &= \frac{3-e}{4}\, k + \frac{1+e}{4}\, |k|\sigma = k - k_-\,. \end{aligned} \tag{3.17}$$

Proof.- Using the weak formulation (3.6) with test function $\varphi(v) = \exp(-i(k\cdot v))$ for any $k\in\mathbb{R}^3$, we get

$$<\varphi, \tilde{Q}_e^+(f,f)>=\frac{1}{4\pi}\int_{\mathbb{R}^6} f(v)f(w)\exp\Big\{-i(k\cdot\tilde{u})-i\frac{1-e}{4}(k\cdot u)\Big\}F(k,u)dv\, dw,$$

with $\tilde{u} = \frac{1}{2}(v+w)$, $u = (v-w)$ and

$$F(k,u) = \int_{S^2} \exp\Big\{-\frac{1+e}{4}i(k\cdot\sigma)|u|\Big\}\, d\sigma\ .$$

In fact, the product $(k \cdot |u|\sigma)$ can be replaced by $(u \cdot |k|\sigma)$ because the function $F(k,u)$ is isotropic and only depends on the values of $|k|$ and $|u|$. This statement follows by observing the existence of an isometry from S^2 to S^2 that maps $\frac{k}{|k|}$ onto $\frac{u}{|u|}$. Proceeding with the replacement and interchanging the order of integration we obtain

$$< \varphi, \tilde{Q}_e^+(f,f) >= \frac{1}{4\pi} \int_{S^2 \times \mathbb{R}^3 \times \mathbb{R}^3} f(v) f(w) \exp\{H(k,v,w)\}\, dv\, dw\, d\sigma$$

with

$$\begin{aligned} H(k,v,w) &= \\ &= \quad -iv\left(\tfrac{k}{2} + \tfrac{1-e}{4}k + \tfrac{1+e}{4}|k|\sigma\right) + iw\left(-\tfrac{k}{2} + \tfrac{1-e}{4}k + \tfrac{1+e}{4}|k|\sigma\right) \end{aligned}$$

from which the stated formula follows. □

In this way, the homogeneous IME (3.15) can be written in Fourier variables as

$$\frac{\partial \hat{f}}{\partial t} = \frac{B\sqrt{\theta(t)}}{4\pi} \int_{S^2} \left\{ \hat{f}(t,k_-)\hat{f}(t,k_+) - \hat{f}(t,0)\hat{f}(t,k) \right\} d\sigma. \tag{3.18}$$

This form is particularly adapted to show by the classical methods of Wild sums the well-posedness of the Cauchy problem for the IME (3.15). In the case of the elastic Boltzmann equation $e = 1$, it was already done in [18, Section 13] and the references therein, based on ideas of Morgenstern and Wild. A simple adaptation to the present case can be obtained. A review on the application of Wild sums to the Boltzmann equation can be found in a recent paper of the Porto Ercole summer school lectures [36].

Let us define $f(t,v) \in C([0,\infty), \mathcal{P}_2(\mathbb{R}^3))$ to be a solution of (3.15) if its Fourier transform in v is a characteristic function for any $t \geq 0$, $\hat{f}(t,k)$ is continuous in (t,k), twice-differentiable in k and differentiable in time and solves (3.18).

Proposition 3.5 (Well-posedness of IME)**.** [18, Section 13] *Given an initial data $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$, the Cauchy problem for* (3.15) *has a unique solution in $f \in C([0,\infty), \mathcal{P}_2(\mathbb{R}^3))$.*

Proof.- Since the temperature will finally be given by

$$\theta(t) = \frac{4\theta(0)}{(\frac{1-e^2}{4} B\sqrt{\theta(0)}t + 2)^2}$$

due to (3.16), then we rescale in time by defining

$$\tilde{t} = B \int_0^t \sqrt{\theta(s)}\, ds,$$

obtaining

$$\frac{\partial \hat{f}}{\partial \tilde{t}} = \frac{1}{4\pi} \int_{S^2} \left\{ \hat{f}(\tilde{t}, k_-)\hat{f}(\tilde{t}, k_+) - \hat{f}(\tilde{t}, 0)\hat{f}(\tilde{t}, k) \right\} d\sigma.$$

Performing again another time change variable defined by

$$\tau = 1 - \exp(-\tilde{t}) \ , \qquad \hat{f}(\tilde{t}, k) = \exp(-\tilde{t})\Phi(\tau, k),$$

then (3.18) leads to

$$\frac{\partial \Phi}{\partial \tau} = \frac{1}{4\pi} \int_{S^2} \Phi(\tau, k_-)\Phi(\tau, k_+)\, d\sigma = \mathcal{B}(\Phi, \Phi)$$

with $\Phi(k, 0) = \hat{f}_0(k)$. Solutions in power series expansion of the type

$$\Phi(\tau, k) = \sum_{n=0}^{\infty} \Phi_n(k)\tau^n$$

are given by a simple recurrent sequence of identities

$$\begin{aligned} \Phi_0 &= \hat{f}_0 \\ \Phi_{n+1} &= \frac{1}{n+1} \sum_{k=0}^{n} \mathcal{B}(\Phi_k, \Phi_{n-k}) \ , \qquad n \geq 0 \end{aligned} \tag{3.19}$$

Noting that $|\hat{f}_0| \leq 1$, we obtain $|\Phi_n| \leq 1$ for any $n \geq 0$. Then the series (3.19), called the Wild's sum in the Fourier representation, converges uniformly on $\tau \in [0, 1)$ together with their time derivatives. Levy's continuity theorem [56] will ensure that the obtained series defines a characteristic function for all times. Moreover, the solution verifies the dissipation of temperature

$$\theta(\tau) = \exp\left(-\frac{1 - e^2}{4}\tau\right)\theta(0)$$

giving (3.16) in the original time variable. □

4. Nontrivial Asymptotics for Dissipative Boltzmann Equations

As we have pointed out in Proposition 3.2 and Corollary 3.3, the asymptotic behavior of solutions of homogeneous dissipative models reflects the intuition from basic modelling: without external source of energy a uniform cooling of the granular gas occurs. If particles dissipates energy, after certain time possibly infinite, particles move uniformly at their mean speed. The main issues now are:

- Can we give a more detailed picture of the cooling process apart from this "boring" behavior of the particles?

- On the other hand, if we introduce some source of energy onto the system, can we expect to achieve non-trivial steady states?

Regarding the first question, we can ask ourselves what it the typical asymptotic cooling profile, that is, if there is a typical profile for the solutions of the system to cool down. The situation is quite similar to diffusion equations in which the opposite behavior happens, i.e., the increase of temperature or heating. In these situations, the first trial is to look for self-similar solutions of the equations cooling down at the dissipation rate of the system. For instance, look for solutions of the inelastic Boltzmann equation IBE (3.11) of the form:

$$f_{hc}(t,v) = \rho\, \theta_{hc}^{-\frac{3}{2}}(t)\, g_\infty((v-u)\, \theta_{hc}^{-\frac{1}{2}}(t)) \tag{4.1}$$

with $\theta_{hc}(t)$ the temperature of the solution $f_{hc}(t,v)$ itself that will follow the dissipation of energy (3.12) and g_∞ the searched cooling profile. These self-similar solutions are called **homogeneous cooling states**. The existence of cooling profiles for the inelastic Boltzmann equation (3.11) has been obtained in [85] although its uniqueness and stability properties are open problems. The analysis of the homogeneous cooling states for the IMM (3.15) has been done in [20, 57, 58, 22, 25, 13, 29, 23, 24] and we will elaborate on this in Subsection 7.3.

Concerning the second question, we can introduce some sources of kinetic energy in the system expecting that a compensation in the struggle between dissipation-cooling due to inelastic collisions and excitation-heating due to some mechanism leads to a possible stationary state. This principle was applied in other simplified granular media models as in [7, 8, 43, 44] with success.

In these notes, we will treat two different heating mechanisms: **stochastic heating** and a **thermal bath of particles**. In the stochastic heating we assume that particles follow Brownian motion between inelastic collisions rather than moving freely. This assumption at the level of the kinetic Boltzmann-type equation results in adding $\Delta_v f$ on the equation, that is,

$$\frac{\partial f}{\partial t} = Q_e(f,f) + \theta_b \Delta_v f. \tag{4.2}$$

with $\theta_b > 0$ related to the temperature or variance of the stochastic process. This term changes the dissipation of temperature to

$$\theta'(t) \leq -\sqrt{3}\frac{1-e^2}{8}\, \theta(t)^{3/2} + 2\theta_b$$

where we observe the possible compensation leading to a steady value of the temperature. The Cauchy problem for equation (4.2) has been analysed in

[63] in which it is proved the existence of smooth solutions of the stationary problem answering affirmatively the existence of non-trivial equilibria. However, the uniqueness and its asymptotic stability remain open problems too. In the particular case of the inelastic Maxwell model, we can consider the equation

$$\frac{\partial f}{\partial t} = \tilde{Q}_e(f,f) + \theta_b \Delta_v f \tag{4.3}$$

for which the dissipation of temperature becomes the identity:

$$\theta'(t) = -\frac{1-e^2}{4} B\, \theta(t)^{3/2} + 2\theta_b$$

which has the unique steady value of the temperature

$$\theta_\infty = \left(\frac{8\theta_b}{B(1-e^2)} \right)^{2/3} \tag{4.4}$$

allowing for the possibility of the existence of non-trivial steady states. This problem for (4.3) has been treated in [51, 21, 12, 29].

Another choice to heat the system up has recently been proposed in [14] based on linear dissipative granular equations introduced in [98]. Here the particles of the granular gas of mass m, called test particles assumed to be rarefied, are immersed in a host medium of particles, called field particles assumed to be denser, in such a way that they undergo inelastic collisions both with test and field particles but with possibly different restitution coefficients. The distribution of field particles of mass m_b is assumed to be given by a normalized Maxwellian distribution

$$M_b = M(v, m_b, U_b, \theta_b) = \left(\frac{m_b}{2\pi\theta_b} \right)^{3/2} \exp\left[-\frac{m_b}{2\theta_b} |v - U_b|^2 \right]$$

with fixed mean velocity U_b and temperature θ_b. The interaction between the test particles, whose evolution we are interested in, and the field particles or particle's bath is modelled through a linear collision kernel of the form

$$\mathcal{L}_{e_b}(f) = \frac{1}{\pi\lambda} \int_{\mathbb{R}^3} \int_{S^2_+} ((v-w)\cdot n) \left[\frac{1}{e_b^2} f(v_L^*)\, M_b(w_L^*) - f(v)\, M_b(w) \right] dw\, dn$$

with $\lambda > 0$ being the ratio between the mean free paths of field and test particles, e_b the restitution coefficient of test/field particle interaction and (v_L^*, w_L^*) are the pre–collisional velocities of the so–called inverse collision, which results in (v, w) as post–collisional velocities (4.7). Test particles exchange momentum and energy with the background even in the elastic case $e_b = 1$, and this effect depends on the mass ratio appearing in the

pre-collisional velocities. Mass ratio will be described by the dimensionless parameter

$$R_m = \frac{m_b}{m + m_b}, \tag{4.5}$$

where $0 < R_m < 1$ excluding thus the peculiarities of the limiting cases of Lorenz and Rayleigh gas.

The associated weak formulation of the linear dissipative collision operator becomes as before

$$< \varphi, \mathcal{L}_{e_b}(f) > = \frac{1}{2\pi\lambda} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} |u \cdot n| f(v) M_b(w) \Big[\varphi(v'_L) - \varphi(v)\Big] dn\, dv\, dw \tag{4.6}$$

with $u = v - w$ and the post-collisional velocities given by

$$\begin{aligned} v'_L &= v - R_m(1 + e_b)(u \cdot n)n \\ w'_L &= w + (1 - R_m)(1 + e_b)(u \cdot n)n. \end{aligned} \tag{4.7}$$

The linear dissipative collision kernel preserves mass while mean velocity and energy change even in the elastic case $e_b = 1$ due to the different masses of particles. The evolution equation for the distribution of test particles in the homogeneous case becomes

$$\frac{\partial f}{\partial t} = Q_e(f, f) + \mathcal{L}_{e_b}(f). \tag{4.8}$$

The Cauchy problem and properties of this equation have not been studied yet to our knowledge apart from the moment equations developed in [14]. It is not known if this linear dissipative operator may prevent the complete cooling of the system as for the stochastic bath case.

As for the nonlinear operator a further reduction is possible on the linear dissipative operator by means of an approximation of the collision frequency. The pseudo-Maxwellian approximation consists in replacing the relative velocity u in the collision kernel $|u \cdot n|$ by a different vector $\tilde{U}\,\Omega$, where Ω is the unit vector in the direction of $v - w$, whereas $\tilde{U}$ is a parameter independent of the integration variables but possibly dependent on macroscopic variables. Actually, due to consistency with the approximation done in the nonlinear case we will take as typical collision frequency a multiple of the thermal speed, i.e., $\tilde{U} \simeq \tilde{u}\sqrt{\theta(t)}$ where $\theta(t)$ is the temperature of the distribution itself. In this case, the simplified linear dissipative operator reads as

$$< \varphi, \tilde{\mathcal{L}}_{e_b}(f) > = \frac{\sqrt{\theta(t)}}{2\pi\tilde{\lambda}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} |\Omega \cdot n| f(v) M_1(w) \Big[\varphi(v'_L) - \varphi(v)\Big] dn\, dv\, dw \tag{4.9}$$

with $\tilde{\lambda} = \frac{\lambda}{\bar{u}}$. It is also straightforward to write its Fourier version using Lemma 3.4 obtaining

$$\widehat{\tilde{\mathcal{L}}_{e_b}(f)} = \frac{\sqrt{\theta(t)}}{2\pi\tilde{\lambda}} \int_{S^2} |\omega \cdot n| \left[\hat{f}(k_+^L)\hat{M}_b(k_-^L) - \hat{f}(k)\hat{M}_b(0) \right] dn. \tag{4.10}$$

with $\omega = k/|k|$ and

$$\begin{cases} k_+^L = k - R_m(1+e_b)(k \cdot n)n \\ k_-^L = R_m(1+e_b)(k \cdot n)n, \end{cases} \tag{4.11}$$

while $\hat{M}_b(k)$ is the Fourier transform of the background Maxwellian distribution,

$$\hat{M}_b(k) = \exp\left\{ -iU_b \cdot k - \frac{\theta_b}{2m_b}|k|^2 \right\}.$$

Therefore, the Maxwellian approximation for the particles bath heating equation (4.8) reads as

$$\frac{\partial f}{\partial t} = \tilde{Q}_e(f,f) + \tilde{\mathcal{L}}_{e_b}(f). \tag{4.12}$$

The study of the well-posedness of the Cauchy problem for $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ of the Inelastic Maxwell Models (4.3) and (4.12) can be done analogously to Proposition 3.5 obtaining the existence and uniqueness of solution in $f \in C([0,\infty), \mathcal{P}_2(\mathbb{R}^3))$.

Another important characteristic of pseudo-maxwell models is that moment equations are always explicit. Thus, we can write the collision changes of momentum and energy giving the evolution for the mean velocity and the temperature of solutions to (4.12)

$$\frac{dU}{dt} = -\sqrt{\theta}\,\frac{R_m(1+e_b)}{2\tilde{\lambda}}(U - U_b) \tag{4.13}$$

$$\begin{aligned} \frac{d\theta}{dt} &= -\frac{1-e^2}{4} B\,\theta^{3/2} + \sqrt{\theta}\,\frac{mR_m^2}{6\tilde{\lambda}}(1+e_b)^2|U-U_b|^2 \\ &\quad - \sqrt{\theta}\,\frac{R_m(1+e_b)}{2\tilde{\lambda}} \left\{ [2 - R_m(1+e_b)]\,\theta - (1-R_m)(1+e_b)\theta_b \right\}. \end{aligned} \tag{4.14}$$

with

$$U = \int_{\mathbb{R}^3} v\, f(v)\, dv \quad \text{and} \quad \theta = \frac{m}{3}\int_{\mathbb{R}^3} |v-U|^2\, f(v)\, dv$$

for normalized densities to unit number density. We observe clearly the existence of a unique stationary value for mean velocity $U = U_b$ and temperature $\theta = \theta^{\#}$ determined by

$$\left\{ \frac{1-e^2}{4} B + \frac{R_m(1+e_b)}{2\tilde{\lambda}} \left[2 - R_m(1+e_b)\right] \right\} \theta^{\#} = \frac{R_m(1-R_m)(1+e_b)^2}{2\tilde{\lambda}} \theta_b \tag{4.15}$$

As for the stochastic heating, this allows the existence of nontrivial steady states in the particle's bath case.

Let us finally mention that in the elastic case, $e = e_b = 1$, this particle's bath thermostat was considered in [26] as a particular case of binary mixtures in the weak coupling case. There, the existence of asymptotically stable stationary solutions is obviously given by the Maxwellian with the equilibrium temperature. The authors devote their findings to study the existence of self-similar profiles of the system converging towards these stationary Maxwellians.

These questions: homogenous cooling states (4.1) for the Inelastic Maxwell Model without external source of energy (3.15), stationary states for the stochastic (4.3) and the particles bath heating (4.8) equations and their asymptotic stability; will be discussed for the Inelastic Maxwell Models in next sections by using the tools and techniques of probability metrics reviewed in section 2. Next section will be devoted to one dimensional reduced dissipative Maxwell models used not only in physics but also in economic models [88].

5. One-dimensional dissipative Maxwell models

In one–dimension of velocity space, while in an elastic binary collision particles simply exchange their velocities and the Bolzmann collision operator for elastic collisions disappears, the dissipative Maxwellian type Boltzmann collision operator $\tilde{Q}_e(f,f)$ is still a well-defined dissipative collision mechanism. Assuming that the coefficient in front of the integral in $\tilde{Q}_e$ is equal to one, the weak form (3.14) now simplifies to

$$< \varphi, \tilde{Q}_e(f,f) >= \int_{\mathbb{R}} f(v) f(w) \Big[\varphi(v') - \varphi(v)\Big] \, dv \, dw. \tag{5.1}$$

In (5.1) the post-collisional velocities are given by:

$$v' = \frac{1}{2}(v+w) + \frac{e}{2}(v-w); \qquad w' = \frac{1}{2}(v+w) - \frac{e}{2}(v-w). \tag{5.2}$$

A more suitable form of (5.2) can be obtained by setting the coefficient of restitution $e = 1 - 2\gamma$, where now $0 < \gamma < 1/2$ is the dissipation parameter.

In terms of γ, the dissipative collision reads

$$v' = (1-\gamma)v + \gamma w; \qquad w' = \gamma v + (1-\gamma)w. \tag{5.3}$$

5.1. **General one-dimensional mixing models.** In form (5.3), the dissipative collision is a particular case of the more general rule

$$v' = pv + qw, \quad w' = qv + pw; \quad p > q > 0, \tag{5.4}$$

where the positive constants p and q represent the interacting parameters, namely the portion of the pre–collisional velocities (v, w) which generate the post–collisional ones (v', w'). A one-dimensional Boltzmann type equation of the form

$$\frac{\partial f}{\partial t} = Q_{p,q}(f, f) \tag{5.5}$$

based on this binary interaction has been first considered in [6], and subsequently studied in [88]. The case $p = 1-\gamma$, $q = \gamma$, however has been first introduced and studied in [5] in connection with dissipative kinetic theory.

Without loss of generality, we can fix the initial density $f_0(v) \in \mathcal{P}_2(\mathbb{R})$, with the normalization conditions

$$\int_{\mathbb{R}} v f_0(v)\, dv = 0 \quad \text{and} \quad \int_{\mathbb{R}} v^2 f_0(v)\, dv = 1. \tag{5.6}$$

The Boltzmann equation (5.5) in weak form reads

$$\frac{d}{dt}\int_{\mathbb{R}} \varphi(v) f(v,t)\, dv = \int_{\mathbb{R}^2} f(v) f(w) \Big[\varphi(v') - \varphi(v)\Big]\, dv\, dw. \tag{5.7}$$

One can alternatively use the symmetric form

$$\frac{d}{dt}\int_{\mathbb{R}} f(v)\phi(v)\, dv = \frac{1}{2}\int_{\mathbb{R}^2} f(v) f(w)(\varphi(v') + \varphi(w') - \varphi(v) - \varphi(w)) dv\, dw\,. \tag{5.8}$$

A remarkable fact is that equations (5.7) and (5.8) can be studied for all values of the mixing parameters p and q. Due to the fact that the nonlinear kinetic equation (5.7) is one-dimensional in the velocity space, it allows a (relatively) easy discussion, which will help to clarify how the various probability metrics work. For this reason, we will discuss in the following equation (5.7) for general values of p and q, reducing to the dissipative case $p + q = 1$ only when necessary.

Choosing $\varphi(v) = v$, (respectively $\varphi(v) = v^2$) shows that

$$m(t) = \int_{\mathbb{R}} v f(v,t)\, dv = m(0) \exp\left\{(p+q-1)t\right\}. \tag{5.9}$$

Hence, since the initial density f_0 satisfies (5.6), $m(0) = 0$ and $m(t) = 0$ for all $t > 0$. Analogously,

$$\theta(t) = \int_{\mathbb{R}} v^2 f(v,t)\, dv = \exp\left\{(p^2 + q^2 - 1)t\right\}. \tag{5.10}$$

Higher order moments can be evaluated recursively, remarking that they obey a closed hierarchy of equations [9].

Note that the second moment of the solution is not conserved, unless the collision parameters satisfy

$$p^2 + q^2 = 1.$$

If this is not the case, the energy can grow to infinity or decrease to zero, depending on the sign of $p^2 + q^2 - 1$. In both cases, however, stationary solutions of finite energy do not exist, and the large–time behavior of the system can at best be described by self-similar solutions. The standard way to look for self–similarity is to scale the solution according to the ansatz

$$g(v,t) = \sqrt{\theta(t)} f\left(v\sqrt{\theta(t)}, t\right). \tag{5.11}$$

This scaling fixes the second moment

$$\int_{\mathbb{R}} v^2 g(v,t)\, dv = 1$$

for all $t \geq 0$. Elementary computations show that $g = g(v,t)$ satisfies

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \varphi(v) g(v,t)\, dv =& \frac{1}{2}\left(p^2 + q^2 - 1\right)\int_{\mathbb{R}} \varphi(v)\frac{\partial}{\partial v}\left(vg\right)\, dv \\ &+ \int_{\mathbb{R}^2} g(v)g(w)(\varphi(v') - \varphi(v))dvdw. \end{aligned}\tag{5.12}$$

Assuming that φ vanishes at infinity, we can integrate by parts the first integral on the right–hand side of (5.12) to obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \varphi(v) g(v,t)\, dv =& -\frac{1}{2}\left(p^2 + q^2 - 1\right)\int_{\mathbb{R}} \frac{\partial \varphi}{\partial v} v g(v)\, dv \\ &+ \int_{\mathbb{R}^2} g(v)g(w)(\varphi(v^*) - \varphi(v))dvdw. \end{aligned}\tag{5.13}$$

Instead of working with the weak form (5.7) for f ((5.12) for g) one can equivalently use the Fourier transform of the equation (5.5) following Bobylev's change of variables [19]:

$$\frac{\partial \hat{f}(k,t)}{\partial t} = \widehat{Q}\left(\hat{f}, \hat{f}\right)(k,t) = \widehat{Q}\left(\hat{f}, \hat{f}\right)(k) = \hat{f}(pk)\hat{f}(qk) - \hat{f}(k)\hat{f}(0). \tag{5.14}$$

The initial conditions (5.6) turn into $\hat{f}(0) = 1$, $\hat{f}'(0) = 0$ and $\hat{f}''(0) = -1$ with $\hat{f} \in C^2(\mathbb{R})$. Hence equation (5.14) can be rewritten as

$$\frac{\partial \hat{f}(k,t)}{\partial t} + \hat{f}(k,t) = \hat{f}(pk)\hat{f}(qk). \tag{5.15}$$

Using the same arguments, one concludes that the Fourier transform of g satisfies

$$\frac{\partial \hat{g}(k,t)}{\partial t} - \frac{p^2+q^2-1}{2}\, k\frac{\partial \hat{g}}{\partial k} + \hat{g}(k) = \hat{g}(pk)\hat{g}(qk) \tag{5.16}$$

Both equations (5.15) and (5.16) have been considered by Bobylev and Cercignani in [22]. In particular, it can be easily verified that two cases are special from others. The first one is the *elastic* case $p^2 + q^2 = 1$. In this case $\theta(t)$ remains constant, and the steady state (of temperature $\theta = 1$) of equation (5.15) is the function $\hat{f}_\infty(k) = \exp\{-k^2/2\}$, namely the Fourier transform of the Maxwellian function

$$M(v) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{v^2}{2}\right\}.$$

The second distinguished case is the *dissipative* case $p + q = 1$. Under this assumption equation (5.16) has the explicit steady state (of temperature $\theta = 1$)

$$\hat{g}_\infty(k) = (1 + |k|)\exp\{-|k|\}. \tag{5.17}$$

One can easily verify by direct inspection that the function (5.17) satisfies equation (5.16). This solution has been found independently by Bobylev and Cercignani in [22] by owing to the Fourier transform of the Boltzmann equation, and one year before by Baldassarri, Marini Bettolo Marconi and Puglisi as similarity solution of the Ulam model in [5]. The explicit form of the steady state $\hat{g}_\infty(k)$ in the velocity variable can be obtained by remarking that its second derivative satisfies the equality

$$\hat{g}_\infty''(k) = (-1 + |k|)\mathrm{e}^{-|k|} = -2\mathrm{e}^{-|k|} + \hat{g}_\infty(k). \tag{5.18}$$

Now, recalling that $e^{-|k|}$ is the Fourier transform of $\big(\pi(1+v^2)\big)^{-1}$, called the Cauchy density, (5.18) implies

$$(1+v^2)g_\infty(v) = \frac{2}{\pi(1+v^2)},$$

that is

$$g_\infty(v) = \frac{2}{\pi(1+v^2)^2}.$$

5.2. **The evolution of Wasserstein metric.** The Fourier version (5.15) suggests that the positive part of the collision operator is a convolution operator. In fact, the Boltzmann equation (5.7) can be rewritten as

$$\frac{\partial f}{\partial t} = f_p * f_q - f,$$

where we used the shorthand $f_p(v) = (1/p)f(v/p)$. Using an argument first introduced in [27], let us consider an explicit Euler approximation to equation (5.15)

$$\hat{f}(k, t + \Delta t) = \Delta t \hat{f}(pk, t)\hat{f}(qk, t) + (1 - \Delta t)\hat{f}(k, t).$$

In the physical space

$$f(v, t + \Delta t) = \Delta t(f_p * f_q)(v, t) + (1 - \Delta t)f(v, t). \tag{5.19}$$

Thanks to the convexity property *vi)* of Proposition 2.1, given two initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R})$, with the normalization conditions (5.6), equality (5.19) implies

$$\begin{aligned} W_2^2(f_1(t + \Delta t), f_2(t + \Delta t)) \leq \Delta t\, W_2^2((f_1)_p(t) * (f_1)_q(t), (f_2)_p(t) * (f_2)_q(t)) \\ + (1 - \Delta t)W_2^2(f_1(t), f_2(t)). \end{aligned} \tag{5.20}$$

Moreover, thanks to the convolution property *vii)* and to the scaling property *v)* of Proposition 2.1,

$$W_2^2((f_1)_p(t)*(f_1)_q(t), (f_2)_p(t)*(f_2)_q(t)) \leq (p^2+q^2)W_2^2(f_1(t), f_2(t)). \tag{5.21}$$

Using (5.21) into (5.20) gives

$$W_2^2(f_1(t + \Delta t), f_2(t + \Delta t)) \leq \big(1 + (p^2 + q^2 - 1)\Delta t\big)\, W_2^2(f_1(t), f_2(t)),$$

which implies

$$W_2^2(f_1(t), f_2(t)) \leq W_2^2(f_1^0, f_2^0)\, \exp\big\{(p^2 + q^2 - 1)t\big\}. \tag{5.22}$$

Equation (5.22) implies the contractivity of the Wasserstein metric in case $p^2 + q^2 - 1 < 0$. Note that this is the case, among others, of the granular Boltzmann equation, where $p + q = 1$. On the other hand, since the scaled density $g(v, t)$ satisfying equation (5.13) is obtained from $f(v, t)$ through the scaling (5.11), the scaling property *v)* of Proposition 2.1 implies that Wasserstein metric is non expanding also along solutions to equation (5.13) for all values of the mixing parameters p and q, i.e.,

$$W_2(g_1(s), g_2(s)) \leq W_2(g_1(t), g_2(t)), \quad 0 \leq s \leq t. \tag{5.23}$$

Let us point out that the contractivity property obtained in (5.21) for the gain operator of (5.7) can be obtained by a different method. In fact,

the gain operator acting on weak form can be written as:

$$(\varphi, Q^+_{p,q}(f,f)) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(v)\, f(w)\, (\varphi, \delta_{pv+qw})\, dv\, dw$$

where δ_{pv+qw} is the Delta Dirac at the post-collisional velocity $v' = pv+qw$. In probabilistic terms, the gain operator is defined as an expectation:

$$Q^+_{p,q}(f,f) = f_p * f_q = \mathbb{E}\left[\delta_{pV+qW}\right]$$

where V and W are independent random variables with law f.

Let us take two independent pairs of random variables (V,X) and (W,Y) such that V and W have law f_1 and X and Y have law f_2. From the convexity of W_2^2 and the independency of the pairs, it follows that

$$W_2^2(Q^+_{p,q}(f_1,f_1), Q^+_{p,q}(f_2,f_2)) \le \mathbb{E}\left[W_2^2(\delta_{pV+qW}, \delta_{pX+qY})\right]$$

for any probability densities $f_1, f_2 \in \mathcal{P}_2(\mathbb{R})$. Now, the last term is directly computed as the euclidean distance of the two points $pV+qW$ and $pX+qY$, and thus,

$$W_2^2(Q^+_{p,q}(f_1,f_1), Q^+_{p,q}(f_2,f_2)) \le \mathbb{E}\left[|p(V-X)+q(W-Y)|^2\right].$$

Using independency of the pairs and taking the pairs to be optimal couples for the $W_2(f_1,f_2)$ in the probabilistic definition (2.2), we deduce finally the contractivity property

$$W_2^2(Q^+_{p,q}(f_1,f_1), Q^+_{p,q}(f_2,f_2)) \le (p^2+q^2) W_2^2(f_1,f_2)$$

as in (5.21).

Remark 5.1 (Convergence without rate in W_2)**.** *As remarked by Murata and Tanaka* [86] *and others, a functional having the contractivity property* (5.23) *can be used for several applications. In particular, it can be used to obtain convergence (without rate) towards the (unique) stationary solution for equation* (5.16)*. An application for the convergence to the Gaussian density in the central limit problem can be found in* [103]*. This argument will be used later on to solve the Ernst-Brito conjecture on global stability of homogeneous cooling states to* (3.15) *without rate* [29].

5.3. **Strong contractivity in Fourier based metrics.** The analysis of Section 5.2 showed that contractivity in Wasserstein metric for the solution to equation (5.16) can be obtained as a simple consequence of the various properties of the metric, outlined in Proposition 2.1. Thus, the same conclusion can be drawn for the Fourier based metric d_2, on the basis of the same properties (see Proposition 2.9). In this case, however, we can obtain stronger estimates, thanks to the convolution property *v)* of the same Proposition, which now holds for any metric d_s.

Let $s > 0$ be fixed. Given two initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R})$, with the normalization conditions (5.6), let us assume that they have equal moments up to $[s]$ if $s \notin \mathbb{N}$, or equal moments up to $s-1$ if $s \in \mathbb{N}$. We consider again the explicit Euler approximation to equation (5.15) given by

$$f(k, t+\Delta t) = \Delta t \hat{f}(pk, t)\hat{f}(qk, t) + (1-\Delta t)\hat{f}(k, t).$$

Thanks to the convexity property *iv)* of Proposition 2.9, equality (5.19) implies

$$\begin{aligned} d_s(f_1(t+\Delta t), f_2(t+\Delta t)) \leq \Delta t\, d_s((f_1)_p(t) * (f_1)_q(t), (f_2)_p(t) * (f_2)_q(t)) \\ + (1-\Delta t) d_s(f_1(t), f_2(t)). \end{aligned} \tag{5.24}$$

Moreover, thanks to the convolution property *v)* and to the scaling property *iii)* of Proposition 2.9,

$$d_s((f_1)_p(t) * (f_1)_q(t), (f_2)_p(t) * (f_2)_q(t)) \leq (p^s + q^s) d_s(f_1(t), f_2(t)). \tag{5.25}$$

Using (5.25) into (5.24) gives

$$d_s(f_1(t+\Delta t), f_2(t+\Delta t)) \leq (1 + (p^s + q^s - 1)\Delta t)\, d_s(f_1(t), f_2(t)),$$

which implies

$$d_s(f_1(t), f_2(t)) \leq d_s(f_1^0, f_2^0)\, \exp\left\{(p^s + q^s - 1)t\right\}. \tag{5.26}$$

As for the case of the Wasserstein metric, (5.26) implies the contractivity in Fourier metric as soon as case $p^s + q^s - 1 < 0$, that among others, if $s > 1$ is the case of the granular Boltzmann equation, where $p + q = 1$. But now we have a flexibility in the choice of the constant s. Since the scaled density $g(v, t)$ satisfying equation (5.13) is obtained from $f(v, t)$ through the scaling (5.11), the scaling property *iii)* of Proposition (2.9) implies that

$$d_s(g_1(t), g_2(t)) \leq d_s(f_1^0, f_2^0)\, \exp\left\{(p^s + q^s - 1)t\right\} \theta(t)^{-(s/2)} \tag{5.27}$$

Using (5.10) into (5.27), we finally conclude that, if $g_1(t)$ and $g_2(t)$ are two solutions of the scaled Boltzmann equation (5.16), corresponding to initial values f_1^0 and f_2^0 satisfying conditions (5.6) and such that $d_s(f_1^0, f_2^0)$ is bounded, then for all times $t \geq 0$,

$$d_s(g_1(t), g_2(t)) \leq \exp\left\{\left[(p^s + q^s - 1) - \frac{s}{2}(p^2 + q^2 - 1)\right] t\right\} d_s(f_1^0, f_2^0).$$

Let us define

$$\mathcal{S}_{p,q}(s) = p^s + q^s - 1 - \frac{s}{2}\left(p^2 + q^2 - 1\right). \tag{5.28}$$

Then, the sign of $\mathcal{S}_{p,q}$ determines the behavior of the distance $d_s(g_1(t),g_2(t))$. In particular, if there exists an interval in which $\mathcal{S}_{p,q}(\delta) < 0$, we can conclude that $d_s(g_1(t), g_2(t))$ converges exponentially to zero. Note that, by construction, $\mathcal{S}_{p,q}(2) = 0$. The function (5.28) was first considered by Bobylev and Cercignani in [22]. The sign of $\mathcal{S}_{p,q}$, however was studied mainly for

$p = 1 - q$, namely the case of the dissipative Boltzmann equation. In Figure 2, a numerical evaluation of the region where the minimum of the function $\mathcal{S}_{p,q}$ is negative for $p, q \in [0, 2]$ is reported.

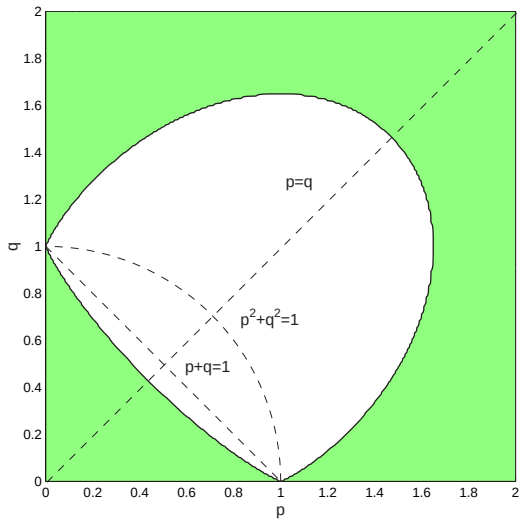

FIGURE 2. The white domain represents the region where the minimum of the function $\mathcal{S}_{p,q}$ is negative for $p, q \in [0, 2]$.

An almost complete study of the behavior of the convex function (5.28) has been done in [88]. The main result reads

Theorem 5.2 (Contraction in d_s for 1D Scaled Dissipative Models)**.** *Let $g_1(t)$ and $g_2(t)$ be two solutions of the one dimensional scaled Boltzmann equation* (5.16)*, corresponding to initial values f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R})$, satisfying conditions* (5.6)*. Then, there exists a constant $\bar{\delta} > 0$ such that, if $2 < s < 2 + \bar{\delta}$, for all times $t \geq 0$,*

$$d_s(g_1(t), g_2(t)) \leq \exp\{-C_s t\}\, d_s(f_1^0, f_2^0). \tag{5.29}$$

The constant $C_s = -\mathcal{S}_{p,q}(s)$ is strictly positive, and the distance d_s is contracting exponentially in time.

Remark 5.3 (Granular Boltzmann Equation)**.** *In the main case of the granular Boltzmann equation, where $p + q = 1$, it can be easily verified by direct inspection that $\mathcal{S}_{p,q}(3) = 0$. This fact together with the convexity of $\mathcal{S}_{p,q}(s)$ implies that $\mathcal{S}_{p,q}(s) < 0$ for $2 < s < 3$. In this case the Fourier metric d_s is contracting exponentially in time for all values of s in this interval.*

These principles that we have seen working in one dimensional models will be at the basis of the main ideas to derive contraction estimates in probability metrics for more complicated IMMs.

6. Contraction of metrics for IMMs

This section is devoted to show the main properties of the gain operators, both for the nonlinear $\tilde{Q}_e(f,f)$ and the linear $\tilde{\mathcal{L}}_{e_b}(f)$ dissipative kernels, regarding their contractivity with respect to probability distances introduced in Section 2, the Euclidean Wasserstein distance W_2 and the Fourier-based distances d_s, $s>0$.

The results related to contractions in W_2 are included in [29] while the ones concerning the Fourier-based metrics are developed in [12, 13, 98]. Here, we make a summary of the main ideas while improving and developping new results compared to those references.

6.1. **Contractions in W_2.** Given a probability measure f on $\mathbb{R}^3$, the gain operator is in fact a probability measure $\tilde{Q}_e^+(f,f)$ defined by

$$(\varphi, \tilde{Q}_e^+(f,f)) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)\, f(w)\, (\varphi, \mathcal{U}_{v,w})\, dv\, dw$$

where $\mathcal{U}_{v,w}$ is the uniform probability distribution on the sphere $S_{v,w}$ with center $c_{v,w} = \frac{1}{2}(v+w) + \frac{1-e}{4}(v-w)$ and radius $r_{v,w} = \frac{1+e}{4}|v-w|$. The geometry of this representation is sketched in Figure 6.1. In probabilistic terms, the gain operator is defined as an expectation:

$$\tilde{Q}_e^+(f,f) = \mathbb{E}\left[\mathcal{U}_{V,W}\right]$$

where V and W are independent random variables with law f.

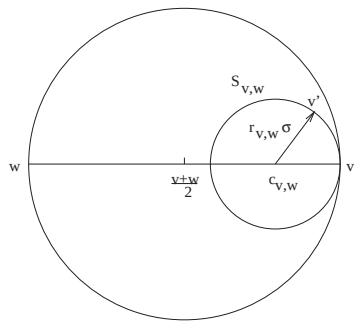

Figure 3. Gain operator as expectation over spheres.

Theorem 6.1 (Contraction of $\tilde{Q}_e^+(f,f)$ in W_2)**.** [29] *Given f and g in $\mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity, then*

$$W_2(\tilde{Q}_e^+(f,f),\tilde{Q}_e^+(g,g)) \leq \sqrt{\frac{3+e^2}{4}}\, W_2(f,g).$$

Moreover, assume f and g belong to $\mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity and temperature, where g is absolutely continuous with respect to Lebesgue measure with positive density such that

$$W_2(\tilde{Q}_e^+(f,f),\tilde{Q}_e^+(g,g)) = \sqrt{\frac{3+e^2}{4}}\, W_2(f,g).$$

for some restitution coefficient $0 < e \leq 1$, then $f = g$.

Proof.- The main steps of the proof can be summarized as follows: Let us take two independent pairs of random variables (V,X) and (W,Y) such that V and W have law f and X and Y have law g.

- **Step 1.-** Convexity of W_2^2 implies

$$W_2^2(\tilde{Q}_e^+(f,f),\tilde{Q}_e^+(g,g)) = W_2^2(\mathbb{E}\left[\mathcal{U}_{V,W}\right],\mathbb{E}\left[\mathcal{U}_{X,Y}\right]) \leq \mathbb{E}\left[W_2^2(\mathcal{U}_{V,W},\mathcal{U}_{X,Y})\right] \tag{6.1}$$

  where the expectation is taken with respect to the joint probability density in $\mathbb{R}^{12}$ of the four random variables. Here, the independency of the pairs of random variables has been used.

- **Step 2.-** The W_2^2 distance between the uniform distributions on the sphere with center O and radius r, $\mathcal{U}_{O,r}$, and on the sphere with center O' and radius r', $\mathcal{U}_{O',r'}$, in $\mathbb{R}^3$ is bounded by $|O'-O|^2+(r'-r)^2$.

  This is an estimate over the euclidean cost of transporting one sphere onto the other made by explicitly constructing a map T transporting them, $\mathcal{U}_{O',r'} = T\#\mathcal{U}_{O,r}$. Then, the transport plan $\Pi_T = (1_{\mathbb{R}^N} \times T)\#\mathcal{U}_{O,r}$ given by

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \eta(v,w)\, d\Pi_T(v,w) = \int_{\mathbb{R}^3} \eta(v,T(v))\, d\mathcal{U}_{O,r}(v)$$

  for all test functions $\eta(v,w)$, is used in the definition of the Euclidean Wasserstein distance (2.1) to conclude

$$W_2^2(\mathcal{U}_{O,r},\mathcal{U}_{O',r'}) \leq \int_{\mathbb{R}^3} |v-T(v)|^2\, d\mathcal{U}_{O,r}(v). \tag{6.2}$$

  Precisely, we define the map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ transporting the sphere of center O and radius $r > 0$ onto the sphere with center

$O' \neq O$ and radius $r' > r$ in the following way: consider the point $\Omega \in \mathbb{R}^3$ given by

$$\Omega = O + \frac{r}{r'-r}(O'-O).$$

Then we let T be the dilation with factor $\frac{r'}{r}$ centered at Ω, that is, we let $T(v) = \Omega + \frac{r'}{r}(v-\Omega)$. The other cases, $O' = O$ or $r' = r$, are done by simple translations or dilations. We show in Figure 4 a sketch of the construction of the map T in the case of non-interior spheres.

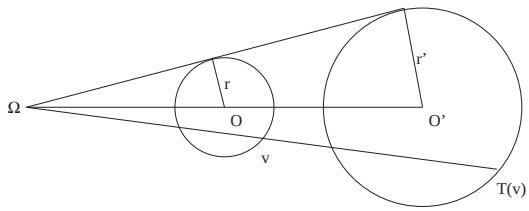

FIGURE 4. Scheme of the transport map between spheres.

Inserting this definition of the map T in (6.2), we deduce

$$W_2^2(\mathcal{U}_{O,r}, \mathcal{U}_{O',r'}) \leq \left(\frac{r'-r}{r}\right)^2 \int_{\mathbb{R}^3} |v-\Omega|^2 \, d\mathcal{U}_{O,r}(v)$$

that can be computed explicitly, giving

$$W_2^2(\mathcal{U}_{O,r}, \mathcal{U}_{O',r'}) \leq |O'-O|^2 + (r'-r)^2$$

and finishing the proof.

- **Step 3.-** We now estimate the right-hand side of (6.1) by using the formulas of the center and radii of the spheres given in (3.7) to deduce

$$W_2^2(\tilde{Q}_e^+(f,f), \tilde{Q}_e^+(g,g)) \leq \frac{5-2\,e+e^2}{8}\,\mathbb{E}\left[|V-X|^2\right] + \frac{(1+e)^2}{8}\,\mathbb{E}\left[|W-Y|^2\right] + \frac{1-e^2}{4}\,\mathbb{E}\left[(V-X)\cdot(W-Y)\right]$$

where the Cauchy-Schwartz inequality has been used.

- **Step 4.-** Finally, we take both pairs (V,X) and (W,Y) as independent pairs of variables with each of them being an optimal couple

for the $W_2(f,g)$ in the probabilistic definition (2.2) of W_2 to obtain

$$W_2^2(\tilde{Q}_e^+(f,f),\tilde{Q}_e^+(g,g)) \leq \frac{3+e^2}{4}\, W_2^2(f,g) + \frac{1-e^2}{4}\, \mathbb{E}\underbrace{[(V-X)\cdot(W-Y)]}_{=0}$$

due to independency and having equal mean velocity.

- **Step 5.-** The identity case is equivalent to say equality in the Cauchy-Schwartz inequality used in Step 3, and thus,

$$\frac{V-W}{|V-W|} = \frac{X-Y}{|X-Y|}$$

almost surely in the above notation. Then, since g is absolutely continuous with respect to Lebesgue measure with positive density, one can proceed as in [101, Lemma 9.1] to show that $f=g$.

We refer for a more detailed proof to [29]. □

Let us deduce an analogous property in the case of the linear operator $\tilde{\mathcal{L}}_{e_b}(f)$. In order to do this, it is better to rewrite the weak formulation with a different parametrization of the collisions like in the nonlinear case. In fact, coming back to its weak formulation in (4.9) and using (3.5), we deduce

$$< \varphi, \tilde{\mathcal{L}}_{e_b}(f) >= \tag{6.3}$$

$$= \frac{\sqrt{\theta}}{2\pi\tilde{\lambda}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |\Omega\cdot n| f(v) M_b(w) \Big[\varphi(v_L') - \varphi(v)\Big] dn\, dv\, dw$$

$$= \frac{\sqrt{\theta}}{\pi\tilde{\lambda}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2_+} \left(\frac{u}{|u|}\cdot n\right) f(v) M_b(w) \Big[\varphi(v_L') - \varphi(v)\Big] dn\, dv\, dw$$

$$= \frac{\sqrt{\theta}}{4\pi\tilde{\lambda}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v) M_b(w) \Big[\varphi(v_L') - \varphi(v)\Big] d\sigma\, dv\, dw \tag{6.4}$$

with

$$v_L' = (1-\chi)v + \chi w + \chi|u|\sigma$$

with $\chi = \frac{R_m(1+e_b)}{2}$. Now, splitting the operator as $\frac{\sqrt{\theta}}{\tilde{\lambda}}(\tilde{\mathcal{L}}_{e_b}^+(f) - \tilde{\mathcal{L}}_{e_b}^-(f))$, the gain part $\tilde{\mathcal{L}}_{e_b}^+(f)$ can be written as

$$(\varphi, \tilde{\mathcal{L}}_{e_b}^+(f)) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)\, M_b(w)\, (\varphi, \mathcal{U}_{v,w})\, dv\, dw$$

where $\mathcal{U}_{v,w}$ is the uniform probability distribution on the sphere $S_{v,w}$ with center $c_{v,w} = (1-\chi)v + \chi w$ and radius $r_{v,w} = \chi|u|$. Again, we can express

the probability measure $\tilde{\mathcal{L}}^+_{e_b}(f)$ as an expectation given by

$$\tilde{\mathcal{L}}^+_{e_b}(f) = \mathbb{E}\left[\mathcal{U}_{V,W}\right]$$

where V and W are independent random variables with laws f and M_b respectively.

Theorem 6.2 (Contraction of $\tilde{\mathcal{L}}^+_{e_b}(f)$ in W_2)**.** *Given f and g in $\mathcal{P}_2(\mathbb{R}^3)$, then*

$$W_2^2(\tilde{\mathcal{L}}^+_{e_b}(f), \tilde{\mathcal{L}}^+_{e_b}(g)) \leq \frac{1 + \left[1 - R_m(1+e_b)\right]^2}{2}\, W_2^2(f,g).$$

Proof.- This result follows the same steps as in Theorem 6.1. Let us now take two pairs of independent random variables (V,X) and (W,Y) such that V has law f, X law g and W and Y with law given by the fixed background Maxwellian distribution M_b. Again, convexity of W_2^2 implies

$$W_2^2(\tilde{\mathcal{L}}^+_{e_b}(f), \tilde{\mathcal{L}}^+_{e_b}(g)) = W_2^2(\mathbb{E}\left[\mathcal{U}_{V,W}\right], \mathbb{E}\left[\mathcal{U}_{X,Y}\right]) \leq \mathbb{E}\left[W_2^2(\mathcal{U}_{V,W}, \mathcal{U}_{X,Y})\right]. \tag{6.5}$$

Step 2 in Theorem 6.1 implies that the right-hand side of (6.5) can be estimated as

$$\begin{aligned} W_2^2(\tilde{\mathcal{L}}^+_{e_b}(f), \tilde{\mathcal{L}}^+_{e_b}(g)) \leq & (1 - 2\chi + 2\chi^2)\, \mathbb{E}\left[|V-X|^2\right] + 2\chi^2\, \mathbb{E}\left[|W-Y|^2\right] \\ & + (\chi - 3\chi^2)\, \mathbb{E}\left[(V-X)\cdot(W-Y)\right] \end{aligned}$$

where the Cauchy-Schwartz inequality has been used.

Now, let us take (V,X) an optimal couple for the $W_2(f,g)$ in the probabilistic definition (2.2) of W_2 and (W,Y) an optimal couple for the $W_2(M_b,M_b) = \mathbb{E}\left[|W-Y|^2\right] = 0$. By independency, we deduce

$$\mathbb{E}\left[(V-X)\cdot(W-Y)\right] = 0$$

since W and Y have the same law given by M_b. Summarizing, we have shown that

$$W_2^2(\tilde{\mathcal{L}}^+_{e_b}(f), \tilde{\mathcal{L}}^+_{e_b}(g)) \leq (1 - 2\chi + 2\chi^2)\, W_2^2(f,g)$$

as desired. □

6.2. **Contractions in d_s.** Now, we will obtain analogous properties with the Fourier-based distance d_2. Taking into account that the scaling and convexity properties of W_2^2 and d_2 are the same from Propositions 2.1 and 2.9, it would be natural to get similar constants for the contractions in d_2 as for W_2^2.

Theorem 6.3 (Contraction of $\tilde{Q}^+_e(f,f)$ in d_2)**.** [12] *Given f and g in $\mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity, then*

$$d_2(\tilde{Q}^+_e(f,f), \tilde{Q}^+_e(g,g)) \leq \frac{3+e^2}{4}\, d_2(f,g).$$

Proof.- The main steps of the proof are:

- **Step 1.-** Using the Fourier representation formula in Lemma 3.4, we deduce

$$\frac{\widehat{\tilde{Q}_e^+(f,f)}(k) - \widehat{\tilde{Q}_e^+(g,g)}(k)}{|k|^2} = \frac{1}{4\pi}\int_{S^2}\left[\frac{\hat{f}(k_-)\hat{f}(k_+) - \hat{g}(k_-)\hat{g}(k_+)}{|k|^2}\right] d\sigma$$

  for all $k \in \mathbb{R}^3_o$ with

$$k_- = \frac{1+e}{4}(k - |k|\sigma) \quad \text{and} \quad k_+ = \frac{3-e}{4}\,k + \frac{1+e}{4}\,|k|\sigma\,.$$

- **Step 2.-** We now estimate the integrand as

$$\begin{aligned}\left|\frac{\hat{f}(k_-)\hat{f}(k_+) - \hat{g}(k_-)\hat{g}(k_+)}{|k|^2}\right| &\le \sup_{k\in\mathbb{R}^3_o}\left\{\frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^2}\right\}\left(\frac{|k_-|^2 + |k_+|^2}{|k|^2}\right)\\ &= d_2(f,g)\left(\frac{|k_-|^2 + |k_+|^2}{|k|^2}\right),\end{aligned}$$

  and thus

$$d_2(\tilde{Q}_e^+(f,f), \tilde{Q}_e^+(g,g)) \le \frac{1}{4\pi}\int_{S^2}\left(\frac{|k_-|^2 + |k_+|^2}{|k|^2}\right) d\sigma\, d_2(f,g).$$

- **Step 3.-** We observe that $\frac{|k_-|^2+|k_+|^2}{|k|^2}$ is a function of the angle between the unit vectors $k/|k|$ and σ and that

$$I := \frac{1}{4\pi}\int_{S^2}\frac{|k_-|^2 + |k_+|^2}{|k|^2}\,d\sigma = \frac{e^2+3}{4}.$$

  In fact, we can compute

$$\begin{aligned}|k_-|^2 &= |k|^2\left(\frac{1+e}{4}\right)^2 2\left(1 - \cos\vartheta\right)\\ |k_+|^2 &= |k|^2\left[\left(\frac{3-e}{4}\right)^2 + \left(\frac{1+e}{4}\right)^2 + 2\left(\frac{3-e}{4}\right)\left(\frac{1+e}{4}\right)\cos\vartheta\right]\end{aligned} \tag{6.6}$$

where ϑ is the angle between the unit vectors $k/|k|$ and σ. Therefore

$$I = \frac{1}{2}\int_0^\pi \left\{ \left(\frac{1+e}{4}\right)^2 2\left(1-\cos\vartheta\right) \right.$$

$$\left. + \left(\frac{3-e}{4}\right)^2 + \left(\frac{1+e}{4}\right)^2 + 2\left(\frac{3-e}{4}\right)\left(\frac{1+e}{4}\right)\cos\vartheta \right\} \sin\vartheta\, d\vartheta$$

$$= \frac{1}{2}\int_0^\pi \left\{ \frac{e^2+3}{4} + \frac{(1+e)(1-e)}{4}\cos\vartheta \right\} \sin\vartheta\, d\vartheta = \frac{e^2+3}{4}. \tag{6.7}$$

Putting together previous estimates we get the contraction in d_2 with the same constant as W_2^2 as desired. □

Now, let us see that we can also control Fourier-based distances with exponent $2+\alpha$, with $\alpha > 0$.

Theorem 6.4 (Contraction of $\tilde{Q}_e^+(f,f)$ in $d_{2+\alpha}$)**.** [12] *Given $f, g \in \mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ with equal moments up to order $2+[\alpha]$, then there exists an explicit constant $A(\alpha,e) > 0$, $A(\alpha,e) \nearrow \frac{3+e^2}{4}$ as $\alpha \to 0$, such that*

$$d_{2+\alpha}(\tilde{Q}_e^+(f,f), \tilde{Q}_e^+(g,g)) \le A(\alpha,e)\, d_{2+\alpha}(f,g).$$

Proof.- As in the proof of the previous theorem, we compute

$$\left| \frac{\widehat{\tilde{Q}_e^+(f,f)}(k) - \widehat{\tilde{Q}_e^+(g,g)}(k)}{|k|^{2+\alpha}} \right| = \frac{1}{4\pi}\left| \int_{S^2} \frac{\hat{f}(k^+)\hat{f}(k^-) - \hat{g}(k^+)\hat{g}(k^-)}{|k|^{2+\alpha}}\, d\sigma \right|$$

$$\le A(\alpha,e) \sup_{k\in\mathbb{R}^3_s} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^{2+\alpha}}$$

where $A(\alpha,e)$ is given by

$$A(\alpha,e) := \frac{1}{4\pi}\int_{S^2} \frac{|k_+|^{2+\alpha} + |k_-|^{2+\alpha}}{|k|^{2+\alpha}}\, d\sigma \tag{6.8}$$

and $A(\alpha,e) \le A(0,e) = (e^2+3)/4 < 1$ for each restitution coefficient $e \neq 1$. In fact, it can be checked by inserting the expressions of k_- and k_+ into

(6.8) that

$$A(\alpha,e) = \frac{1}{2}\int_0^\pi \left\{ \left[\left(\frac{1+e}{4}\right)^2 2(1-\cos\vartheta) \right]^{\frac{2+\alpha}{2}} \right.$$
$$\left. + \left[\left(\frac{3-e}{4}\right)^2 + \left(\frac{1+e}{4}\right)^2 + 2\left(\frac{3-e}{4}\right)\left(\frac{1+e}{4}\right)\cos\vartheta \right]^{\frac{2+\alpha}{2}} \right\} \sin\vartheta\, d\vartheta$$
$$= \frac{2}{4+\alpha}\left[\left(\frac{1+e}{2}\right)^{2+\alpha} + \frac{1-\left(\frac{1-e}{2}\right)^{4+\alpha}}{1-\left(\frac{1-e}{2}\right)^2} \right],$$

giving the value of the contraction constant. □

Finally, let us analyse the contraction properties in Fourier distances of the linear operator $\tilde{\mathcal{L}}^+_{e_b}(f)$.

Theorem 6.5 (Contraction of $\tilde{\mathcal{L}}^+_{e_b}(f)$ in d_s)**.** *Given f and g in $\mathcal{P}_s(\mathbb{R}^3)$, with equal moments up to order $[s]$, then there exists an explicit constant $B(s,e_b,R_m)$ such that*

$$d_s(\tilde{\mathcal{L}}^+_{e_b}(f), \tilde{\mathcal{L}}^+_{e_b}(g)) \le B(s,e_b,R_m)\, d_s(f,g).$$

Proof.- As in the proof of previous theorems, we compute

$$\left| \frac{\widehat{\tilde{\mathcal{L}}^+_{e_b}(f)}(k) - \widehat{\tilde{\mathcal{L}}^+_{e_b}(g)}(k)}{|k|^s} \right| = \frac{1}{2\pi}\left| \int_{S^2} |\omega\cdot n| \frac{\hat{f}(k^L_+) - \hat{g}(k^L_+)}{|k|^s} \hat{M}_b(k^L_-)\, dn \right|$$
$$\le \left(\frac{1}{2\pi}\int_{S^2} |\omega\cdot n| \frac{|k^L_+|^s}{|k|^s}\, dn \right) \sup_{k\in\mathbb{R}^3_o} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^s}$$
$$= B(s,e_b,R_m)\, d_s(f,g) \tag{6.9}$$

with $\omega = k/|k|$ and

$$\begin{cases} k^L_+ = k - R_m(1+e_b)(k\cdot n)n \\ k^L_- = R_m(1+e_b)(k\cdot n)n, \end{cases}$$

from which

$$\frac{|k^L_+|^s}{|k|^s} = \left(1 - 4\chi(1-\chi)|\omega\cdot n|^2\right)^{s/2}$$

with the shortcut $\chi = \frac{R_m(1+e_b)}{2}$. Inserting this into (6.9) we find

$$\begin{aligned} B(s, e_b, R_m) &= \int_0^\pi |\cos\vartheta| \left(1 - 4\chi(1-\chi)\cos^2\vartheta\right)^{s/2} \sin\vartheta \, d\vartheta \\ &= \frac{2}{4\chi(1-\chi)(s+2)} \left[1 - (1 - 4\chi(1-\chi))^{(s+2)/2}\right] \end{aligned}$$

since $4\chi(1-\chi) \leq 1$. □

Let us remark that in the previous theorem we find for $s = 2$ that

$$B(2, e_b, R_m) = \frac{1 + [1 - R_m(1+e_b)]^2}{2},$$

and thus, the contraction constant for d_2 of $\tilde{\mathcal{L}}^+_{e_b}(f)$ coincides with the one for W_2^2 obtained in Theorem 6.2.

Remark 6.6 (Open Problem)**.** *In view of the preceding discussion on contractions for metrics of order $d_{2+\alpha}$, it naturally arises this question for Wasserstein metrics of higher order $W_{2+\alpha}$. Even in the case of even natural numbers, we do not know if the gain operators are contractive for $W_{2+\alpha}$. Actually, it was already mentioned in Remark* 2.10 *the difference between the convolution properties of d_4 and W_4. The computation in that remark is at the heart of why a similar proof to Theorem* 6.1 *cannot be done for W_4, and we leave the reader to check it.*

7. Consequences on solutions of IMMs

In this section, we will use the contraction estimates on the gain operators to obtain information about the asymptotic behavior for the different Inelastic Maxwell Models introduced in Section 4: equations (3.15), (4.3) and (4.12). In fact, the global strategy that we will use can be summarized as follows:

Step 1.- Contractions in the distances W_2 or d_2 for solutions of these equations together with uniform in time propagation of certain moments imply the existence of steady states in these models.

Step 2.- As a first consequence, they also imply a rate of decay towards the steady state in the probability metrics W_2 or d_2, and thus in a weak convergence setting taking into account the equivalences discussed in Section 2.

Step 3.- Uniform in time propagation of regularity and decay in d_2 imply decay in L^2 and homogeneous Sobolev norms by the interpolation inequality in Proposition 2.23.

Step 4.- Uniform in time propagation of moments and decay in L^2 imply decay in L^1 by using Lemma 2.24.

This whole strategy was applied in [12] for the IMM with stochastic forcing (4.3) using the d_2 distance as starting point. The first two steps of the strategy were developed in [13] in the case of homogeneous cooling states (4.1) for the IMM without external source of energy (3.15). Finally, this whole strategy is shown here to be applicable in the case of the particle's bath heating equation (4.12). Let us finally mention that this strategy had its roots in [62] and subsequent works, and it was already mentioned in the nice survey on the mathematics of granular materials by C. Villani [107]. A nice review of related techniques can be found in another paper of the Porto Ercole summer school lectures [36]. Finally, let us mention that the contractions in optimal mass transport and Fourier-based metrics can also be obtained for the inelastic Kac model introduced in [91], and similar conclusions on the asymptotic stability can be drawn, see [91, 29] for full details.

7.1. **Stochastic heating IMM.**

7.1.1. *Existence, Uniqueness and Stability of Steady States.* Let us consider the diffusive version (4.3) of (3.15). The first observation is that the temperature of solutions of (4.3) is given by the ODE:

$$\theta'(t) = -\frac{1-e^2}{4} B\, \theta(t)^{3/2} + 2\theta_b, \tag{7.1}$$

and thus, it depends on the initial data only through the initial temperature θ_o. Thus any two solutions with equal initial temperature will have the same temperature for all times. Now, we change the time variable to

$$\tau = \frac{B}{E} \int_0^t \sqrt{\theta(w)}\, dw$$

with $E = \dfrac{8}{1-e^2}$ and we are reduced to analyze

$$\frac{\partial f}{\partial \tau} = E\, [\tilde{Q}_e^+(f,f) - f] + \Theta^2(\tau)\, \Delta_v f \tag{7.2}$$

with

$$\Theta^2(\tau) = \frac{E\, \theta_b}{B} \left[\theta(\tau)\right]^{-1/2}.$$

Let us show the contraction for solutions in W_2.

Theorem 7.1 (Contraction in W_2)**.** [29] *If f_1 and f_2 are two solutions to* (7.2) *for the respective initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity and temperature, then*

$$W_2^2(f_1(\tau), f_2(\tau)) \leq \mathrm{e}^{-2\tau}\, W_2^2(f_1^0, f_2^0) \tag{7.3}$$

for all $\tau \geq 0$.

Proof.- The main steps of the proof are:

- **Step 1.-** We write a Duhamel's representation of the solutions with a fixed temperature evolution. It is not difficult to deduce by a standard Fourier transform procedure that the solutions satisfy

$$f(\tau,v) = \mathrm{e}^{-E\,\tau}\,(f^0 * \Gamma_{2\Sigma(\tau)})(v) + E\int_0^\tau \mathrm{e}^{-E(\tau-s)}\,(F(s) * \Gamma_{2(\Sigma(\tau)-\Sigma(s))})(v)\,ds$$

$$:= \mathrm{e}^{-E\,\tau}\,\tilde f(\tau,v) + E\int_0^\tau \mathrm{e}^{-E(\tau-s)}\,\tilde F(\tau,s,v)\,ds,$$

where $F = \tilde Q_e^+(f,f)$,

$$\Sigma(\tau) = \int_0^\tau \Theta^2(s)\,ds \qquad \text{and} \qquad \Gamma_\alpha(v) = \frac{1}{(2\pi\alpha)^{3/2}}\,\mathrm{e}^{-|v|^2/2\alpha}$$

is the centered Maxwellian with temperature $\alpha > 0$.

- **Step 2.-** We use the convexity of the squared Wasserstein distance and its non-increasing character by convolution with a given measure, see Proposition 2.1, to imply that

$$\begin{aligned} &W_2^2(f_1(\tau), f_2(\tau)) \\ &\quad \le \mathrm{e}^{-E\tau} W_2^2(\tilde f_1(\tau), \tilde f_2(\tau)) + E\int_0^\tau \mathrm{e}^{-E(\tau-s)} W_2^2(\tilde F_1(\tau,s), \tilde F_2(\tau,s))ds \\ &\quad \le \mathrm{e}^{-E\tau}\,W_2^2(f_1^0, f_2^0) + E\int_0^\tau \mathrm{e}^{-E(\tau-s)}\,W_2^2(F_1(s), F_2(s))\,ds. \end{aligned}$$

- **Step 3.-** We now use the contraction of the gain operator in W_2 obtained in Theorem 6.3 to deduce

$$\begin{aligned} &W_2^2(f_1(\tau), f_2(\tau)) \\ &\quad \le \quad \mathrm{e}^{-E\tau}\,W_2^2(f_1^0, f_2^0) + E\int_0^\tau \mathrm{e}^{-E(\tau-s)}\,\frac{3+e^2}{4}\,W_2^2(f_1(s), f_2(s))\,ds. \end{aligned}$$

- **Step 4.-** Finally, the function $y(\tau) = \mathrm{e}^{E\tau}\,W_2^2(f_1(\tau), f_2(\tau))$ satisfies the inequality

$$y(\tau) \le y(0) + \frac{3+e^2}{4}\,E\int_0^\tau y(s)\,ds$$

and then $y(\tau) \le y(0)\,\mathrm{e}^{\gamma E\tau}$ by Gronwall's lemma with $\gamma = (3+e^2)/4$.

This concludes the argument since $(1-\gamma)\,E = 2$. □

The previous result implies a contraction in original variables.

Corollary 7.2 (Contraction in W_2 in original variables)**.** *If f_1 and f_2 are two solutions to* (4.3) *for the respective initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity and temperature, then*

$$W_2^2(f_1(t), f_2(t)) \leq \mathrm{e}^{-\frac{1-e^2}{4} C_1 t} W_2^2(f_1^0, f_2^0)$$

for all $t \geq 0$ with C_1 depending on the initial temperature $\theta(0)$ and θ_∞.

Proof.- It is straightforward based on (7.1) that the temperature $\theta(t)$ of any solution f in the original time variable t converges towards

$$\theta_\infty = \Big(\frac{8}{B(1-e^2)}\Big)^{2/3}$$

as $t \to \infty$, and satisfies $\theta(t) \geq \min(\theta(0), \theta_\infty)$. In particular

$$\tau = \frac{B}{E} \int_0^t \sqrt{\theta(s)}\, ds \geq \frac{C_1}{E}\, t$$

if $C_1 = B\, \min(\theta(0), \theta_\infty)^{1/2}$. Writing (7.3) in the original variable t for initial data with equal mean velocity and temperature, we recover the contraction property

$$W_2(f_1(t), f_2(t)) \leq W_2(f_1^0, f_2^0)\, \mathrm{e}^{-\frac{1-\mathrm{e}^2}{4} \mathrm{C}_1 \mathrm{t}}$$

for the solutions of (4.3). □

Let us know use this contraction to deduce the existence of stationary states by a dynamical proof. In fact, we can state the following abstract lemma.

Lemma 7.3 (Dynamic proof of existence of Steady States)**.** *Given a complete metric space $(\mathcal{M}, d)$ and a continuous semigroup $\{\mathcal{T}(t)\}_{t\geq 0} : (\mathcal{M}, d) \longrightarrow (\mathcal{M}, d)$, for which there exists $0 < L(t) < 1$, for all $t > 0$, such that*

$$d(\mathcal{T}(t)(x), \mathcal{T}(t)(y)) \leq L(t)\, d(x, y)$$

for all $t > 0$ and $x, y \in \mathcal{M}$. Then, there exists a unique stationary point $x_\infty \in \mathcal{M}$, i.e., $\mathcal{T}(t)(x_\infty) = x_\infty$ for all $t \geq 0$.

Proof.- A direct application of Banach fixed point theorem ensures the existence of a unique fixed point $x_\infty(t) \in \mathcal{M}$, eventually dependent on t, to each $\mathcal{T}(t)$ for all $t > 0$.

Let us define $x_\infty = x_\infty(1)$. The semigroup property implies easily by uniqueness of the fixed point for each $t > 0$ that $x_\infty(t) = x_\infty$ for all rational numbers $t > 0$. Finally, the continuity in time of the semigroup shows that x_∞ should be a fixed point for all $t > 0$. Again by uniqueness of the fixed point for each $t > 0$, we achieve the result. □

Let us remark that a variant of this lemma allowing the use of Schauder-Tychonoff's fixed point theorem has also been used in the proof of the existence of steady states for hard-spheres inelastic Boltzmann and coagulation equations by a number of authors [2, 4, 59, 63, 85] and references therein.

Corollary 7.4 (Existence, Uniqueness & Stability of Stationary States)**.** *Equation* (4.3) *has a unique steady state* f_∞ *in* $\mathcal{P}_2(\mathbb{R}^3)$ *with zero mean velocity and temperature*

$$\theta_\infty = \Big(\frac{8}{B(1-e^2)}\Big)^{2/3}.$$

Moreover, given f *any solution to* (4.3) *for the initial data* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ *with equal mean velocity and temperature to* f_∞*, then*

$$W_2^2(f(t), f_\infty) \leq \mathrm{e}^{-\frac{1-e^2}{4}\, C_1\, t}\, W_2^2(f_0, f_\infty)$$

for all $t \geq 0$ *with* C_1 *depending on the initial temperature* $\theta(0)$ *and* θ_∞*.*

Proof.- Remark 2.3 shows that the set

$$\mathcal{M}_{\theta_\infty} = \left\{\mu \in \mathcal{P}_2(\mathbb{R}^3) \text{ such that } \int_{\mathbb{R}^3} |v|^2 df(v) = 3\theta_\infty\right\},$$

endowed with the distance W_2 is a complete metric space. It is easy to see that the subset

$$\tilde{\mathcal{M}}_{\theta_\infty} = \left\{\mu \in \mathcal{P}_2(\mathbb{R}^3) \text{ such that } \int_{\mathbb{R}^3} |v|^2 df(v) = 3\theta_\infty \text{ and } \int_{\mathbb{R}^3} v\, df(v) = 0\right\},$$

is a closed subset in W_2 using Proposition 2.1. Then, let us take as complete metric space $(\mathcal{M}, d) = (\tilde{\mathcal{M}}_{\theta_\infty}, W_2)$. Let us consider the flow map of (4.3), i.e.,

$$\mathcal{T}(t) : (\tilde{\mathcal{M}}_{\theta_\infty}, W_2) \longrightarrow (\mathcal{P}_2(\mathbb{R}^3), W_2),$$

for any time $t > 0$, given by $\mathcal{T}(t)(f_0) = f(t)$ with $f(t)$ the unique solution at time t of (4.3) with initial datum $f_0 \in \tilde{\mathcal{M}}_{\theta_\infty}$. Then, $\mathcal{T}(t)$ is a continuous semigroup from $\tilde{\mathcal{M}}_{\theta_\infty}$ onto itself due to the adaptation of Theorem 3.5 to (4.3), the conservation of mean velocity and the conservation of the steady value of the temperature θ_∞.

Theorem 7.2 proves that $\mathcal{T}(t)$ is a uniform contraction from the complete metric space $(\tilde{\mathcal{M}}_{\theta_\infty}, W_2)$ into itself with contraction constant

$$L(t) = \mathrm{e}^{-\frac{1-\mathrm{e}^2}{4}\mathrm{C}_1\mathrm{t}} < 1.$$

Therefore, Lemma 7.3 ensures the existence and uniqueness of a unique steady state in $(\tilde{\mathcal{M}}_{\theta_\infty}, W_2)$. The last assertion is a simple consequence of

Corollary 7.1 by taking one of the solutions the stationary state we just obtained. □

As a simple corollary using the average control in Corollary 2.4, we deduce:

Corollary 7.5 (Exponential Convergence of Averages)**.** *Given f any solution to* (4.3) *for the initial data $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity and temperature to f_∞, then*

$$\left| \int_{\mathbb{R}^N} \varphi(v)(f(t,v) - f_\infty(v))\, dv \right| \leq L\, \mathrm{e}^{-\frac{1-e^2}{4} C_1\, t}\, W_2^2(f_0, f_\infty)$$

for all $t \geq 0$ and any $\varphi \in Lip(\mathbb{R}^3)$ with Lipschitz constant L.

We would like to mention that the existence and uniqueness of stationary states for the IMM with stochastic heating (4.3) was first addressed in [51] by direct fixed point arguments. Later, it was obtained in [21] with spectral techniques using a detailed study of the linearized operator in Fourier variables.

7.1.2. *Moment bounds.* Further properties of the solutions and in particular of the stationary state can be obtained. Let us start by controlling the tails of the distribution.

The exact evolution equations for moments of order higher than two has been done in [21], see also [20], only in the simpler case of isotropic solutions. For non isotropic moments formulas were written in [21] up to third order. The computation of the evolution of the moment of order 4 can be found in [29, Appendix]. All non-isotropic moments are in principle explicitly computable but they give rise to cumbersome recursive formulas. The objective of this part is to show uniform in time control up to infinity of all moments of the solutions.

Lemma 7.6 (Evolution of second moments)**.** [39, 21] *Let f be a solution to* (4.3) *with unit mass, zero mean velocity and initial second order moments bounded, then f has finite second order moments for any $t > 0$, and*

$$\frac{d}{dt} \int_{\mathbb{R}^3} f(v) v_i v_j\, dv = - \frac{(1+e)(3-e)}{8} B \sqrt{\theta(t)} \int_{\mathbb{R}^3} f(v) v_i v_j\, dv$$
$$+ \delta_{ij}\, 2 \left(\frac{1+e}{4} \right)^2 B\, \theta(t)^{\frac{3}{2}} + \delta_{ij}\, 2\, \theta_b \tag{7.4}$$

for any $t \geq 0$.

Proof.- By multiplying (4.3) by $v_i v_j$ and integrating we get

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f(v)v_i v_j dv = \frac{B}{4\pi}\sqrt{\theta_f(t)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)f(w)\Big[v_i' v_j' - v_i v_j\Big]\, d\sigma\, dv\, dw$$
$$+\,\theta_b\int_{\mathbb{R}^3} v_i v_j \Delta_v f\, dv.$$

Using the post-collisional velocities (3.7), we have

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f(v)v_i v_j\, dv =$$
$$= \frac{B}{4\pi}\sqrt{\theta_f(t)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)f(w)\left\{\left[\left(\frac{3-e}{4}\right)^2 - 1\right] v_i v_j\right.$$
$$+\left(\frac{1+e}{4}\right)^2\Big(w_i w_j + |v-w|^2\sigma_i\sigma_j + |v-w|(w_i\sigma_j + w_j\sigma_i)\Big)$$
$$\left.+\left(\frac{3+2e-e^2}{16}\right)\Big(v_i w_j + w_i v_j + |v-w|(v_i\sigma_j + v_j\sigma_i)\Big)\right\} d\sigma dv dw$$
$$+\,\delta_{ij}\, 2\,\theta_b.$$

Evolution equation (7.4) is then obtained imposing zero mean velocity and unit mass. □

We show an explicit inequality for the time evolution of moments of any order for general solutions, which leads to a uniform bound in time of these moments, in terms of the moments of the initial value. To simplify notations, in what follows we denote

$$m_{2r}(t) = \int_{\mathbb{R}^3} f(v,t)|v|^{2r}\, dv,$$

for any $r \in \mathbb{N}$.

Lemma 7.7 (Uniform in time moment estimates)**.** [12] *Let $f(t,v)$ be the solution to equation* (4.3)*, where the initial distribution $f_0(v)$ with zero mean velocity satisfies $m_{2r}(0) < \infty$ for some $r \geq 2$. Then, $m_{2r}(t)$ satisfies the*

following differential inequality

$$\frac{d}{dt}m_{2r}(t) \leq -B\sqrt{\frac{m_2(t)}{3}}\left[\frac{1-e^{2r}}{4}\left(m_{2r}(t)+m_{2(r-1)}(t)m_2(t)\right)\right.$$
$$\left.-\frac{1}{2}\sum_{l=1}^{r-1}\left(\begin{array}{c} r \\ l \end{array}\right)m_{2(r-l)}(t)m_{2l}(t)\right] \tag{7.5}$$
$$+\,\theta_b\,(2r+4r^2)m_{2(r-1)}(t).$$

Consequently, $m_{2r}(t)$ is uniformly bounded in time provided $m_{2r}(0) < \infty$. As a consequence, all moments of stationary solutions are bounded.

Proof.- Elementary computations using the collision mechanism (3.7) show that

$$0 \geq |v'|^2 + |w'|^2 - |v|^2 - |w|^2 = -\frac{1-e^2}{4}\left[|v-w|^2 - |v-w|(v-w)\cdot\sigma\right], \tag{7.6}$$

and

$$|v'|^2 + |w'|^2 \geq e^2\left(|v|^2+|w|^2\right). \tag{7.7}$$

Inequality (7.7) follows from (7.6). In fact, since

$$0 \leq \left[|v-w|^2 - |v-w|(v-w)\cdot\sigma\right] \leq 2|v-w|^2,$$

we obtain

$$|v'|^2 + |w'|^2 \geq |v|^2 + |w|^2 - \frac{1-e^2}{2}|v-w|^2 =$$

$$= \frac{1}{2}|v+w|^2 + \frac{e^2}{2}|v-w|^2 \geq e^2\left(|v|^2+|w|^2\right).$$

Choosing $\varphi(v) = |v|^{2r}$, $r \geq 2$, in the weak formulation of the operator $\tilde{Q}_e(f,f)$ in (3.14), we obtain

$$\begin{aligned}
&\langle |v|^{2r}, \tilde{Q}_e(f,f)\rangle = \\
&\quad = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)f(w)\Big[|v'|^{2r} + |w'|^{2r} - |v|^{2r} + |w|^{2r}\Big] d\sigma\, dv\, dw \\
&\quad = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)f(w)\Big[(|v'|^2 + |w'|^2)^r - (|v|^2 + |w|^2)^r\Big] d\sigma\, dv\, dw \\
&\qquad + \sum_{l=1}^{r-1}\begin{pmatrix} r \\ l \end{pmatrix} \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)f(w)\Big[|v|^{2(r-l)}|w|^{2l} \\
&\qquad\qquad - |v'|^{2(r-l)}|w'|^{2l}\Big] d\sigma\, dv\, dw
\end{aligned}$$

from which

$$\begin{aligned}
&\langle |v|^{2r}, \tilde{Q}_e(f,f)\rangle = \qquad (7.8)\\
&\quad =\leq \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v)f(w)\Big[(|v'|^2 + |w'|^2)^r - (|v|^2 + |w|^2)^r\Big] d\sigma\, dv\, dw \\
&\qquad + \frac{1}{2}\sum_{l=1}^{r-1}\begin{pmatrix} r \\ l \end{pmatrix} m_{2(r-l)} m_{2l}. \qquad (7.9)
\end{aligned}$$

Taking into account (7.7), we obtain

$$\begin{aligned}
I :&= \frac{1}{8\pi}\int_{S^2}\Big[(|v'|^2 + |w'|^2)^r - (|v|^2 + |w|^2)^r\Big] d\sigma \\
&= \frac{1}{8\pi}\int_{S^2}\Big[|v'|^2 + |w'|^2 - (|v|^2 + |w|^2)\Big]\sum_{l=0}^{r-1}\Big[|v'|^2 + |w'|^2\Big]^{r-l-1}\Big[|v|^2 + |w|^2\Big]^l d\sigma
\end{aligned}$$

from which

$$\begin{aligned}
I &\leq \frac{1}{8\pi}\int_{S^2}\Big[|v'|^2 + |w'|^2 - (|v|^2 + |w|^2)\Big]\sum_{l=0}^{r-1} e^{2(r-l-1)}\Big[|v|^2 + |w|^2\Big]^{r-1} d\sigma \\
&= -\frac{1-e^{2r}}{4}\Big[|v|^2 + |w|^2\Big]^{r-1}\frac{1}{8\pi}\int_{S^2}\Big[|v-w|^2 - |v-w|(v-w)\cdot\sigma\Big]\, d\sigma \\
&= -\frac{1-e^{2r}}{8}\Big[|v|^2 + |w|^2\Big]^{r-1}|v-w|^2. \qquad (7.10)
\end{aligned}$$

Thus, inserting (7.10) into (7.9) we obtain the inequality

$$\langle |v|^{2r}, \tilde{Q}_e(f,f)\rangle \leq -\frac{1-e^{2r}}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)f(w)\Big[|v|^2+|w|^2\Big]^{r-1}|v-w|^2 dv\,dw$$

$$+\frac{1}{2}\sum_{l=1}^{r-1}\begin{pmatrix} r \\ l \end{pmatrix} m_{2(r-l)}m_{2l}.$$

Finally, since

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)f(w)\Big[|v|^2+|w|^2\Big]^{r-1}|v-w|^2 dv\,dw$$

$$\geq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)f(w)\Big[|v|^{2(r-1)}+|w|^{2(r-1)}\Big]|v-w|^2 dvdw = 2m_{2r}+2m_{2(r-1)}m_2$$

where zero mean velocity of the solutions has been used, we get

$$\langle |v|^{2r}, \tilde{Q}_e(f,f)\rangle \leq -\frac{1-e^{2r}}{4}\left(m_{2r}+m_{2(r-1)}m_2\right)+\frac{1}{2}\sum_{l=1}^{r-1}\begin{pmatrix} r \\ l \end{pmatrix} m_{2(r-l)}m_{2l}. \tag{7.11}$$

To conclude the proof of the differential inequality for moments, consider that

$$\int_{\mathbb{R}^3} |v|^{2r}\Delta_v f\,dv = (2r+4r^2)m_{2(r-1)}. \tag{7.12}$$

Then, in order to show that $m_{2r}(t)$ is uniformly bounded in time, one rewrites inequality (7.5) as

$$\frac{d}{dt}m_{2r}(t) \leq -A_2(t)m_{2r}(t)+B_r(t),$$

where

$$A_2(t) = B\frac{1-e^{2r}}{4}\sqrt{\frac{m_2(t)}{3}}$$

and $B_r(t)$ depend on moments up to the order $2(r-1)$.

The proof follows by induction. The second moment $m_2(t)$ is uniformly bounded away from 0 and from ∞ in time since the temperature is so due to (7.1). This implies that $0 < A_* \leq A_2(t) < A^*$ for all $t \geq 0$. Moreover, $B_1(t)$ is uniformly bounded in time, $B_1(t) \leq B_1$, and, consequently, $M_4(t)$ is uniformly bounded,

$$m_4(t) \leq \max\left\{m_4(0), \frac{B_1}{A_*}\right\}.$$

Recursively, the proof can be extended to any $r > 2$.

The last assertion regarding the boundedness of all moments for stationary solutions to (4.3) follows from previous arguments by neglecting the time derivatives. □

Now, let us show a further contraction property of the solutions to (4.3).

Proposition 7.8 (Contraction in $d_{2+\alpha}$)**.** [12] *Given $\alpha \geq 0$, f_1 and f_2 solutions to* (4.3) *for the initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$ with equal moments up to order $2+[\alpha]$ if $\alpha \notin \mathbb{N}$ or $\alpha = 0$, or equal moments up to order $1+\alpha$ if $\alpha \in \mathbb{N}$ with $\alpha \geq 1$, then*

$$d_{2+\alpha}(f_1(t), f_2(t)) \leq d_{2+\alpha}(f_1^0, f_2^0)\, \mathrm{e}^{-(1-A(\alpha,e))C_1 t},$$

for all $t \geq 0$.

Proof.- The main steps of the proof are:

- **Step 1.-** Let us remind that the temperature of both solutions $f_1(t)$ and $f_2(t)$ is equal to $\theta(t)$ solution to (7.1), since they are equal initially. Using the Fourier transform expression of $\tilde{Q}_e(f,f)$ in (3.18), equation (4.3) can be written in Fourier as

$$\frac{\partial \hat{f}}{\partial t} = \frac{B\sqrt{\theta(t)}}{4\pi} \int_{S^2} \left\{ \hat{f}(t,k_-)\hat{f}(t,k_+) - \hat{f}(t,0)\hat{f}(t,k) \right\} d\sigma - \theta_b\, |k|^2 \hat{f}, \quad (7.13)$$

  or equivalently

$$\frac{\partial \hat{f}}{\partial t} = B\sqrt{\theta(t)} \left[\widehat{\tilde{Q}_e^+(f,f)} - \hat{f} \right] - \theta_b\, |k|^2 \hat{f}.$$

- **Step 2.-** Taking the expressions of the two solutions $\hat{f}_1(t)$ and $\hat{f}_2(t)$, subtracting them and dividing by $|k|^{2+\alpha}$ with $k \in \mathbb{R}^3_o$, we get

$$\frac{\partial}{\partial t} \frac{\hat{f}_1(k) - \hat{f}_2(k)}{|k|^{2+\alpha}} = B\sqrt{\theta(t)}\, \frac{\widehat{\tilde{Q}_e^+(f_1,f_1)}(k) - \widehat{\tilde{Q}_e^+(f_2,f_2)}(k)}{|k|^{2+\alpha}}$$

$$- B\sqrt{\theta(t)} \frac{\hat{f}_1(k) - \hat{f}_2(k)}{|k|^{2+\alpha}} - \theta_b\, |k|^2 \frac{\hat{f}_1(k) - \hat{f}_2(k)}{|k|^{2+\alpha}}.$$

- **Step 3.-** Let us set

$$h(t,k) = \frac{\hat{f}_1(k) - \hat{f}_2(k)}{|k|^{2+\alpha}},$$

  the last identity implies

$$\left| \partial_t h(t,k) + \left(B\sqrt{\theta(t)} + \theta_b\, |k|^2 \right) h(t,k) \right| \leq B\sqrt{\theta(t)}\, A(\alpha,e)\, d_{2+\alpha}(f_1,f_2)$$

where Theorem 6.4 has been used. This is equivalent to

$$\left|\partial_t \left[h(t,k)\,\exp\left(B\int_0^t \sqrt{\theta(s)}\,ds + |k|^2\theta_b\,t\right)\right]\right|$$
$$\leq B\sqrt{\theta(t)}\,A(\alpha,e)\,d_{2+\alpha}(f_1,f_2)\exp\left(B\int_0^t \sqrt{\theta(s)}\,ds + |k|^2\theta_b\,t\right).$$

Integrating from 0 to t, we deduce

$$\left|h(t,k)\exp\left(B\int_0^t \sqrt{\theta(s)}\,ds + |k|^2\theta_b\,t\right)\right| \leq |h(0,k)| +$$

$$B\,A(\alpha,e)\int_0^t \sqrt{\theta(\tau)}\,d_{2+\alpha}(f_1(\tau),f_2(\tau))\exp\left(B\int_0^\tau \sqrt{\theta(s)}\,ds + |k|^2\theta_b\,\tau\right)d\tau,$$

from which

$$\exp\left(B\int_0^t \sqrt{\theta(s)}\,ds\right)|h(t,k)| \leq |h(0,k)| +$$

$$B\,A(\alpha,e)\int_0^t \sqrt{\theta(\tau)}\,d_{2+\alpha}(f_1(\tau),f_2(\tau))\exp\left(B\int_0^\tau \sqrt{\theta(s)}\,ds\right)d\tau.$$

Since the above inequality holds for all values of the variable $k \in \mathbb{R}^3_o$, we finally conclude

$$\exp\left(B\int_0^t \sqrt{\theta(s)}\,ds\right)d_{2+\alpha}(f_1(t),f_2(t)) \leq d_{2+\alpha}(f_1^0,f_2^0) +$$

$$B\,A(\alpha,e)\int_0^t \sqrt{\theta(\tau)}\,d_{2+\alpha}(f_1(\tau),f_2(\tau))\exp\left(B\int_0^\tau \sqrt{\theta(s)}\,ds\right)d\tau.$$

- **Step 4.-** The final step is to use a Gronwall's like lemma. Denoting by

$$w(t) := \exp\left(B\int_0^t \sqrt{\theta(s)}\,ds\right)d_{2+\alpha}(f_1(t),f_2(t)),$$

we write the last inequality as

$$w(t) \leq w(0) + B\,A(\alpha,e)\int_0^t \sqrt{\theta(\tau)}\,w(\tau)\,d\tau.$$

By the generalized Gronwall inequality, this implies

$$w(t) \leq w(0)\exp\left(B\,A(\alpha,e)\int_0^t \sqrt{\theta(\tau)}\,d\tau\right),$$

namely

$$d_{2+\alpha}(f_1(t), f_2(t)) \leq d_{2+\alpha}(f_1^0, f_2^0) \exp\left(-(1 - A(\alpha, e))B \int_0^t \sqrt{\theta(\tau)}\, d\tau\right).$$

As before $\theta(t) \geq \min(\theta(0), \theta_\infty)$ from which the desired result follows. □

Remark 7.9 (Existence of Stationary States)**.** *Corollary* 7.4 *can also be obtained from previous Proposition* 7.8 *for* $\alpha = 0$ *and moment bounds of order* $2 + \alpha$ *from Lemma* 7.7*. Let us consider the metric space* $\mathcal{X}_{\alpha,M}$ *given by the set of probability measures* $f \in \mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ *such that*

$$\int_{\mathbb{R}^3} v\, df(v) = 0 \qquad , \qquad \int_{\mathbb{R}^3} v_i\, v_j\, df(v) = \theta_\infty \delta_{ij}$$

and

$$\int_{\mathbb{R}^N} |v|^{2+\alpha}\, df(v) \leq M.$$

Proposition 2.7 *shows that the set* $\mathcal{X}_{\alpha,M}$ *endowed with the distance* d_2 *is a complete metric space. Lemma* 7.6 *and Proposition* 7.8 *for* $\alpha = 0$ *with well chosen* M *shows that* $\mathcal{X}_{\alpha,M}$ *is invariant along the semigroup of the flow map associated to* (4.3)*. Using Lemma* 7.3 *analogously to Corollary* 7.4 *we conclude.*

A simple consequence is the asymptotic stability in $d_{2+\alpha}$ distances.

Corollary 7.10 (Decay rates in $d_{2+\alpha}$)**.** *Given* $\alpha \geq 0$ *and* f *any solution to* (4.3) *for the initial data* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ *with equal moments up to order* $2 + [\alpha]$ *if* $\alpha \notin \mathbb{N}$ *or* $\alpha = 0$*, or equal moments up to order* $1 + \alpha$ *if* $\alpha \in \mathbb{N}$ *with* $\alpha \geq 1$*, then*

$$d_{2+\alpha}(f(t), f_\infty) \leq d_{2+\alpha}(f_0, f_\infty)\, \mathrm{e}^{-(1-A(\alpha,e))C_1 t},$$

for all $t \geq 0$*.*

Let us remark that the previous result shows that as long as the initial data has more common moments with f_∞ the decay rate improves since $A(\alpha, e) \searrow 0$ as $\alpha \nearrow \infty$. This phenomena happens in other situation like the diffusion equations both linear and nonlinear, cf. [68, 41, 35].

Corollary 7.11 (Evolution of any moment)**.** *Given* $\alpha > 0$*,* f_1 *and* f_2 *solutions to* (4.3) *for the initial data* f_1^0 *and* f_2^0 *in* $\mathcal{P}_2(\mathbb{R}^3)$ *with equal moments up to order* $2 + [\alpha]$ *if* $\alpha \notin \mathbb{N}$ *or equal moments up to order* $1 + \alpha$ *if* $\alpha \in \mathbb{N}$*, then*

$$\int_{\mathbb{R}^3} v^\beta\, df_1(t, v) = \int_{\mathbb{R}^3} v^\beta\, df_2(t, v)$$

for all $t \geq 0$*, and any multi-index* β *with* $|\beta| \leq 2 + [\alpha]$ *if* $\alpha \notin \mathbb{N}$ *or* $|\beta| \leq 1 + \alpha$ *if* $\alpha \in \mathbb{N}$*.*

The previous result is a simple consequence of the contraction estimate in Proposition 7.8 since the distance $d_{2+\alpha}(f_1, f_2)$ is not finite unless moments of f_1 and f_2 are equal up to order $2+[\alpha]$ if $\alpha \notin \mathbb{N}$ or $1+\alpha$ if $\alpha \in \mathbb{N}$. Now, we can improve the decay distance in Proposition 7.8 for d_2 allowing different initial temperature.

Corollary 7.12 (Improved decay of the d_2 distance)**.** [12] *Any solution $f(t,v)$ of* (4.3) *corresponding to an initial density with unit mass, zero mean velocity and finite initial temperature, converges exponentially towards the steady state $f_\infty(v)$ in d_2 distance. More precisely, there exist constants $C_1, C_2, C_3 > 0$ such that*

$$d_2(f(t), f_\infty) \leq d_2(f_0, f_\infty)\, \mathrm{e}^{-\frac{1-\mathrm{e}^2}{4}\, \mathrm{C}_1\, \mathrm{t}} + \mathrm{C}_2 \mathrm{e}^{-\mathrm{C}_3\, \mathrm{t}}, \tag{7.14}$$

for all $t \geq 0$.

Proof.- Since now the evolution of the temperature for both solutions is different we define $z(t)$ by the relation $\theta(t) = \theta_\infty\, z(t)$. The proof follows the same steps as in Proposition 7.8 and we just sketch it here leaving the details to the reader. Given $k \in \mathbb{R}^3_o$, we write

$$\frac{\partial}{\partial t} \frac{\hat{f}(k) - \hat{f}_\infty(k)}{|k|^2} = \frac{B}{4\pi} \theta_\infty^{\frac{1}{2}}\, z^{\frac{1}{2}}(t) \int_{S^2} \left[\frac{\hat{f}(k_-)\hat{f}(k_+) - \hat{f}_\infty(k_-)\hat{f}_\infty(k_+)}{|k|^2} \right.$$

$$\left. - \frac{\hat{f}(0)\hat{f}(k) - \hat{f}_\infty(0)\hat{f}_\infty(k)}{|k|^2} \right] d\sigma$$

$$- \theta_b\, |k|^2 \frac{\hat{f}(k) - \hat{f}_\infty(k)}{|k|^2} + \theta_b \left(z^{\frac{1}{2}}(t) - 1 \right) \hat{f}_\infty(k).$$

Using the same arguments as in Proposition 7.8, we deduce

$$\left| \partial_t h(t,k) + \left(B\, \theta_\infty^{\frac{1}{2}}\, z^{\frac{1}{2}}(t) + \theta_b\, |k|^2 \right) h(t,k) \right| \leq$$

$$\leq \frac{3+e^2}{4}\, B\, \theta_\infty^{\frac{1}{2}}\, z^{\frac{1}{2}}(t)\, \|h(t,\cdot)\|_\infty + \varphi(t)$$

where

$$h(t,k) = \frac{\hat{f}(k) - \hat{f}_\infty(k)}{|k|^2} \qquad \text{and} \qquad \varphi(t) = \theta_b \left| z^{\frac{1}{2}}(t) - 1 \right|.$$

Proceeding again as in Proposition 7.8, we obtain

$$\exp\left(B\,\theta_\infty^{\frac{1}{2}}\int_0^t z^{\frac{1}{2}}(s)\,ds\right)\|h(t,\cdot)\|_\infty \le \|h(0,\cdot)\|_\infty + \Phi(t)$$
$$+\,\frac{3+e^2}{4}\,B\,\theta_\infty^{\frac{1}{2}}\int_0^t z^{\frac{1}{2}}(\tau)\,\|h(\tau,\cdot)\|_\infty \exp\left(B\,\theta_\infty^{\frac{1}{2}}\int_0^\tau z^{\frac{1}{2}}(s)\,ds\right)d\tau,$$

where

$$\Phi(t) = \int_0^t \varphi(\tau)\exp\left(B\,\theta_\infty^{\frac{1}{2}}\int_0^\tau z^{\frac{1}{2}}(s)\,ds\right)d\tau.$$

By the generalized Gronwall lemma, denoting by $w(t)$ the same quantity as in Proposition 7.8, we finally conclude that

$$w(t) \le w(0)\exp\left(\frac{3+e^2}{4}\,B\,\theta_\infty^{\frac{1}{2}}\int_0^t z^{\frac{1}{2}}(\tau)\,d\tau\right)$$
$$+\int_0^t \exp\left(\frac{3+e^2}{4}\,B\,\theta_\infty^{\frac{1}{2}}\int_\tau^t z^{\frac{1}{2}}(s)\,ds\right)\varphi(\tau)\,\exp\left(B\,\theta_\infty^{\frac{1}{2}}\int_0^\tau z^{\frac{1}{2}}(s)\,ds\right)d\tau\,.$$

Hence

$$\|h(t,\cdot)\|_\infty \le \|h(0,\cdot)\|_\infty \exp\left(-\frac{1-e^2}{4}\,B\,\theta_\infty^{\frac{1}{2}}\int_0^t z^{\frac{1}{2}}(s)\,ds\right) + \Psi(t)$$

where

$$\Psi(t) = \int_0^t \varphi(\tau)\exp\left(-\frac{1-e^2}{4}\,B\,\theta_\infty^{\frac{1}{2}}\int_\tau^t z^{\frac{1}{2}}(s)\,ds\right)d\tau.$$

To finish the proof, it suffices to show that $\Psi(t)$ decays exponentially fast as $t\to\infty$. This is just an exercise in calculus and estimates on the solutions of the differential evolution for the temperature (7.1), see [12] for full details. □

7.1.3. *Propagation of Regularity.* We will need to estimate the evolution of moments in Fourier variables. Let us remind we follow the notations introduced in Section 2.6 for homogeneous Sobolev spaces. Given a solution to (4.3), we compute the evolution of the quantity

$$\|f(t)\|^2_{\dot H^r(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |k|^{2r}|\hat f(t,k)|^2\,dk,$$

to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3}|k|^{2r}|\hat{f}(k)|^2\,dk = 2B\sqrt{\theta(t)}\left[\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{S^2}|k|^{2r}\hat{f}(k_-)\hat{f}(k_+)\hat{f}^c(k)\,d\sigma\,dk\right.$$
$$\left.-\int_{\mathbb{R}^3}|k|^{2r}|\hat{f}(k)|^2\,dk\right] - 2\theta_b\int_{\mathbb{R}^3}|k|^{2r+2}|\hat{f}(k)|^2\,dk, \tag{7.15}$$

where z^c is the complex conjugate of z. Let us start by estimating the contribution of the first term.

Lemma 7.13 (Estimate on Regularity contribution of $\tilde{Q}_e^+(f,f)$)**.** *Given a function $f \in \dot{H}^r(\mathbb{R}^3)$, then there exists $C(r,e) > 1$ such that*

$$\left|\int_{\mathbb{R}^3}\int_{S^2}|k|^{2r}\hat{f}(k_-)\hat{f}(k_+)\hat{f}^c(k)\,d\sigma\,dk\right| \le 4\pi\,C(r,e)\int_{\mathbb{R}^3}|k|^{2r}|\hat{f}(k)|^2\,dk.$$

Proof.- We obviously have

$$\left|\int_{\mathbb{R}^3}\int_{S^2}|k|^{2r}\hat{f}(k_-)\hat{f}(k_+)\hat{f}^c(k)\,d\sigma\,dk\right| \le$$
$$\le \int_{\mathbb{R}^3}\int_{S^2}|k|^{2r}|\hat{f}(k_-)||\hat{f}(k_+)||\hat{f}^c(k)|\,d\sigma\,dk$$
$$\le \int_{\mathbb{R}^3}\int_{S^2}|k|^{2r}|\hat{f}(k_+)||\hat{f}^c(k)|\,d\sigma\,dk$$
$$\le \sqrt{4\pi}\|f\|_{\dot{H}^r(\mathbb{R}^3)}\left[\int_{\mathbb{R}^3}\int_{S^2}|k|^{2r}|\hat{f}(k_+)|^2\,d\sigma\,dk\right]^{\frac{1}{2}}.$$

Now, we can change variables from k to k_+ to get the identity

$$\int_{\mathbb{R}^3}\int_{S^2}|k|^{2r}|\hat{f}(k_+)|^2\,d\sigma\,dk = \int_{\mathbb{R}^3}\int_{S^2}|k|^{2r}|\hat{f}(k_+)|^2\,\frac{dk}{dk_+}\,d\sigma\,dk_+.$$

Thanks to the formula

$$\frac{dk_+}{dk} = \left(\frac{3-e}{4}\right)^2\left(\frac{3-e}{4}+\frac{1+e}{4}\frac{k\cdot\sigma}{|k|}\right)$$

we deduce

$$\left(\frac{3-e}{4}\right)^2\left(\frac{1-e}{2}\right) \le \frac{dk_+}{dk} \le \left(\frac{3-e}{4}\right)^2.$$

This estimate together with $|k_+| \geq \frac{1-e}{2}|k|$ from (6.6) imply

$$\int_{\mathbb{R}^3}\int_{S^2} |k|^{2r}|\hat{f}(k_+)|^2 \frac{dk}{dk_+}\, d\sigma\, dk_+ \leq$$
$$\leq \left(\frac{2}{1-e}\right)^{2r} \int_{\mathbb{R}^3}\int_{S^2} |k_+|^{2r}|\hat{f}(k_+)|^2 \frac{dk}{dk_+}\, d\sigma\, dk_+$$
$$\leq \left(\frac{2}{1-e}\right)^{2r} \frac{128\pi}{(3-e)^2(1-e)}\, \|f\|^2_{\dot{H}^r(\mathbb{R}^3)}.$$

Putting together the above estimates, we get the desired result with

$$C(r,e) = \frac{4}{3-e}\left(\frac{2}{1-e}\right)^{r+1/2} > \frac{4}{3},$$

for all values of $r \geq 0$ and $0 < e < 1$. □

Using previous lemma in (7.15), we obtain the differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}^3} |k|^{2r}|\hat{f}(k)|^2\, dk \leq 2B\sqrt{\theta(t)}[C(r,e)-1]\int_{\mathbb{R}^3} |k|^{2r}|\hat{f}(k)|^2\, dk$$
$$- 2\theta_b \int_{\mathbb{R}^3} |k|^{2r+2}|\hat{f}(k)|^2\, dk.$$

Hence, if $Z_r(t) = \|f(t)\|^2_{\dot{H}^r}$, $Z_r(t)$ satisfies the inequality

$$\frac{dZ_r(t)}{dt} \leq 2B\sqrt{\theta(t)}\,[C(r,e)-1]\, Z_r(t) - 2\theta_b Z_{r+1}(t). \tag{7.16}$$

The desired result follows from (7.16) by virtue of the following Nash-type inequality:

Lemma 7.14 (Nash-type inequality)**.** *Let $f \in \dot{H}^{r+1}(\mathbb{R}^3) \cap \mathcal{P}(\mathbb{R}^3)$ with $r \geq 0$, then $f \in \dot{H}^r(\mathbb{R}^3)$ and*

$$\|f\|_{\dot{H}^{r+1}(\mathbb{R}^3)} \geq c_r \left(\|f\|_{\dot{H}^r(\mathbb{R}^3)}\right)^{(2r+5)/(2r+3)} \tag{7.17}$$

where

$$c_r = \left(\frac{1}{2\pi}\right)^{2/(2r+3)} \left(\frac{2r+3}{2r+5}\right)^{(2r+5)/(2r+3)}.$$

Proof.- For any constant $R > 0$, we obtain the bound

$$\int_{\mathbb{R}^3} |k|^{2r}|\hat{f}(k)|^2\, dk \leq \int_{|k|\leq R} |k|^{2r}|\hat{f}(k)|^2\, dk + \frac{1}{R^2}\int_{|k|>R} |k|^{2r+2}|\hat{f}(k)|^2\, dk.$$

Since f is a probability density, $|\hat{f}(k)| \leq 1$. Hence

$$\int_{|k|\leq R} |k|^{2r} |\hat{f}(k)|^2 \, dk \leq \int_{|k|\leq R} |k|^{2r} \, dk = 4\pi \frac{R^{2r+3}}{2r+3}.$$

By hypothesis, f belongs to $\dot{H}^{r+1}(\mathbb{R}^3)$. This implies the inequality

$$\|f\|^2_{\dot{H}^r(\mathbb{R}^3)} \leq 4\pi \frac{R^{2r+3}}{2r+3} + \frac{1}{R^2} \|f\|^2_{\dot{H}^{r+1}(\mathbb{R}^3)}. \tag{7.18}$$

Optimizing in R now yields the result. □

We use inequality (7.17) into (7.16) to obtain

$$\frac{dZ_r(t)}{dt} \leq 2B\sqrt{\theta(t)}\,[C(r,e)-1]\,Z_r(t) - 2\theta_b c_r \left(Z_r(t)\right)^{(2r+5)/(2r+3)}. \tag{7.19}$$

Previous inequality can we written as

$$\frac{dZ_r(t)}{dt} \leq 2B\sqrt{\theta(t)}\,[C(r,e)-1]\,Z_r(t)\left\{1 - \frac{\theta_b \theta(t)^{-1/2} c_r}{B[C(r,e)-1]} \left(Z_r(t)\right)^{2/(2r+3)}\right\}.$$

Considering that the temperature $\theta(t)$ is bounded uniformly in time both from above, $\max(\theta(0), \theta_\infty) \geq \theta(t)$, we get

$$\frac{dZ_r(t)}{dt} \leq$$
$$\leq 2B\sqrt{\theta(t)}\,[C(r,e)-1]\,Z_r(t)\left\{1 - \frac{\theta_b \max(\theta(0),\theta_\infty)^{-1/2} c_r}{B[C(r,e)-1]} \left(Z_r(t)\right)^{2/(2r+3)}\right\}.$$

that gives the bound

$$Z_r(t) \leq \max\left\{Z_r(0); \left[\frac{B[C(r,e)-1]}{\theta_b \max(\theta(0),\theta_\infty)^{-1/2} c_r}\right]^{(2r+3)/2}\right\}. \tag{7.20}$$

We summarize it as:

Theorem 7.15 (Propagation of smoothness)**.** [12] *Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap \dot{H}^r(\mathbb{R}^3)$ be any initial datum for equation* (4.3)*. Then, the solution $f(t,v)$ of* (4.3) *is bounded in $\dot{H}^r(\mathbb{R}^3)$, and there is a universal constant C_r so that, for all $t > 0$,*

$$\|f(t)\|_{\dot{H}^r(\mathbb{R}^3)} \leq \max\left\{\|f_0\|_{\dot{H}^r(\mathbb{R}^3)}, C_r\right\}.$$

Moreover, the stationary solutions to (4.3) *belongs to $H^\infty(\mathbb{R}^3)$.*

The last part of the previous theorem follows from (7.19) since for the stationary solutions we obtain

$$Z_r^\infty \leq \left[\frac{B[C(r,e)-1]}{\theta_b \max(\theta(0),\theta_\infty)^{-1/2} c_r}\right]^{(2r+3)/2}.$$

where $Z_r^\infty = \|f_\infty\|^2_{\dot{H}^r}$.

7.1.4. *Decay Estimates in Sobolev and L^1 spaces.* To finish the program stated at the beginning of this section, we use the interpolation inequalities obtained in Section 2.6 to deduce decay rates in several classical spaces.

Corollary 7.16 (Decay rate in Sobolev norms)**.** *Given an initial data $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap \dot{H}^{r+\epsilon}(\mathbb{R}^3)$, $r \geq 0$ and $\epsilon > 0$, with zero mean velocity, then the solution $f(t,v)$ of* (4.3) *corresponding to f_0 converges exponentially towards the steady state f_∞ in the $\dot{H}^r(\mathbb{R}^3)$-norm. More precisely, there exist explicitly computable constants $h(r,\epsilon,e,\theta(0),\|f_0\|_{\dot{H}^r(\mathbb{R}^3)},f_\infty) > 0$ and $0 < \beta(r,\epsilon) < 1$ such that*

$$\|f(t) - f_\infty\|_{\dot{H}^r(\mathbb{R}^3)} \leq h \, \exp\left[-\max\left(\frac{1-e^2}{4}\, C_1, C_3\right)\,(1-\beta)\, t\right],$$

for all $t \geq 0$. Moreover, $\beta \to 1$ as $\epsilon \to 0$ and $\beta \to 0$ as $\epsilon \to \infty$.

Proof.- This result follows directly from the combination of the decay result of d_2 in (7.14) given in Corollary 7.12 together with the interpolation inequality in Proposition 2.23 and the propagation of regularity obtained in Theorem 7.15. We need just to remark that by taking β_1 and β_2 appropiately in terms of r and ϵ, we can assure that r_1 and r_2 in Proposition 2.23 are smaller than $r + \epsilon$. □

Finally, we can make use of the propagation of moments to show decay estimates in L^1.

Corollary 7.17 (Decay rate in L^1)**.** *Given initial data $f_0 \in \mathcal{P}_{2l}(\mathbb{R}^3) \cap \dot{H}^\epsilon(\mathbb{R}^3)$, $l \in \mathbb{N}$, $l \geq 1$, $\epsilon > 0$, with zero mean velocity, then the solution $f(t,v)$ of* (4.3) *corresponding to f_0 converges exponentially towards the steady state f_∞ in the $L^1(\mathbb{R}^3)$-norm.*

Proof.- This result follows directly from the combination of the decay result in $L^2(\mathbb{R}^3)$ obtained in Corollary 7.16 for $r = 0$ together with the interpolation inequality in Lemma 2.24 and the propagation of moments obtained in Lemma 7.7. More precisely, previous corollary for $r = 0$ ensures that

$$\|f(t) - f_\infty\|_{L^2(\mathbb{R}^3)} \leq h \, \exp\left[-\max\left(\frac{1-e^2}{4}\, C_1, C_3\right)\,(1-\beta)\, t\right],$$

with suitable h and $0 < \beta < 1$. Now, Lemma 2.24 implies that

$$\begin{aligned}
&\|f(t) - f_\infty\|_{L^1(\mathbb{R}^3)} \leq \\
&\quad \leq C\,\|f(t) - f_\infty\|_{L^2(\mathbb{R}^3)}^{4l/(3+4l)} \left(\int_{\mathbb{R}^3} |v|^{2l} |f(t,v) - f_\infty(v)|\, dv \right)^{3/(3+4l)} \\
&\quad \leq C\,\|f(t) - f_\infty\|_{L^2(\mathbb{R}^3)}^{4l/(3+4l)} \left(m_{2l}(f)(t) + m_{2l}(f_\infty) \right)^{3/(3+4l)}
\end{aligned}$$

from which the result follows due to the uniform in time bound of moments obtained in Lemma 7.7. □

Remark 7.18 (Spectral Gap Estimates)**.** *Previous results give somehow quantitative estimates on the spectral gap of the linearized operator corresponding to the IMM with stochastic forcing around f_∞ in the space of smooth $H^\infty(\mathbb{R}^3)$ functions with zero mean velocity, Corollary* 7.16 *assert that the spectral gap estimate is as good as the decay rate in d_2 or W_2^2.*

With these results we have achieved to apply the global strategy to the IMM with stochastic forcing. The result cannot be better since we have achieved the whole study of the asymptotic stability in all interesting spaces for this equation.

7.2. **Particle's bath heating IMM.** This subsection will apply the same strategy used for the stochastic heating equation (4.3) to the case of the particle's bath heating equation (4.12):

$$\frac{\partial f}{\partial t} = \tilde{Q}_e(f,f) + \tilde{\mathcal{L}}_{e_b}(f),$$

with $\tilde{Q}_e(f,f)$ and $\tilde{\mathcal{L}}_{e_b}(f)$ giving by the weak formulations in (3.14) and (4.9) respectively.

7.2.1. *Existence, Uniqueness and Stability of Steady States.* We start by reminding the reader that the evolution for the mean velocity and temperature form a closed system of ODE's given by

$$\begin{aligned}
\frac{dU}{dt} &= -\sqrt{\theta}\,\frac{R_m(1+e_b)}{2\tilde{\lambda}}(U - U_b) \\
\frac{d\theta}{dt} &= -\frac{1-e^2}{4} B\,\theta^{3/2} + \sqrt{\theta}\,\frac{mR_m^2}{6\tilde{\lambda}}(1+e_b)^2 |U - U_b|^2 \\
&\quad - \sqrt{\theta}\,\frac{R_m(1+e_b)}{2\tilde{\lambda}} \left\{ [2 - R_m(1+e_b)]\,\theta - (1-R_m)(1+e_b)\theta_b \right\}.
\end{aligned}$$

with

$$U = \int_{\mathbb{R}^3} v\, f(v)\, dv \quad \text{and} \quad \theta = \frac{m}{3} \int_{\mathbb{R}^3} |v - U|^2\, f(v)\, dv$$

for normalized densities to unit number density, from which the existence of a unique stationary value for mean velocity $U = U_b$ and temperature $\theta = \theta^{\#}$ follows where $\theta^{\#}$ is the unique solution of (4.15). This information implies that given any two solutions with equal initial mean velocity and temperature will have equal mean velocity and temperature for all subsequent times. As before, we change the time variable to

$$\tau = \frac{B}{E} \int_0^t \sqrt{\theta(w)}\, dw \tag{7.21}$$

with $E = \dfrac{8}{1-e^2}$ and we are reduced to analyze

$$\frac{\partial f}{\partial \tau} = E\,[\tilde{Q}_e^+(f,f) - f] + \Theta\,[\tilde{\mathcal{L}}_{e_b}^+(f) - f].$$

with

$$\Theta = \frac{E}{B\,\tilde{\lambda}}.$$

Theorem 7.19 (Contraction in W_2)**.** *If f_1 and f_2 are two solutions to* (4.12) *for the respective initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity and temperature, then*

$$W_2^2(f_1(\tau), f_2(\tau)) \le \mathrm{e}^{-\eta\tau}\, W_2^2(f_1^0, f_2^0) \tag{7.22}$$

for all $\tau \ge 0$, with

$$\eta = 2 + \Theta \frac{1 - [1 - R_m(1+e_b)]^2}{2}.$$

Proof.- The main steps of the proof are:

- **Step 1.-** We write a Duhamel's representation of the solutions given by

$$f_i(\tau) = \mathrm{e}^{-(E+\Theta)\tau}\, f_i^0 + E \int_0^\tau \mathrm{e}^{-(E+\Theta)(\tau-s)}\, \tilde{Q}_e^+(f_i(s), f_i(s))\, ds + \Theta \int_0^\tau \mathrm{e}^{-(E+\Theta)(\tau-s)}\, \tilde{\mathcal{L}}_{e_b}^+(f_i(s))\, ds, \qquad i = 1, 2.$$

- **Step 2.-** Using the notation

$$F_i(s) = \frac{1}{E+\Theta} \left(E\tilde{Q}_e^+(f_i(s), f_i(s)) + \Theta \tilde{\mathcal{L}}_{e_b}^+(f_i(s)) \right).$$

  and the convexity of the squared Wasserstein distance, see Proposition 2.1, we infer that

$$W_2^2(f_1(\tau), f_2(\tau)) \leq \mathrm{e}^{-(E+\Theta)\tau}\, W_2^2(f_1^0, f_2^0) + (E+\Theta) \int_0^\tau \mathrm{e}^{-(E+\Theta)(\tau-s)}\, W_2^2(F_1(s), F_2(s))\, ds,$$

- **Step 3.-** We now use the contraction of the gain operators in W_2 obtained in Theorem 6.1 and 6.2 with the notation

$$\gamma = \frac{E}{E+\Theta}\frac{3+e^2}{4} + \frac{\Theta}{E+\Theta}\frac{1+[1-R_m(1+e_b)]^2}{2}$$

together with the convexity property in Proposition 2.1, to deduce

$$W_2^2(f_1(\tau), f_2(\tau)) \leq \mathrm{e}^{-(E+\Theta)\tau}\, W_2^2(f_1^0, f_2^0) + (E+\Theta) \int_0^\tau \mathrm{e}^{-(E+\Theta)(\tau-s)}\, \gamma\, W_2^2(f_1(s), f_2(s))\, ds,$$

- **Step 4.-** Finally, the function $y(\tau) = \mathrm{e}^{(E+\Theta)\tau}\, W_2^2(f_1(\tau), f_2(\tau))$ satisfies the inequality

$$y(\tau) \leq y(0) + \gamma\,(E+\Theta) \int_0^\tau y(s)\, ds$$

and then $y(\tau) \leq y(0)\, \mathrm{e}^{\gamma(E+\Theta)\tau}$ by Gronwall's lemma.

This concludes the argument since $(1-\gamma)\,(E+\Theta) = \eta$. □

The previous result implies a contraction in original variables.

Corollary 7.20 (Contraction in W_2 in original variables)**.** *If f_1 and f_2 are two solutions to* (4.12) *for the respective initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$ with equal mean velocity and temperature, then*

$$W_2^2(f_1(t), f_2(t)) \leq \mathrm{e}^{-\tilde{\eta}\sqrt{R}\,t}\, W_2^2(f_1^0, f_2^0)$$

for all $t \geq 0$ with R depending on the initial temperature $\theta(0)$, U_b and $\theta^\#$, with

$$\tilde{\eta} = B\,\frac{1-e^2}{4} + \frac{1-[1-R_m(1+e_b)]^2}{2\tilde{\lambda}}.$$

Proof.- It is not difficult based on the system of ODE's (4.13)-(4.14) to show that the temperature $\theta(t)$ of any solution f in the original time variable t converges towards $\theta^\#$ as $t \to \infty$, and satisfies $\theta(t) \geq R := R(\theta(0), \theta^\#, U_b) > 0$. In fact, it is an easy consequence of Liapunov functional techniques for systems of differential equations, that we leave the reader to

check as an exercise, to show there exist suitable constants $\lambda_1, \lambda_2, C > 0$ such that

$$\frac{d}{dt}\mathcal{F}(t) \leq -C\mathcal{F}(t) \quad \text{with} \quad \mathcal{F}(t) = \lambda_1 \left(\sqrt{\theta} - \sqrt{\theta^\#}\right)^2 + \lambda_2 |U - U_b|^2.$$

In particular,

$$\tau = \frac{B}{E}\int_0^t \sqrt{\theta(s)}\, ds \geq \frac{B\sqrt{R}}{E}\, t.$$

Writing (7.22) in the original variable t for initial data with equal mean velocity and temperature, we recover the desired contraction property for solutions of the equation (4.12). □

As in the case of the stochastic heating, the previous contraction property shows the existence and uniqueness of stationary states using Lemma 7.3.

Corollary 7.21 (Existence, Uniqueness & Stability of Stationary States)**.** *Equation* (4.12) *has a unique steady state* f_∞ *in* $\mathcal{P}_2(\mathbb{R}^3)$ *with mean velocity* U_b *and temperature* $\theta^\#$*. Moreover, given* f *any solution to* (4.12) *for the initial data* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ *with mean velocity* U_b *and temperature* $\theta^\#$*, then*

$$W_2^2(f(t), f_\infty) \leq \mathrm{e}^{-\tilde{\eta}\sqrt{R}\,t}\, W_2^2(f_0, f_\infty)$$

for all $t \geq 0$ *with* R *depending on the initial temperature* $\theta(0)$*,* $\theta^\#$ *and* U_b*.*

Proof.- Let us define the complete metric space

$$\tilde{\mathcal{M}} = \left\{\mu \in \mathcal{P}_2(\mathbb{R}^3) \text{ such that } \int_{\mathbb{R}^3} |v|^2 df(v) = 3\theta^\# \text{ and } \int_{\mathbb{R}^3} v\, df(v) = U_b\right\},$$

endowed with W_2, see Remark 2.3 and Proposition 2.1. Given the flow map of (4.12), i.e.,

$$\mathcal{T}(t) : (\tilde{\mathcal{M}}, W_2) \longrightarrow (\mathcal{P}_2(\mathbb{R}^3), W_2),$$

for any time $t > 0$, given by $\mathcal{T}(t)(f_0) = f(t)$ with $f(t)$ the unique solution at time t of (4.12) with initial datum $f_0 \in \tilde{\mathcal{M}}$. Then, $\mathcal{T}(t)$ is a continuous semigroup from $\tilde{\mathcal{M}}$ onto itself due to the adaptation of Theorem 3.5 to (4.12), the conservation of the steady value of the mean velocity U_b and the conservation of the steady value of the temperature $\theta^\#$.

Corollary 7.20 proves that $\mathcal{T}(t)$ is a uniform contraction from the complete metric space $(\tilde{\mathcal{M}}, W_2)$ into itself with contraction constant

$$L(t) = \mathrm{e}^{-\tilde{\eta}\,\mathrm{C}_1\mathrm{t}} < 1.$$

Therefore, Lemma 7.3 ensures the existence and uniqueness of a unique steady state in $(\tilde{\mathcal{M}}, W_2)$. The last assertion is a simple consequence of Corollary 7.20 by taking one of the solutions the stationary state we just obtained. □

As a simple corollary using the average control in Corollary 2.4, we deduce an exponential convergence of averages of solutions of (4.12) $f(t)$ towards those of f_∞ analogous to Corollary 7.5.

Remark 7.22 (Convergence to Equilibrium: General initial data)**.** *The exponential convergence towards equilibrium in Theorem* 7.21 *can be generalized to initial data with different temperature and mean velocity respect to the steady state. Actually, using the proof of Corollary* 7.20*, one can show that*

$$\max\{|\theta - \theta^{\#}|, |U - U_b|\} \leq A_1 \, \mathrm{e}^{-A_2 t} \tag{7.23}$$

with explicitly computable constants A_1 and A_2. On the other hand, one can repeat the proof of the contraction of the gain operators in W_2, obtained in Theorems 6.1 *and* 6.2*, without the assumption of equal mean velocity. In this case, additional terms appears in the contraction estimates. For instance, the estimate in Theorem* 6.1 *will look like*

$$W_2(\tilde{Q}_e^+(f,f), \tilde{Q}_e^+(g,g)) \leq \sqrt{\frac{3+e^2}{4}}\, W_2(f,g) + \frac{1-e^2}{4} \left| <f> - <g> \right|^2,$$

for $f, g \in \mathcal{P}_2(\mathbb{R}^3)$ with different mean velocity. Here, $<f>$ is the mean velocity of f. With these two ingredients, one can repeat easily the arguments in Theorem 7.19 *for two different solutions with equal initial temperature and different initial mean velocity. Finally, to deal with different initial temperature one has to proceed analogously to Theorem* 7.12*, by controlling the additional terms appearing in the computation in terms of the difference of the temperatures, that decay exponentially due to* (7.23)*. We leave all details, as a good exercise for the reader, stating the final result: given f any solution to* (4.12) *for the initial data $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$, then there exist explicitly computable constants R_1 and R_2 such that*

$$W_2(f(t), f_\infty) \leq R_1 \, \mathrm{e}^{-R_2 \, t}$$

for all $t \geq 0$. As a result of Proposition 2.12*, we also show that: given f any solution to* (4.12) *for the initial data $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ with mean velocity U_b, then there exist explicitly computable constants R_3 and R_4 such that*

$$d_2(f(t), f_\infty) \leq R_3 \, \mathrm{e}^{-R_4 \, t}$$

for all $t \geq 0$.

7.2.2. *Contraction on Fourier-Based Distances.* Let us improve on the stability properties of the unique steady state f_∞ obtained in previous subsection.

Proposition 7.23 (Contraction in d_s)**.** *Given $s > 0$, f_1 and f_2 solutions to* (4.12) *for the initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$ with equal moments up to order $2 + [\alpha]$ if $\alpha \notin \mathbb{N}$ or $\alpha = 0$, or equal moments up to order $1 + \alpha$ if $\alpha \in \mathbb{N}$ with $\alpha \geq 1$, then*

$$d_{2+\alpha}(f_1(t), f_2(t)) \leq d_{2+\alpha}(f_1^0, f_2^0)\, \mathrm{e}^{-\tilde{\eta}(\alpha)\sqrt{R}\, t},$$

for all $t \geq 0$ with R depending on the initial temperature $\theta(0)$, $\theta^{\#}$ and U_b.

Proof.- The main steps of the proof are:

- **Step 1.-** Let us remind that the mean velocity and temperature of both solutions $f_1(t)$ and $f_2(t)$ is equal to $U(t)$ and $\theta(t)$ respectively, solution to the ODE system (4.13)-(4.14), since they are equal initially. Using the Fourier transform expression of $\tilde{Q}_e(f, f)$ and $\tilde{\mathcal{L}}_{e_b}(f)$ in (4.10) and (4.12) respectively, equation (4.12) can be written in Fourier as

$$\begin{aligned}\frac{\partial \hat{f}}{\partial t} = & \frac{B\sqrt{\theta(t)}}{4\pi} \int_{S^2} \left\{ \hat{f}(t, k_-)\hat{f}(t, k_+) - \hat{f}(t, 0)\hat{f}(t, k) \right\} d\sigma \\ & + \frac{\sqrt{\theta(t)}}{2\pi\tilde{\lambda}} \int_{S^2} |\omega \cdot n| \left[\hat{f}(k_+^L)\hat{M}_b(k_-^L) - \hat{f}(k)\hat{M}_b(0) \right] dn. \end{aligned} \tag{7.24}$$

  or equivalently

$$\frac{\partial \hat{f}}{\partial t} = B\sqrt{\theta(t)} \left[\widehat{\tilde{Q}_e^+(f, f)} - \hat{f} \right] + \frac{\sqrt{\theta(t)}}{\tilde{\lambda}} \left[\widehat{\tilde{\mathcal{L}}_{e_b}^+(f)} - \hat{f} \right].$$

- **Step 2.-** Taking the expressions of the two solutions $\hat{f}_1(t)$ and $\hat{f}_2(t)$, subtracting them and dividing by $|k|^{2+\alpha}$ with $k \in \mathbb{R}^3_o$, we get

$$\begin{aligned}\frac{\partial}{\partial t} \frac{\hat{f}_1(k) - \hat{f}_2(k)}{|k|^{2+\alpha}} = & B\sqrt{\theta(t)}\, \frac{\widehat{\tilde{Q}_e^+(f_1, f_1)}(k) - \widehat{\tilde{Q}_e^+(f_2, f_2)}(k)}{|k|^{2+\alpha}} \\ & + \frac{\sqrt{\theta(t)}}{\tilde{\lambda}}\, \frac{\widehat{\tilde{\mathcal{L}}_{e_b}^+(f_1)}(k) - \widehat{\tilde{\mathcal{L}}_{e_b}^+(f_2)}(k)}{|k|^{2+\alpha}} \\ & - \left(B + \frac{1}{\tilde{\lambda}} \right) \sqrt{\theta(t)} \frac{\hat{f}_1(k) - \hat{f}_2(k)}{|k|^{2+\alpha}}.\end{aligned}$$

- **Step 3.-** Let us set

$$h(t, k) = \frac{\hat{f}_1(k) - \hat{f}_2(k)}{|k|^{2+\alpha}},$$

the last identity implies

$$\left|\partial_t h(t,k) + D_1\sqrt{\theta(t)}h(t,k)\right| \le D_2\sqrt{\theta(t)}\, d_{2+\alpha}(f_1,f_2)$$

with $D_1 = B + \frac{1}{\tilde\lambda}$ and $D_2 = BA(\alpha,e) + \frac{1}{\tilde\lambda}B(2+\alpha,e_b,R_m)$ where Theorems 6.4 and 6.5 have been used. Proceeding similarly to Steps 3 and 4 in the proof of Theorem 7.8, we deduce

$$\exp\left(D_1\int_0^t \sqrt{\theta(s)}\,ds\right) d_{2+\alpha}(f_1(t),f_2(t)) \le d_{2+\alpha}(f_1^0,f_2^0) +$$

$$D_2\int_0^t \sqrt{\theta(\tau)}\, d_{2+\alpha}(f_1(\tau),f_2(\tau)) \exp\left(D_1\int_0^\tau \sqrt{\theta(s)}\,ds\right) d\tau.$$

- **Step 4.-** Denoting by

$$w(t) := \exp\left(D_1\int_0^t \sqrt{\theta(s)}\,ds\right) d_{2+\alpha}(f_1(t),f_2(t)),$$

we write the last inequality as

$$w(t) \le w(0) + D_2\int_0^t \sqrt{\theta(\tau)}\, w(\tau)\, d\tau.$$

By the Gronwall lemma, this implies

$$w(t) \le w(0)\exp\left(D_2\int_0^t \sqrt{\theta(\tau)}\,d\tau\right),$$

namely

$$d_{2+\alpha}(f_1(t),f_2(t)) \le d_{2+\alpha}(f_1^0,f_2^0)\exp\left(-\tilde\eta(\alpha)\int_0^t \sqrt{\theta(\tau)}\,d\tau\right).$$

with

$$\tilde\eta(\alpha) = B[1-A(\alpha,e)] + \frac{1}{\tilde\lambda}[1-B(2+\alpha,e_b,R_m)].$$

As in the proof of Corollary 7.20, we have $\theta(t) \ge R := R(\theta(0),\theta^\#,U_b) > 0$ from which the desired result follows. □

Remark 7.24 (Rates in d_2)**.** *We point out that the rate in previous theorem corresponding to $\alpha = 0$ is exactly $\tilde\eta(0) = \tilde\eta$, i.e., the rate of convergence in W_2^2 obtained in Corollary* 7.20.

Simple consequences as for the case of stochastic heating are the asymptotic stability in $d_{2+\alpha}$ distances and equality in the evolution of moments if initially are equal.

Corollary 7.25 (Decay rates in $d_{2+\alpha}$)**.** *Given $\alpha \geq 0$ and f any solution to* (4.12) *for the initial data $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ with equal moments up to order $2+[\alpha]$ if $\alpha \notin \mathbb{N}$ or $\alpha = 0$, or equal moments up to order $1+\alpha$ if $\alpha \in \mathbb{N}$ with $\alpha \geq 1$, then*

$$d_{2+\alpha}(f(t), f_\infty) \leq d_{2+\alpha}(f_0, f_\infty)\, \mathrm{e}^{-\tilde{\eta}(\alpha)\sqrt{R}\, t},$$

for all $t \geq 0$.

As before the previous result shows that as long as the initial data has more common moments with f_∞ the decay rate improves.

Corollary 7.26 (Evolution of any moment)**.** *Given $\alpha > 0$, f_1 and f_2 solutions to* (4.12) *for the initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$ with equal moments up to order $2+[\alpha]$ if $\alpha \notin \mathbb{N}$ or equal moments up to order $1+\alpha$ if $\alpha \in \mathbb{N}$, then*

$$\int_{\mathbb{R}^3} v^\beta \, df_1(t,v) = \int_{\mathbb{R}^3} v^\beta \, df_2(t,v)$$

for all $t \geq 0$, and any multi-index β with $|\beta| \leq 2+[\alpha]$ if $\alpha \notin \mathbb{N}$ or $|\beta| \leq 1+\alpha$ if $\alpha \in \mathbb{N}$.

The previous result is a simple consequence of the contraction estimate in Proposition 7.23 since the distance $d_{2+\alpha}(f_1, f_2)$ is not finite unless moments of f_1 and f_2 are equal up to order $2+[\alpha]$ if $\alpha \notin \mathbb{N}$ or $1+[\alpha]$ if $\alpha \in \mathbb{N}$.

7.2.3. *Moment bounds.* The evolution of moments for the solution to the particle's bath heating IMM can be easily obtained by owing to the analogous results described in subsection 7.1.2. In particular, the result of Lemma 7.7 can be used to derive the evolution of moments for the dissipative nonlinear Boltzmann operator. Thus, only the analysis of the linear collision operator is needed. To this aim, let us evaluate, for any given $r \in \mathbb{N}$, $r > 1$

$$\begin{aligned}\langle |v|^{2r}, \tilde{\mathcal{L}}_{e_b}(f)\rangle &= \frac{\sqrt{\theta(t)}}{2\pi\tilde{\lambda}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |\Omega\cdot n| f(v) M_1(w) \Big[|v_L'|^{2r} - |v|^{2r}\Big] dn\, dv\, dw \\ &= \frac{\sqrt{\theta(t)}}{4\pi\tilde{\lambda}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v) M_1(w) \Big[|v_L'|^{2r} - |v|^{2r}\Big] d\sigma\, dv\, dw.\end{aligned} \tag{7.25}$$

In our case, the post-collision velocity v_L' is given by the first identity in (4.7):

$$v_L' = v - R_m(1+e_b)(u\cdot n)n,$$

where, as usual, we denoted $u = v - w$. Since $r \in \mathbb{N}$, we can write

$$|v_L'|^{2r} - |v|^{2r} = \big(|v_L'|^2 - |v|^2\big) \sum_{l=0}^{r-1} |v_L'|^{2(r-1-l)} |v|^{2l}. \tag{7.26}$$

Moreover

$$|v_L'|^2 - |v|^2 = R_m^2(1+e_b)^2(u\cdot n)^2 - 2R_m(1+e_b)(u\cdot n)(v\cdot n) \tag{7.27}$$

$$= R_m^2(1+e_b)^2(u\cdot n)^2 - 2R_m(1+e_b)(u\cdot n)^2 + R_m(1+e_b)(u\cdot n)(w\cdot n)$$

$$= R_m(1+e_b)[R_m(1+e_b)-2]\,(u\cdot n)^2 + 2R_m(1+e_b)\big[(v\cdot n)(w\cdot n)-(w\cdot n)^2\big]\,.$$

Since $R_m(1+e_b) < 2$, the first term in the final expression of (7.27) has negative sign, while the second term is dominated by $2R_m(1+e_b)|v\cdot n||w\cdot n|$. Finally, (7.26) implies

$$|v_L'|^{2r} - |v|^{2r} \le -\,R_m(1+e_b)\,(2-R_m(1+e_b))\,(u\cdot n)^2 \sum_{l=0}^{r-1} |v_L'|^{2(r-1-l)}|v|^{2l}$$
$$+\,2R_m(1+e_b)|v\cdot n||w\cdot n| \sum_{l=0}^{r-1} |v_L'|^{2(r-1-l)}|v|^{2l}. \tag{7.28}$$

It follows that the highest power in $|v|$ of the expansion (7.28) has the negative sign. Therefore, there exist suitable constants α_r and β_r such that the following inequality holds

$$\langle |v|^{2r}, \tilde{\mathcal{L}}_{e_b}(f)\rangle \le \sqrt{\theta(t)}\,[-\alpha_r m_{2r}(f) + \beta_r m_{2r-1}(f) + B_r]\,,$$

where B_r depends on moments of the Maxwellian bath, i.e., $m_{2l}(M_b)$, $l = 1,\dots,r$. Now, the proof of boundedness of moments follows as in Lemma 7.7 leaving further details to the reader.

Lemma 7.27. [Uniform in time moment estimates] *Let $f(t,v)$ be the solution to particle's bath heating equation* (4.12)*, where the initial distribution $f_0(v)$ with zero mean velocity satisfies $m_{2r}(0) < \infty$ for some $r \ge 2$. Then, $m_{2r}(t)$ satisfies the following differential inequality*

$$\frac{d}{dt}m_{2r}(t) \le -B\sqrt{\frac{m_2(t)}{3}}\left[\frac{1-e^{2r}}{4}\big(m_{2r}(t) + m_{2(r-1)}(t)m_2(t)\big)\right.$$
$$\left. -\frac{1}{2}\sum_{l=1}^{r-1}\binom{r}{l} m_{2(r-l)}(t)m_{2l}(t)\right] \tag{7.29}$$
$$+\sqrt{m_2(t)}\,[-\alpha_r m_{2r}(f) + \beta_r m_{2r-1}(f) + B_r]\,.$$

Consequently, $m_{2r}(t)$ is uniformly bounded in time provided $m_{2r}(0) < \infty$. As a consequence, all moments of stationary solutions are bounded.

7.2.4. *Propagation of regularity and Decay Rates in Sobolev and L^1 spaces.* Now, we attack the problem of evolution of moments in Fourier variables. Remember that with the notation

$$\|f(t)\|^2_{\dot{H}^r(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |k|^{2r} |\hat{f}(t,k)|^2 \, dk,$$

we can work on equation (4.12) in Fourier variables (7.24), to obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3} |k|^{2r}|\hat{f}(k)|^2\,dk &= \frac{B\sqrt{\theta(t)}}{2\pi}\int_{\mathbb{R}^3}\int_{S^2} |k|^{2r}\hat{f}(k_-)\hat{f}(k_+)\hat{f}^c(k)\,d\sigma\,dk \\
&\quad + \frac{\sqrt{\theta(t)}}{\pi\tilde{\lambda}}\int_{\mathbb{R}^3}\int_{S^2} |\omega\cdot n||k|^{2r}\hat{M}_b(k_-^L)\hat{f}(k_+^L)\hat{f}^c(k)\,dn\,dk \\
&\quad - 2\sqrt{\theta(t)}\left(B+\frac{1}{\tilde{\lambda}}\right)\int_{\mathbb{R}^3}|k|^{2r}|\hat{f}(k)|^2\,dk \\
&:= \frac{B\sqrt{\theta(t)}}{2\pi}I_1 + \frac{\sqrt{\theta(t)}}{\pi\tilde{\lambda}}I_2 - 2\sqrt{\theta(t)}\left(B+\frac{1}{\tilde{\lambda}}\right)Z_r(t)
\end{aligned} \tag{7.30}$$

with the notation $Z_r(t) = \|f(t)\|^2_{\dot{H}^r}$ and where z^c is the complex conjugate of z as above. Let us improve our estimate on the first term of the previous expression obtained in Lemma 7.13. In order to do that, let us start by the following technical lemma.

Lemma 7.28 (Control of Weighted Fourier L^∞-norms)**.** *Given an initial data $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ for equation* (4.12)*, then there exists $0<\delta<1$ and $A_\delta>0$ depending on $\tilde{\lambda}$, such that if the initial data satisfies $\||k|^\delta \hat{f}_0(k)\|_{L^\infty(\mathbb{R}^3)} < \infty$, the solution to equation* (4.12) *verifies:*

$$\||k|^\delta \hat{f}(t,k)\|_{L^\infty(\mathbb{R}^3)} \le \max\left\{\||k|^\delta \hat{f}_0(k)\|_{L^\infty(\mathbb{R}^3)}, A_\delta\right\},$$

for all $t \ge 0$.

Proof.- Taking the Fourier expression of equation (4.12), changing the time variable similar to (7.21), i.e.,

$$\tau = \int_0^t \sqrt{\theta(w)}\,dw$$

and multiplying by $|k|^\delta$, we get

$$|\partial_\tau h(\tau,k) + D_1\, h(\tau,k)| \le \frac{B}{4\pi}\int_{S^2} |k|^\delta |\hat f(\tau,k_-)||\hat f(\tau,k_+)|\, d\sigma + \frac{1}{2\pi\tilde\lambda}\int_{S^2} |\omega\cdot n||k|^\delta |\hat f(\tau,k_+^L)||\hat M_b(k_-^L)|\, dn \tag{7.31}$$

with $h(\tau,k) := |k|^\delta \hat f(\tau,k)$ and $D_1 = B + \frac{1}{\tilde\lambda}$.

Using that $|k_-^L| = 2\chi|\omega\cdot n||k|$, we can estimate the second term on the right-hand side of (7.31) as

$$\frac{1}{2\pi\tilde\lambda}\int_{S^2} |\omega\cdot n||k|^\delta |\hat f(k_+^L)||\hat M_b(k_-^L)|\, dn \le A_1 \int_{S^2} |\omega\cdot n|^{1-\delta}|k_-^L|^\delta |\hat M_b(k_-^L)|\, dn \le A_2\left(\sup_{k\in\mathbb{R}^3} |k|^\delta |\hat M_b(k)|\right) := C_\delta$$

In order to estimate the first term in the right-hand side of (7.31), we split the integral over the sphere in two sets: let us define

$$\mathcal{S}_\epsilon = \left\{\sigma\in S^2 \text{ such that } |k_+|\ge(1-\epsilon)|k| \text{ or } |k_-|\ge(1-\epsilon)|k|\right\}$$

and its complementary set $\bar{\mathcal{S}}_\epsilon$ on S^2 for any given $0<\epsilon<\frac14$. Actually, the complementary set can be characterized as

$$\bar{\mathcal{S}}_\epsilon = \left\{\sigma\in S^2 \text{ such that } |k_+|\le(1-\epsilon)|k| \text{ and } |k_-|\le(1-\epsilon)|k|\right\}.$$

Moreover, given any $\sigma\in\bar{\mathcal{S}}_\epsilon$, we have that $|k_+|\ge\epsilon|k|$ and $|k_-|\ge\epsilon|k|$ by contradiction. Assume it does not hold that $|k_+|\ge\epsilon|k|$, then

$$|k_+|+|k_-| < \epsilon|k| + |k_-| \le \epsilon|k| + (1-\epsilon)|k| = |k| = |k_+ + k_-| \le |k_+| + |k_-|.$$

Therefore, given any $\sigma\in\bar{\mathcal{S}}_\epsilon$, we deduce that

$$\epsilon|k| \le \min(|k_+|,|k_-|) \le \max(|k_+|,|k_-|) \le (1-\epsilon)|k|,$$

and thus,

$$\epsilon \le \min\left(\frac{|k_+|}{|k|},\frac{|k_-|}{|k|}\right) \le \max\left(\frac{|k_+|}{|k|},\frac{|k_-|}{|k|}\right) \le 1-\epsilon.$$

Now, we may use on the set $\bar{\mathcal{S}}_\epsilon$ the inequality

$$(a+b)^\delta \le c_\epsilon(\delta)(a^\delta + b^\delta)$$

for all $\epsilon\le a\le b\le 1-\epsilon$, $0<\delta<1$, with

$$c_\epsilon(\delta) := \frac{1}{\epsilon^\delta + (1-\epsilon)^\delta}.$$

Actually, this inequality is equivalent to the straightforward inequality

$$(1+x)^\delta \le c_\epsilon(\delta)(1+x^\delta)$$

for $\frac{\epsilon}{1-\epsilon} \leq x \leq 1$, left to the reader. Thus, the first term in the right-hand side of (7.31) can be estimated as

$$\begin{aligned}\frac{B}{4\pi}\int_{S^2}|k|^\delta|\hat{f}(\tau,k_-)||\hat{f}(\tau,k_+)|\,d\sigma &= \frac{B}{4\pi}\left(\int_{\mathcal{S}_\epsilon}|k|^\delta|\hat{f}(\tau,k_-)||\hat{f}(\tau,k_+)|\,d\sigma\right.\\ &\quad\left.+\int_{\bar{\mathcal{S}}_\epsilon}|k|^\delta|\hat{f}(\tau,k_-)||\hat{f}(\tau,k_+)|\,d\sigma\right)\\ &\leq \frac{B}{4\pi}\left[\frac{1}{(1-\epsilon)^\delta}\int_{\mathcal{S}_\epsilon}d\sigma\left(\sup_{k\in\mathbb{R}^3}|k|^\delta|\hat{f}(\tau,k)|\right)\right.\\ &\quad\left.+\frac{2}{\epsilon^\delta+(1-\epsilon)^\delta}\int_{\bar{\mathcal{S}}_\epsilon}d\sigma\left(\sup_{k\in\mathbb{R}^3}|k|^\delta|\hat{f}(\tau,k)|\right)\right]\end{aligned}$$

since on $\mathcal{S}_\epsilon$ we have $|k_+| \geq (1-\epsilon)|k|$ or $|k_-| \geq (1-\epsilon)|k|$, and on $\bar{\mathcal{S}}_\epsilon$, we have

$$|k_+ + k_-|^\delta \leq (|k_+| + |k_-|)^\delta = |k|^\delta\left(\frac{|k_+|}{|k|}+\frac{|k_-|}{|k|}\right)^\delta \leq c_\epsilon(\delta)(|k_+|^\delta + |k_-|^\delta).$$

Thus, we deduce

$$\frac{B}{4\pi}\int_{S^2}|k|^\delta|\hat{f}(\tau,k_-)||\hat{f}(\tau,k_+)|d\sigma \leq B\left(\frac{\alpha}{1-\epsilon}+\frac{2(1-\alpha)}{\epsilon^\delta+(1-\epsilon)^\delta}\right)\|h(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)}$$

with

$$\alpha = \frac{1}{4\pi}\int_{\mathcal{S}_\epsilon}d\sigma < 1.$$

Now, let us choose ϵ such that

$$\frac{1}{1-\epsilon} \leq 1 + \frac{3}{4B\tilde{\lambda}},$$

and $0 < \delta < 1$ such that

$$\frac{2}{\epsilon^\delta+(1-\epsilon)^\delta} \leq 1 + \frac{3}{4B\tilde{\lambda}},$$

which is possible since the limit of the left-hand side is 1 as $\delta \to 0^+$. Then, we deduce that

$$\frac{B}{4\pi}\int_{S^2}|k|^\delta|\hat{f}(\tau,k_-)||\hat{f}(\tau,k_+)|\,d\sigma \leq \left(B+\frac{3}{4\tilde{\lambda}}\right)\|h(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)}$$

and coming back to (7.31), we get

$$|\partial_\tau h(\tau,k) + D_1\,h(\tau,k)| \leq \left(B+\frac{3}{4\tilde{\lambda}}\right)\|h(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)} + C_\delta. \tag{7.32}$$

with $h(\tau,k) := |k|^\delta\hat{f}(\tau,k)$ and $D_1 = B + \frac{1}{\tilde{\lambda}}$. Proceeding as in the proof of Proposition 7.23, it is easy to deduce by Gronwall's like arguments that

$$\||k|^\delta\hat{f}(\tau,k)\|_{L^\infty(\mathbb{R}^3)} \leq \max\left\{\||k|^\delta\hat{f}_0(k)\|_{L^\infty(\mathbb{R}^3)}, A_\delta\right\},$$

for all $\tau \geq 0$, with $A_\delta = 4C_\delta \tilde{\lambda}$. A final change of time variables gives the desired result taking into account that the temperature $\theta(t)$ is bounded above and below away from zero. We leave these final details to the interested reader. □

Remark 7.29 (Necessity of the Particle's bath)**.** *We point out that the bound on* $\||k|^\delta \hat{f}(t,k)\|_{L^\infty(\mathbb{R}^3)}$ *for suitably small* $0 < \delta < 1$ *will be of paramount importance for the propagation of regularity below and cannot be obtained without the presence of the linear operator.*

Now, let us come back to our original question, i.e., the propagation of regularity to equation (4.12).

Step 1.- Estimate of I_1: Coming back to estimate the first term in (7.30), we can use Lemma 7.28 and Hölder's inequality to obtain

$$|I_1| \leq \||k|^\delta \hat{f}(t,k)\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} \int_{S^2} \frac{|k|^\delta}{|k_-|^\delta} |k|^{2r-\delta} |\hat{f}(k_+)||\hat{f}^c(k)| \, d\sigma \, dk$$

$$\leq \||k|^\delta \hat{f}(t,k)\|_{L^\infty(\mathbb{R}^3)} \left(\int_{\mathbb{R}^3} \int_{S^2} |k|^{2r-\delta} |\hat{f}(k_+)|^2 \, d\sigma \, dk \right)^{1/2} I_3^{1/2} \quad (7.33)$$

for $0 < \delta < 1$ given by Lemma 7.28 and

$$I_3 := \int_{\mathbb{R}^3} \int_{S^2} \frac{|k|^{2\delta}}{|k_-|^{2\delta}} |k|^{2r-\delta} |\hat{f}(k)|^2 \, d\sigma \, dk.$$

Now, taking into account (6.6), we have that

$$\frac{|k|^{2\delta}}{|k_-|^{2\delta}} = C(e) \, \frac{1}{(1-\cos\vartheta)^{2\delta}}$$

where ϑ is the angle between the unit vectors $k/|k|$ and σ. As a consequence, the last integral becomes

$$I_3 = C(e) \int_{\mathbb{R}^3} |k|^{2r-\delta} |\hat{f}(k)|^2 \int_{S^2} \frac{1}{(1-\cos\vartheta)^{2\delta}} \, d\sigma \, dk = C \, Z_{r-\delta/2}(t)$$

since the integral over S^2 is convergent for $0 < \delta < \frac{1}{2}$ and does not depend on k. Now, coming back to (7.33) and using the same arguments as in the proof of Lemma 7.13, we get

$$|I_1| \leq C(r,e) \, \||k|^\delta \hat{f}(t,k)\|_{L^\infty(\mathbb{R}^3)} \, Z_{r-\delta/2}(t).$$

Step 2.- Estimate of I_2: given $r > 1/2$, let us estimate the second term I_2 as

$$I_2 \leq \left(\sup_{k\in\mathbb{R}^3} |k||\hat{M}_b(k)|\right) \int_{\mathbb{R}^3}\int_{S^2_+} |\omega\cdot n| \frac{|k|}{|k^L_-|} |k|^{2r-1} |\hat{f}(k^L_+)||\hat{f}^c(k)|\, dn\, dk.$$

As already seen in Theorem 6.5, we have

$$\frac{|k^L_-|}{|k|} = 2\chi|\omega\cdot n| \quad \text{and} \quad \frac{|k^L_+|}{|k|} = [1-4\chi(1-\chi)|\omega\cdot n|^2]^{1/2} \geq [1-4\chi(1-\chi)]^{1/2} \tag{7.34}$$

with the shortcut $\chi = \frac{R_m(1+e_b)}{2}$. Thus, the estimate in the second term reads as

$$I_2 \leq \left(\sup_{k\in\mathbb{R}^3} |k||\hat{M}_b(k)|\right) \int_{S^2_+}\int_{\mathbb{R}^3} |k|^{2r-1} |\hat{f}(k^L_+)||\hat{f}^c(k)|\, dk\, dn.$$

Hölder's estimate and changing variables from k to k^L_+ through the linear transformation $k^L_+ = k - 2\chi(k\cdot n)n$ gives

$$\begin{aligned}
\left(\int_{\mathbb{R}^3} |k|^{2r-1} |\hat{f}(k^L_+)||\hat{f}^c(k)|\, dk\right)^2 \leq \\
&\leq \left(\int_{\mathbb{R}^3} |k|^{2r-1} |\hat{f}(k)|^2\, dk\right)\left(\int_{\mathbb{R}^3} |k|^{2r-1} |\hat{f}(k^L_+)|^2\, dk\right) \\
&\leq Z_{r-1/2}(t) \left(\int_{\mathbb{R}^3} |k^L_+|^{2r-1} |\hat{f}(k^L_+)|^2\, dk^L_+\right) h(n) \\
&:= Z^2_{r-1/2}(t)\, h(n)
\end{aligned}$$

where (7.34) has been used and with

$$h(n) = \det(I - 2\chi(n\otimes n)))^{-1}[1-4\chi(1-\chi)]^{-(2r-1)/2}.$$

Step 3.- Conclusion: Coming back to the evolution of $Z_r(t)$ in (7.30), we have obtained using previous steps and Lemma 7.28 that

$$\frac{dZ_r(t)}{dt} \leq$$

$$\leq -2\sqrt{\theta(t)}\left(B + \frac{1}{\tilde{\lambda}}\right) Z_r(t) + 2\frac{\sqrt{\theta(t)}}{\pi\tilde{\lambda}}\, H_1\, Z_{r-1/2}(t) + \frac{B\sqrt{\theta(t)}}{2\pi}\, H_2\, Z_{r-\delta/2}(t),$$

with

$$H_1 := \left(\sup_{k\in\mathbb{R}^3} |k||\hat{M}_b(k)|\right) \int_{S^2_+} h(n)^{1/2}\, dn$$

and

$$H_2 := C(r,e)\ \max\left\{ \||k|^\delta \hat{f}_0(k)\|_{L^\infty(\mathbb{R}^3)}, A_\delta \right\}$$

and small enough $0 < \delta < \frac{1}{2}$. Finally, using an analogous of the Nash-type inequality in Lemma 7.14, left to the reader, we get

$$Z_{r-1/2}(t) \le C_r\, Z_r^{\alpha_1}(t) \quad \text{and} \quad Z_{r-\delta/2}(t) \le C_{r,\delta}\, Z_r^{\alpha_2}(t),$$

with $\alpha_1 = (2r+2)/(2r+3) < 1$ and $\alpha_2 = (2r+3-\delta)/(2r+3) < 1$, from which

$$\frac{dZ_r(t)}{dt} \le -2\sqrt{\theta(t)}\left(B + \frac{1}{\tilde{\lambda}}\right) Z_r(t) + 2\frac{\sqrt{\theta(t)}}{\pi\tilde{\lambda}} H_1 Z_r^{\alpha_1}(t) + \frac{B\sqrt{\theta(t)}}{2\pi} H_2\, Z_r^{\alpha_2}(t).$$

Taking into account that the temperature is bounded away from zero and infinity, we deduce the final result:

Theorem 7.30 (Propagation of smoothness)**.** *Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap \dot{H}^r(\mathbb{R}^3)$ with $\||k|^\delta \hat{f}_0(k)\|_{L^\infty(\mathbb{R}^3)} < \infty$, $0 < \delta < 1$, be any initial datum for equation* (4.12) *with $r > \frac{1}{2}$. Then the solution $f(t,v)$ of* (4.12) *is bounded in $\dot{H}^r(\mathbb{R}^3)$, and there is a universal constant A_r so that, for all $t > 0$,*

$$\|f(t)\|_{\dot{H}^r(\mathbb{R}^3)} \le \max\left\{ \|f_0\|_{\dot{H}^r(\mathbb{R}^3)}, A_r \right\}.$$

Moreover, the stationary solutions to (4.12) *belongs to $H^\infty(\mathbb{R}^3)$.*

Remark 7.31 (L^2-bounds)**.** *Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ with $\||k|^\delta \hat{f}_0(k)\|_{L^\infty(\mathbb{R}^3)} < \infty$, $0 < \delta < 1$, be any initial datum for equation* (4.12)*. Previous theorem together with the Nash inequality in Lemma* 7.14 *implies that the solution $f(t,v)$ of* (4.12) *is bounded in $L^2(\mathbb{R}^3)$, and there is a universal constant C_2 so that, for all $t > 0$,*

$$\|f(t)\|_{L^2(\mathbb{R}^3)} \le \max\left\{ \|f_0\|^{3/5}_{\dot{H}^1(\mathbb{R}^3)}, C_2 \right\}.$$

Moreover, the stationary solutions to (4.12) *belongs to $L^2(\mathbb{R}^3)$.*

We can now apply the last points of the strategy explained at the beginning of this section by using again the interpolation inequalities obtained in Section 2.6 together with the convergence in d_2 for general initial data in Remark 7.22 and the propagation of moments and regularity shown above. Proceeding as in the stochastic heating case, we deduce the following results:

Corollary 7.32 (Decay rate in Sobolev norms)**.** *Given an initial data $f_0 \in \mathcal{P}_2(\mathbb{R}^3) \cap \dot{H}^{r+\epsilon}(\mathbb{R}^3)$ with $\||k|^\delta \hat{f}_0(k)\|_{L^\infty(\mathbb{R}^3)} < \infty$, $0 < \delta < 1$, $r > \frac{1}{2}$ and $\epsilon > 0$, with mean velocity U_b, then the solution $f(t,v)$ of* (4.12) *corresponding to f_0 converges exponentially towards the steady state f_∞ in the $\dot{H}^r(\mathbb{R}^3)$-norm.*

Corollary 7.33 (Decay rate in L^1)**.** *Given initial data $f_0 \in \mathcal{P}_{2l}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$, $l \in \mathbb{N}$, $l \geq 1$, $\epsilon > 0$, with $\||k|^\delta \hat{f}_0(k)\|_{L^\infty(\mathbb{R}^3)} < \infty$, $0 < \delta < 1$, and with mean velocity U_b, then the solution $f(t,v)$ of* (4.12) *corresponding to f_0 converges exponentially towards the steady state f_∞ in the $L^1(\mathbb{R}^3)$-norm.*

Remark 7.34 (Spectral Gap Estimates)**.** *Previous results give somehow quantitative estimates on the spectral gap of the linearized operator corresponding to the IMM with stochastic forcing around f_∞ in the space of smooth $H^\infty(\mathbb{R}^3)$ functions with $\||k|^\delta \hat{f}_0(k)\|_{L^\infty(\mathbb{R}^3)} < \infty$, $0 < \delta < 1$, and with mean velocity U_b. Corollary* 7.16 *assert that the spectral gap estimate is as good as the decay rate in d_2 or W_2^2.*

Remark 7.35 (Sufficient Conditions for Regularity)**.** *At a first view, the condition $\||k|^\delta \hat{f}_0(k)\|_{L^\infty(\mathbb{R}^3)} < \infty$ seems quite difficult to verify. To show that this is not the case, we make use of an analogous argument used in* [79] *in connection with the central limit theorem. The argument of* [79] *was indeed related to show that this condition hold with $\delta = 1$ whenever the Fisher information of f_0 is bounded* [16]*.*

Let $g = \sqrt{f}$. Then, the Fourier transform of f can be written as the convolution of g with itself, $\hat{f}(k) = (\hat{g} * \hat{g})(k)$. Now, the boundedness of the Fisher information of f,

$$\int_{\mathbb{R}^3} \left|\nabla \sqrt{f(v)}\right|^2 dv,$$

coupled with the boundedness of mass,

$$\int_{\mathbb{R}^3} \left(\sqrt{f(v)}\right)^2 dv,$$

implies that $\sqrt{f} \in H^1(\mathbb{R}^3)$, and

$$\int_{\mathbb{R}^3} |k|^2 |\hat{g}(k)|^2 \, dk < \infty. \tag{7.35}$$

Using (7.35) *we obtain*

$$\begin{aligned} |k||\hat{f}(k)| = |k| \left| \int_{\mathbb{R}^3} \hat{g}(k-k_*)\hat{g}(k_*) \, dk_* \right| \leq \\ \leq \int_{\mathbb{R}^3} (|k-k_*| + |k_*|) \, |\hat{g}(k-k_*)||\hat{g}(k*)| \, dk_* \\ \leq 2 \left(\int_{\mathbb{R}^3} |k|^2 |\hat{g}(k)|^2 \, dk \right)^{1/2} \left(\int_{\mathbb{R}^3} |\hat{g}(k)|^2 \, dk \right)^{1/2}. \end{aligned}$$

Hence, the boundedness of Fisher information of a density function f_0 is a sufficient easily-to-check condition to ensure the boundedness of $|k||\hat{f}_0(k)|$.

Proceeding in the same way, since $|a+b|^\delta \le |a|^\delta + |b|^\delta$ *for all* $\delta \le 1$*, one obtains*

$$|k|^\delta |\hat{f}(k)| \le 2 \left(\int_{\mathbb{R}^3} |k|^{2\delta} |\hat{g}(k)|^2 \, dk \right)^{1/2} \left(\int_{\mathbb{R}^3} |\hat{g}(k)|^2 \, dk \right)^{1/2}. \tag{7.36}$$

Finally, for any given $\delta \le 1$*, a sufficient condition for the boundedness of* $|k|^\delta |\hat{f}_0(k)|$ *is given by* $\sqrt{f}_0 \in H^\delta(\mathbb{R}^3)$*.*

7.3. **Homogeneous Cooling States for the IMM.** In this last subsection, we treat the free cooling of the Inelastic Maxwell Model (3.15):

$$\frac{\partial f}{\partial t} = \tilde{Q}_e(f, f).$$

As proved in Corollary (3.3), all probability density solutions with zero mean velocity to (3.15) converges to the mono-kinetic distribution δ_0 as $t \to \infty$ due to Haff's law of cooling:

$$\theta'(t) = -\frac{1-e^2}{4} B \, \theta(t)^{3/2}.$$

We would like to make more precise this cooling behavior for large times by giving a typical cooling profile in the form of self-similar solutions:

$$f_{hc}(t, v) = \rho \, \theta_{hc}^{-\frac{3}{2}}(t) \, g_\infty((v-u) \, \theta_{hc}^{-\frac{1}{2}}(t)) \tag{7.37}$$

where $\theta_{hc}(t)$ the temperature of the solution $f_{hc}(t, v)$ itself that will follow the dissipation of energy (3.12) and g_∞ the searched cooling profile. These self-similar solutions are called homogeneous cooling states. The objective of this subsection is summarized the state of the art in this question for the IMM showing some of their properties.

This question was considered by M. Ernst and R. Brito [57, 58] where the authors conjectured the existence of a self-similar solution g_∞ attracting a large set of initial data (first part of the EB conjecture) and moreover, these self-similar solutions should have polynomial decay at ∞ (second part of the EB conjecture). The existence of a self-similar solution with a precise number of moments bounded g_∞, second part of the EB conjecture, was shown in [22] by A.V. Bobylev and C. Cercignani. The techniques used are based on spectral properties of the linearized operator in Fourier space and they have been reviewed, improved and applied to generalized IMM models in [23, 24].

Concerning the first part of the EB conjecture, the proof of convergence towards g_∞ was obtained in [25] for initial data which has moments of order $2+\alpha$ bounded with $\alpha > 0$. Later, in [13], these results were readdressed in terms of probability metrics improving certain aspects of the convergence properties. Finally, in [29] the convergence without rate for initial data

with only finite kinetic energy has been obtained. In these notes, we only pretend to discuss the properties of the contraction of probability metrics for this question and the consequences that can be concluded from this fact referring for many other results to the quoted literature above.

7.3.1. *Existence, Uniqueness and Stability of Steady States.* Given any solution $f(t)$ to the Boltzmann equation (3.15), after time scaling defined by

$$\tau = \frac{B}{E} \int_0^t \sqrt{\theta(f(w))}\, dw$$

with $E = \dfrac{8}{1-e^2}$ as above, we get a function denoted again $f(\tau)$ for simplicity, solution to

$$\frac{\partial f}{\partial \tau} = E\,[\tilde{Q}_e^+(f,f) - f]. \tag{7.38}$$

Proceeding as in Theorems 7.1 and 7.19, we obtain:

Theorem 7.36 (Strict Contraction in W_2)**.** [29] *If f_1 and f_2 are two solutions to* (3.15) *with respective initial data f_1^0 and f_2^0 in $\mathcal{P}_2(\mathbb{R}^3)$ and zero mean velocity, then*

$$W_2^2(f_1(\tau), f_2(\tau)) \le \mathrm{e}^{-2\tau}\, W_2^2(f_1^0, f_2^0)$$

for all $\tau \ge 0$.

Proof.- Duhamel's formula for (7.38) reads as

$$f_i(\tau) = \mathrm{e}^{-E\tau}\, f_i^0 + E \int_0^\tau \mathrm{e}^{-E(\tau-s)}\, \tilde{Q}_e^+(f_i(s), f_i(s))\, ds, \qquad i = 1, 2.$$

As before, the convexity of the squared Wasserstein distance in Proposition 2.1 and the contraction of the gain operator in Theorem 6.1 imply

$$\begin{aligned}
&W_2^2(f_1(\tau), f_2(\tau)) \\
&\le \mathrm{e}^{-E\tau} W_2^2(f_1^0, f_2^0) + E \int_0^\tau \mathrm{e}^{-E(\tau-s)} W_2^2\big(\tilde{Q}_e^+(f_1(s), f_1(s)), \tilde{Q}_e^+(f_2(s), f_2(s))\big) ds \\
&\le \mathrm{e}^{-E\tau} W_2^2(f_1^0, f_2^0) + E \frac{3+e^2}{4} \int_0^\tau \mathrm{e}^{-E(\tau-s)} W_2^2(f_1(s), f_2(s)) ds.
\end{aligned}$$

Therefore, the function $y(\tau) = \mathrm{e}^{E\tau}\, W_2^2(f_1(\tau), f_2(\tau))$ satisfies the inequality

$$y(\tau) \le y(0) + E\, \frac{3+e^2}{4} \int_0^\tau y(s)\, ds$$

and thus, $y(\tau) \le y(0)\, \mathrm{e}^{\gamma E \tau}$ by Gronwall's lemma with $\gamma = (3+e^2)/4$. This concludes the argument since $(1-\gamma)\, E = 2$. □

Remark 7.37 (Optimality of the contraction in W_2)**.**

(1) *Without further assumptions on the initial data, this result is optimal in the following sense: If $f_2^0 = \delta_0$, then the contraction estimate is actually an equality for all τ by the temperature equation.*
(2) *In terms of the original time variable t in* (3.15)*, if f_1^0 and f_2^0 are two initial data with the same initial temperature θ_0 and zero mean velocity, then the temperatures of the corresponding solutions f_1 and f_2 to* (3.15) *follow the law* (3.16)*, and hence are both equal to*

$$\theta(t) = \left(\theta_0^{-1/2} + \frac{1-e^2}{8} Bt\right)^{-2}.$$

Then, the contraction estimate reads as

$$W_2^2(f_1(t), f_2(t)) \leq \frac{\theta(t)}{\theta_0} W_2^2(f_1^0, f_2^0) \tag{7.39}$$

that gives the typical decay towards δ_0 of all solutions.

Now, let us look for the behavior of solutions with initial zero mean velocity in a different scaling. Let us rescale solutions with their own temperature, that is, let us define $g(\tau, v)$ by

$$g(\tau, v) = \theta^{3/2}(f(\tau))\, f(\tau, \theta^{1/2}(f(\tau))\, v). \tag{7.40}$$

It is easy to check that they give rise to solutions of equation

$$\frac{\partial g}{\partial \tau} + \nabla \cdot (g\, v) = E\, [\tilde{Q}_e^+(g, g) - g] \tag{7.41}$$

with unit temperature and zero mean velocity. The contraction property in (7.36) reads as

$$W_2(g_1(\tau), g_2(\tau)) \leq W_2(g_1^0, g_2^0) \tag{7.42}$$

in these rescaled variables. This non-strict contraction of the rescaled equation (7.41) with respect to W_2 does not give any information about the existence of typical cooling profiles of the system. Let us remark that the strict contraction (7.36) does not help to find nontrivial stationary states since for equation (7.38) the only stationary state is for zero temperature, i.e., the delta dirac δ_0 while the self-similar variables allows the unit temperature as stationary value of the dissipation of energy for (7.41).

Let us look for the contraction properties with respect to Fourier-based distances with larger exponent. Actually, the situation is quite similar to the one dimensional models studied in Section 5. The following result follows similar arguments to Propositions 7.8 and 7.23.

Theorem 7.38 (Strict contraction for (7.41) in $d_{2+\alpha}$)**.** [13] *Let g_1 and g_2 be two solutions to* (7.41) *corresponding to initial values g_1^0, g_2^0 with unit*

mass, zero mean velocity and unit pressure tensor, i.e.,

$$\int_{\mathbb{R}^3} v_i v_j \, g_1^0(v)\, dv = \int_{\mathbb{R}^3} v_i v_j \, g_2^0(v)\, dv = \delta_{ij}. \tag{7.43}$$

Then $d_{2+\alpha}(g_1^0, g_2^0) < \infty$, $0 < \alpha < 1$, *and there exists an explicit constant* $C(\alpha, e) > 0$, $C(\alpha, e) \searrow 0$ *as* $\alpha \to 0$, *such that*

$$d_{2+\alpha}(g_1(\tau), g_2(\tau)) \le d_{2+\alpha}(g_1^0, g_2^0)\, \mathrm{e}^{-C(\alpha,e)\tau}, \tag{7.44}$$

for any $\tau \ge 0$.

Proof.- The main steps of the proof are:

- **Step 1.-** Using Lemma 7.6 we deduce that all moments up to order 2 are equal between g_1 and g_2 since they are equal initially. This implies that the distance $d_{2+\alpha}$, $0 < \alpha < 1$, between g_1 and g_2 is well-defined. Now, the Fourier expression of equation (7.41) is given by

$$\frac{\partial \hat{g}}{\partial \tau} - \Big(k \cdot \nabla_k\Big)\hat{g} = E\left[\frac{1}{4\pi}\int_{S^2} \hat{g}(k_+)\hat{g}(k_-)dn - \hat{g}\right] = E\Big[\widehat{\tilde{Q}_e^+(g,g)} - \hat{g}\Big]$$

whose solution can be written in terms of the characteristics as

$$\hat{g}\Big(\tau, k\,\mathrm{e}^{-\tau}\Big) = \mathrm{e}^{-E\tau}\hat{g}(0,k) + E\int_0^\tau \mathrm{e}^{-E(\tau-s)}\widehat{\tilde{Q}_e^+(g,g)}\Big(s, k\,\mathrm{e}^{-s}\Big)\, ds\,. \tag{7.45}$$

- **Step 2.-** Taking the expressions of two solutions $\hat{g}_1(\tau)$ and $\hat{g}_2(\tau)$ in (7.45), subtracting them and dividing by $|k|^{2+\alpha}$ with $k \in \mathbb{R}^3_o$, we get

$$\mathrm{e}^{E\tau}\frac{(\hat{g}_1 - \hat{g}_2)(\tau, k\,\mathrm{e}^{-\tau})}{|k|^{2+\alpha}} = \mathrm{e}^{(E-(2+\alpha))\tau}\frac{(\hat{g}_1 - \hat{g}_2)(\tau, k\,\mathrm{e}^{-\tau})}{|k\,\mathrm{e}^{-\tau}|^{2+\alpha}} =$$

$$\frac{\hat{g}_1(0,k) - \hat{g}_2(0,k)}{|k|^{2+\alpha}} + E\int_0^\tau \mathrm{e}^{(E-(2+\alpha))s}\frac{\Big(\widehat{\tilde{Q}_e^+(g_1,g_1)} - \widehat{\tilde{Q}_e^+(g_2,g_2)}\Big)\Big(s, k\mathrm{e}^{-s}\Big)}{|k\mathrm{e}^{-s}|^{2+\alpha}}\, ds.$$

- **Step 3.-** Using Theorem 6.4 and taking the supremum in $k \in \mathbb{R}^3_o$, we obtain

$$\begin{aligned}\mathrm{e}^{(E-(2+\alpha))\tau} d_{2+\alpha}(\hat{g}_1, \hat{g}_2)(\tau) \quad &\le \quad d_{2+\alpha}\big(\hat{g}_1(0), \hat{g}_2(0)\big) \\ &\quad + A(\alpha,e)E\int_0^\tau \mathrm{e}^{(E-(2+\alpha))s} d_{2+\alpha}(\hat{g}_1, \hat{g}_2)(s) ds.\end{aligned}$$

Let us set $w(\tau) = \mathrm{e}^{(E-(2+\alpha))\tau} d_{2+\alpha}(\hat{g}_1, \hat{g}_2)(\tau)$. Then

$$w(\tau) \le w(0) + A(\alpha,e)E\int_0^\tau w(s)\, ds,$$

which, by Gronwall inequality implies $w(\tau) \le w(0)\, \mathrm{e}^{A(\alpha,e)E\tau}$.

As a conclusion, we deduce

$$d_{2+\alpha}(\hat{g}_1, \hat{g}_2) \leq d_{2+\alpha}\big(\hat{g}_1(0), \hat{g}_2(0)\big)\, \mathrm{e}^{-C(\alpha,e)\tau},$$

with

$$C(\alpha, e) = E(1 - A(\alpha, e)) - (2 + \alpha) = E(1 - G(\alpha, e)) \tag{7.46}$$

where $G(\alpha, e) = A(\alpha, e) + \frac{1-e^2}{8}(2 + \alpha)$. One finally has to check that $C(\alpha, e) > 0$, a detailed analysis of this fact can be seen in [13]. □

As consequence, we will show that equation (7.41) has a unique steady state g_∞ which belongs to the set of probability measures with unit mass, zero mean velocity and unit pressure tensor. With this objective, we need to find a suitable invariant set of the flow with respect to the distance $d_{2+\alpha}$. According to Proposition 2.7 and Remark 2.8, we need to control uniformly in time a larger moment than 2.

Proposition 7.39 (Uniform Control of 4th moment)**.** [29] *If g^0 is a Borel probability measure on $\mathbb{R}^3$ such that*

$$\int_{\mathbb{R}^3} |v|^4\, g^0(v)\, dv < \infty,$$

then the solution g to (7.41) *with initial datum g^0 verifies*

$$\sup_{\tau \geq 0} \int_{\mathbb{R}^3} |v|^4\, g(\tau, v)\, dv < \infty.$$

Proof.- Without loss of generality we can assume that g^0, and hence $g(\tau)$ for all $\tau \geq 0$, has zero mean velocity. We let

$$m_4(\tau) = \int_{\mathbb{R}^3} |v|^4\, g(\tau, v)\, dv$$

denote the fourth order moment of $g(\tau)$. Then, using the weak formulation of the inelastic Boltzmann equation, we have:

$$\frac{dm_4(\tau)}{d\tau} = \int_{\mathbb{R}^3} \nabla(|v|^4) \cdot v\, g(\tau, v)\, dv + E \int_{\mathbb{R}^3} |v|^4\, \tilde{Q}_e^+(g(\tau), g(\tau))(v)\, dv. \tag{7.47}$$

While the first term in the right hand side is simply $4\, m_4(\tau)$, the second term is computed by

Lemma 7.40 (4th Moment of the Collision Operator)**.** *There exist some constants μ_1 and μ_2, depending only on e, such that*

$$\int_{\mathbb{R}^3} |v|^4\, \tilde{Q}_e^+(g, g)(v)\, dv = -\lambda \int_{\mathbb{R}^3} |v|^4\, g(v)\, dv + \mu_1 \Big(\int_{\mathbb{R}^3} |v|^2\, g(v)\, dv \Big)^2 + \mu_2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (v \cdot w)^2\, g(v)\, g(w)\, dv\, dw$$

for any probability measure g on $\mathbb{R}^3$ with finite moment of order 4 and zero mean velocity, where

$$\lambda = \frac{1}{3}(1 + 4\,\epsilon - 7\,\epsilon^2 + 4\,\epsilon^3 - 2\,\epsilon^4) \qquad \textit{and} \qquad \epsilon = \frac{1-e}{2}\cdot$$

With this lemma in hand, (7.47) reads

$$\frac{dm_4(\tau)}{d\tau} = \Big(4 - E\,\lambda\Big) m_4(\tau) + m(\tau) \tag{7.48}$$

where $m(\tau)$ is a combination of second order moments, which are bounded in time since the kinetic energy is preserved by equation (7.41). Moreover one can check from the expression of E and λ in terms of $\epsilon = (1-e)/2$ that

$$4 - E\,\lambda = \frac{2}{3\,\epsilon(1-\epsilon)}\,[-1 + 2\,\epsilon + \epsilon^2 - 4\,\epsilon^3 + 2\,\epsilon^4]$$

which is negative for any $0 < \epsilon < 1/2$, that is, for any $0 < e < 1$. By Gronwall's lemma this ensures that $m_4(\tau)$ is bounded uniformly in time if initially finite, and concludes the argument to Proposition 7.39. □

Now, we are in position to show the existence of stationary states by the dynamical proof approach.

Corollary 7.41 (Existence, Uniqueness & Stability of Stationary States)**.** *Equation* (7.41) *has a unique steady state g_∞ in $\mathcal{P}_2(\mathbb{R}^3)$ with zero mean velocity and unit pressure tensor. Moreover, given g any solution to* (7.41) *for the initial data $g_0 \in \mathcal{P}_2(\mathbb{R}^3)$ with zero mean velocity and unit pressure tensor, then*

$$d_{2+\alpha}(g(\tau), g_\infty) \le d_{2+\alpha}(g_0, g_\infty)\,\mathrm{e}^{-C(\alpha,e)\tau},$$

for all $\tau \ge 0$, $0 < \alpha < 1$.

Proof.- Let us define the complete metric space $\tilde{\mathcal{M}}$ of measures $\mu \in \mathcal{P}_{2+\alpha}(\mathbb{R}^3)$, $0 < \alpha < 1$, such that

$$\int_{\mathbb{R}^3} v_i\, v_j\, df(v) = \delta_{ij}\,, \int_{\mathbb{R}^3} v\, df(v) = 0 \ \text{ and } \int_{\mathbb{R}^3} |v|^{2+\alpha}\, df(v) \le M,$$

endowed with $d_{2+\alpha}$, see Proposition 2.7, with M chosen below. Given the flow map of (4.12), i.e.,

$$\mathcal{T}(\tau) : (\tilde{\mathcal{M}}, d_{2+\alpha}) \longrightarrow (\mathcal{P}_2(\mathbb{R}^3), d_{2+\alpha}),$$

for any time $t > 0$, given by $\mathcal{T}(t)(g_0) = g(t)$ with $g(t)$ the unique solution at time t of (7.41) with initial datum $g_0 \in \tilde{\mathcal{M}}$. Then, $\mathcal{T}(t)$ is a continuous semigroup from $\tilde{\mathcal{M}}$ onto itself due to the adaptation of Theorem 3.5 to (7.41), the conservation of the moments up to order 2 and the uniform bound on the 4th moment and thus, on the moment of order $2 + \alpha$ shown

in Proposition 7.39 that chooses M. Let us make precise this last point, coming back to the evolution of the fourth moment in (7.48) and taking into account that we deal with distributions with unit pressure tensor, we deduce that its evolution is given by

$$\frac{dm_4(\tau)}{d\tau} = \Big(4 - E\,\lambda\Big) m_4(\tau) + 9(\mu_1 + \mu_2)$$

with the notation of Proposition 7.39. Thus, by choosing

$$M_4 = \frac{9(\mu_1 + \mu_2)}{4 - E\lambda}$$

the initial data with 4th moment less than M_4 gives rise to solutions with 4th moment less than M_4 for all times. Thus, by the inequality

$$\int_{\mathbb{R}^3} |v|^{2+\alpha}\, df(v) \le \int_{|v|\le 1} df(v) + \int_{|v|\ge 1} |v|^4\, df(v) \le 1 + M_4 := M$$

we find the choice of M that makes $\tilde{\mathcal{M}}$ invariant through $\mathcal{T}(t)$.

Corollary 7.38 proves that $\mathcal{T}(t)$ is a uniform contraction from the complete metric space $(\tilde{\mathcal{M}}, d_{2+\alpha})$ into itself with contraction constant

$$L(\tau) = \mathrm{e}^{-\mathrm{C}(\alpha,\mathrm{e})\tau} < 1.$$

Therefore, Lemma 7.3 ensures the existence and uniqueness of a unique steady state in $(\tilde{\mathcal{M}}, d_{2+\alpha})$. The last assertion is a simple consequence of Corollary 7.38 by taking one of the solutions the stationary state we just obtained. □

7.3.2. *Moment Behavior.* Let us review the properties of moments for solutions of the scaled equation. Actually, the first part of the Ernst-Brito conjecture concerns precisely the number of moments that the unique stationary probability solution g_∞ with zero mean velocity and unit pressure tensor of (7.41) has. The following result has been proven in [22] and generalized to other IMMs in [23, 24].

Theorem 7.42 (Thick Tails of HCS)**.** [22, 23, 24] *The unique stationary solution g_∞ with zero mean velocity and unit pressure tensor of* (7.41) *has moments*

$$\int_{\mathbb{R}^3} |v|^{2+\alpha} g_\infty(v)\, dv < \infty,$$

with $\alpha > 0$, if and only if

$$2 + \alpha < 2\, r_{EB}(e) \quad \Longleftrightarrow \quad C(\alpha, e) > 0 \quad \Longleftrightarrow \quad G(\alpha, e) < 1.$$

The function $r_{EB}(e)$ is characterized as the unique solution r to the equation

$$\frac{1-e^2}{4}\,r = 1 - A(2r-2,e) = 1 - \frac{1}{1+r}\left[\left(\frac{1+e}{2}\right)^{2r} + \frac{1-\left(\frac{1-e}{2}\right)^{2r+2}}{1-\left(\frac{1-e}{2}\right)^{2}}\right]. \tag{7.49}$$

This equation obtained in [58, Equation 3.13] for capturing the high energy tails of the distribution function was also given in [9, 10].

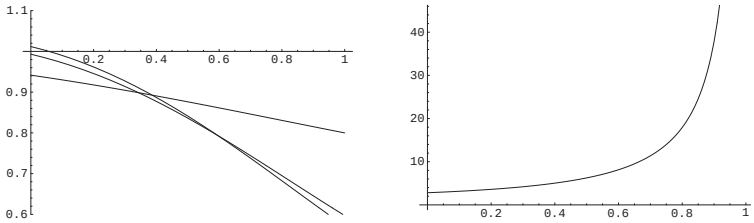

FIGURE 5. Left Figure: The functions $G(\alpha,e)$ for values $\alpha=1$, $\alpha=2.7$ and $\alpha=3$. Right Figure: The largest zero α_e of $C(\alpha,e)=0 \iff G(\alpha,e)=1$ as a function of e, for instance $\alpha_0 = 2.81307$, and thus, $r_{EB}(0) = 2.40653$ by Newton-Raphson method.

In Figure 5, we show the largest root α_e of $C(\alpha,e)=0$ in terms of e, which corresponds to compute $r_{EB}(e) = 1+\frac{\alpha_e}{2}$. In fact, taking into account [22, Theorem 7.2] and [23, 26], we obtain the following corollary by following the same procedure as in Proposition 7.38.

Corollary 7.43 (Optimality of the contraction result)**.** [13] *The flow map for equation* (7.41) *is a strict contraction for the distance* $d_{2+\alpha}$ *if and only if*

$$G(\alpha,e) < 1 \quad\iff\quad C(\alpha,e) > 0 \quad\iff\quad 2+\alpha < 2\,r_{EB}(e),$$

or equivalently if and only if the moments of order $2+\alpha$ *of the homogeneous cooling state* g_∞ *are bounded.*

Now, let us proceed to study some more properties of the moments of solutions of the Cauchy problem. The first observation is that moments are propagated if initially bounded. Similar arguments to those done in Proposition 7.7 allow to show:

Lemma 7.44 (Time-dependent moment estimates)**.** *Let* $g(\tau,v)$ *be the solution to equation* (7.41)*, where the initial distribution* $g_0(v)$ *is such that*

$m_{2r}(g_0) < +\infty$ for some $r > 1$. Then, $m_{2r}(\tau)$ satisfies the following differential inequality

$$\frac{d}{d\tau} m_{2r}(\tau) \leq -E\left[\frac{1-e^{2r}}{4}\Big(m_{2r}(\tau) + m_{2(r-1)}(\tau)m_2(\tau)\Big)\right.$$
$$\left. -\frac{1}{2}\sum_{l=1}^{r-1}\begin{pmatrix} r \\ l \end{pmatrix} m_{2(r-l)}(\tau)m_{2l}(\tau)\right] + 2rm_{2r}(\tau). \tag{7.50}$$

Consequently, $m_{2r}(\tau) < \infty$, for all $\tau > 0$, and bounded in $[0,T]$, for all $T > 0$.

With the information above, we can now describe the asymptotic behavior of moments for the Cauchy problem.

Proposition 7.45 (Asymptotic Behavior of Moments)**.** *Given any solution $g(\tau, v)$ of equation (7.41) with zero mean velocity and unit pressure tensor, where the initial distribution $g_0(v)$ is such that $m_{2r}(g_0) < +\infty$ for some $r > 1$. Then,*

i) *If $r < r_{EB}(e)$, $r \in \mathbb{N}$ and all moments of g_0 are equal to those of g_∞ up to order $2r-1$, then*

$$\sup_{\tau \geq 0} \int_{\mathbb{R}^3} |v|^{2r}\, g(\tau, v)\, dv < \infty.$$

ii) *If $r \geq r_{EB}(e)$, then*

$$\lim_{\tau \to \infty} \int_{\mathbb{R}^3} |v|^{2r}\, g(\tau, v)\, dv = \infty.$$

Proof.- Due to Lemma 7.44, we know solutions propagate the finiteness of moments if initially are bounded. Now, let us start with the case $r < r_{EB}(e)$. Using Corollary 7.43, we know that the distance d_{2r} is contractive and moreover, $m_{2r}(g_\infty) < +\infty$. Now, the control of moments in terms of the distance d_{2r} in Proposition 2.9 finishes the proof, since

$$\left|\int_{\mathbb{R}^N} v^\beta\, dg(\tau, v) - \int_{\mathbb{R}^N} v^\beta\, dg_\infty(v)\right| \leq C\, d_{2r}(g(\tau), g_\infty) \leq C\, d_{2r}(g_0, g_\infty) < \infty,$$

for all multi-indices β with $|\beta| = 2r$ and using the assumption on moments of order less than $2r-1$ on the initial data.

Concerning the case $r \geq r_{EB}(e)$. Let us proceed by contradiction. Assume that there exists a sequence $\{\tau_n\} \nearrow \infty$ such that

$$\int_{\mathbb{R}^3} |v|^{2r}\, g(\tau_n, v)\, dv \leq M < \infty.$$

Now, Theorem 7.38 implies that

$$d_{2+\alpha}(g(\tau_n), g_\infty) \leq d_{2+\alpha}(g_0, g_\infty)\, \mathrm{e}^{-C(\alpha,e)\tau_n} \to 0 \qquad \text{as} \qquad n \to \infty$$

for $0 < \alpha < 1$ such that $2 + \alpha < 2r$. Using the properties of $d_{2+\alpha}$ in Proposition 2.9, we deduce that $g(\tau_n) \rightharpoonup g_\infty$ weakly-* as measures. It is now a simple consequence of the uniform bound $m_{2r}(g(\tau_n)) \leq M$ that

$$\int_{\mathbb{R}^3} |v|^{2r}\, g_\infty\, dv \leq M < \infty$$

which is in contradiction with $r \geq r_{EB}(e)$ due to Corollary 7.43. □

Remark 7.46 (Open Problem for Moments)**.** *It is clear that the first part of the last result is a bit deceptive. It is intuitive based on the computation of the 4th moment in Proposition* 7.39 *to expect that*

$$\sup_{\tau \geq 0} \int_{\mathbb{R}^3} |v|^{2r}\, g(\tau, v)\, dv < \infty,$$

whenever $r < r_{EB}(e)$ *and* $m_{2r}(g_0) < \infty$ *independently of being* r *natural and how many moments the initial data* g_0 *has in common with* g_∞*. Concerning the second part, we will improve it below by allowing any initial value of the kinetic energy. On the other hand, we are not able to deduce an exponential divergence of these moments, expected from the explicit computation of the sixth moment in the spirit of Lemma* 7.40*, left to the reader as an exercise. An explicit recursive formula for non isotropic moments will certainly answer this question.*

7.3.3. *Improved Convergence.* In this subsection, we plan to get rid of the assumption of equal second moments in Theorem 7.38, in order to prove the exponential convergence of each solution $f(\tau, v)$ of equation (3.15) corresponding to a general initial datum, towards the corresponding similarity solution $f_{hc}(\tau)$ in d_2. Let us remark that neither Theorem 7.38 nor the results contained in [25] give any decay rate in the case of d_2. In fact, equation (7.41) is a non-strict contraction for d_2, i.e.,

$$d_2(g_1(\tau), g_2(\tau)) \leq d_2(g_1^0, g_2^0) \tag{7.51}$$

for any $\tau \geq 0$ and any g_1, g_2 solutions to (7.41) corresponding to initial data with unit mass, zero mean velocity and second moment bounded. Defining for $i \neq j$ the quantity

$$p_{ij}(\tau) = \int_{\mathbb{R}^3} v_i v_j f(v, \tau)\, dv,$$

its evolution is governed by the equation

$$\frac{dp_{ij}}{d\tau} = -\frac{(1+e)(3-e)}{8}\, E\, p_{ij} \tag{7.52}$$

due to Lemma 7.6. If $\hat{\Phi}(k,\tau)$ is defined as

$$\hat{\Phi}(k,\tau) = \begin{cases} -\dfrac{1}{2}\displaystyle\sum_{i\neq j} p_{ij}(\tau)k_ik_j & if \quad |k| \leq 1 \\ 0 & if \quad |k| > 1 \end{cases}, \tag{7.53}$$

we will show that the contraction in $d_{2+\alpha}$ of the non-isotropic part $\hat{f}(\tau) - \hat{\Phi}(\tau)$ together with the decay of the pressure tensor of the solution towards the pressure tensor of the HCS f_{hc} is enough to ensure the convergence of the solution towards the HCS in d_2. In the proof we shall resort to the contraction in $d_{2+\alpha}$, $\alpha > 0$, and thus, we need an additional assumption on the initial data, i.e., to have the corresponding moment of order $2+\alpha$ finite. For the proof of this result, we refer to [25, Section 5] or [13, Appendix], although it is a good exercise based on the proofs above for Proposition 7.8, 7.23 and 7.38.

In the following results, we will denote by $d_s(\hat{f},\hat{g})$ the same quantity as in the definition of the distance d_s but applied to the Fourier transform of two probability distribution for the sake of clarity of exposition.

Theorem 7.47. [General Decay Rate towards self-similarity [13] *Let $f(\tau,v)$ be the solution of the time-scaled inelastic Maxwell equation* (7.38) *corresponding to the initial datum f_0 with unit mass, zero mean velocity such that $d_{2+\alpha}(\hat{f}(0) - \hat{\Phi}(0), \hat{f}_{hc}(0)) < \infty$, where f_{hc} denotes the corresponding self–similar solution. Then there exists $C_1 > 0$ such that*

$$d_{2+\alpha}(\hat{f}(\tau)-\hat{\Phi}(\tau),\hat{f}_{hc}(\tau)) \leq \Big[2d_{2+\alpha}(\hat{f}(0)-\hat{\Phi}(0),\hat{f}_{hc}(0))+C_1\Big]\,\mathrm{e}^{-(1-A(\alpha,e))E\tau}$$

for any $0 < \alpha < 1$.

Now, we can improve to get the exponential decay in d_2 without the assumption of unit pressure tensor.

Theorem 7.48 (Decay Rate towards self-similarity)**.** [13] *Let $f(\tau,k)$ be the solution of the time-scaled inelastic Maxwell equation* (7.38) *corresponding to the initial datum f_0 with unit mass, zero mean velocity and moment of order $2+\alpha$, $0 < \alpha < 1$, bounded. Then there exist explicit constants C_1, $C_2 > 0$, depending on second moments of the initial data, such that*

$$\begin{aligned} d_2(f(\tau),f_{hc}(\tau)) \leq C_{2\,,\,2+\alpha} \Big\{ \Big[2\, d_{2+\alpha}(\hat{f}(0) - \hat{\Phi}(0), \hat{f}_{hc}(0)) + C_1\Big] \\ \times \exp\Big\{ -(1-A(\alpha,e))E\,\tau\Big\}\Big\}^{\frac{2}{2+\alpha}} + C_2 \exp\Big\{ -\frac{3-e}{1-e}\,\tau\Big\}. \end{aligned} \tag{7.54}$$

Proof.- The distance $d_2(f(\tau), f_{hc}(\tau))$ can be split as

$$d_2(f(\tau), f_{hc}(\tau)) \leq \sup_{k\in\mathbb{R}^3_o} \frac{|\hat{f}(k,\tau) - \hat{\Phi}(k,\tau) - \hat{f}_{hc}(k,\tau)|}{|k|^2} + \sup_{k\in\mathbb{R}^3_o} \frac{|\hat{\Phi}(k,\tau)|}{|k|^2} =$$

$$= d_2(\hat{f}(\tau) - \hat{\Phi}(\tau), \hat{f}_{hc}(\tau)) + \sup_{|k|\leq 1} \frac{|\hat{\Phi}(k,\tau)|}{|k|^2}\,. \tag{7.55}$$

Using the interpolation of d_s metrics in Proposition 2.9, we get

$$d_2(\hat{f}(\tau) - \hat{\Phi}(\tau), \hat{f}_{hc}(\tau)) \leq C_{2\,,\,2+\alpha} \Big[d_{2+\alpha}(\hat{f}(\tau) - \hat{\Phi}(\tau), \hat{f}_{hc}(\tau)) \Big]^{\frac{2}{2+\alpha}},$$

hence from Theorem 7.47 we get the first term in the right-hand side of (7.54). Owing to the definition of $\hat{\Phi}(k,\tau)$, the last term of (7.55) can be estimated by means of the law (7.52) which describes the evolution of the pressure tensor:

$$\begin{aligned} \sup_{|k|\leq 1} \frac{|\hat{\Phi}(k,\tau)|}{|k|^2} &\leq \frac{1}{2}\left(\max_{i\neq j} |p_{ij}(\tau)|\right) \sup_{|k|\leq 1} \left\{ \frac{1}{|k|^2} \sum_{i\neq j} |k_i\, k_j| \right\} \\ &\leq \left(\max_{i\neq j} |p_{ij}(0)|\right) \exp\left\{ -\frac{(1+e)(3-e)}{8}\, E\,\tau \right\} \end{aligned}$$

and this concludes the proof. □

Remark 7.49 (Exponential decay result in scaled variables)**.** *Given g a solution to* (7.41) *corresponding to the initial value g_0 with unit mass, zero mean velocity and moment of order $2+\alpha$, $0<\alpha<1$, bounded, then*

$$\begin{aligned} d_2(g(\tau), g_\infty) \leq \frac{C_{2\,,\,2+\alpha}}{\theta_0} \Big[2\, d_{2+\alpha}(\hat{g}(0) - \hat{\Phi}(0), \hat{g}_\infty) + C_1 \Big]^{2/(2+\alpha)} \\ \times \exp\Big\{ -\frac{2}{2+\alpha}\, C(\alpha,e)\,\tau \Big\} + \frac{C_2}{\theta_0} \exp\Big\{ -\frac{1+e}{1-e}\,\tau \Big\}. \end{aligned} \tag{7.56}$$

This is a direct consequence of $\hat{g}(\tau,k) = \hat{f}(k\theta^{\tau,-\frac{1}{2}}(\tau))$, the scaling property of d_2, in Proposition 2.9*, and the evolution of the temperature for* (7.38)*, $\theta(\tau) = \theta_0\, \mathrm{e}^{-2\,\tau}$.*

Remark 7.50 (Algebraic decay result in original variables)**.** *The evolution equation* (3.16) *yields $\theta(t) = (\theta_0^{-1/2} + \frac{1-e^2}{8} Bt)^{-2}$,. Hence, the time scaling in* (7.38) *is nothing but $\tau = \log[1 + (B/(E\theta_0^{-1/2}))\, t]$. Therefore, to any exponential decay in the variable τ, there corresponds an algebraic decay in t.*

Remark 7.51 (Divergence of Moments for general initial data)**.** *Based on the previous result and Remarks, we can improve over the hypotheses of the initial data to deduce the divergence of moments as in the second part of Proposition* 7.45*. More precisely, given any solution $g(\tau, v)$ of equation* (7.41) *with zero mean velocity and bounded kinetic energy where the initial distribution $g_0(v)$ is such that $m_{2r}(g_0) < +\infty$ for some $r > 1$ with $r \geq r_{EB}(e)$, then*

$$\lim_{\tau\to\infty} \int_{\mathbb{R}^3} |v|^{2r}\, g(\tau, v)\, dv = \infty.$$

Finally, let us get rid of the assumption of moments of order $2+\alpha$ bounded on the initial data, as a payoff we will obtain only a convergence without rate.

Theorem 7.52 (Convergence without rate in W_2)**.** [29] *Let g_1^0 and g_2^0 be two probability measures on $\mathbb{R}^3$ with zero mean velocity and unit kinetic energy, and let $g_1(\tau)$ and $g_2(\tau)$ be the solutions to* (7.41) *with respective initial data g_1^0 and g_2^0. Then the map $\tau \mapsto W_2(g_1(\tau), g_2(\tau))$ is non-increasing and tends to* 0 *as τ goes to infinity.*

By taking as one solution, in this theorem, the homogeneous cooling state in scaled variables, i.e., the stationary solution g_∞ of (7.41), we improve over the first part of the Ernst-Brito conjecture shown in [25, 13].

In terms of the original variables, the scaling properties of W_2 given in Proposition 2.1 and the convergence result

$$\lim_{\tau\to\infty} W_2(g(\tau), g_\infty) = 0$$

have the following direct consequence, which improves over the decay towards the Dirac mass estimate given in Corollary 3.3 and (7.39).

Corollary 7.53 (Intermediate Asymptotics)**.** *Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ with zero mean velocity and let $f(t)$ be the solution to* (3.15) *with initial datum f_0, then*

$$\lim_{t\to\infty} \theta(f(t))^{-1/2} W_2(f(t), f_{hc}(t)) = 0$$

where the homogeneous cooling state f_{hc} is given by

$$f_{hc}(t) = \theta^{-\frac{3}{2}}(f(t))\, g_\infty(v\, \theta^{-\frac{1}{2}}(f(t))).$$

Sketch of the proof of Theorem 7.52.- It is based on the argument in [106] to Tanaka's theorem. The first statement is a simple consequence of (7.42). Then we turn to the second part of the theorem which by triangular inequality for the W_2 distance is enough to prove when g_2^0, and hence $g_2(\tau)$, is the unique stationary state g_∞ to (7.41) with zero mean velocity and unit

kinetic energy. A density argument allows us to reduce to the case in which the fourth moment of the initial datum is bounded, i.e.,

$$\int_{\mathbb{R}^3} |v|^4 \, g_1^0(v) \, dv < \infty.$$

The proof is now done by a typical dynamical systems argument. First, one uses the extra compactness given by the uniform in time bound of the 4th moments in Proposition 7.39 to show compactness in W_2 of the trajectories of the dynamical system. On the other hand, the characterization of the ω-limit set is done by using carefully the equality case in the proof of Theorem 6.1 showing finally that the ω-limit set is reduced to the stationary point g_∞. □

We close these notes with some open question related to the argument treated in these notes. Among others, we outline two of them, that we retain of great interest to people working in this field.

Remark 7.54 (Open Problem for Contractions in EB conjecture)**.** *Let us point out that a natural question related to the fact that equation* (7.41) *is a strict contraction with respect to $d_{2+\alpha}$ is whether a Wassertein distance with larger index, for instance W_4, could be strictly contractive for* (7.41)*. Of course, a similar scheme as in Theorem* 6.1 *can be performed to verify it, but there is one term we cannot control in the transport of spheres argument and we cannot conclude, see also Remark* 2.10*. It is an open problem to prove or disprove this claim, even for a non-strict contraction in the elastic case.*

Remark 7.55 (Open Problem about propagation of regularity)**.** *As showed by Bobylev and Cercignani* [22]*, in scaled variables the self-similar solution satisfies the bounds*

$$\exp\{-|k|^2\} \le |\hat{g}_\infty(|k|)| \le \exp\{-|k|\}(1+|k|). \tag{7.57}$$

In particular, the upper bound in (7.57) *guarantees that the steady state $g_\infty(v)$ is smooth. In fact, by using the homogeneous Sobolev space norms given by* (2.28)*, namely*

$$\|f\|^2_{\dot{H}^r(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |k|^{2r} |\hat{f}(k)|^2 \, dk,$$

one sees at once that, for all $r > 0$,

$$\|g_\infty\|^2_{\dot{H}^r(\mathbb{R}^3)} \le \int_{\mathbb{R}^3} |k|^{2r} (1+|k|)^2 \exp\{-2|k|\} \, dk < \infty \, .$$

The regularity of the steady state to the scaled Boltzmann equation could suggest that convergence towards the steady solution takes place in stronger spaces (it is usual to think in L^1-convergence). The proof of such result

requires the knowledge of the eventual propagation of regularity for the solution to equation (7.41)*. By showing uniform propagation of regularity for the solution to equation* (7.41) *one could use interpolation inequalities like in Subsections* 7.1 *and* 7.2 *to obtain convergence towards the homogeneous cooling state in the strong* L^1*–norm, as well as in various Sobolev norms. Unlikely, the uniform propagation of regularity for equation* (7.41) *is not presently known. We remind here that the analogous result for the elastic Boltzmann equation for Maxwellian molecules has been proven in* [38]*. The proof takes essential advantage from the knowledge of the validity of the* H*-theorem, which is not known to hold in the scaled inelastic case.*

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