ON THE CRITICAL PERIODS OF PERTURBED ISOCHRONOUS CENTERS

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ABSTRACT. Consider a family of planar systems $\dot{x} = X(x, \varepsilon)$ having a center at the origin and assume that for $\varepsilon = 0$ they have an isochronous center. Firstly, we give an explicit formula for the first order term in ϵ of the derivative of the period function. We apply this formula to prove that, up to first order in ϵ, at most one critical period bifurcates from the periodic orbits of isochronous quadratic systems when we perturb them inside the class of quadratic reversible centers. Moreover necessary and sufficient condition for the existence of this critical period are explicitly given. From the tools developed in this paper we also provide a new characterization of planar isochronous centers.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

A well known method for obtaining limit cycles for planar autonomous vector fields consists in perturbing vector fields having a continuum of periodic orbits, like for instance Hamiltonian systems. When the unperturbed Hamiltonian vector field is polynomial, and the perturbation is polynomial as well, the determination of the number of remaining periodic orbits leads, among other problems, to the study of the Abelian integrals and to the famous weak Hilbert sixteenth problem, see for instance the survey [16].

A continuum of periodic orbits of a planar vector fields can be parameterized by a curve $\gamma =: \{\gamma(h) \,|\, h \in (h_0, h_1)\}$, transversal to them. Then *the period function* $T(h)$ is defined as the period of the periodic orbit that passes through the point $\gamma(h)$. A *critical period* of the period function means a value h_0 such that $T'(h_0) = 0$. It is possible to prove that this definition does not depend on the choice of $\gamma(h)$. Recall that the behavior of the period function plays sometimes an important role in the study of several Abelian

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integral, see for instance [3, 12, 13]. Moreover it is also important in the study of other dynamical problems, see [1, 5, 6, 8, 15].

In a similar manner that in the study of perturbations of continua of periodic orbits, in this paper we consider a vector field which has an isochronous center and study the problem of how many critical periods bifurcate from its periodic orbits when it is perturbed keeping the critical point as a center. This problem has been already treated in [9] when the characterization of the isochronous center of the unperturbed vector field X is made in terms of the existence of a commutator of X or in [7] when X is the linear vector field. Here we extend the method to other characterizations of isochronous centers.

Our first result lists three well known characterizations of smooth planar isochronous centers together with two new characterizations (iv) and (v). In this theorem, $\mathbf{x}(t) := \mathbf{x}(t;q)$ denotes the flow of X satisfying $\mathbf{x}(0;q) = q$ and $\mathcal{D} \setminus \{p\}$ the period annulus of the center p.

Theorem 1. *Let X be an analytic planar vector field X having a center at $p \in \mathcal{D}$. Then p is an isochronous center if and only if:*

(i) There exists a smooth change of coordinates in a neighbourhood of p that linearizes X.

(ii) There exists a transversal vector field U in $\mathcal{D}\setminus\{p\}$ commuting with X, i.e. $[X, U] = DU\,X - DX\,U = 0$.

(iii) There exists a transversal vector field U and a scalar function α such that $[X, U] = \alpha\, X$ and

$$\int_0^{T_\gamma} \alpha(\mathbf{x}(t))\ dt = 0,$$

where $\gamma = \{\mathbf{x}(t), t \in [0, T_\gamma]\}$ is any periodic orbit of X in $\mathcal{D}\setminus\{p\}$ and T_γ is its period.

(iv) There exists a transversal vector field U in $\mathcal{D}\setminus\{p\}$ such that

$$\int_0^{T_\gamma} \alpha(\mathbf{x}(t)) e^{-\int_0^t \beta(\mathbf{x}(s))ds} dt = 0,$$

where α and β are given by the expression $[X, U] = \alpha X + \beta U$, $\gamma = \{\mathbf{x}(t), t \in [0, T_\gamma]\}$ is any periodic orbit of X in $\mathcal{D}\setminus\{p\}$ and T_γ is its period.

(v) There exists a transversal vector field U in $\mathcal{D}\setminus\{p\}$ and a scalar function β such that $[X, U] = \beta U$.

The first characterization is a classical result due to Poincaré; (ii) is proved in [17, 20] and (iii) is given in [9]. Characterization (iv) is an straightforward consequence of [18, Thm 1]. For the sake of completeness we also include in next section a self contained and short proof of this theorem, see Theorem 4. As far as we know, characterization (v) is a new one. We will use it along this paper. Note that it can be seen as a relaxation of characterization (ii).

Next, we consider an one parametric family of vector fields $X+\varepsilon Y$ having a center at p for all $|\varepsilon|$ small enough. By using again Theorem 4 we obtain the following result that allows to study the critical periods of $X+\varepsilon Y$.

Theorem 2. *Let X be a smooth planar vector field X having an isochronous center of period T_0 at $p\in\mathcal{D}$. Let U be a transversal vector in $\mathcal{D}\backslash\{p\}$ such that $[X,U]=\beta U$ for some smooth function β.*

Consider the family of vector fields $X+\varepsilon Y$ and assume that for $|\varepsilon|$ small enough all them have a center at p. Write $Y=aX+bU$ for some scalar functions a and b. Fix a point $q\in\mathcal{D}\backslash\{p\}$, and let $\mathbf{y}(h;q)$ denote the flow of U satisfying $\mathbf{y}(0;q)=q$ and let $\mathbf{x}(t,h):=\mathbf{x}(t;\mathbf{y}(h,q))$ be a parameterization of the the periodic orbits of X. Then,

(i) The period function $T(h,\varepsilon)$ of the periodic orbit $\mathbf{x}_\varepsilon(t;\mathbf{y}(h,q))$ of $X+\varepsilon Y$ is

$$T(h,\varepsilon)=T_0+\varepsilon T_1(h)+O(\varepsilon^2),$$

where

$$\begin{aligned} T_1'(h) &= -\int_0^{T_0} U(a)(\mathbf{x}(t,h))e^{-\int_0^t \beta(\mathbf{x}(s,h))ds}dt \qquad (1)\\ &= -\int_0^{T_0} \nabla a(\mathbf{x}(t,h))\cdot U(\mathbf{x}(t,h))e^{-\int_0^t \beta(\mathbf{x}(s,h))ds}dt. \end{aligned}$$

(ii) If h^* is a simple zero of $T_1'(h)$ then for $|\varepsilon|$ small enough there is exactly one critical period of $X+\varepsilon Y$ corresponding to a value of h that tends to h^* as ε tends to zero.

The above result is a similar result to Theorem 1 of [9], but using characterization (v) of Theorem 1 instead of characterization (ii). In Theorem 5 it is also extended to characterization (iv).

Finally we apply Theorem 2 to prove that, up to first order in ϵ, at most one critical period bifurcates from the periodic orbits of isochronous quadratic systems when we perturb them inside the class of quadratic reversible centers. Moreover we also give explicitly conditions for the existence of such critical period. More precisely, we take the unperturbed system and its perturbation in the Loud form,

$$\left\{\begin{array}{ll} \dot{x} &= -y+(\tilde{B}+\varepsilon B)xy,\\ \dot{y} &= x+(\tilde{D}+\varepsilon D)x^2+(\tilde{F}+\varepsilon F)y^2. \end{array}\right. \qquad (2)$$

It is well known that system $(2)_{\varepsilon=0}$, with $|\tilde{B}| + |\tilde{D}| + |\tilde{F}| \neq 0$, has an isochronous center at the origin if and only if $\tilde{B} \neq 0$ and $(\tilde{D}/\tilde{B}, \tilde{F}/\tilde{B}) \in \{(0,1), (-1/2, 1/2), (0, 1/4), (-1/2, 2)\}$. In fact for the perturbations of first order in ε of the Loud form (2) we can assume, without loss of generality, that $B = 0$ because (2) and (3) are equivalent when we consider only terms of order ε, by the transformation

$$(x, y) \longrightarrow \frac{\tilde{B}}{\tilde{B} + \varepsilon B}(x, y).$$

Hence from now on we consider the system

$$\begin{cases} \dot{x} &= -y + \tilde{B}xy, \\ \dot{y} &= x + (\tilde{D} + \varepsilon D)x^2 + (\tilde{F} + \varepsilon F)y^2. \end{cases} \tag{3}$$

By using Theorem 2 we know that when we perturb an isochronous center of a planar vector field X, by $X + \varepsilon Y$, keeping the critical point always as a center, then the new period function writes as $T(h, \varepsilon) = T_0 + \varepsilon T_1(h) + O(\varepsilon^2)$ for a suitable parameterization given by h. Motivated by (ii) of Theorem 2 when $T_1(h)$ has at most k zeros, being all of the simple, we will say that, *up to first order in ε, the number of critical periods bifurcating from the periodic orbits of X is k.*

Theorem 3. *Consider system* (3) *with $|\tilde{B}| + |\tilde{D}| + |\tilde{F}| \neq 0$. Fix a compact set K in the region filled by the periodic orbits of* $(3)|_{\varepsilon=0}$. *Then for $|\varepsilon|$ small enough, up to first order in ε, at most one critical period bifurcates from the periodic orbits of system* (3) *contained in K. Moreover,*

(a) *When $(\tilde{D}/\tilde{B}, \tilde{F}/\tilde{B}) = (0, 1)$, the critical period can appear if and only if $-\frac{1}{3} < \frac{D}{F} < 0$.*
(b) *When $(\tilde{D}/\tilde{B}, \tilde{F}/\tilde{B}) = (0, \frac{1}{4})$, the critical period can appear if and only if $0 < \frac{D}{F} < 2$.*
(c) *When $(\tilde{D}/\tilde{B}, \tilde{F}/\tilde{B}) = (-\frac{1}{2}, \frac{1}{2})$ no critical periods appear. Furthermore for $|\varepsilon|$ small enough, the period function T is increasing (resp. decreasing) as the closed orbit run away from the center* $(0, 0)$ *if $\varepsilon(D + F)\tilde{B} < 0$ (resp. $\varepsilon(D + F)\tilde{B} > 0$).*
(d) *When $(\tilde{D}/\tilde{B}, \tilde{F}/\tilde{B}) = (-\frac{1}{2}, 2)$, the critical period can appear if and only if $-\frac{3}{2} < \frac{F}{D} < 0$.*

We also notice that from the proof of the above theorem, in all the situations where no critical periods appear, as for instance in case (c), the fact that the period function in K is increasing or decreasing can also be established.

The main tool to prove the above result is the study of the function $T_1(h)$ given in Theorem 2, associated to system (3). After some cumbersome

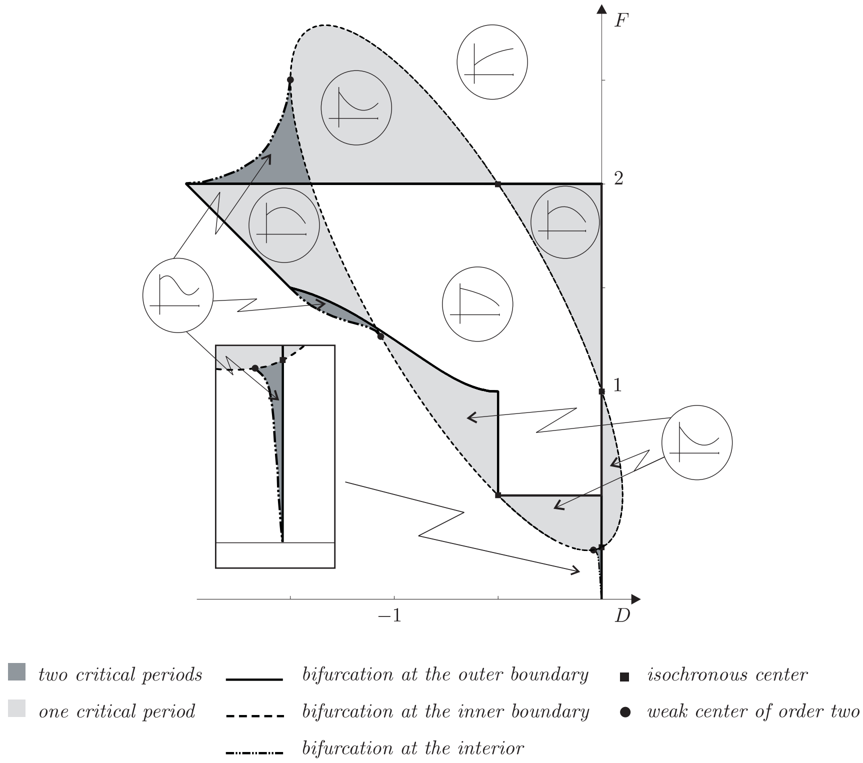

Figure 1. Conjectured diagram for the period function for system $\dot{x} = -y + xy, \quad \dot{y} = x + Dx^2 + Fy^2$, made in [14].

computations $T_1(h)$ is explicitly obtained in all cases. As we will see, in three of the cases it is an elementary function and the proof of the result is quite straightforward. On the other hand, in case (d) the function $T_1(h)$ is given in terms of elliptic functions and the proof that it has at most one zero turns out to be more complicated. We notice that the result of the above theorem for case (a) has been already obtained in [10].

To end this introduction we interpret Theorem 3 under the light of other known results. The main interest for studying system (3) is the conjecture made in [4]. Indeed a more detailed conjecture is made in [14]. See Fig. 1, borrowed from Fig. 3 of that paper. Notice that the boundary conditions to ensure the existence of at least one critical point for the period function given in Theorem 3 coincide with the tangent lines of some of the curves of the bifurcation diagram plotted in Fig. 1. We include these lines in Fig. 2. Notice also that according to the results in [14], there should be

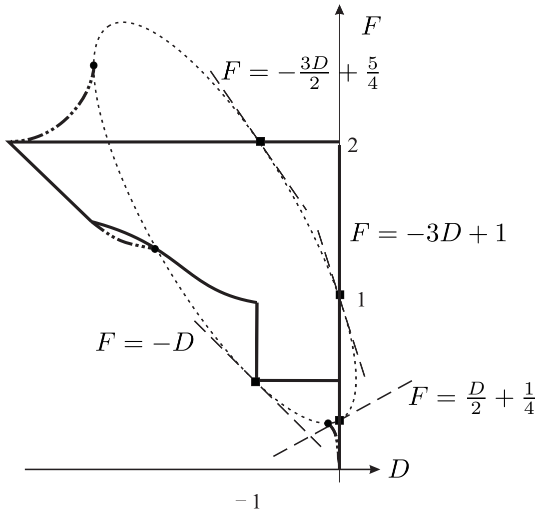

Figure 2. Boundaries for existence of at least one critical period given in Theorem 3 near the isochronous centers.

an even number of critical points for the period function in some places of the bifurcation diagram, for example when $0 < \frac{D}{F} < 2$ and $\varepsilon < 0$, $|\varepsilon|$ small, in Case (b) of our Theorem 3. This fact is not contradiction with our results because the critical points can appear far from the origin. So in this situation there are at least two critical points, one located in some bounded compact region, and another coming from the outer boundary of the period annulus.

2. Proofs of Theorems 1 and 2

To make this paper self contained, we start this section giving a new proof of [18, Thm 1], stated in our work as Theorem 4. This result is an extension of [10, Thm 1] where the function β appearing in its statement is assumed to be identically zero.

Theorem 4. *Assume that a vector field X has a center at p with period annulus $\mathcal{D} \setminus \{p\}$. Take any vector field $U \in C^1(\mathcal{D})$ transversal to X in $\mathcal{D}\setminus\{p\}$. Then $[X, U] = \alpha X + \beta U$ for some smooth functions α and β. Fix a point $q \in \mathcal{D}\setminus\{p\}$, and let $\mathbf{y}(h; q)$ denote the flow of U satisfying $\mathbf{y}(0; q) = q$ and let $\mathbf{x}(t, h) := \mathbf{x}(t; \mathbf{y}(h, q))$ be a parameterization of the the periodic orbits* *of X. Then*

$$T'(h) = \int_0^{T(h)} \alpha(\mathbf{x}(t,h)) e^{-\int_0^t \beta(\mathbf{x}(s,h))ds} dt, \tag{4}$$

where $T(h)$ is the period of $\mathbf{x}(t,h)$. It also holds that $\int_0^{T(h)} \beta(\mathbf{x}(t,h))dt \equiv 0$.

Proof : Let $\gamma = \{\mathbf{x}(t) := \mathbf{x}(t;q) \,:\, \mathbf{x}(0) = \mathbf{x}(T) = q\}$ be a periodic orbit of X with period T. Take a transversal section Σ, which is a part arc of the orbit $\mathbf{y}(h) := \mathbf{y}(h;q)$ of the vector fields U, transversal to X in $\mathcal{D} \setminus p$, such that $\mathbf{y}(0) = q$. Hence we define the return map on Σ as

$$\pi : \ \Sigma_0 \subset \Sigma \to \Sigma,$$

and we have

$$\pi(\mathbf{y}(h)) = \mathbf{x}(T + \tau(h), \mathbf{y}(h)),$$

where $T(h) := T + \tau(h)$ is the period of a closed orbit of X passing through $\mathbf{y}(h)$.

Consider the variational equation of X along the periodic orbit $\mathbf{x}(t)$,

$$\frac{d\eta}{dt} = DX(\mathbf{x}(t))\eta.$$

Let us see that the following function

$$\eta(t) = U(\mathbf{x}(t)) e^{-\int_0^t \beta(\mathbf{x}(s))ds} - X(\mathbf{x}(t)) \int_0^t \alpha(\mathbf{x}(u)) e^{-\int_0^u \beta(\mathbf{x}(s))ds} du, \tag{5}$$

is one of its solutions. Since $[X,U] = DU\,X - DX\,U = \alpha X + \beta U$, it is easy to be checked that

$$\begin{aligned} \frac{d}{dt}\eta(t) \quad = \quad & (DU\,X - \beta U - \alpha X)(\mathbf{x}(t)) e^{-\int_0^t \beta(\mathbf{x}(s)))ds} - \\ & (DX\,X)(\mathbf{x}(t)) \int_0^t \alpha(\mathbf{x}(u)) e^{-\int_0^t \beta(\mathbf{x}(s))ds} du = DX(\mathbf{x}(t))\eta(t). \end{aligned}$$

Notice that the solutions of X in the neighborhood of $\mathbf{x}(t)$ are a family of periodic orbits. The monodromy matrix of the variational equation of the return map in the basis $\{X(q), U(q)\}$ is

$$\begin{pmatrix} 1 & -T'(0) \\ 0 & 1 \end{pmatrix},$$

see for instance [19, pp. 231-232]. From (5) it follows that

$$\eta(0) = U(q), \quad \eta(T) = U(q) e^{-\int_0^T \beta(\mathbf{x}(s))ds} - X(q) \int_0^T \alpha(\mathbf{x}(u)) e^{-\int_0^u \beta(\mathbf{x}(s))ds} du.$$

Hence in the basis of $\{X(q), U(q)\}$, we have

$$\begin{pmatrix} -\int_0^T \alpha(\mathbf{x}(u))e^{-\int_0^u \beta(\mathbf{x}(s))ds}du \\ e^{-\int_0^T \beta(\mathbf{x}(s))ds} \end{pmatrix} = \begin{pmatrix} 1 & -T'(0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which implies the desired result. That the integral over β identically vanishes is also a consequence of the above formula. □

Proof of Theorem 1 : Recall that characterizations (i), (ii) and (iii) of the isochronous centers are well known results, see [9, 17, 20]. On the other hand (iv) is an straightforward consequence of Theorem 4.

So we proceed with the proof of characterization (v). That the vanishing of the integral is a necessary condition follows again from Theorem 4. For the sufficiency recall that if p is an isochronous center from (ii) we know that there exists a transversal vector field U_1 to X in $\mathcal{D}\backslash\{p\}$ such that $[X, U_1] = 0$. Taking the new vector field $U_2 = fU_1$, where f a nonzero function in $\mathcal{D}$ and such that $X(f)$ is not identically zero we get

$$[X, U_2] = [X, fU_1] = f[X, U_1] + X(f)U_1 = \frac{X(f)}{f}U_2,$$

as we wanted to prove. □

Proof of Theorem 2 : (i) From the equality $Y = aX + bU$, we have

$$X = \frac{1}{1+\varepsilon a}(X + \varepsilon Y - \varepsilon bU).$$

Hence

$$\begin{aligned} [X + \varepsilon Y, U] &= -\varepsilon U(a)X + ((1+\varepsilon a)\beta - \varepsilon U(b))U \\ &= \tilde{\alpha}(X + \varepsilon Y) + \tilde{\beta}U, \end{aligned}$$

where

$$\tilde{\alpha} = -\varepsilon\frac{U(a)}{1+\varepsilon a}, \quad \tilde{\beta} = (1+\varepsilon a)\beta - \varepsilon U(b) + \varepsilon^2\frac{bU(a)}{1+\varepsilon a}.$$

By (4) in Theorem 4, we have

$$\begin{aligned} \frac{\partial T(h,\varepsilon)}{\partial h} &= \varepsilon T_1'(h) + O(\varepsilon^2) = \int_0^{T_\varepsilon(h)} \tilde{\alpha}(\mathbf{x}_\varepsilon(t,h))e^{-\int_0^t \tilde{\beta}(\mathbf{x}_\varepsilon(s,h))ds}dt \\ &= -\varepsilon\int_0^{T_\varepsilon(h)} U(a)e^{-\int_0^t \beta(\mathbf{x}_\varepsilon(s,h))ds}\Big(1 - \varepsilon\int_0^t (a\beta - U(b))(\mathbf{x}_\varepsilon(s,h))ds\Big)dt + O(\varepsilon^2) \\ &= -\varepsilon\int_0^{T_0} U(a)(\mathbf{x}(t,h))e^{-\int_0^t \beta(\mathbf{x}(s,h))ds}dt + O(\varepsilon^2), \end{aligned}$$

as we wanted to prove.

(ii) The proof of this part is a straightforward consequence of the Implicit Function Theorem. □

Next results extends the above theorem to the more general case where $[X, U] = \alpha X + \beta U$. As we can see from its statement it is more difficult to be applied than Theorem 4, so when we are interested in studying the period function of a perturbed isochronous center X it is better to chose, whenever is possible, an U that satisfies characterization (ii) or (v) of Theorem 1 instead of characterization (v).

Theorem 5. *Let X be a smooth planar vector field having an isochronous center of period T_0 at $p \in \mathcal{D}$. Let U be a transversal vector in $\mathcal{D}\backslash\{p\}$ such that $[X, U] = \alpha X + \beta U$ for some smooth functions α and β.*

Consider the family of vector fields $X + \varepsilon Y$ and assume that for $|\varepsilon|$ small enough all them have a center at p. Write $Y = aX + bU$ for some scalar functions a and b. Fix a point $q \in \mathcal{D}\backslash\{p\}$, and let $\mathbf{y}(h) := \mathbf{y}(h; q)$ denote the flow of U satisfying $\mathbf{y}(0; q) = q$ and let $\mathbf{x}(t, h) := \mathbf{x}(t; \mathbf{y}(h; q))$ be a parameterization of the the periodic orbits of X. Then,

The period function $T(h, \varepsilon)$ of the periodic orbit $\mathbf{x}_\varepsilon(t; \mathbf{y}(h, q))$ of $X + \varepsilon Y$ is

$$T(h, \varepsilon) = T_0 + \varepsilon T_1(h) + O(\varepsilon^2),$$

where $T_1(h)$ satisfies the linear differential equation

$$T_1'(h) = \alpha(\mathbf{y}(h))\, T_1(h) + \delta(h),$$

being $\delta(h)$ the function

$$\delta(h) = \delta_0(h) + \delta_1(h) + \delta_2(h), \tag{6}$$

where

$$\begin{aligned}
&\delta_0(h) = \\
&= \int_0^{T_0} e^{-\int_0^t \beta(\mathbf{x}(s,h))ds} \left[\nabla\alpha(\mathbf{x}(t,h))\mathbf{x}_1(t,h) - \alpha(\mathbf{x}(t,h)) \int_0^t \nabla\beta(\mathbf{x}(s,h))\mathbf{x}_1(s,h) ds \right] dt, \\
&\delta_1(h) \quad = \quad \int_0^{T_0} \alpha(\mathbf{x}(t,h)) e^{-\int_0^t \beta(\mathbf{x}(s,h))ds} \left[\int_0^t (U(b) + b\alpha - a\beta)(\mathbf{x}(s,h)) ds \right] dt, \\
&\delta_2(h) \quad = \quad -\int_0^{T_0} U(a)(\mathbf{x}(t,h)) e^{-\int_0^t \beta(\mathbf{x}(s,h))ds} dt,
\end{aligned}$$

where $\mathbf{x}_1(t)$ satisfies

$$\frac{\partial \mathbf{x}_1(t,h)}{\partial t} = DX(\mathbf{x}(t,h))\mathbf{x}_1(t,h) + Y(\mathbf{x}(t,h)), \quad \mathbf{x}_1(0,h) = 0. \tag{7}$$

Proof : From the equality $Y = aX + bU$, we have

$$X = \frac{1}{1+\varepsilon a}(X + \varepsilon Y - \varepsilon b U).$$

Hence

$$\begin{aligned}[X+\varepsilon Y, U] &= [\alpha + \varepsilon(a\alpha - U(a))]X + [\beta + \varepsilon(a\beta - U(b))]U \\ &= \tilde{\alpha}(X+\varepsilon Y) + \tilde{\beta}U,\end{aligned}$$

where

$$\begin{aligned}\tilde{\alpha} &= \frac{\alpha + \varepsilon(a\alpha - U(a))}{1+\varepsilon a} = \alpha - \varepsilon U(a) + O(\varepsilon^2), \\ \tilde{\beta} &= \beta + \varepsilon(a\beta - U(b)) - \varepsilon b \frac{\alpha + \varepsilon(a\alpha - U(a))}{1+\varepsilon a} = \\ &= \beta + \varepsilon(a\beta - b\alpha - U(b)) + O(\varepsilon^2).\end{aligned}$$

Now let $\mathbf{x}_\varepsilon(t,h) = \mathbf{x}(t,h) + \varepsilon \mathbf{x}_1(t,h) + O(\varepsilon^2)$ with $\mathbf{x}_\varepsilon(0,h) = \mathbf{x}(0,h) = \mathbf{y}(h)$ be a solution of

$$\dot{\mathbf{x}}(t) = (X + \varepsilon Y)(\mathbf{x}(t)).$$

Hence $\mathbf{x}_1(t,h)$ satisfies the variational equation (7). Recall that by characterization (iv) of Theorem 1 it holds that

$$\int_0^{T_0} \alpha(\mathbf{x}(t,h)) e^{-\int_0^t \beta(\mathbf{x}(s,h))ds} dt \equiv 0.$$

Moreover, by using (4) of Theorem 4 we have that

$$\begin{aligned}\frac{\partial T(h,\varepsilon)}{\partial h} &= \varepsilon T_1'(h) + O(\varepsilon^2) = \int_0^{T_\varepsilon(h)} \tilde{\alpha}(\mathbf{x}_\varepsilon(t,h)) e^{-\int_0^t \tilde{\beta}(\mathbf{x}_\varepsilon(s,h))ds} dt \\ &= \Delta_0(h,\varepsilon) + \varepsilon(\delta_1(h) + \delta_2(h)) + O(\varepsilon^2), \qquad (8)\end{aligned}$$

where

$$\Delta_0(h,\varepsilon) = \int_0^{T_\varepsilon(h)} \alpha(\mathbf{x}_\varepsilon(t,h)) e^{-\int_0^t \beta(\mathbf{x}_\varepsilon(s,h))ds} dt.$$

Differentiating once at the both sides of (8) with respect to ε, we have

$$T_1'(h) = \frac{\partial}{\partial \varepsilon}\left[\frac{\partial T(h,\varepsilon)}{\partial h}\right]_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\left[\Delta_0(h,\varepsilon)\right]_{\varepsilon=0} + \delta_1(h) + \delta_2(h).$$

It is easy to deduce

$$\frac{\partial}{\partial \varepsilon}\left[\Delta_0(h,\varepsilon)\right]_{\varepsilon=0} = \alpha(\mathbf{y}(h))\, T_1(h) + \delta_0(h),$$

and so the theorem follows. □

3. Proof of Theorem 3

Before proving our result notice that through a change of scale of the variables x and y it is possible to take as unperturbed system any triple of the form $\lambda(\tilde{B}, \tilde{D}, \tilde{F})$ for $\lambda \in \mathbb{R}\backslash\{0\}$ and $(\tilde{B}, \tilde{D}, \tilde{F})$ satisfying the isochronicity conditions.

Proof of Theorem 3, parts (a) and (b) :

(a) In this case we consider as unperturbed isochronous system the one corresponding to $(\tilde{B}, \tilde{D}, \tilde{F}) = (1, 0, 1)$, *i.e.*

$$\begin{cases} \dot{x} &= -y + xy, \\ \dot{y} &= x + y^2, \end{cases} \tag{9}$$

and call X its associated vector field. It has the following first integral

$$H(x, y) = \frac{2x + y^2 - 1}{(1 - x)^2}.$$

Moreover $\{H(x, y) = \ell, \ -1 < \ell < 0\}$ is the family of closed orbits surrounding the origin. If we take the vector field $U = (x, y)$, it is transversal to X in the period annulus and $[X, U] = -y\,U$. Now, consider the following perturbed system (3),

$$\begin{cases} \dot{x} &= -y + xy, \\ \dot{y} &= x + \varepsilon D x^2 + (1 + \varepsilon F) y^2. \end{cases}$$

In order to apply Theorem 2, writing $Y := (0, Dx^2 + Fy^2)$, then $Y = aX + bU$, where

$$a(x, y) = \frac{(Fy^2 + Dx^2)x}{x^2 + y^2}, \quad b(x, y) = \frac{(Fy^2 + Dx^2 - Dx^3 - Fxy^2)y}{x^2 + y^2}.$$

Let $\mathbf{y}(h) = (e^h, 0)$ for $h < -\ln 2$, be a trajectory of U. Hence the periodic orbit of X starting at $\mathbf{y}(h)$ can be written as $\mathbf{x}(t, \mathbf{y}(h)) = (x(t, h), y(t, h))$, where

$$x(t, h) = \frac{m \cos t}{1 - m + m \cos t}, \quad y(t, h) = \frac{m \sin t}{1 - m + m \cos t}, \quad 0 < m = e^h < \frac{1}{2}. \tag{10}$$

Note that from the first equations of (9) and (10), we have

$$\exp\left(\int_0^t y(s) ds\right) = \frac{1}{1 - m + m \cos t}.$$

Hence, from Theorem 2 and (10),

$$\begin{aligned} T_1'(h) &= \int_0^{2\pi} \frac{(Fy^2(t)+Dx^2(t))x(t)}{x^2(t)+y^2(t)} \exp\left(\int_0^t y(s)ds\right) dt \\ &= \int_0^{2\pi} \frac{m\cos t(D\cos^2 t + F\sin^2 t)}{(1-m+m\cos t)^2} dt \\ &= \frac{\pi}{m^2(1-2m)^{3/2}}(d(m)D + f(m)F). \end{aligned}$$

where

$$\begin{aligned} d(m) &= 4(1-2m)(m-1)M(m) - 2(m^2+4m-2)(m-1)^2 \\ f(m) &= 2(2m-1)(2(m-1)M(m)+m^2-4m+2),\ M(m) = \sqrt{1-2m}. \end{aligned}$$

Take

$$Z(m) = -\frac{f(m)}{d(m)} = \frac{D}{F}.$$

It is easy to check that $Z(0) = -1/3$, $Z(1/2) = 0$. Moreover

$$Z'(m) = \frac{2m^3(g_0(m) - g_1(m)M(m))}{(m-1)^2((m-1)(m^2+4m-2) - (1-2m)M(m))^2 M(m)}$$

where

$$g_1(m) = (m-1)(m^2-8m+4), \quad g_0(m) = (m-2)(2m-1)(3m-2).$$

Hence, from the inequality

$$g_1^2(m)M - g_0^2(m) = m^6(1-2m) \neq 0.$$

and using that $Z(1/4) > 0$, we get that $Z'(m) > 0$ for $0 < m < 1/2$. Thus $T_1'(h) = 0$ can have some solution if and only if $-\frac{1}{3} < \frac{D}{F} < 0$, as we wanted to prove.

(b) In this case we take as X, the vector field with $(\tilde{B}, \tilde{D}, \tilde{F}) = (4, 0, 1)$:

$$\begin{cases} \dot{x} &= -y + 4xy, \\ \dot{y} &= x + y^2. \end{cases} \tag{11}$$

It has the first integral

$$H(x,y) = \frac{1 - 2x + 2y^2}{2\sqrt{1-4x}},$$

defined in the region $\{(x,y) : x < \frac{1}{4}\}$. Again its origin is an isochronous center and $\{H(x,y) = \ell,\ \ell \geq 1\}$ is the family of closed curves around it. Taking the new vector field $U := (x + y^2, y)$ we have that

$$\det(X,U) = -y^2 - (x - y^2)^2,$$

and so it is transversal to X in its period annulus. Moreover $[X,U] = -2yU$.

As in case (a), we consider the perturbed system (3),

$$\begin{cases} \dot{x} &= -y + 4xy, \\ \dot{y} &= x + \varepsilon D x^2 + (1+\varepsilon F) y^2. \end{cases}$$

Then writing $Y = (0, Dx^2 + Fy^2)$ we have that $Y = aX + bU$, where

$$\begin{aligned} a(x,y) &= \frac{Dx^3 + Fxy^2 + Dx^2y^2 + Fy^4}{(x-y^2)^2 + y^2}, \\ b(x,y) &= \frac{(-4Dx^3 + Dx^2 - 4Fxy^2 + Fy^2)y}{(x-y^2)^2 + y^2}. \end{aligned}$$

Take the trajectory of U, $\mathbf{y}(h) = (\frac{4m}{8m+1}, 0)$, for $0 < m = \frac{e^h}{4(1-2e^h)} < 1/8$. Hence the periodic orbits of X starting at $\mathbf{y}(h)$ can be written as $\mathbf{x}(t, \mathbf{y}(h)) = (x(t,h), y(t,h))$, where

$$x(t,h) = \frac{4m(4m + (1+4m)\cos^2 t)}{(8m\cos t + 1)^2}, \quad y(t,h) = \frac{4m \sin t}{8m \cos t + 1}.$$

Note that from the first equation of (11), we have

$$\exp\left(\int_0^t 2y(s)ds\right) = \frac{8m+1}{8m\cos t + 1}.$$

From Theorem 2,

$$T_1'(h) = \int_0^{2\pi} \frac{Dx^3 + Fxy^2 + 5Dx^2y^2 + 2Dxy^4 + 3Fy^4}{(x-y^2)^2 + y^2} \exp\left(2\int_0^t y(s)ds\right) dt$$
$$= (8m+1)\int_0^{2\pi}\left(t_0(t,m) + \frac{t_1(t,m)}{z(t,m)} + \frac{t_2(t,m)}{z^2(t,m)} + \frac{t_3(t,m)}{z^3(t,m)} + \frac{t_4(t,m)}{z^4(t,m)} + \frac{t_5(t,m)}{z^5(t,m)}\right) dt$$

where $z(t,m) = 8m\cos t + 1$ and

$$\begin{aligned} t_0(t,m) &= \frac{(4F-D)\cos t}{64m} - \frac{F}{32m^2}, \\ t_1(t,m) &= \frac{(64m^2-3)(6F+D)}{256m^2}, \\ t_2(t,m) &= \frac{(64m^2-1)(5D+14F)}{256m^2}, \\ t_3(t,m) &= \frac{(64m^2-1)(192Dm^2 + 512m^2F + D - 8F)}{512m^2}, \\ t_5(t,m) &= \frac{4t_4(t,m)}{9} = \frac{D(64m^2-1)^3}{128m^2}. \end{aligned}$$

By direct calculations we obtain

$$T_1'(h) = \frac{8m+1}{32(1-64m^2)^{9/2}}(d(m)D + f(m)F), \tag{12}$$

being

$$d(m) = 256m^4(64m^2-1)^3 \text{ and } f(m) = (64m^2-1)^4(1-32m^2-\sqrt{1-64m^2}).$$

Take

$$Z(m) = -\frac{f(m)}{d(m)} = \frac{D}{F}.$$

It is easy to check that $Z(0) = 2$, $Z(1/8) = 0$. Moreover

$$Z'(m) = \frac{g_0(m) - g_1(m)\sqrt{1-64m^2}}{64m^5}$$

where $g_1(m) = 16m^2 - 1$ and $g_0(m) = 48m^2 - 1$. Hence, from the inequality

$$g_1^2(m)(1-64m^2) - g_0^2(m) = -16384m^6 \neq 0.$$

and using that $Z(1/10) < 0$, we get that $Z'(m) < 0$ for $0 < m < 1/8$, *i.e.* $Z(m)$ is decreasing for $0 < m < 1/8$. Hence $T_1'(h) = 0$ if and only if $0 < \frac{D}{F} < 2$. Thus the period function T can have a unique critical point in any compact region only when $0 < \frac{D}{F} < 2$. □

Before proving parts (c) and (d) we need two preliminary results. The first one follows from straightforward computations.

Lemma 6. *For $a > 1$ and $\alpha \in \mathbb{R}$ define*

$$I_\alpha(a) = \int_0^{2\pi} \frac{dt}{(a+\cos t)^\alpha}.$$

Then

$$\begin{array}{lll} I_1(a) = \frac{2\pi}{\sqrt{a^2-1}}, & I_2(a) = \frac{2\pi a}{(a^2-1)^{3/2}}, & I_3(a) = \frac{\pi(2a^2+1)}{(a^2-1)^{5/2}}, \\ I_{1/2}(a) = \frac{4\mathcal{K}(\zeta)}{\sqrt{a+1}}, & I_{-1/2}(a) = 4\sqrt{a+1}\mathcal{E}(\zeta), & \\ I_{3/2}(a) = \frac{4\mathcal{E}(\zeta)}{\sqrt{a+1}(a-1)}, & I_{-3/2}(a) = -\frac{4\sqrt{a+1}}{3}((a-1)\mathcal{K}(\zeta) - 4a\mathcal{E}(\zeta)), & \end{array}$$

where $\zeta = \zeta(a) = \frac{\sqrt{2}}{\sqrt{a+1}}$.

Recall that the above funcions, $\mathcal{K}$ and $\mathcal{E}$ are defined as

$$\mathcal{K}(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad \text{and} \quad \mathcal{E}(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta$$

and are the complete normal elliptic integrals of the first and second kinds, respectively, see [2].

Theorem 7. *The function* $Z(m) = -\dfrac{d(m)}{f(m)}$, *where*

$$d(m) = $$
$$= 16\left((m-1)^2(m+1)\mathcal{E}\left(\frac{2\sqrt{2m}}{m+1}\right) - \frac{(m^2+1)(m^2-4m+1)}{m+1}\mathcal{K}\left(\frac{2\sqrt{2m}}{m+1}\right)\right),$$
$$f(m) \quad = \quad 3\pi(m-1)\left((m-1)^2 + \frac{m^4 - 4m^3 - 2m^2 - 4m + 1}{\sqrt{m^2 - 6m + 1}(m+1)}\right),$$

is monotone decreasing in (m_0, ∞), *where* $m_0 = 3 + 2\sqrt{2}$ *is the biggest solution of* $m^2 - 6m + 1 = 0$. *Moreover*

$$\lim_{m \to m_0^+} Z(m) = 0 \quad \textit{and} \quad \lim_{m \to +\infty} Z(m) = -3/2. \tag{13}$$

Proof : The conditions (13) are easy to obtain. Moreover it holds that $Z(m) > -3/2$ for m big enough. Notice that $Z = -3/2$ is indeed a horizontal asymptote of $Z(m)$. In order to prove the monotonicity of $Z(m)$ define

$$\phi(m) = d(m)D + f(m)F.$$

By using that the functions $\mathcal{K}(k)$ and $\mathcal{E}(k)$ satisfy the following Picard-Fuchs equations

$$\frac{d\mathcal{K}}{dk} = \frac{\mathcal{E} - (1-k^2)\mathcal{K}}{k(1-k^2)}, \qquad \frac{d\mathcal{E}}{dk} = \frac{\mathcal{E} - \mathcal{K}}{k},$$

see again [2], it is not difficult to obtain that it satisfies

$$p_2(m)\frac{d^2\phi}{dm^2} - p_1(m)\frac{d\phi}{dm} - p_0(m)\phi - p(m) = 0, \tag{14}$$

where

$$\begin{aligned}
p_2(m) &= m(m^2-1)^3(\sqrt{m^2-6m+1})^5,\\
p_1(m) &= (m^4 - 10m^3 - 10m^2 + 14m - 3)(m+1)^2(\sqrt{m^2-6m+1})^3,\\
p_0(m) &= 3(m+1)^2(m^3 + m^2 + 5m - 3)(\sqrt{m^2-6m+1})^3,\\
p(m) &= 16\pi m(q_0(m) - q_1(m)\sqrt{m^2-6m+1})F,\\
q_0(m) &= 3m^8 - 24m^7 + 12m^6 + 120m^5 - 350m^4 + 120m^3 + 12m^2 - 24m + 3,\\
q_1(m) &= 3(m^2 - 6m + 1)(-1+m)^2(m+1)^3.
\end{aligned}$$

Notice that the functions $p_0(m)$ and $p_2(m)$ are positive in (m_0, ∞). Let us prove that the function $p(m)$, when $F \neq 0$, never vanishes in (m_0, ∞). Consider the following polynomial

$$\begin{aligned}
q_0^2(m) - (m^2-6m+1)q_1^2(m) &= -64m^4(39m^8 - 312m^7 + 228m^6 + 1272m^5\\
&\quad -2710m^4 + 1272m^3 + 228m^2 - 312m + 39).
\end{aligned}$$

By using its Sturm sequence it can be proved that it has only one root in (m_0, ∞), which approximately is $m = 6.5859\cdots$. Since this root satisfies $q_0(m)+q_1(m)\sqrt{m^2-6m+1} = 0$, instead of $q_0(m)-q_1(m)\sqrt{m^2-6m+1} = 0$, the assertion follows.

Now we can show that $Z(m)$ is decreasing in (m_0, ∞). By assuming the contrary we know that it should exist a horizontal line $Z = z_0$, $-3/2 < z_0 < 0$ such that the curve $Z(m)$ intersects this line in at least three points. Thus for some values of D and $F \neq 0$ the function ϕ also has at least three zeroes in (m_0, ∞). Hence there exist at least two values of m in (m_0, ∞), say m_1 and m_2 such that:

$$\begin{aligned} &\phi(m_1) \geq 0, \quad \phi'(m_1) = 0 \quad \text{and} \quad \phi''(m_1) \leq 0, \\ &\phi(m_2) \leq 0, \quad \phi'(m_2) = 0 \quad \text{and} \quad \phi''(m_2) \geq 0. \end{aligned} \tag{15}$$

On the other hand for each of these values it holds that

$$\phi''(m_i) = \frac{p_0(m_i)\phi(m_i) + p(m_i)}{p_2(m_i)}, \quad i = 1, 2. \tag{16}$$

Assume for instance that $p(m_i) > 0$ for $i = 1, 2$, (the case where both values are negative follows similarly). By using (16) we obtain that

$$\phi''(m_1) = \frac{p_0(m_1)\phi(m_1) + p(m_1)}{p_2(m_1)} > 0,$$

in contradiction with (15). So $Z(m)$ is decreasing, as we wanted to see. □

Proof of Theorem 3, parts (c) and (d) :

(c) In this case we start with the Loud isochronous center X, with $(\tilde{B}, \tilde{D}, \tilde{F}) = (2, -1, 1)$

$$\begin{cases} \dot{x} &= -y + 2xy, \\ \dot{y} &= x - x^2 + y^2. \end{cases} \tag{17}$$

It is easy to check that it has the first integral

$$H = \frac{1 - 2x + 4x^2 + 4y^2}{4(2x-1)}.$$

Thus the $(0, 0)$ is surrounded by the closed curves $\{H(x, y) = \ell,\ \ell \leq -1/4\}$. Notice that if we consider the vector field $U := (x - x^2 + y^2, (1-2x)y)$ then

$$\det(X, U) = -(x^2 + y^2)((x-1)^2 + y^2).$$

Hence it is transversal to X in the period annulus of the origin and moreover $[X, U] = 0$. Consider the following perturbed system

$$\begin{cases} \dot{x} &= -y + 2xy, \\ \dot{y} &= x + (\varepsilon D - 1)x^2 + (1 + \varepsilon F)y^2. \end{cases} \tag{18}$$

If $Y = (0, Dx^2 + Fy^2)$, then $Y = aX + bU$, where

$$a(x,y) = \frac{D(x^3 - x^4) + Fxy^2 + (D-F)x^2y^2 + Fy^4}{(x^2+y^2)((x-1)^2+y^2)},$$
$$b(x,y) = -\frac{(2Dx^3 - Dx^2 + 2Fxy^2 - Fy^2)y}{(x^2+y^2)((x-1)^2+y^2)}.$$

Take the trajectory of U, $\mathbf{y}(h) = (\frac{1}{1+\exp(-h)}, 0)$ for $h < 0$. Hence parameterizing by h, the periodic orbits of X starting at $\mathbf{y}(h)$ can be written as $\mathbf{x}(t, \mathbf{y}(h)) = (x(t,h), y(t,h))$ where

$$x(t,h) = \frac{m\cos t + 1}{m^2+1+2m\cos t}, \quad y(t,h) = \frac{m \sin t}{m^2+1+2m\cos t}, \quad \text{and} \quad m = e^{-h}.$$

From Theorem 2, we have

$$\begin{aligned} T_1'(h) &= -\int_0^{2\pi} \frac{M(x,y)}{(x^2+y^2)((x-1)^2+y^2)} dt \\ &= -\int_0^{2\pi} \frac{1}{4m}\left(g_0(t,m) + \frac{g_1(t,m)}{a(m)+\cos t} + \frac{g_2(t,m)}{(a(m)+\cos t)^2} + \frac{g_3(t,m)}{(a(m)+\cos t)^3}\right) dt, \end{aligned}$$

where $a(m) = (m^2+1)/(2m)$ and

$$\begin{aligned} M(x,y) &= D(x^3 - x^4) + Fxy^2 + (5D - 3F)x^2y^2 + \\ &\quad 2(F - 3D)x^3y^2 + 2(D - 3F)xy^4 + 3Fy^4, \\ g_0(t,m) &= 2(D-F)m^4 + 2D + 2F + (D-F)(1-m^2)\cos t, \\ g_1(t,m) &= \frac{1-m^2}{4m^2}[9(D-F)m^4 + 2(D-3F)m^2 - 3(D+3F)], \\ g_2(t,m) &= \frac{(1-m^2)^3[7(F-D)m^2 + 7F - 3D)}{8m^2}, \\ g_3(t,m) &= \frac{(m^2-1)^5(F-D)}{4m^2}. \end{aligned}$$

By using Lemma 6 we get that

$$T_1'(h) = -\frac{2\pi}{m^2}(D+F). \tag{19}$$

Hence, if $D+F \neq 0$, period function T is monotone in K. Specifically, there is $\varepsilon_0 > 0$, such that for $0 < |\varepsilon| < \varepsilon_0$, the period function in K is increasing as the closed orbit run away from the center $(0,0)$ if $\varepsilon(D+F) < 0$. When $\varepsilon(D+F) > 0$ it is decreasing.

(d) In this last case we consider the parameters $(\tilde{B}, \tilde{D}, \tilde{F}) = (2, -1, 4)$. Then the quadratic isochronous center X is

$$\begin{cases} \dot{x} &= -y + 2xy, \\ \dot{y} &= x - x^2 + 4y^2. \end{cases} \tag{20}$$

It has the first integral

$$H = \frac{16y^2 - 1 + 8x - 8x^2}{16(2x-1)^4},$$

and $(0,0)$ is surrounded by the closed curves $\{H(x,y) = \ell\}$ for $-1/16 < \ell \le 0$. Taking $U := (x(1-x), y - 2xy)$, since

$$\det(X,U) = -x^2(x-1)^2 - y^2,$$

it is transversal to X in the period annulus. We also have $[X,U] = -2y\,U$. Now, we consider the following perturbed system

$$\begin{cases} \dot{x} &= -y + 2xy, \\ \dot{y} &= x + (\varepsilon D - 1)x^2 + (4 + \varepsilon F)y^2. \end{cases} \tag{21}$$

Writing $Y = (0, Dx^2 + Fy^2)$, we have $Y = aX + bU$, where

$$\begin{aligned} a(x,y) &= -\frac{x(Dx^3 - Dx^2 - Fy^2 + Fxy^2)}{x^2(x-1)^2 + y^2}, \\ b(x,y) &= -\frac{(2Dx^3 + Dx^2 + 2Fxy^2 - Fy^2)y}{x^2(x-1)^2 + y^2}. \end{aligned}$$

Take the trajectory of U, $\mathbf{y}(h) = (\frac{1}{1+\exp(-h)}, 0)$ for $h < 0$. Hence the periodic orbit of X starting at $\mathbf{y}(h)$ can be written as $\mathbf{x}(t, \mathbf{y}(h)) = (x(t,h), y(t,h))$, where

$$x(t,h) = \frac{1}{2} - \frac{m-1}{2M(t,m)}, \quad y(t,h) = \frac{m \sin t}{M^2(t,m)},$$

being $M(t,m) = \sqrt{(1-m)^2 + 4m\cos t}$ and $m = \exp(-h) > 0$. Again, by using the first equation of (20), we have

$$\exp\left(\int_0^t 2y(s)ds\right) = \exp\left(2\int_0^t \frac{dx(s)}{2x(s)-1}\right) = \frac{1-m}{M(t,m)}. \tag{22}$$

From Theorem 2 and (22),

$$\begin{aligned} T_1'(h) &= \int_0^{2\pi} \frac{x(1-x)(Dx^2 - Fxy^2 + Fy^2)}{x^2(x-1)^2 + y^2} \exp\left(2\int_0^t y(s)\,ds\right) dt \\ &= \frac{m+1}{64m^2} \int_0^{2\pi} \Big(4DM^3(t,m) + (1-m)(8D+F)M^2(t,m) - \\ &\quad (1-m)^3(3F+8D) - \frac{4D(1-m)^4}{M(t,m)} + \frac{g_1(m)F}{M^2(t,m)} - \frac{g_2(m)F}{M^4(t,m)}\Big) dt, \end{aligned}$$

where

$$\begin{aligned} g_1(m) &= (1-m)(3m^4 - 12m^3 + 2m^2 - 12m + 3), \\ g_2(m) &= (1-m)^3(m^2 - 6m + 1)(m+1)^2. \end{aligned}$$

By integrating the above equation we obtain

$$T_1'(h) = \frac{m+1}{48m^2}(d(m)D + f(m)F),$$

where $d(m)$ and $f(m)$ are the functions introduced in the statement of Theorem 7.

In order to study the zeros of T_1' for $m > 3+2\sqrt{2}$, we take

$$Z(m) = -\frac{d(m)}{f(m)} = \frac{F}{D}. \tag{23}$$

By Theorem 7 we know that $Z(m)$ tends to $-3/2$ as $m \to +\infty$ and tends to zero as $m \to 3+2\sqrt{2}$. Moreover, $Z(m)$ is monotone decreasing for $m > 3+2\sqrt{2}$.

Hence we get that for $|\varepsilon|$ small enough the period function T has at most one critical point in any compact region and it can exists only when $-3/2 < F/D < 0$. □

4. Some final remarks

(i) In Theorem 3 we only consider the critical periods that bifurcate from one of the centers of the Loud systems, the origin. Notice that in some of the cases the unperturbed system has two isochronous centers. For instance in case (d), system (20) has the two isochronous centers, $(0,0)$ and $(1,0)$. Consider its perturbation (21) in the family of quadratic reversible centers. Theorem 3.(d) implies that there is at most one critical point bifurcating from the periodic orbits of surrounding $(0,0)$ and that it can appear if and only if $-3/2 < \frac{F}{D} < 0$. We claim that simultaneously, no critical periods appear from the period annulus of the other center. To prove this, notice that for the perturbed system (21), the new center is $(1/1-\varepsilon D, 0)$. In fact, after the transformation $\bar{x} = 1/(1-\varepsilon D) - x$ and $\bar{y} = y$, it is moved to $(0,0)$ and the new differential equation writes as

$$\begin{cases} \dot{\bar{x}} &= -c_0\bar{y} + 2\bar{x}\bar{y}, \\ \dot{\bar{y}} &= \bar{x} + (\varepsilon D - 1)\bar{x}^2 + (4+\varepsilon F)\bar{y}^2, \end{cases}$$

where $c_0 = \frac{2}{1-\varepsilon D} - 1$. Then taking the affine transformation $(t, \bar{x}, \bar{y}) \longrightarrow (\frac{t}{\sqrt{c_0}}, c_0 x, \sqrt{c_0} y)$, we have

$$\begin{cases} \dot{x} &= -y + 2xy, \\ \dot{y} &= x + (-1 - \varepsilon D + O(\varepsilon^2))x^2 + (4+\varepsilon F)y^2. \end{cases} \tag{24}$$

Hence, as we wanted to see, by Theorem 3.(d), we know that when $-3/2 < \frac{F}{D} < 0$ no critical periods bifurcate from its period annulus, because for the above system $-\frac{F}{D} > 0$.

(ii) In Theorem 3 the period function of all the quadratic Loud isochronous centers, perturbed in the world of reversible quadratic centers, is studied by using Theorem 4. Notice that in this theorem the linear center, *i.e.* when $|\tilde{B}| + |\tilde{D}| + |\tilde{F}| = 0$ in (3), is excluded from the study. By using the same tools that in the proof of this theorem it is not difficult to check that in this case $T_1(h) \equiv 0$ and so, up to first order in ε, no critical periods appear when we study the first order perturbation of the linear isochronous center. See [7] for an study of this problem up to higher order terms.

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