TWISTED REIDEMEISTER TORSION FOR TWIST KNOTS

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ABSTRACT. The aim of this paper is to give an explicit formula for the $SL_2(\mathbb{C})$-twisted Reidemeister torsion as defined in [6] in the case of twist knots. For hyperbolic twist knots, we also prove that the twisted Reidemeister torsion at the holonomy representation can be expressed as a rational function evaluated at the cusp shape of the knot. Tables given approximations of the twisted Reidemeister torsion for twist knots on some concrete examples are also enclosed.

1. INTRODUCTION

Twist knots form a family of special two–bridge knots which include the trefoil knot and the figure eight knot. The knot group of a two–bridge knot has a particularly nice presentation with only two generators and a single relation. One could find our interest in this family of knots in the following facts: first, twist knots except the trefoil knot are hyperbolic, and second twist knots are not fibered except the trefoil knot and the figure eight knot (see Remark 2 of the present paper for details).

In [5], the first author introduced the notion of the twisted Reidemeister torsion in the adjoint representation associated to an irreducible representation of a knot group. In [6, Main Theorem], one can find an "explicit" formula which gives the value of this torsion for fibered knots in terms of the map induced by the monodromy of the knot at the level of the character variety of the knot exterior. In particular, a practical formula of the twisted Reidemeister torsion for torus knots is presented in [6, Section 6.2]. One can also find an explicit formula for the twisted Reidemeister torsion for the figure eight knot in [6, Section 7]. More recently, the last author found [26, Theorem 3.1.2] an interpretation of the twisted Reidemeister torsion in terms of the twisted Reidemeister torsion polynomial and gave an explicit formula of the twisted torsion for the twist knot 5_2.

In the present paper we give an explicit formula of the twisted Reidemeister torsion for all twist knots. Since twist knots are particular two–bridge knots, this

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paper is a first step in the understanding of the twisted Reidemeister torsion for two–bridge knots.

Organization

We recall some properties of twist knots in Section 2. In Section 3, we give a recursive description of the character variety of twist knots and an explicit formula for the cusp shape of hyperbolic twist knots. In Section 4, we recall the definition of the twisted Reidemeister torsion for a knot and an algebraic description of this invariant. We give formulas for the twisted Reidemeister torsion for twist knots in Section 5. In particular, we show in Subsection 5.3 that the twisted Reidemeister torsion for a hyperbolic twist knot at its holonomy representation is expressed by using the cusp shape of the hyperbolic structure of the knot complement. The last part of the paper (Subsection 5.5) deals with some remarks on the behavior of the sequence of twisted Reidemeister torsions for twist knots at the holonomy indexed by the number of crossings. The Appendix contains concrete examples and tables of the values of the twisted torsion for some explicit twist knots.

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2. Twist knots

Notation. According to the notation of [8], the twist knots are written $J(\pm 2, n)$, where n is an integer. The n crossings are right–handed when $n > 0$ and left–handed when $n < 0$.

Here is some important facts about twist knots.

(1) By definition, if $n \in \{0, 1, -1\}$ then $J(\pm 2, n)$ is the unknot. In all this paper, we focus on the knots $J(\pm 2, n)$ with $|n| \geqslant 2$.
(2) If we rotate the diagram of $J(2, n)$ by a 90 degrees angle clockwise then we get a diagram of a rational knot in the sense of Conway. In rational knot notation, $J(2, n)$, $n > 0$, is represented by the continued fraction

$$[n, -2] = \frac{1}{-2 + \frac{1}{n}} = \frac{-n}{2n-1}.$$

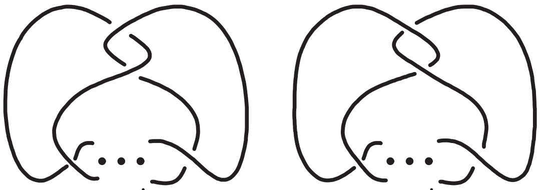

FIGURE 1. The diagrams of $J(2,n)$ and $J(-2,n)$, $n > 0$.

Therefore in two–bridge knot notation, for $n > 0$, we have

$$J(2,n) = b(2n-1,-n) = b(2n-1,n-1).$$

Similarly, the knot $J(2,-n)$, $n > 0$, is represented by the continued fraction $[-n,-2]$, therefore $J(2,-n) = b(2n+1,n+1)$.

On the other hand, the knot $J(-2,n)$, $n > 0$, is $b(2n+1,n)$ and $J(-2,-n)$, $n > 0$, is $b(2n-1,n)$.

(3) Another important observation is the following: the twist knot $J(\pm 2, 2m+1)$ is isotopic to $J(\mp 2, 2m)$ (see [8]) and moreover $J(\pm 2, n)$ is the mirror image of $J(\mp 2, -n)$.

As a consequence, *we will only consider the twist knots* $J(2,n)$, *where* n *is an integer such that* $|n| \geqslant 2$. From now on, we adopt in the sequel the following terminology: a twist knot $J(2,n)$ is said to be *even* (resp. *odd*) if n is even (resp. odd).

Example. Note that in Rolfsen's table [18], the trefoil knot $3_1 = J(2,2) = b(3,1)$, the figure eight knot $4_1 = J(2,-2) = b(5,3)$, $5_2 = J(2,4)$ and $6_1 = J(2,-4)$ etc.

Notation. For a knot K in S^3, we let E_K (resp. $\Pi(K)$) denote the exterior (resp. the group) of K, i.e. $E_K = S^3 \setminus N(K)$, where $N(K)$ is an open tubular neighborhood of K (resp. $\Pi(K) = \pi_1(E_K)$).

Convention. Suppose that S^3 is oriented. The exterior of a knot is thus oriented and we know that it is bounded by a 2-dimensional torus T^2. This boundary inherits an orientation by the convention *"the inward pointing normal vector in the last position"*. Let $\mathrm{int}(\cdot,\cdot)$ be the intersection form on the boundary torus induced by its orientation. The peripheral subgroup $\pi_1(T^2)$ is generated by the meridian–longitude system (μ,λ) of the knot. If we suppose that the knot is oriented, then μ is oriented by the convention that the linking number of the knot with μ is $+1$. Next, λ is the oriented preferred longitude using the rule $\mathrm{int}(\mu,\lambda) = +1$. These orientation conventions will be used in the definition of the twisted sign–refined Reidemeister torsion.

Twist knots live in the more general family of two-bridge knots. The group of such a knot admits a particularly nice Wirtinger presentation with only two generators and a single relation. Such Wirtinger presentations of groups of twist knots are given in the two following facts (see for example [18] or [8] for a proof). We distinguish even and odd cases and suppose that $m \in \mathbb{Z}$.

Fact 1. *The knot group of* $J(2, 2m)$ *admits the following presentation:*

$$\Pi(J(2,2m)) = \langle x, y \mid w^m x = y w^m \rangle \tag{1}$$

where w *is the word* $[y, x^{-1}] = yx^{-1}y^{-1}x$*.*

Fact 2. *The knot group of* $J(2, 2m+1)$ *admits the following presentation:*

$$\Pi(J(2,2m+1)) = \langle x, y \mid w^m x = y w^m \rangle \tag{2}$$

where w *is the word* $[x, y^{-1}] = xy^{-1}x^{-1}y$*.*

One can easily describe the peripheral–system (μ, λ) of a twist knot. It is expressed in the knot group as:

$$\mu = x \text{ and } \lambda = (\overleftarrow{w})^m w^m,$$

where we let $\overleftarrow{w}$ denote the word obtained from w by reversing the order of the letters.

Remark 1. The knot group of a two–bridge knot K admits a distinguished Wirtinger presentation of the following form:

$$\Pi(K) = \langle x, y \mid \Omega x = y \Omega \rangle \text{ where } \Omega = x^{\epsilon_1} y^{\epsilon_n} x^{\epsilon_2} y^{\epsilon_{n-1}} \cdots x^{\epsilon_n} y^{\epsilon_1},\ \epsilon_i = \pm 1.$$

With the above notation, for $K = J(2, 2m)$, $m \in \mathbb{Z}^*$, the word Ω is:

$$\Omega_m = \begin{cases} w^m & \text{if } m < 0, \\ x^{-1}(\overline{w})^{m-1} y^{-1} & \text{if } m > 0. \end{cases} \tag{3}$$

Here $\overline{w} = (\overleftarrow{w})^{-1}$, i.e. the word $\overline{w}$ is obtain from w by changing each of its letters by its reverse. Of course this choice is strictly equivalent to presentation (1). But in a sense, when $m > 0$ the word w^m does not give a "reduced" relation (some cancelations are possible in $w^m x w^{-m} y^{-1}$) which is not the case for Ω_m.

Some more elementary properties of twist knots are discussed in the following remark.

Remark 2.
1. The knot groups $\Pi(J(2, 2m+1))$ and $\Pi(J(2, -2m))$ are isomorphic by interchanging x and y. Therefore it is enough to consider the case of even twist knots.
2. *The genus of a twist knot is* 1 ([1, p. 99]). Thus, the only torus knot which is a twist knot is the trefoil knot 3_1.

(3) *Twist knots are hyperbolic knots* except in the case of the trefoil knot (see [13]).

(4) It is well known (see for example [18]) that *the Alexander polynomial of the twist knot* $J(2, 2m)$ is given by

$$\Delta_{J(2,2m)}(t) = mt^2 + (1 - 2m)t + m.$$

Moreover, using the mirror image invariance of the Alexander polynomial, one has $\Delta_{J(2,2m+1)}(t) = \Delta_{J(2,-2m)}(t)$. Thus the Alexander polynomial becomes monic if and only if m is ± 1. As a consequence, *the knot* $J(2, 2m)$ *is not fibered* (since its Alexander polynomial is not monic) except for $m = \pm 1$, that is to say except for the trefoil knot and the figure eight knot, which are known to be fibered knots.

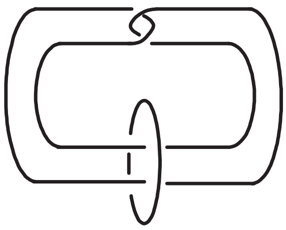

FIGURE 2. The Whitehead link.

(5) Twist knot exteriors can be obtained by surgery on the trivial component of the Whitehead link $\mathcal{W}$ (see Figure 2). More precisely, $E_{J(2,-2m)} = \mathcal{W}(1/m)$ *is obtained by a surgery of slope* $1/m$ on the trivial component of the Whitehead link $\mathcal{W}$, see [18, p. 263] for a proof. As a consequence, twist knots are *all virtually fibered*, see [12].

3. ON THE $SL_2(\mathbb{C})$-CHARACTER VARIETY AND NON-ABELIAN REPRESENTATIONS

3.1. **Review on the** $SL_2(\mathbb{C})$**-character variety of knot groups.** Given a finitely generated group π, we let

$$R(\pi; SL_2(\mathbb{C})) = \mathrm{Hom}(\pi; SL_2(\mathbb{C}))$$

denote the space of $SL_2(\mathbb{C})$-representations of π. As usual, this space is endowed with the compact–open topology. Here π is assumed to have the discrete topology and the Lie group $SL_2(\mathbb{C})$ is endowed with the usual one.

A representation $\rho\colon \pi \to SL_2(\mathbb{C})$ is called *abelian* if $\rho(\pi)$ is an abelian subgroup of $SL_2(\mathbb{C})$. A representation ρ is called *reducible* if there exists a *proper* subspace $U \subset \mathbb{C}^2$ such that $\rho(g)(U) \subset U$, for all $g \in \pi$. Of course, any abelian representation is reducible (while the converse is false in general). A non reducible representation is called *irreducible*.

The group $\mathrm{SL}_2(\mathbb{C})$ acts on the representation space $R(\pi;\mathrm{SL}_2(\mathbb{C}))$ by conjugation, but the naive quotient $R(\pi;\mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is not Hausdorff in general. Following [4], we will focus on the *character variety* $X(\pi) = X(\pi;\mathrm{SL}_2(\mathbb{C}))$ which is the set of *characters* of π. Associated to the representation $\rho \in R(\pi;\mathrm{SL}_2(\mathbb{C}))$, its character $\chi_\rho\colon \pi \to \mathbb{C}$ is defined by $\chi_\rho(g) = \mathrm{tr}(\rho(g))$, where tr denotes the trace of matrices. In some sense $X(\pi)$ is the "algebraic quotient" of $R(\pi;\mathrm{SL}_2(\mathbb{C}))$ by the action by conjugation of $\mathrm{PSL}_2(\mathbb{C})$. It is well known that $R(\pi,\mathrm{SL}_2(\mathbb{C}))$ and $X(\pi)$ have the structure of complex algebraic affine varieties (see [4]).

Let $R^{\mathrm{irr}}(\pi;\mathrm{SL}_2(\mathbb{C}))$ denote the subset of irreducible representations of π, and $X^{\mathrm{irr}}(\pi)$ denote its image under the map $R(\pi;\mathrm{SL}_2(\mathbb{C})) \to X(\pi)$. Note that two irreducible representations of π in $\mathrm{SL}_2(\mathbb{C})$ with the same character are conjugate by an element of $\mathrm{SL}_2(\mathbb{C})$, see [4, Proposition 1.5.2]. Similarly, we write $X^{\mathrm{nab}}(\Pi(K))$ for the image of the set $R^{\mathrm{nab}}(\Pi(K))$ of non–abelian representations in $X(\Pi(K))$. Note that $X^{\mathrm{irr}}(\Pi(K)) \subset X^{\mathrm{nab}}(\Pi(K))$ and observe that this inclusion is strict in general.

3.2. **Review on the character varieties of two–bridge knots.** Here we briefly review Riley's method [17] for describing the non–abelian part of the representation space of two–bridge knot groups.

The knot group of a two–bridge knot K admits a presentation of the following form:

$$\Pi(K) = \langle x, y \,|\, \Omega x = y\Omega \rangle \text{ where } \Omega = x^{\epsilon_1} y^{\epsilon_n} x^{\epsilon_2} y^{\epsilon_{n-1}} \cdots x^{\epsilon_n} y^{\epsilon_1},\ \epsilon_i = \pm 1. \tag{4}$$

We use the following notation:

$$C = \begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix},\ D = \begin{pmatrix} t & 0 \\ -tu & 1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix},\ D_1 = \begin{pmatrix} t & 0 \\ -u & t^{-1} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} t & t^{-1} \\ 0 & t^{-1} \end{pmatrix},\ D_2 = \begin{pmatrix} t & 0 \\ -tu & t^{-1} \end{pmatrix}.$$

Remark 3. Note that C and D can be obtained by conjugating tC_1 and tD_1 by the diagonal matrix $U = \left(\begin{smallmatrix} t^{-1/2} & 0 \\ 0 & t^{1/2} \end{smallmatrix}\right)$, and replacing t^2 by t. We also note that C_2 and D_2 are conjugate to C_1 and D_1 via U.

Fact 3. *If M_1, M_2 are non–commuting elements in* $\mathrm{SL}_2(\mathbb{C})$ *with same traces, then there exists a pair $(t,u) \in \mathbb{C}^2$ such that M_1 and M_2 are conjugated to C_2 and D_2 respectively.*

Combining Fact 3 and Remark 3, we obtain:

Claim 4. *If M_1, M_2 are non-commuting elements in* $\mathrm{SL}_2(\mathbb{C})$ *with same traces, then M_1 and M_2 are simultaneously conjugated to C_1 and D_1 respectively.*

For a two–bridge knot K, x and y are conjugate elements in $\Pi(K)$ and represent meridians of the knot, therefore $\rho(x)$ and $\rho(y)$ have same traces. If $\rho(x)$ and $\rho(y)$ do not commute, i.e. if ρ is non–abelian, then up to a conjugation one can assume that $\rho(x)$ and $\rho(y)$ are the matrices C_1 and D_1 respectively.

Proposition 5. *The homomorphism $\rho\colon \Pi(K) \to \mathrm{GL}_2(\mathbb{C})$ defined by $\rho(x) = C$ and $\rho(y) = D$ is a non–abelian representation of $\Pi(K)$ if and only if the pair $(t, u) \in \mathbb{C}^2$ satisfies the following equation*

$$w_{1,1} + (1 - t)w_{1,2} = 0 \tag{5}$$

where $W = \rho(\Omega) = \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix}$.

Conversely, every non–abelian representation is conjugated to a representation satisfying Equation (5).

Proof. A direct matrix computation shows that the requirement $WC = DW$ is equivalent to the following two equations:

$$w_{1,1} = (t - 1)w_{1,2} \text{ and } w_{2,1} = -tuw_{1,2}.$$

The second equation is just a consequence of the fact that W is palindromic and therefore is not really a requirement at all. Indeed, using W^T to denote the transpose of the matrix W, we have $W^T = D^{T\epsilon_1}C^{T\epsilon_n}D^{T\epsilon_2}C^{T\epsilon_{n-1}} \cdots D^{T\epsilon_n}C^{T\epsilon_1}$. Now $C^T = VDV^{-1}$ and $D^T = VCV^{-1}$ where $V = \left(\begin{smallmatrix} (-tu)^{1/2} & 0 \\ 0 & (-tu)^{-1/2} \end{smallmatrix}\right)$. This provides $V^{-1}W^TV = W$, which immediately gives the second equation above. □

Notation. We let $\phi_K(t, u) = w_{1,1} + (1 - t)w_{1,2}$ denote the left hand side of Equation (5) and call it the *Riley polynomial* of K.

The same proof provides a sometimes more convenient result:

Proposition 6. *The homomorphism $\rho\colon \Pi(K) \to \mathrm{SL}_2(\mathbb{C})$ defined by $\rho(x) = C_1$ and $\rho(y) = D_1$ is a non–abelian representation of $\Pi(K)$ if and only if the pair $(t, u) \in \mathbb{C}^2$ satisfies the following equation:*

$$w_{1,1} + (t^{-1} - t)w_{1,2} = 0 \tag{6}$$

where $W = \rho(\Omega)$.

Conversely, every non–abelian representation is conjugated to a representation satisfying Equation (6).

Similarly, if $\rho(x) = C_2$ and $\rho(y) = D_2$, then Riley's equation is

$$w_{1,1} + (1 - t^2)w_{1,2} = 0.$$

3.3. The holonomy representation of a hyperbolic twist knot.

3.3.1. *Some generalities.* It is well known that the complete hyperbolic structure of a hyperbolic knot complement determines a unique discrete faithful representation of the knot group in $\mathrm{PSL}_2(\mathbb{C})$, called the *holonomy representation*. It is proved [20, Proposition 1.6.1] that such a representation lifts to $\mathrm{SL}_2(\mathbb{C})$ and determines two representations in $\mathrm{SL}_2(\mathbb{C})$.

The trace of the peripheral–system at the holonomy is ± 2 because their images by the holonomy are parabolic matrices. More precisely, Calegari proved [2] that the trace of the longitude at the holonomy is always -2 and the trace of the meridian at the holonomy is ± 2, depending on the choice of the lift. We summarize all this in the following important fact.

Fact 7. *Let ρ_0 be one of the two lifts of the discrete and faithful representation associated to the complete hyperbolic structure of a hyperbolic knot K and let T^2 denote the boundary of the knot exterior. The restriction of ρ_0 to $\pi_1(T^2)$ is conjugate to the parabolic representation such that*

$$\mu \mapsto \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \lambda \mapsto \begin{pmatrix} -1 & \mathfrak{c} \\ 0 & -1 \end{pmatrix}.$$

Here $\mathfrak{c} = \mathfrak{c}(\lambda, \mu) \in \mathbb{C}$ is called the cusp shape of K.

Remark 4. The universal cover of the exterior of a hyperbolic knot is the hyperbolic 3-space $\mathbb{H}^3$. The cusp shape can be seen as the ratio of the translations of the parabolic isometries of $\mathbb{H}^3$ induced by projections to $\mathrm{PSL}_2(\mathbb{C})$. Of course, the cusp shape $\mathfrak{c} = \mathfrak{c}(\lambda, \mu)$ depends on the choice of the basis (μ, λ) for $\pi_1(T^2)$. A change in the basis of $\pi_1(T^2)$ shifts $\mathfrak{c}$ by an integral Möbius transformation.

3.3.2. *Holonomy representations of twist knots.* This subsection is concerned with the $\mathrm{SL}_2(\mathbb{C})$-representations which are lifts of the holonomy representation in the special case of (hyperbolic) twist knots. Especially, we want to precise the images, up to conjugation, of the group generators x and y (see the group presentation (4)).

Lemma 8. *Let K be a hyperbolic two–bridge knot and suppose that its knot group admits the following presentation $\Pi(K) = \langle x, y \,|\, \Omega x = y\Omega \rangle$. If ρ_0 denotes a lift in $\mathrm{SL}_2(\mathbb{C})$ of the holonomy representation, then ρ_0 is given by, up to conjugation,*

$$x \mapsto \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \pm \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix},$$

where u is a root of Riley's equation $\phi_K(1, u) = 0$ of K.

Proof. It follows from Fact 7 that each lift of the holonomy representation maps the meridian to $\pm \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. It is known that the lifts of the holonomy representation are irreducible $\mathrm{SL}_2(\mathbb{C})$-representations, in particular, non–abelian ones. Hence we can construct the $\mathrm{SL}_2(\mathbb{C})$-representations which are conjugate to the lifts of the holonomy representation by using roots of Riley's equation. Using Section 3.2, if

x is sent to $\pm\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ then y is sent to $\pm\left(\begin{smallmatrix}1&0\\-u&1\end{smallmatrix}\right)$, where u is a root of Riley's equation $\phi_K(1,u) = 0$. □

Notation. If we let A be an element of $\mathrm{SL}_2(\mathbb{C})$, then the adjoint actions of A and $-A$ are same. So, we use the $\mathrm{SL}_2(\mathbb{C})$-representation such that

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} \quad \text{(where u is such that $\phi_K(1,u) = 0$)}$$

as a lift of the holonomy representation and we improperly call it the *holonomy representation*.

3.4. **On parabolic representations of twist knot groups.** In this subsection, we are interested in the *parabolic representations* of (hyperbolic) two–bridge knot groups and especially twist knot groups. The holonomy representation is one of them. Lemma 8 characterizes the holonomy representation algebraically and says that it corresponds to a root of Riley's equation $\phi_K(1,u) = 0$. A natural and interesting question is the following: whose roots of Riley's equation $\phi_K(1,u) = 0$ correspond to the holonomy representation? Here we will give a geometric characterization of such roots.

3.4.1. *Crucial remarks.* We begin this section by some elementary but important remarks on the roots of Riley's equation $\phi_K(1,u) = 0$ corresponding to holonomy representations.

(1) One can first notice that such roots are necessarily *complex numbers which are not real*, because the discrete and faithful representation is irreducible and *not* conjugate to a real representation (i.e. a representation such that the image of each element is a matrix with real entries).
(2) One can also observe that holonomy representations correspond to *a pair of complex conjugate roots* of Riley's equation $\phi_K(1,u) = 0$ as it is easy to see.

3.4.2. *Generalities: the case of two–bridge knots.* Let K be a hyperbolic two–bridge knot. Suppose that a presentation of the knot group $\Pi(K)$ is given as in Equation (4) by

$$\Pi(K) = \langle x, y \,|\, \Omega x = y\Omega \rangle, \text{ where Ω is a word in x, y.}$$

The longitude of K is of the form: $\lambda = \overleftarrow{\Omega}\,\Omega\, x^n$. Here n is an integer such that the sum of the exponents in the word λ is 0 and we repeat that $\overleftarrow{\Omega}$ denotes the word obtained from Ω by reversing the order of the letters.

Let $\rho\colon \Pi(K) \to \mathrm{SL}_2(\mathbb{C})$ be a representation such that:

$$x \mapsto \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \pm\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}$$

where u is necessarily a root of Riley's equation $\phi_K(1, u) = 0$. Suppose that

$$\rho(\Omega) = \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix}$$

where $w_{i,j}$ is a polynomial in u for all $i, j \in \{1, 2\}$.

Riley's method gives us the following identities (see Section 3.2):

$$w_{1,1} = 0 \text{ and } uw_{1,2} + w_{2,1} = 0.$$

Thus

$$\rho(\Omega) = \begin{pmatrix} 0 & w_{1,2} \\ -uw_{1,2} & w_{2,2} \end{pmatrix}.$$

The fact that $\rho(w) \in \mathrm{SL}_2(\mathbb{C})$ further gives the following equation:

$$uw_{1,2}^2 = 1. \tag{7}$$

The crucial point to compute $\rho(\lambda)$ is to express $\rho(\overleftarrow{w})$ with the help of $\rho(w)$. Consider the diagonal matrix:

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}),$$

where i stands for a square root of -1. Let Ad denote the adjoint representation of the Lie group $\mathrm{SL}_2(\mathbb{C})$. Then the following identities hold:

$$\rho(x^{-1}) = Ad_{\mathbf{i}}(\rho(x)),\ \rho(y^{-1}) = Ad_{\mathbf{i}}(\rho(y)) \text{ and } \rho(\Omega^{-1}) = Ad_{\mathbf{i}}(\rho(\overleftarrow{\Omega})).$$

Thus, we have

$$\rho(\overleftarrow{\Omega}) = Ad_{\mathbf{i}}(\rho(\Omega^{-1})) = \begin{pmatrix} w_{2,2} & w_{1,2} \\ w_{2,1} & w_{1,1} \end{pmatrix}.$$

Next, a direct computation gives:

$$\rho(\lambda) = \rho(\overleftarrow{\Omega})\rho(\Omega)\rho(x)^n = \begin{pmatrix} -uw_{1,2}^2 & -nuw_{1,2}^2 + 2w_{1,2}w_{2,2} \\ 0 & -uw_{1,2}^2 \end{pmatrix}. \tag{8}$$

Combining Equations (7) and (8), we obtain

$$\rho(\lambda) = \begin{pmatrix} -1 & -n + 2w_{1,2}w_{2,2} \\ 0 & -1 \end{pmatrix}. \tag{9}$$

And we conclude that the *cusp shape of K* is

$$\mathfrak{c} = n - 2w_{1,2}w_{2,2}. \tag{10}$$

Remark 5. In particular, Equation (9) gives us, by an elementary and direct computation, Calegari's result [2]: $\operatorname{tr}\rho_0(\lambda) = -2$ for the discrete faithful representation ρ_0 associated to the complete hyperbolic structure of the exterior of a (hyperbolic) two–bridge knot.

3.4.3. *The special case of twist knots.* In the case of hyperbolic twist knots, we can further estimate $w_{i,j}$ in Equation (10). In fact, we only consider the case where $K = J(2, 2m)$ in what follows. The group $\Pi(K)$ of such a knot has the following presentation:

$$\Pi(K) = \langle x, y \,|\, w^m x = y w^m \rangle,$$

where w is the commutator $[y, x^{-1}]$ (see Fact 1, Section 2). A direct computation of the commutator $\rho(w) = [\rho(y), \rho(x)^{-1}]$ gives:

$$W = \rho(w) = \begin{pmatrix} 1-u & -u \\ u^2 & u^2+u+1 \end{pmatrix} \quad \text{(where } u \text{ is such that } \phi_K(1,u) = 0\text{)}.$$

Using the Cayley–Hamilton identity, it is easy to obtain the following recursive formula for the powers of the matrix W:

$$W^k - (u^2+2)W^{k-1} + W^{k-2} = 0, \; k \geqslant 2. \tag{11}$$

Equation (11) implies

$$w_{2,2} = -(u+2)w_{1,2}.$$

Since $n = 0$ and $uw_{1,2}^2 = 1$, the *cusp shape of the twist knot K* is:

$$\mathfrak{c} = n + (2u+4)w_{1,2}^2 = \frac{2u+4}{u}.$$

In other words, the root u_0 of Riley's equation $\phi_K(1, u) = 0$ corresponding to the holonomy representation satisfies the following equation:

$$u_0 = \frac{4}{\mathfrak{c} - 2}, \tag{12}$$

where $\mathfrak{c}$ is the cusp shape of the knot exterior.

Remark 6. Equation (12) gives a geometric characterization of the (pair of complex conjugate) roots of Riley's equation $\phi_K(1, u) = 0$ associated to the holonomy representation in terms of the cusp shape, a geometric quantity associated to each cusped hyperbolic 3-dimensional manifold.

3.5. **The character varieties of twist knots: a recursive description.** T. Le [11] gives a recursive description of the $SL_2(\mathbb{C})$-character variety of two–bridge knots and apply it to obtain an *explicit* description of the $SL_2(\mathbb{C})$-character variety of torus knots. Here we apply his method to obtain an explicit recurrent description of the $SL_2(\mathbb{C})$-character variety of twist knots.

Let $n = 2m$ or $2m + 1$, recall that $\Pi(J(2, n)) = \langle x, y \,|\, \Omega_m x = y\Omega_m \rangle$, where Ω_m is a word in x, y (see Facts 1–2).

Notation. Let $\gamma \in \Pi(J(2, n))$. Following a notation introduced in [3], we let

$$I_\gamma : X(\Pi(J(2, n))) \to \mathbb{C}$$

be the trace–function defined by $I_\gamma : \rho \mapsto \mathrm{tr}(\rho(\gamma))$.

Let $a = I_x$, $b = I_{xy}$ and recall the following useful formulas for $A, B, C \in \mathrm{SL}_2(\mathbb{C})$:

$$\operatorname{tr}(A^{-1}) = \operatorname{tr}(A) \text{ and } \operatorname{tr}(AB) = \operatorname{tr}(BA), \tag{13}$$

$$\operatorname{tr}(AB) + \operatorname{tr}(A^{-1}B) = \operatorname{tr}(A)\operatorname{tr}(B), \tag{14}$$

$$\operatorname{tr}(ABA^{-1}B^{-1}) = -2 - \operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(AB) + (\operatorname{tr}(A))^2 + (\operatorname{tr}(B))^2 + (\operatorname{tr}(AB))^2. \tag{15}$$

As x and y are conjugate elements in $\Pi(J(2,n))$, we have $I_y = a = I_x$. If γ is a word in the letters x and y, then I_γ can always be expressed as a polynomial function in a and b. For example, combining the usual Formulas (13), (14) and (15), one can easily observe that for $w = [y, x^{-1}] = yx^{-1}y^{-1}x$:

$$I_w = -2 - a^2b + 2a^2 + b^2. \tag{16}$$

The character variety of $\Pi(J(2,n))$ is thus parametrized by a and b. Here is a practical description of it:

(1) We first consider the abelian part of the character variety. It is easy to see that the equation $a^2 - b - 2 = 0$ determines the abelian part of the character variety of any knot group.
(2) Next, consider the non-abelian part of the character variety of $\Pi(J(2,n))$, suppose that the length of the word Ω_m is $2k+2$ (we know that the length of Ω_m is even). According to [11, Theorem 3.3.1], the non–abelian part of the character variety of $\Pi(J(2,n))$, $n = 2m$ or $2m+1$, is determined by the polynomial equation:

$$\mathbf{\Phi}_m(a,b) = 0,$$

where

$$\mathbf{\Phi}_m(a,b) = I_{\Omega_m} - I_{\Omega'_m} + \cdots + (-1)^k I_{\Omega_m^{(k)}} + (-1)^{k+1}.$$

Here we adopt the following notation: if Λ is a word then Λ' denotes the word obtained from Λ by deleting the two end letters.

Let us give the two simplest examples to illustrate this general result and find again some well–known facts.

Example 1. The trefoil knot 3_1 is the twist knot $J(2,2)$. With the above notation, one has $\Omega_1 = x^{-1}y^{-1}$. Thus applying the above method, the non–abelian part of the character variety is given by the polynomial equation:

$$\mathbf{\Phi}_1(a,b) = I_{x^{-1}y^{-1}} - 1 = 0,$$

which reduces to

$$b = 1.$$

Example 2. The figure eight knot 4_1 is the twist knot $J(2,-2)$. With the above notation, one has $\Omega_{-1} = x^{-1}yxy^{-1}$. Thus, the non–abelian part of the character variety of the group of the figure eight knot is given by the polynomial equation:

$$\mathbf{\Phi}_{-1}(a,b) = I_{x^{-1}yxy^{-1}} - I_{yx} + 1 = 0,$$

which reduces, using Equation (16), to:

$$2a^2 + b^2 - a^2b - b = 1.$$

Now, we turn back to the general case and only consider the twist knot $J(2,2m)$ (see Item (2) of Remark 2). Recall that (see Remark 1):

(17)
$$\Pi(J(2,2m)) = \langle x, y \mid \Omega_m x = y\Omega_m\rangle, \text{ where } \Omega_m = \begin{cases} w^m & \text{if } m < 0,\\ x^{-1}(\overline{w})^{m-1}y^{-1} & \text{if } m > 0.\end{cases}$$

Here $w = [y, x^{-1}]$ and $\overline{w} = [y^{-1}, x]$ and observe that the length of the word Ω_m is $4m$, if $m < 0$, and $4m - 2$, if $m > 0$.

Our method is based on the fact that the word Ω_m in the distinguished Wirtinger presentation (17) of $\Pi(J(2,2m))$ presents a particularly nice "periodic" property. This property is discussed in the following obvious claim.

Claim 9. *For $m \in \mathbb{Z}^*$, we have*

$$\Omega_m^{(4)} = \begin{cases} w^{m+2} & \textit{if } m \leqslant -2,\\ x^{-1}(\overline{w})^{m-3}y^{-1} & \textit{if } m \geqslant 3.\end{cases}$$

Based on Claim 9, for $m \geqslant 0$, we adopt the following notation:

$$S_m^+ = I_{\Omega_{m+2}},\ T_m^+ = I_{(\Omega_{m+2})'},\ U_m^+ = I_{(\Omega_{m+2})''} \text{ and } V_m^+ = I_{(\Omega_{m+2})'''},$$

and similarly, for $m \leqslant 0$,

$$S_m^- = I_{\Omega_{m-2}},\ T_m^- = I_{(\Omega_{m-2})'},\ U_m^- = I_{(\Omega_{m-2})''} \text{ and } V_m^- = I_{(\Omega_{m-2})'''}.$$

The Cayley–Hamilton identity applied to the matrix $A^2 \in \mathrm{SL}_2(\mathbb{C})$ gives

$$A^m\left(A^4 - (\operatorname{tr} A^2)A^2 + \mathbf{1}\right) = 0.$$

Write $c(a,b) = I_{w^2}$; thus for $m \geqslant 0$, we have

$$S_{m+4}^+ - c(a,b)S_{m+2}^+ + S_m^+ = 0$$

and same relations for T_m^+, U_m^+ and V_m^+ hold. Similarly, we have

$$S_{m-4}^- - c(a,b)S_{m-2}^- + S_m^- = 0$$

and same relations for T_m^-, U_m^- and V_m^- also hold.

If we write $R_m^\pm = S_m^\pm - T_m^\pm + U_m^\pm - V_m^\pm$ for $m \in \mathbb{Z}$, then above computations can be summarized in the following claim which give us a recursive relation for $(R_m^\pm)_{m\in\mathbb{Z}}$.

Claim 10. *The sequence of polynomials* $(R^{\pm}_m(a,b))_{m\in\mathbb{Z}}$ *satisfies the following recursive relation:*

$$R^{\pm}_{m\pm4} - c(a,b)R^{\pm}_{m\pm2} + R^{\pm}_m = 0. \tag{18}$$

In Equation (18), using Formula (14) and Equation (16), we have:

$$c(a,b) = I_{w^2} = (I_w)^2-2 = 2+4a^2b-8a^2-4b^2+a^4b^2-4a^4b-2a^2b^3+4a^4+4a^2b^2+b^4.$$

Let v be such that:

$$v + v^{-1} = c(a,b). \tag{19}$$

For $m \in \mathbb{Z}^*$, we distinguish four cases to derive helpful formulas for $\mathbf{\Phi}_m$ in the case of twist knots.

- *Case 1:* $m > 0$ is even.

  Let $m = 2l$, with $l > 0$, and set $r_i = R_{2i}$. Then
  $$r_{i+2} = c(a,b)r_{i+1} - r_i, \text{ for } i \geqslant 0.$$
  As we have supposed that $v + v^{-1} = c(a,b)$ and following a standard argument in combinatorics (see e.g. [14, p. 322]), we have the general formula (which can also be proved by induction) $r_i = Mv^i + Nv^{-i}$, where M and N are determined by the initial conditions:
  $$\begin{cases} r_0 = R^+_0 = M^+_0 + N^+_0 \\ r_1 = R^+_2 = M^+_0 v + N^+_0 v^{-1} \end{cases}$$
  Further observe that:
  $$r_0 = I_{\Omega_2} - I_{\Omega'_2} + I_{\Omega''_2} - I_{\Omega'''_2} = I_{x^{-1}\overline{w}y^{-1}} - I_w + b - 1.$$
  So, we have
  $$\begin{aligned} \mathbf{\Phi}_{m+2} &= R^+_m + R^+_{m-2} + \cdots + R^+_0 \\ &= r_l + \cdots + r_0 \\ &= \sum_{i=0}^{l}(M^+_0 v^i + N^+_0 v^{-i}) \\ &= M^+_0 \frac{v^{l+1}-1}{v-1} + N^+_0 \frac{v^{-l-1}-1}{v^{-1}-1}. \end{aligned}$$
- *Case 2:* $m < 0$ is even.

  Let $m = -2l$, with $l > 0$, and set $r_i = R^-_{-2i}$, $i \geqslant 0$. Similar to the first case,
  $$r_{i+2} = c(a,b)r_{i+1} - r_i, \text{ for } i \geqslant 0.$$
  The initial conditions are
  $$\begin{cases} r_0 = R^-_0 = M^-_0 + N^-_0 \\ r_1 = R^-_{-2} = M^-_0 v + N^-_0 v^{-1} \end{cases}$$

Further observe that:

$$r_0 = I_{\Omega_{-2}} - I_{\Omega'_{-2}} + I_{\Omega''_{-2}} - I_{\Omega'''_{-2}} = I_{w^2} - I_{y(\overline{w})^{-1}x} + I_w - b.$$

Thus, we have

$$\begin{aligned}\mathbf{\Phi}_{m-2} &= R^-_m + R^-_{m+2} + \cdots + R^-_0 + 1\\ &= r_l + \cdots + r_0 + 1\\ &= \sum_{i=0}^{l}(M_0^- v^i + N_0^- v^{-i}) + 1\\ &= M_0^- \frac{v^{l+1}-1}{v-1} + N_0^- \frac{v^{-l-1}-1}{v^{-1}-1} + 1.\end{aligned}$$

- *Case 3:* $m > 0$ is odd.

  Let $m = 2l+1$, with $l > 0$, and set $r_i = R^+_{2i+1},\ i \geqslant 0$. Similar to the first case,

$$r_{i+2} = c(a,b)r_{i+1} - r_i, \text{ for } i \geqslant 0.$$

The initial conditions are

$$\begin{cases} r_0 = R_1^+ = M_1^+ + N_1^+\\ r_1 = R_3^+ = M_1^+ v + N_1^+ v^{-1}\end{cases}$$

Further observe that:

$$r_0 = I_{\Omega_3} - I_{\Omega'_3} + I_{\Omega''_3} - I_{\Omega'''_3} = I_{x^{-1}(\overline{w})^2y^{-1}} - I_{(\overline{w})^2} + I_{xwy} - I_w.$$

Thus,

$$\begin{aligned}\mathbf{\Phi}_{m+2} &= R^+_m + R^+_{m-2} + \cdots + R^+_1 + b - 1\\ &= r_l + \cdots + r_0 + b - 1\\ &= \sum_{i=0}^{l}(M_1^+ v^i + N_1^+ v^{-i}) + b - 1\\ &= M_1^+ \frac{v^{l+1}-1}{v-1} + N_1^+ \frac{v^{-l-1}-1}{v^{-1}-1} + b - 1.\end{aligned}$$

- *Case 4:* $m < 0$ is odd.

  Let $m = -2l-1$, with $l > 0$, and set $r_i = R^-_{-2i-1},\ i \geqslant 0$. Similar to the first case,

$$r_{i+2} = c(a,b)r_{i+1} - r_i, \text{ for } i \geqslant 0.$$

The initial conditions are

$$\begin{cases} r_0 = R_{-1}^- = M_1^- + N_1^-\\ r_1 = R_{-3}^- = M_1^- v + N_1^- v^{-1}\end{cases}$$

Further observe that:

$$r_0 = I_{\Omega_{-3}} - I_{\Omega'_{-3}} + I_{\Omega''_{-3}} - I_{\Omega'''_{-3}} = I_{w^3} - I_{y(\overline{w})^{-2}x} + I_{w^2} - I_{y^{-1}w^{-1}x^{-1}}.$$

Similarly to the previous case, we have:

$$\begin{aligned}
\boldsymbol{\Phi}_{m-2} &= R^-_m + R^-_{m+2} + \cdots + R^-_{-1} + I_w - b + 1 \\
&= r_l + \cdots + r_0 + I_w - b + 1 \\
&= \sum_{i=0}^{l} (M^-_1 v^i + N^-_1 v^{-i}) + I_w - b + 1 \\
&= M^-_1 \frac{v^{l+1} - 1}{v - 1} + N^-_1 \frac{v^{-l-1} - 1}{v^{-1} - 1} - a^2 b + 2a^2 + b^2 - b - 1.
\end{aligned}$$

If we adopt the following notation:

$$M^{\pm}_j(v, l) = M^{\pm}_j \frac{v^{l+1} - 1}{v - 1} \text{ and } N^{\pm}_j(v, l) = N^{\pm}_j \frac{v^{-l-1} - 1}{v^{-1} - 1}, \quad j = 0, 1, \tag{20}$$

then we summarize our computations in the following proposition.

Proposition 11. *The polynomial equation which describes the character variety of the group of the twist knot $J(2, n)$, where $n = 2m$ or $2m+1$, is given by $\boldsymbol{\Phi}_m(a, b) = 0$ where the sequence $(\boldsymbol{\Phi}_m(a, b))_{m\in\mathbb{Z}}$ is recursively defined as follows:*

$$\boldsymbol{\Phi}_0(a, b) = 1, \quad \boldsymbol{\Phi}_1(a, b) = b - 1, \quad \boldsymbol{\Phi}_{-1}(a, b) = -a^2 b + 2a^2 + b^2 - b - 1,$$

and for $m > 1$

$$\boldsymbol{\Phi}_{m+2}(a, b) = \begin{cases} M^+_0(v, l) + N^+_0(v, l) & \text{if } m = 2l \text{ is even,} \\ M^+_1(v, l) + N^+_1(v, l) + b - 1 & \text{if } m = 2l + 1 \text{ is odd,} \end{cases} \tag{21}$$

and

$$\boldsymbol{\Phi}_{-m-2}(a, b) = \begin{cases} M^-_0(v, l) + N^-_0(v, l) + 1 & \text{if } m = 2l \text{ is even,} \\ M^-_1(v, l) + N^-_1(v, l) - a^2 b + 2a^2 + b^2 - b - 1 & \text{if } m = 2l + 1 \text{ is odd.} \end{cases} \tag{22}$$

Here v is defined in Equation (19) and $M^{\pm}_k(v, l), N^{\pm}_k(v, l)$ in Equation (20).

Remark 7. Observe that in Equations (21) and (22), the part $M^{\pm}_k(v, l) + N^{\pm}_k(v, l)$ is the "recursive" part. In Equation (21) the part $b - 1$ corresponds to $\boldsymbol{\Phi}_1(a, b)$, and in Equation (22) the part $-a^2 b + 2a^2 + b^2 - b - 1$ corresponds to $\boldsymbol{\Phi}_{-1}(a, b)$, see Examples 1 and 2.

4. Review on the twisted Reidemeister torsion and twisted polynomial torsion

4.1. **Preliminaries: the sign-determined torsion of a CW-complex.** We review the basic notions and results about the sign–determined Reidemeister torsion introduced by Turaev which are needed in this paper. Details can be found in Milnor's survey [15] and in Turaev's monograph [23].

Torsion of a chain complex. Let $C_* = (0 \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \longrightarrow 0)$ be a chain complex of finite dimensional vector spaces over $\mathbb{C}$. Choose a basis $\mathbf{c}^i$ for C_i and a basis $\mathbf{h}^i$ for the i-th homology group $H_i = H_i(C_*)$. The torsion of C_* with respect to these choices of bases is defined as follows.

Let $\mathbf{b}^i$ be a sequence of vectors in C_i such that $d_i(\mathbf{b}^i)$ is a basis of $B_{i-1} = \mathrm{im}(d_i\colon C_i \to C_{i-1})$ and let $\widetilde{\mathbf{h}}^i$ denote a lift of $\mathbf{h}^i$ in $Z_i = \ker(d_i\colon C_i \to C_{i-1})$. The set of vectors $d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i$ is a basis of C_i. Let $[d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i/\mathbf{c}^i] \in \mathbb{C}^*$ denote the determinant of the transition matrix between those bases (the entries of this matrix are coordinates of vectors in $d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i$ with respect to $\mathbf{c}^i$). The *sign-determined Reidemeister torsion* of C_* (with respect to the bases $\mathbf{c}^*$ and $\mathbf{h}^*$) is the following alternating product (see [22, Definition 3.1]):

$$\mathrm{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) = (-1)^{|C_*|} \cdot \prod_{i=0}^{n} [d_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^i\mathbf{b}^i/\mathbf{c}^i]^{(-1)^{i+1}} \in \mathbb{C}^*. \tag{23}$$

Here

$$|C_*| = \sum_{k\geqslant 0} \alpha_k(C_*)\beta_k(C_*),$$

where $\alpha_i(C_*) = \sum_{k=0}^{i} \dim C_k$ and $\beta_i(C_*) = \sum_{k=0}^{i} \dim H_k$.

The torsion $\mathrm{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*)$ does not depend on the choices of $\mathbf{b}^i$ and $\widetilde{\mathbf{h}}^i$. Note that if C_* is acyclic (i.e. if $H_i = 0$ for all i), then $|C_*| = 0$.

Torsion of a CW-complex. Let W be a finite CW-complex and ρ be a $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(W)$. We define the $\mathfrak{sl}_2(\mathbb{C})_\rho$-twisted chain complex of W to be

$$C_*(W; \mathfrak{sl}_2(\mathbb{C})_\rho) = C_*(\widetilde{W}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(W)]} \mathfrak{sl}_2(\mathbb{C})_\rho.$$

Here $C_*(\widetilde{W}; \mathbb{Z})$ is the complex of the universal cover with integer coefficients which is in fact a $\mathbb{Z}[\pi_1(W)]$-module (via the action of $\pi_1(W)$ on $\widetilde{W}$ as the covering group), and $\mathfrak{sl}_2(\mathbb{C})_\rho$ denotes the $\mathbb{Z}[\pi_1(W)]$-module via the composition $Ad \circ \rho$, where $Ad\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C})), A \mapsto Ad_A$, is the adjoint representation. The chain complex $C_*(W; \mathfrak{sl}_2(\mathbb{C})_\rho)$ computes the $\mathfrak{sl}_2(\mathbb{C})_\rho$-twisted homology of W which we denote as $H_*^\rho(W) = H_i(W; Ad \circ \rho)$.

Let $\left\{e_1^{(i)}, \ldots, e_{n_i}^{(i)}\right\}$ be the set of i-dimensional cells of W. We lift them to the universal cover and we choose an arbitrary order and an arbitrary orientation for the cells $\left\{\tilde{e}_1^{(i)}, \ldots, \tilde{e}_{n_i}^{(i)}\right\}$. If $\mathcal{B} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is an orthonormal basis of $\mathfrak{sl}_2(\mathbb{C})$, then we consider the corresponding basis over $\mathbb{C}$

$$\mathbf{c}^i_{\mathcal{B}} = \left\{\tilde{e}_1^{(i)} \otimes \mathbf{a}, \tilde{e}_1^{(i)} \otimes \mathbf{b}, \tilde{e}_1^{(i)} \otimes \mathbf{c}, \ldots, \tilde{e}_{n_i}^{(i)} \otimes \mathbf{a}, \tilde{e}_{n_i}^{(i)} \otimes \mathbf{b}, \tilde{e}_{n_i}^{(i)} \otimes \mathbf{c}\right\}$$

of $C_i(W; \mathfrak{sl}_2(\mathbb{C})_\rho) = C_*(\widetilde{W}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(W)]} \mathfrak{sl}_2(\mathbb{C})_\rho$. Now choosing for each i a basis $\mathbf{h}^i$ for the $\mathfrak{sl}_2(\mathbb{C})_\rho$-twisted homology $H_i^\rho(W)$, we can compute the torsion

$$\mathrm{Tor}(C_*(W; \mathfrak{sl}_2(\mathbb{C})_\rho), \mathbf{c}^*_{\mathcal{B}}, \mathbf{h}^*) \in \mathbb{C}^*.$$

The cells $\left\{\tilde{e}_j^{(i)}\right\}_{0\leqslant i\leqslant \dim W, 1\leqslant j\leqslant n_i}$ are in one–to–one correspondence with the cells of W, their order and orientation induce an order and an orientation for the cells $\left\{e_j^{(i)}\right\}_{0\leqslant i\leqslant \dim W, 1\leqslant j\leqslant n_i}$. Again, corresponding to these choices, we get a basis c^i over $\mathbb{R}$ for $C_i(W;\mathbb{R})$.

Choose an *homology orientation* of W, which is an orientation of the real vector space $H_*(W;\mathbb{R}) = \bigoplus_{i\geqslant 0} H_i(W;\mathbb{R})$. Let $\mathfrak{o}$ denote this chosen orientation. Provide each vector space $H_i(W;\mathbb{R})$ with a reference basis h^i such that the basis $\left\{h^0,\dots,h^{\dim W}\right\}$ of $H_*(W;\mathbb{R})$ is positively oriented with respect to $\mathfrak{o}$. Compute the sign–determined Reidemeister torsion $\mathrm{Tor}(C_*(W;\mathbb{R}), c^*, h^*) \in \mathbb{R}^*$ of the resulting based and homology based chain complex and consider its sign

$$\tau_0 = \mathrm{sgn}\,(\mathrm{Tor}(C_*(W;\mathbb{R}), c^*, h^*)) \in \{\pm 1\}.$$

We define the twisted (sign–refined) Reidemeister torsion of W to be

$$\mathrm{TOR}(W;\mathfrak{sl}_2(\mathbb{C})_\rho, \mathbf{h}^*, \mathfrak{o}) = \tau_0 \cdot \mathrm{Tor}(C_*(W;\mathfrak{sl}_2(\mathbb{C})_\rho), \mathbf{c}^*_{\mathcal{B}}, \mathbf{h}^*) \in \mathbb{C}^*. \tag{24}$$

This definition only depends on the combinatorial class of W, the conjugacy class of ρ, the choice of $\mathbf{h}^*$ and the cohomology orientation $\mathfrak{o}$. It is independent of the orthonormal basis $\mathcal{B}$ of $\mathfrak{sl}_2(\mathbb{C})$, of the choice of the lifts $\tilde{e}_j^{(i)}$, and of the choice of the positively oriented basis of $H_*(W;\mathbb{R})$. Moreover, it is independent of the order and the orientation of the cells (because they appear twice).

One can prove that TOR is invariant under cellular subdivision, homeomorphism and simple homotopy equivalences. In fact, it is precisely the sign $(-1)^{|C_*|}$ in Equation (23) which ensures all these important invariance properties to hold.

4.2. **Regularity for representations.** In this subsection, we briefly review two notions of regularity (see [6], [7] and [16]). In the sequel $K \subset S^3$ denotes an oriented knot.

We say that $\rho \in R^{\mathrm{irr}}(\Pi(K);\mathrm{SL}_2(\mathbb{C}))$ is *regular* if $\dim H_1^\rho(E_K) = 1$. This notion is invariant by conjugation and thus it is well defined for irreducible characters.

Example 3. For the trefoil knot and for the figure eight knot, one can prove that each irreducible representation of its group in $\mathrm{SL}_2(\mathbb{C})$ is regular (see [5] and [16])

Note that for a regular representation $\rho\colon \Pi(K) \to \mathrm{SL}_2(\mathbb{C})$, we have

$$\dim H_1^\rho(E_K) = 1,\ \dim H_2^\rho(E_K) = 1 \text{ and } H_j^\rho(E_K) = 0 \text{ for all } j \neq 1, 2.$$

Let γ be a simple closed unoriented curve in ∂E_K. Among irreducible representations we focus on the γ-regular ones. We say that an irreducible representation $\rho : \Pi(K) \to \mathrm{SL}_2(\mathbb{C})$ is *γ-regular*, if (see [16, Definition 3.21]):

(1) the inclusion $\iota\colon \gamma \hookrightarrow E_K$ induces a *surjective* map
$$\iota^*\colon H_1^\rho(\gamma) \to H_1^\rho(E_K),$$
(2) if $\mathrm{tr}(\rho(\pi_1(\partial E_K))) \subset \{\pm 2\}$, then $\rho(\gamma) \neq \pm\mathbf{1}$.

It is easy to see that this notion is invariant by conjugation, thus the notion of γ-regularity is well-defined for irreducible characters. Also observe that a γ-regular representation is necessarily regular (the converse is false in general for an arbitrary curve).

Example 4. For trefoil knot, all irreducible representations of its group in $\mathrm{SL}_2(\mathbb{C})$ are λ-regular (see [5]).

For the figure eight knot, one can prove that each irreducible representation of its group in $\mathrm{SL}_2(\mathbb{C})$ are λ-regular *except* two.

We close this section with an important fact concerning hyperbolic knots

Fact 12 ([16])**.** *Let K be a hyperbolic knot and consider the holonomy representation ρ_0 associated to the hyperbolic structure. Let γ be any simple closed curve in the boundary of E_K such that $\rho_0(\gamma) \neq \pm\mathbf{1}$, then ρ_0 is γ-regular.*

In particular, for a hyperbolic knot the holonomy representation ρ_0 is always μ-regular and λ-regular.

Applying [16, Proposition 3.26] to a hyperbolic knot exterior E_K, we obtain that for any simple closed curve γ, irreducible and non-γ-regular characters are contained in the set of zeros of the differential of the trace–function I_γ.

Remark 8. Since the trace–function I_γ is a regular function on the character variety, the set of irreducible and non-γ-regular characters is discrete on the components where I_γ is nonconstant.

If K is a hyperbolic knot, then the character of a holonomy representation is contained in a 1-dimensional irreducible component $X_0(\Pi(K))$ of $X(\Pi(K))$, which satisfies the following condition: if a simple closed curve γ in ∂E_K represents any nontrivial element of $\Pi(K)$ then the trace–function I_γ is nonconstant on $X_0(\Pi(K))$ (see [20, Corollary 4.5.2]). In particular, irreducible characters near the character of a holonomy representation are μ-regular and λ-regular.

4.3. **Review on the twisted Reidemeister torsion for knot exteriors.**

This subsection gives a detailed review of the constructions made in [5, Section 6]. In particular, we shall explain how to construct distinguished bases for the twisted homology groups of knot exteriors.

Canonical homology orientation of knot exteriors. We equip the exterior of K with its *canonical homology orientation* defined as follows (see [23, Section V.3]). We have

$$H_*(E_K;\mathbb{R}) = H_0(E_K;\mathbb{R}) \oplus H_1(E_K;\mathbb{R})$$

and we base this $\mathbb{R}$-vector space with $\{[\![pt]\!], [\![\mu]\!]\}$. Here $[\![pt]\!]$ is the homology class of a point, and $[\![\mu]\!]$ is the homology class of the meridian μ of K. This reference basis of $H_*(E_K;\mathbb{R})$ induces the so–called canonical homology orientation of E_K. In the sequel, we let $\mathfrak{o}$ denote the canonical homology orientation of E_K.

How to construct natural bases for the twisted homology. Let ρ be a regular $SL_2(\mathbb{C})$-representation of $\Pi(K)$ and fix a generator P^ρ of $H_0^\rho(\partial E_K)$ (i.e. P^ρ is an element in $\mathfrak{sl}_2(\mathbb{C})$ such that $Ad_{\rho(g)}(P^\rho) = P^\rho$ for all $g \in \pi_1(\partial E_K)$).

The canonical inclusion $i\colon \partial E_K \to E_K$ induces (see [5, Lemma 5.2] and [16, Corollary 3.23]) an isomorphism $i_*\colon H_2^\rho(\partial E_K) \to H_2^\rho(E_K)$. Moreover, one can prove that (see [5, Lemma 5.1] and [16, Proposition 3.18])

$$H_2^\rho(\partial E_K) \cong H_2(\partial E_K; \mathbb{Z}) \otimes \mathbb{C}.$$

More precisely, let $[\![\partial E_K]\!] \in H_2(\partial E_K; \mathbb{Z})$ be the fundamental class induced by the orientation of ∂E_K, one has $H_2^\rho(\partial E_K) = \mathbb{C}\,[[\![\partial E_K]\!] \otimes P^\rho]$.

The *reference generator* of $H_2^\rho(E_K)$ is defined by

$$h^\rho_{(2)} = i_*([[\![\partial E_K]\!] \otimes P^\rho]). \tag{25}$$

Let ρ be a λ-regular representation of $\Pi(K)$. The *reference generator* of the first twisted homology group $H_1^\rho(E_K)$ is defined by

$$h^\rho_{(1)}(\lambda) = \iota_*\left([[\![\lambda]\!] \otimes P^\rho]\right). \tag{26}$$

Remark 9. The generator $h^\rho_{(1)}(\lambda)$ of $H_1^\rho(E_K)$ depends on the orientation of λ. If we change the orientation of the longitude λ in Equation (26), then the generator is change into its reverse.

Remark 10. Note that $H_i^\rho(E_K)$ is isomorphic to the dual space of the $\mathfrak{sl}_2(\mathbb{C})_\rho$-twisted cohomology $H^i_\rho(E_K) = H^i(E_K; Ad \circ \rho)$. Reference elements defined in Equations (25) and (26) are dual from the ones defined in [6, Section 3.4].

The Reidemeister torsion for knot exteriors. Let $\rho\colon \Pi(K) \to SL_2(\mathbb{C})$ be a λ-regular representation. The *Reidemeister torsion* $\mathbb{T}^K_\lambda$ at ρ is defined to be

$$\mathbb{T}^K_\lambda(\rho) = \mathrm{TOR}\left(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho, \{h^\rho_{(1)}(\lambda), h^\rho_{(2)}\}, \mathfrak{o}\right) \in \mathbb{C}^*. \tag{27}$$

It is an invariant of knots. Moreover, if ρ_1 and ρ_2 are two λ-regular representations which have the same character then $\mathbb{T}^K_\lambda(\rho_1) = \mathbb{T}^K_\lambda(\rho_2)$. Thus $\mathbb{T}^K_\lambda$ defines a map on the set $X^{\mathrm{irr}}_\lambda(\Pi(K)) = \{\chi \in X^{\mathrm{irr}}(\Pi(K)) \mid \chi \text{ is } \lambda\text{-regular}\}$.

Remark 11. The Reidemeister torsion $\mathbb{T}^K_\lambda(\rho)$ defined in Equation (27) is exactly the inverse of the one considered in [6].

4.4. **Review on the twisted Reidemeister torsion polynomial.** To compute the twisted Reidemeister torsion for twist knots, we use techniques developed by the third author in [27]. In fact, we compute a more general invariant of knots called the twisted Reidemeister torsion polynomial. It is a sort of Alexander polynomial invariant (but with non–abelian twisted coefficients) whose “derivative coefficient” at $t = 1$ is exactly $\mathbb{T}^K_\lambda$.

Definitions. Let W be a finite CW–complex. We regard $\mathbb{Z}$ as a multiplicative group which is generated by one variable t. Let α be the surjective homomorphism from $\pi_1(W)$ to $\mathbb{Z} = \langle t \rangle$.

If ρ is a $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(W)$, we define the $\widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho$-twisted chain complex of W to be

$$C_*(W; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho) = C_*(\widetilde{W}; \mathbb{Z}) \otimes_{Ad\circ\rho\otimes\alpha} (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t)),$$

where $\sigma \cdot \gamma \otimes v \otimes f$ is identified with $\sigma \otimes Ad_{\rho(\gamma)}(v) \otimes f \cdot t^{\alpha(\gamma)}$.

The sign–defined Reidemeister torsion of W with respect to this $\widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho$-twisted chain complex is defined to be (compare with Equation (24))

$$\mathrm{TOR}(W; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \mathbf{h}^*, \mathfrak{o}) = \tau_0 \cdot \mathrm{Tor}(C_*(W; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho), \mathbf{c}^*_{\mathcal{B}}, \mathbf{h}^*) \in \mathbb{C}(t)^*.$$

Note that $\mathrm{TOR}(W; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \mathbf{h}^*, \mathfrak{o})$ is — as the Alexander polynomial — determined up to a factor t^m where $m \in \mathbb{Z}$.

Next we turn back to knots exteriors. From now on, we suppose that the CW–complex W is E_K and that the homomorphism $\alpha : \Pi(K) \to \mathbb{Z}$ is the abelianization. From [26, Proposition 3.1.1], we know that if ρ is λ-regular, then all homology groups $H_*(E_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho)$ vanishes. So if ρ is λ-regular, then we define the twisted Reidemeister torsion polynomial at ρ to be

$$\mathcal{T}^K_\lambda(\rho) = \mathrm{TOR}(W; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho, \emptyset, \mathfrak{o}) \in \mathbb{C}(t)^*. \tag{28}$$

The torsion in Equation (28) is also determined up to a factor t^m where $m \in \mathbb{Z}$. It is also shown in [26, Theorem 3.1.2] that

$$\mathbb{T}^K_\lambda(\rho) = -\lim_{t\to 1} \frac{\mathcal{T}^K_\lambda(\rho)}{(t-1)}.$$

Remark 12. It is shown by T. Kitano [10, Theorem A] that $\mathcal{T}^K_\lambda(\rho)$ agree with the twisted Alexander invariant for K and $Ad \circ \rho$.

How to compute $\mathcal{T}^K_\lambda(\rho)$ from Fox–calculus. Here we review a description of $\mathcal{T}^K_\lambda(\rho)$ from a Wirtinger presentation of $\Pi(K)$. This description comes from some results by T. Kitano [10]. For simplicity, write Φ for $(Ad \circ \rho) \otimes \alpha$. Choose and fix a Wirtinger presentation

$$\Pi(K) = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1} \rangle \tag{29}$$

of $\Pi(K)$. Let W_K be the 2-dimensional CW–complex constructed from the presentation (29) in the usual way. The 0-skeleton of W_K consists in a single 0-cell pt, the 1-skeleton is a wedge of k oriented 1-cells $x_1, \ldots x_k$ and the 2-skeleton consists in $(k-1)$ 2-cells $D_1, \ldots, D_{k-1}$ with attaching maps given by the relations $r_1, \ldots, r_{k-1}$ of presentation (29).

F. Waldhausen proved [24] that the Whitehead group of a knot group is trivial. As a result, W_K has the same simple homotopy type as E_K. So, the CW–complex W_K can be used to compute the twisted Reidemeister torsion polynomial (28).

Therefore it is enough to consider Reidemeister torsion of the $\widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho$-twisted chain complex $C_*\left(W_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho\right)$.

The twisted complex $C_*(W_K; \widetilde{\mathfrak{sl}_2}(\mathbb{C})_\rho)$ thus becomes:

$$0 \longrightarrow (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t))^{k-1} \xrightarrow{\partial_2} (\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t))^k \xrightarrow{\partial_1} \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t) \longrightarrow 0$$

where

$$\partial_1 = (\Phi(x_1 - 1), \Phi(x_2 - 1), \ldots, \Phi(x_k - 1)).$$

and ∂_2 is expressed using the Fox differential calculus and the action given by $\Phi = (Ad \circ \rho) \otimes \alpha$:

$$\partial_2 = \begin{pmatrix} \Phi(\frac{\partial r_1}{\partial x_1}) & \ldots & \Phi(\frac{\partial r_{k-1}}{\partial x_1}) \\ \vdots & \ddots & \vdots \\ \Phi(\frac{\partial r_1}{\partial x_k}) & \ldots & \Phi(\frac{\partial r_{k-1}}{\partial x_k}) \end{pmatrix} \tag{30}$$

Here we briefly denote the l-times direct sum of $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t)$ by $(\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}(t))^l$.

Let $A^1_{K,Ad\circ\rho}$ denote the $3(k-1) \times 3(k-1)$–matrix obtained from matrix (30) of ∂_2 by deleting its first row. The torsion polynomial $\mathcal{T}^K_\lambda(\rho)$ defined in Equation (28) can be described, up to a factor t^m ($m \in \mathbb{Z}$), as follows (for more details see [9, 10]):

$$\mathcal{T}^K_\lambda(\rho) = \tau_0 \cdot \frac{\det A^1_{K,Ad\circ\rho}}{\det(\Phi(x_1 - 1))}. \tag{31}$$

This rational function has the first order zero at $t = 1$ [26, Theorem 3.1.2]. The twisted Reidemeister torsion $\mathbb{T}^K_\lambda(\rho)$ is expressed as

$$\mathbb{T}^K_\lambda(\rho) = -\lim_{t\to 1} \frac{\mathcal{T}^K_\lambda(\rho)}{(t-1)} = -\lim_{t\to 1} \left(\tau_0 \cdot \frac{\det A^1_{K,Ad\circ\rho}}{(t-1)\det(\Phi(x_1 - 1))} \right). \tag{32}$$

Remark 13. From [26, Proposition 4.3.1], we can see that the twisted Reidemeister torsion $\mathbb{T}^K_\lambda$ associated to a two–bridge knot K is a rational function in $s + 1/s$ and u, where (s, u) is a solution of Riley's equation $\phi_K(s, u) = 0$. In particular, if we consider the case for $s = 1$, then the Reidemeister torsion $\mathbb{T}^K_\lambda$ is a rational function of u. The variable u satisfies Riley's equation $\phi_K(1, u) = 0$. Since u is expressed in terms of the cusp shape, the twisted Reidemeister torsion $\mathbb{T}^K_\lambda$ at the holonomy ρ_0 is also a rational function in the cusp shape corresponding to the root u.

5. The twisted Reidemeister torsion for twist knots

In this section, we compute the twisted Reidemeister torsion for twist knots. Since there exists an isomorphism between the knot groups $\Pi(J(2, 2m+1))$ and $\Pi(J(2, -2m))$ (see Remark 2), it is enough for us to make the computation in the case of even twist knots $K = J(2, 2m)$, $m \in \mathbb{Z}$. The method used is the following. We will make the computation at the acyclic level, i.e. compute the torsion polynomial $\mathcal{T}^K_\lambda(\rho)$, and next apply [26, Theorem 3.1.2] to obtain $\mathbb{T}^K_\lambda(\rho)$.

5.1. **The twisted Reidemeister torsion for even twist knots.** We calculate the twisted Reidemeister torsion for even twist knots $J(2,2m)$ where m is an integer.

5.1.1. *Preliminaries.* Following Section 3.2 and using Riley's method we can parametrize a non–abelian $\mathrm{SL}_2(\mathbb{C})$-representation ρ by two parameters u and s as follows:

$$\rho(x) = \begin{pmatrix} \sqrt{s} & 1/\sqrt{s} \\ 0 & 1/\sqrt{s} \end{pmatrix}, \rho(y) = \begin{pmatrix} \sqrt{s} & 0 \\ -\sqrt{s}u & 1/\sqrt{s} \end{pmatrix},$$

where s and u satisfy Riley's equation $\phi_{J(2,2m)}(s,u) = 0$. Besides, the Riley polynomial for twist knots is such that:

$$\phi_{J(2,2m)}(u,s) = \frac{(s+s^{-1}-1-u)(\xi_+^m - \xi_-^m) - (\xi_+^{m-1} - \xi_-^{m-1})}{\xi_+ - \xi_-}, \tag{33}$$

where $\xi_\pm$ are the *eigenvalues* of the matrix $\rho(w) = \rho([y,x^{-1}])$ given by

(34)
$$\xi_\pm = \frac{1}{2}\left[u^2+(2-s-s^{-1})u+2 \pm \sqrt{(u^2+(2-s-s^{-1})u+4)(u^2+(2-s-s^{-1})u)}\right].$$

5.1.2. *Statement of the result.*

Notation. Let $\alpha_1, \alpha_2, \beta_1, \beta_2$, c and t_m be as follows:

$$\begin{aligned}
c &= c(u,s) = u + 1 - s^{-1};\\
\alpha_1 &= (\xi_- - 1)(\xi_+ + s) + (s-1)^2 u - su^2;\\
\alpha_2 &= (1 - su - \xi_+)\,(1 + (\xi_- - s)/c);\\
\beta_1 &= (\xi_+ - 1)(\xi_- + s) + (s-1)^2 u - su^2;\\
\beta_2 &= (1 - su - \xi_-)\,(1 + (\xi_+ - s)/c);\\
t_m &= (\xi_+^m - \xi_-^m)/(\xi_+ - \xi_-).
\end{aligned}$$

Remark 14. Using such notation, the Riley polynomial of the twist knot $J(2,2m)$ becomes:

$$\phi_{J(2,2m)}(u,s) = (s-c)t_m - t_{m-1}.$$

With this notation in mind we can write down the general formula for the twisted Reidemeister torsion for twist knots.

Theorem 13. *Let m be a positive integer.*

(1) *The Reidemeister torsion $\mathbb{T}_\lambda^{J(2,2m)}(\rho)$ satisfies the following formula:*

$$\mathbb{T}_\lambda^{J(2,2m)}(\rho) = \frac{\tau_0}{s+s^{-1}-2}\left[C_1(m)\xi_+^{m-1}t_m + C_2(m)\xi_-^{m-1}t_m + C_3(m)\right]. \tag{35}$$

(2) *Similarly, we have*

$$\mathbb{T}_\lambda^{J(2,-2m)}(\rho) = \frac{\tau_0}{s+s^{-1}-2}\left[-C_1(-m)\xi_+^{-m-1}t_m - C_2(-m)\xi_-^{-m-1}t_m + C_3(-m)\right]. \tag{36}$$

In that two formulas we have:

$$
\begin{aligned}
C_1(m) = \frac{1}{(\xi_+ - \xi_-)^2} &\left\{\frac{1}{s}\alpha_1^2(3m+1) + \frac{m}{s}\beta_1^2(\xi_+^2+1) - m(\xi_+ - \xi_-)^2(s + \frac{1}{s} + 1)\right\} \\
&- \frac{m}{(\xi_+ - \xi_-)^4}\left\{\left(c(1-\xi_+) + \frac{\alpha_1}{s}\right)^2\left((\xi_+ - \xi_-)^2(s + \frac{1}{s} + 1) - \frac{\alpha_1^2+\beta_1^2}{s}\right)\right. \\
&\qquad\left. + \frac{2\alpha_1\beta_2}{s}\left(c(1-\xi_+) + \frac{\alpha_1}{s}\right)\left(c(1-\xi_+) + \frac{\alpha_2}{s}\right)\right\} \\
&- \frac{m}{(\xi_+ - \xi_-)^4}\left\{\frac{\alpha_1^2}{s}\left((\xi_+ - \xi_-)^2(u^2+4u+3) - (c(1-\xi_+) + \frac{\alpha_1}{s})^2\right.\right. \\
&\qquad\qquad \left. -(c(1-\xi_-) + \frac{\beta_1}{s})^2\right) \\
&\qquad\left. + \frac{2\alpha_1\alpha_2}{s}\left(c(1-\xi_+) + \frac{\alpha_1}{s}\right)\left(c(1-\xi_-) + \frac{\beta_2}{s}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
C_2(m) = \frac{1}{(\xi_+ - \xi_-)^2} &\left\{\frac{1}{s}\beta_1^2(3m+1) + \frac{m}{s}\alpha_1^2(\xi_-^2+1) - m(\xi_+ - \xi_-)^2(s + \frac{1}{s} + 1)\right\} \\
&- \frac{m}{(\xi_+ - \xi_-)^4}\left\{\left(c(1-\xi_-) + \frac{\beta_1}{s}\right)^2\left((\xi_+ - \xi_-)^2(s + \frac{1}{s} + 1) - \frac{\alpha_1^2+\beta_1^2}{s}\right)\right. \\
&\qquad\left. + \frac{2\alpha_2\beta_1}{s}\left(c(1-\xi_-) + \frac{\beta_1}{s}\right)\left(c(1-\xi_-) + \frac{\beta_2}{s}\right)\right\} \\
&- \frac{m}{(\xi_+ - \xi_-)^4}\left\{\frac{\beta_1^2}{s}\left((\xi_+ - \xi_-)^2(u^2+4u+3) - (c(1-\xi_+) + \frac{\alpha_1}{s})^2\right.\right. \\
&\qquad\qquad \left. -(c(1-\xi_-) + \frac{\beta_1}{s})^2\right) \\
&\qquad\left. + \frac{2\beta_1\beta_2}{s}\left(c(1-\xi_-) + \frac{\beta_1}{s}\right)\left(c(1-\xi_+) + \frac{\alpha_2}{s}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
C_3(m) = {} & \frac{m}{(\xi_+ - \xi_-)^2}\left\{(\xi_+ - \xi_-)^2(s + \frac{1}{s} + 1) - \frac{\alpha_1^2 + \beta_1^2}{s}\right\} \\
& + \frac{t_m^2}{(\xi_+ - \xi_-)^2}\left\{4(\xi_+ - \xi_-)^2(s + \frac{1}{s} + 1) - \frac{5(\alpha_1^2 + \beta_1^2)}{s}\right\} \\
& - \frac{t_m^2}{(\xi_+ - \xi_-)^4}\left\{\frac{1}{s}\left(c(1-\xi_+)\beta_1 + \frac{\alpha_1\beta_1}{s}\right)^2 - \frac{1}{s}\left(c(1-\xi_+)\beta_2 + \frac{\alpha_2\beta_2}{s}\right)^2\right. \\
& \qquad\qquad \left. + \frac{1}{s}\left(c(1-\xi_-)\alpha_1 + \frac{\alpha_1\beta_1}{s}\right)^2 - \frac{1}{s}\left(c(1-\xi_-)\alpha_2 + \frac{\alpha_2\beta_2}{s}\right)^2\right\} \\
& + m(s + \frac{1}{s} - 2)^2 t_m^2.
\end{aligned}
$$

Remark 15. One can observe that $\mathbb{T}_\lambda^{J(2,2m)}$ is symmetric in $\xi_\pm$. Together with the fact that $\xi_+ \cdot \xi_- = 1$, we can see that $\mathbb{T}_\lambda^{J(2,2m)}$ is in fact a function of $\xi_+ + \xi_- = u^2 + (2 - s - s^{-1})u + 2$.

5.1.3. *Proof of Theorem 13.* We make the detailed proof in the case of $J(2, 2m)$ for $m > 0$.

First, recall that the group of $J(2, 2m)$ admits the following presentation (see Fact 1):

$$\langle x, y \,|\, w^m x = y w^m \rangle.$$

Here w is the word $[y, x^{-1}] = yx^{-1}y^{-1}x$.

Before computations, we give an elementary and useful lemma about trace of matrices in $M_3(\mathbb{C})$.

Lemma 14. *The two following items hold:*

(1) *Let A be in* $\mathrm{GL}(3, \mathbb{C})$. *Set*

$$\sigma_1(A) = \mathrm{tr}\,(A) \text{ and } \sigma_2(A) = \frac{1}{2}\left(\mathrm{tr}^2(A) - \mathrm{tr}\,(A^2)\right).$$

We have

$$\sigma_2(A) = \sigma_1(A^{-1}) \cdot \det(A).$$

(2) *If $A = (a_{i,j})_{1 \le i,j \le 3}$ and $B = (b_{i,j})_{1 \le i,j \le 3}$ are two matrices in $M_3(\mathbb{C})$, then we have*

$$
\begin{aligned}
\mathrm{tr}\,(A)\mathrm{tr}\,(B) - \mathrm{tr}\,(AB) = {} & \begin{vmatrix} a_{1,1} & a_{1,3} \\ b_{3,1} & b_{3,3} \end{vmatrix} + \begin{vmatrix} b_{1,1} & b_{1,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} \\
& + \begin{vmatrix} a_{2,2} & a_{2,3} \\ b_{3,2} & b_{3,3} \end{vmatrix} + \begin{vmatrix} b_{2,2} & b_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} \\
& + \begin{vmatrix} a_{1,1} & a_{1,2} \\ b_{2,1} & b_{2,2} \end{vmatrix} + \begin{vmatrix} b_{1,1} & b_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}.
\end{aligned}
$$

Proof.

(1) From the Cayley–Hamilton identity, we have
$$A^3 - \sigma_1(A)A^2 + \sigma_2(A)A - \det(A)\mathbf{1} = 0.$$
Multiplying this equation by $-\det(A)^{-1}A^{-3}$, we obtain our first claim.

(2) Second item follows from direct calculations.

□

Fox–differential calculus for $2m$*-twist knots.* Since $J(2, 2m)$ is a two–bridge knot, the twisted Reidemeister torsion polynomial $\mathcal{T}^K_\lambda(\rho)$ associated to $J(2, 2m)$ is expressed as (see Equation (31)):

$$\mathcal{T}^K_\lambda(\rho) = \tau_0 \frac{\det \Phi(\frac{\partial}{\partial x} w^m x w^{-m} y^{-1})}{\det \Phi(y - 1)} \tag{37}$$

where $\Phi = Ad \circ \rho \otimes \alpha : \mathbb{Z}[\Pi(J(2, 2m))] \to M_3(\mathbb{C}[t, t^{-1}])$.

The following claim gives us the Fox–differential part in the numerator of Equation (37).

Claim 15. *For $m > 0$, we have:*

$$\frac{\partial}{\partial x}\left(w^m x w^{-m} y^{-1}\right) = w^m \left(1 + (1 - x)(1 + w^{-1} + \cdots + w^{-m+1})(x^{-1} - x^{-1}y)\right). \tag{38}$$

Proof of Claim 15. We have:

$$\begin{aligned}\frac{\partial}{\partial x}\left(w^m x w^{-m} y^{-1}\right) &= \frac{\partial w^m}{\partial x} + w^m + w^m x(-w^{-m})\frac{\partial w^m}{\partial x} \\ &= w^m \left(1 + (1 - x)w^{-m}\frac{\partial w^m}{\partial x}\right).\end{aligned}$$

It is easy to see that

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(yx^{-1}y^{-1}x\right) = yx^{-1}y^{-1} - yx^{-1}.$$

Thus

$$\begin{aligned}w^{-m}\frac{\partial w^m}{\partial x} &= (1 + w^{-1} + \cdots + w^{-m+1})w^{-1}\frac{\partial w}{\partial x} \\ &= (1 + w^{-1} + \cdots + w^{-m+1})(x^{-1} - x^{-1}y),\end{aligned}$$

which gives us Equation (38). □

Let $\{E, H, F\}$ be the following usual $\mathbb{C}$-basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that the adjoint actions of x and y in the basis $\{E, H, F\}$ of $\mathfrak{sl}_2(\mathbb{C})$ are given by the following matrices:

$$X = Ad_{\rho(x)} = \begin{pmatrix} s & -2 & -s^{-1} \\ 0 & 1 & s^{-1} \\ 0 & 0 & s^{-1} \end{pmatrix}, \quad Y = Ad_{\rho(y)} = \begin{pmatrix} s & 0 & 0 \\ su & 1 & 0 \\ -su^2 & -2u & s^{-1} \end{pmatrix}.$$

If $W = Ad_{\rho(w)}$, then $\Phi(\frac{\partial}{\partial x} w^m x w^{-m} y^{-1})$ is given by (see Claim 15):

$$W^m \left(\mathbf{1} + (\mathbf{1} - tX)(\mathbf{1} + W^{-1} + \cdots + W^{-m+1})(t^{-1}X^{-1} - X^{-1}Y)\right).$$

Set $S_m(A) = \mathbf{1} + A + \cdots + A^{m-1}$, for $A \in \mathrm{SL}_2(\mathbb{C})$, we finally obtain:

$$\Phi(\frac{\partial}{\partial x} w^m x w^{-m} y^{-1}) = W^m \left(\mathbf{1} + (\mathbf{1} - tX) S_m(W^{-1})(t^{-1}X^{-1} - X^{-1}Y)\right). \tag{39}$$

Observation about the "second differential" of a determinant. We can compute the twisted Reidemeister torsion for $J(2, 2m)$ combining Equations (32) and (39) as follows:

$$\mathbb{T}_\lambda^{J(2,2m)}(\rho) = -\tau_0 \lim_{t\to 1} \frac{\det(\Phi(\frac{\partial}{\partial x} w^m x w^{-m} y^{-1}))}{(t-1)\det(\Phi(y-1))}.$$

Using the fact that $\det W = 1$, Equation (39) gives:

$$\det \Phi(\frac{\partial}{\partial x} w^m x w^{-m} y^{-1}) = \det\left(\mathbf{1} + (\mathbf{1} - tX) S_m(W^{-1})(t^{-1}X^{-1} - X^{-1}Y)\right). \tag{40}$$

If we write $\det(\mathbf{1} + Z_m)$ for the right hand side of Equation (40), then

$$\mathbb{T}_\lambda^{J(2,2m)}(\rho) = -\tau_0 \lim_{t\to 1} \frac{\det(\mathbf{1} + Z_m)}{(t-1)^2(t^2 - (s + s^{-1})t + 1)} \tag{41}$$

thus

$$\mathbb{T}_\lambda^{J(2,2m)}(\rho) = \frac{\tau_0}{s + s^{-1} - 2} \lim_{t\to 1} \frac{\det(\mathbf{1} + Z_m)}{(t-1)^2}.$$

Moreover we can split $\det(\mathbf{1} + Z_m)$ as follows:

$$\det(\mathbf{1} + Z_m) = 1 + \sigma_1(Z_m) + \sigma_2(Z_m) + \sigma_3(Z_m), \tag{42}$$

where

$$\sigma_1(Z_m) = \mathrm{tr}\,(Z_m), \quad \sigma_2(Z_m) = \frac{1}{2}\left(\mathrm{tr}^2(Z_m) - \mathrm{tr}\,(Z_m^2)\right), \quad \sigma_3(Z_m) = \det(Z_m).$$

Thus

$$\mathbb{T}_\lambda^{J(2,2m)}(\rho) = \frac{\tau_0}{s + s^{-1} - 2} \lim_{t\to 1} \frac{1 + \sigma_1(Z_m) + \sigma_2(Z_m) + \sigma_3(Z_m)}{(t-1)^2}. \tag{43}$$

With the "splitting" of $\mathbb{T}_\lambda^{J(2,2m)}(\rho)$ given in Equation (43) in mind, we compute separately each "second differential" of the $\sigma_i(Z_m)$ ($i = 1, 2, 3$) to obtain the twisted Reidemeister torsion of $J(2, 2m)$.

The "second differential" of $\sigma_3(Z_m)$. We concentrate first on the $\sigma_3(Z_m)$-part of Equation (43), which is the easier term to compute and correspond to the "second differential" of $\sigma_3(Z_m)$.

Claim 16. *We have:*

$$\lim_{t\to 1} \frac{\sigma_3(Z_m)}{(t-1)^2} = (2 - s - s^{-1})^2 \cdot m t_m^2.$$

Proof of Claim 16. By definition $\sigma_3(Z_m) = \det(Z_m)$, thus

$$\begin{aligned}
\sigma_3(Z_m) &= \det((\mathbf{1} - tX)S_m(W^{-1})(t^{-1}X^{-1} - X^{-1}Y)) \\
&= \det(\mathbf{1} - tX)\det(S_m(W^{-1}))\det(t^{-1}X^{-1}Y)\det(Y^{-1} - t\mathbf{1}) \\
&= t^{-3}(t-1)^2(1-ts)(1-ts^{-1})(t-s)(t-s^{-1})\det(S_m(W^{-1})).
\end{aligned} \tag{44}$$

Taking the limit when t goes to 1, we thus obtain:

$$\lim_{t\to 1} \frac{\sigma_3(Z_m)}{(t-1)^2} = (2 - s - s^{-1})^2 \det(S_m(W^{-1})).$$

Note, with Equation (34) in mind, that $\xi_\pm^2$ and 1 are the eigenvalues of $W^{-1} = Ad_{\rho(w)^{-1}}$. It thus follows that

$$\begin{aligned}
\det(S_m(W^{-1})) &= m\frac{(1-\xi_+^{2m})(1-\xi_-^{2m})}{(1-\xi_+^2)(1-\xi_-^2)} \\
&= m\frac{(\xi_-^m - \xi_+^m)(\xi_+^m - \xi_-^m)}{(\xi_- - \xi_+)(\xi_+ - \xi_-)} \\
&= m t_m^2.
\end{aligned}$$

□

If we substitute the result of Claim 16 into Equation (43), we obtain

$$\mathbb{T}_\lambda^{J(2,2m)}(\rho) = \frac{\tau_0}{s + s^{-1} - 2}\left[\lim_{t\to 1} \frac{1 + \sigma_1(Z_m) + \sigma_2(Z_m)}{(t-1)^2} + (s + s^{-1} - 2)^2 \cdot m t_m^2\right].$$

These expression can be easily written again as

$$\mathbb{T}_\lambda^{J(2,2m)}(\rho) = \frac{\tau_0}{s + s^{-1} - 2}\left[\frac{1}{2}\frac{d^2}{dt^2}(\sigma_1(Z_m) + \sigma_2(Z_m))\bigg|_{t=1} + (s + s^{-1} - 2)^2 \cdot m t_m^2\right]. \tag{45}$$

The "second differentials" of $\sigma_1(Z_m)$ and $\sigma_2(Z_m)$. We now focus on the "second differentials" of $\sigma_1(Z_m)$ and $\sigma_2(Z_m)$. If we let

$$\tilde{Z}_m = (t^{-1}X^{-1} - X^{-1}Y)(\mathbf{1} - tX)S_m(W^{-1}),$$

then it follows from the definitions of σ_1 and σ_2 that

$$\sigma_1(Z_m) = \sigma_1(\tilde{Z}_m), \quad \sigma_2(Z_m) = \sigma_2(\tilde{Z}_m).$$

We use $\sigma_1(\tilde{Z}_m)$ and $\sigma_2(\tilde{Z}_m)$ instead of $\sigma_1(Z_m)$ and $\sigma_2(Z_m)$ for our calculations.

Claim 17. *We have:*

$$\frac{1}{2}\left.\frac{d^2}{dt^2}\sigma_1(\tilde{Z}_m)\right|_{t=1} = \frac{1}{2}\lim_{t\to 1}\frac{\sigma_1(\tilde{Z}_m)}{(t-1)^2} = \mathrm{tr}\,(X^{-1}S_m(W^{-1})), \tag{46}$$

$$\frac{1}{2}\left.\frac{d^2}{dt^2}\sigma_2(\tilde{Z}_m)\right|_{t=1} = \frac{1}{2}\lim_{t\to 1}\frac{\sigma_2(\tilde{Z}_m)}{(t-1)^2} \tag{47}$$

$$\begin{aligned} &= 3\sigma_2(X^{-1}S_m(W^{-1})) + \sigma_2(YS_m(W^{-1})W^{-1}) \\ &\quad - \mathrm{tr}\left(X^{-1}S_m(W^{-1})\right)\mathrm{tr}\left((\mathbf{1}+X^{-1}Y)S_m(W^{-1})\right) \\ &\quad + \mathrm{tr}\left(X^{-1}S_m(W^{-1})(\mathbf{1}+X^{-1}Y)S_m(W^{-1})\right). \end{aligned} \tag{48}$$

Proof of Claim 17. Since $\sigma_1(\tilde{Z}_m)$ is the trace of $\tilde{Z}_m$, the only term which remains after taking the “second differential” at $t = 1$ is $\left.\frac{d^2}{dt^2}\frac{1}{t}X^{-1}S_m(W^{-1})\right|_{t=1}$.

Now we consider $\sigma_2(\tilde{Z}_m)$. From the definition of $\sigma_2(\tilde{Z}_m)$, we have

$$\begin{aligned} \frac{1}{2}\left.\frac{d^2}{dt^2}\sigma_2(\tilde{Z}_m)\right|_{t=1} &= \frac{1}{2}\frac{d^2}{dt^2}\frac{1}{2}\left.\left(\mathrm{tr}^2(\tilde{Z}_m) - \mathrm{tr}\,(\tilde{Z}_m^2)\right)\right|_{t=1} \\ &= \frac{1}{4}\left\{\left.\frac{d^2}{dt^2}\mathrm{tr}^2(\tilde{Z}_m)\right|_{t=1} - \left.\frac{d^2}{dt^2}\mathrm{tr}\,(\tilde{Z}_m^2)\right|_{t=1}\right\}. \end{aligned}$$

In $\mathrm{tr}\,(\tilde{Z}_m^2)$, the following three terms are the terms which remain after taking the second differential at $t = 1$:

$$\left.\tfrac{d^2}{dt^2}t^{-2}\mathrm{tr}\left((X^{-1}S_m(W^{-1}))^2\right)\right|_{t=1},$$

$$\left.\tfrac{d^2}{dt^2} - 2t^{-1}\mathrm{tr}\left(X^{-1}S_m(W^{-1})(\mathbf{1}+X^{-1}Y)S_m(W^{-1})\right)\right|_{t=1},$$

$$\left.\tfrac{d^2}{dt^2}t^2\mathrm{tr}\left((YS_m(W^{-1})W^{-1})^2\right)\right|_{t=1}.$$

Hence

$$\begin{aligned} \frac{1}{2}\left.\frac{d^2}{dt^2}\sigma_2(\tilde{Z}_m)\right|_{t=1} &= \frac{3}{2}\left\{\mathrm{tr}^2\left(X^{-1}S_m(W^{-1})\right) - \mathrm{tr}\left((X^{-1}S_m(W^{-1}))^2\right)\right\} \\ &\quad - \mathrm{tr}\left(X^{-1}S_m(W^{-1})\right)\mathrm{tr}\left((\mathbf{1}+X^{-1}Y)S_m(W^{-1})\right) \\ &\quad + \mathrm{tr}\left(X^{-1}S_m(W^{-1})(\mathbf{1}+X^{-1}Y)S_m(W^{-1})\right) \\ &\quad + \frac{1}{2}\left\{\mathrm{tr}^2\left(YS_m(W^{-1})W^{-1}\right) - \mathrm{tr}\left((YS_m(W^{-1})W^{-1})^2\right)\right\}. \end{aligned}$$

□

If we substitute Equations (46) & (48) of Claim 17 into Equation (45), then we obtain the following formula for $\mathbb{T}_\lambda^{J(2,2m)}(\rho)$.

Claim 18. *The twisted Reidemeister torsion for $J(2,2m)$ satisfies the following formula:*

(49)
$$\begin{aligned}\mathbb{T}^{J(2,2m)}_{\lambda}(\rho) = \frac{\tau_0}{s+s^{-1}-2}\Big[&\mathrm{tr}\left(X^{-1}S_m(W^{-1})\right) + 3\sigma_2(X^{-1}S_m(W^{-1}))\\ &+\sigma_2(YS_m(W^{-1})W^{-1}) - \mathrm{tr}\left(X^{-1}S_m(W^{-1})\right)\mathrm{tr}\left(S_m(W^{-1})\right)\\ &+\mathrm{tr}\left(X^{-1}S_m(W^{-1})S_m(W^{-1})\right) - \mathrm{tr}\left(X^{-1}S_m(W^{-1})\right)\mathrm{tr}\left(X^{-1}YS_m(W^{-1})\right)\\ &+\mathrm{tr}\left(X^{-1}S_m(W^{-1})X^{-1}YS_m(W^{-1})\right) + (s+s^{-1}-2)^2\cdot mt_m^2\Big].\end{aligned}$$

More explicit descriptions. To find more explicit expression of the $\mathbb{T}^{J(2,2m)}_{\lambda}(\rho)$, we change our basis of $\mathfrak{sl}_2(\mathbb{C})$ in order to diagonalize the matrix *rho*(w).

The $\mathrm{SL}_2(\mathbb{C})$-matrix $\rho(w)$ can be diagonalized by

$$p = \begin{pmatrix} u+1-s^{-1} & u+1-s^{-1}\\ 1-su-\xi_+ & 1-su-\xi_- \end{pmatrix}.$$

Explicitly, $p^{-1}\rho(w)p$ is the diagonal matrix $\mathrm{diag}(\xi_+,\xi_-)$.

Set $a = 1-su-\xi_+$ and $b = 1-su-\xi_-$. With respect to the basis $\{E,H,F\}$ of $\mathfrak{sl}_2(\mathbb{C})$, the matrix of adjoint action of p becomes as follows:

$$P = \frac{1}{a-b}\begin{pmatrix} -c & 2c & c\\ a & -(a+b) & -b\\ a^2/c & -2ab/c & -b^2/c \end{pmatrix}$$

where $c = u+1-s^{-1}$ is defined in Subsection 5.1.2.

Note that the matrix $P^{-1}WP$ is the diagonal matrix $\mathrm{diag}(\xi_+^2, 1, \xi_-^2)$. Here we repeat that $W = Ad_{\rho(w)}$.

Set

$$\tilde{X} = P^{-1}XP,\quad \tilde{Y} = P^{-1}YP \text{ and } \tilde{W} = P^{-1}WP.$$

Since we have $P^{-1}S_m(W^{-1})P = S_m(\tilde{W}^{-1})$, the matrix $P^{-1}S_m(W^{-1})P$ is the following diagonal matrix

$$P^{-1}S_m(W^{-1})P = \mathrm{diag}(\xi_-^{m-1}t_m, m, \xi_+^{m-1}t_m).$$

Moreover as $\mathrm{tr}\,(X^{-1}S_m(W^{-1})) = \mathrm{tr}\,(\tilde{X}^{-1}S_m(\tilde{W}^{-1}))$, we have the following claim.

Claim 19. *We have*

$$\begin{aligned}&\mathrm{tr}\,(X^{-1}S_m(W^{-1}))\\ &= \frac{1}{(a-b)^2}\left(\frac{\beta_1^2}{s}\xi_-^{m-1}t_m + \left((a-b)^2(s+s^{-1}+1) - \frac{\beta_1^2}{s} - \frac{\alpha_1^2}{s}\right)m + \frac{\alpha_1^2}{s}\xi_+^{m-1}t_m\right).\end{aligned}$$

Proof of Claim 19. By a direct computation, we obtain that the $(1,1)$-component of the matrix $\tilde{X}^{-1}$ is equal to $\beta_1^2/(s(a-b)^2)$ and its $(3,3)$-component is equal to

$\alpha_1^2/(s(a-b)^2)$. We can also find the $(2,2)$-component of $\tilde{X}^{-1}$from $\mathrm{tr}\,(\tilde{X}^{-1}) = s + s^{-1} + 1$. ◻

Now, we compute $\sigma_2(XS_m(W^{-1}))$ and $\sigma_2(YS_m(W^{-1})W^{-1})$ from Lemma 14 as follows.

Claim 20. *The following equalities hold:*

(1) $\sigma_2(X^{-1}S_m(W^{-1}))$

$$= \frac{1}{(a-b)^2}\left(\frac{\beta_1^2}{s}\xi_-^{m-1}mt_m + \left((a-b)^2(s+s^{-1}+1) - \frac{\alpha_1^2}{s} - \frac{\beta_1^2}{s}\right)t_m^2 + \frac{\alpha_1^2}{s}\xi_+^{m-1}mt_m\right).$$

(2) $\sigma_2(YS_m(W^{-1})W^{-1})$

$$= \frac{1}{(a-b)^2}\left(\frac{\alpha_1^2}{s}\xi_-^{m+1}mt_m + \left((a-b)^2(s+s^{-1}+1) - \frac{\alpha_1^2}{s} - \frac{\beta_1^2}{s}\right)t_m^2 + \frac{\beta_1^2}{s}\xi_+^{m+1}mt_m\right).$$

Proof of Claim 20. Using Lemma 14 and because $\det(X) = 1, \det(Y) = 1$ we have:

$$\sigma_2(X^{-1}S_m(W^{-1})) = \mathrm{tr}\,(XS_m(W^{-1})^{-1}) \cdot \det(S_m(W^{-1}))$$

and

$$\sigma_2(YS_m(W^{-1})W^{-1}) = \mathrm{tr}\,(Y^{-1}WS_m(W^{-1})^{-1}) \cdot \det(S_m(W^{-1})).$$

We obtain the above formulas by computing the traces using $\tilde{X}, \tilde{Y}$ and $S_m(\tilde{W}^{-1})$ as in Claim 19. ◻

Finally we calculate the other two terms which are of the following form: $-\mathrm{tr}\,(A)\mathrm{tr}\,(B) + \mathrm{tr}\,(AB)$.

Claim 21. *We have:*

(1) $-\mathrm{tr}\,(X^{-1}S_m(W^{-1}))\mathrm{tr}\,(S_m(W^{-1})) + \mathrm{tr}\,(X^{-1}S_m(W^{-1})S_m(W^{-1}))$

$$= -\frac{1}{(a-b)^2}\Bigg[\left((a-b)^2(s+\frac{1}{s}+1) - \frac{\alpha_1^2}{s}\right)\xi_-^{m-1}mt_m + \left(\frac{\alpha_1^2}{s} + \frac{\beta_1^2}{s}\right)t_m^2 + \left((a-b)^2(s+\frac{1}{s}+1) - \frac{\beta_1^2}{s}\right)\xi_+^{m-1}mt_m.\Bigg]$$

(2) If we set $\tilde{X}^{-1} = (a_{i,j})_{1\le i,j\le 3}$, and $\tilde{X}^{-1}\tilde{Y} = (b_{i,j})_{1\le i,j\le 3}$, then we have

$$\begin{aligned}
&-\mathrm{tr}\,(X^{-1}S_m(W^{-1}))\mathrm{tr}\,(X^{-1}YS_m(W^{-1})) + \mathrm{tr}\,(X^{-1}S_m(W^{-1})X^{-1}YS_m(W^{-1}))\\
&= \left(\begin{vmatrix} a_{1,1} & a_{1,3} \\ b_{3,1} & b_{3,3} \end{vmatrix} + \begin{vmatrix} b_{1,1} & b_{1,3} \\ a_{3,1} & a_{3,3} \end{vmatrix}\right)t_m^2\\
&\quad - \left(\begin{vmatrix} a_{2,2} & a_{2,3} \\ b_{3,2} & b_{3,3} \end{vmatrix} + \begin{vmatrix} b_{2,2} & b_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix}\right)m\xi_+^{m-1}t_m\\
&\quad - \left(\begin{vmatrix} a_{1,1} & a_{1,2} \\ b_{2,1} & b_{2,2} \end{vmatrix} + \begin{vmatrix} b_{1,1} & b_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}\right)m\xi_-^{m-1}t_m.
\end{aligned}$$

Proof of Claim 21. Item (1) follows from above results and Item (2) follows from Lemma 14. ◻

Remark 16. The matrices $\tilde{X}^{-1}$ and $\tilde{X}^{-1}\tilde{Y}$ are described explicitly as follows.

$$\tilde{X}^{-1} = \frac{1}{(a-b)^2}\begin{pmatrix} \frac{\beta_1^2}{s} & -\frac{\beta_1\beta_2}{s} & -\frac{\beta_2^2}{s} \\ \frac{\alpha_2\beta_1}{s} & (a-b)^2(s+\frac{1}{s}+1)-\frac{\alpha_1^2+\beta_1^2}{s} & -\frac{\alpha_1\beta_2}{s} \\ -\frac{\alpha_2}{s} & \frac{2\alpha_1\alpha_2}{s} & \frac{\alpha_1^2}{s} \end{pmatrix},$$

$$(a-b)^2\tilde{X}^{-1}\tilde{Y}$$
$$= \begin{pmatrix} (c(1-\xi_-)+\frac{\beta_1}{s})^2 & -2(c(1-\xi_-)+\frac{\beta_1}{s})(c(1-\xi_-)+\frac{\beta_2}{s}) & -(c(1-\xi_-)+\frac{\beta_2}{s})^2 \\ \begin{array}{c}(c(1-\xi_+)+\frac{\alpha_2}{s})\\ \cdot(c(1-\xi_-)+\frac{\beta_1}{s})\end{array} & \begin{array}{c}(a-b)^2(u^2+4u+3)\\ -(c(1-\xi_+)+\frac{\alpha_1}{s})^2\\ -(c(1-\xi_-)+\frac{\beta_1}{s})^2\end{array} & \begin{array}{c}-(c(1-\xi_+)+\frac{\alpha_1}{s})\\ \cdot(c(1-\xi_-)+\frac{\beta_2}{s})\end{array} \\ -(c(1-\xi_+)+\frac{\alpha_2}{s})^2 & 2(c(1-\xi_+)+\frac{\alpha_1}{s})(c(1-\xi_+)+\frac{\alpha_2}{s}) & (c(1-\xi_+)+\frac{\alpha_1}{s})^2 \end{pmatrix}.$$

One can observe that $(a-b)^2$ is equal to $(\xi_+ - \xi_-)^2$. Thus, if we substitute the results given in Claims 19, 20 and 21 into Equation (49), then we obtain Equation (35) of Theorem 13. This achieves the first part of the proof of Theorem 13.

The computation of $\mathbb{T}_\lambda^{J(2,-2m)}(\rho)$, for $m > 0$, is completely similar and has the following expression:

$$\begin{aligned} \mathbb{T}_\lambda^{J(2,-2m)}(\rho) &= -\tau_0 \lim_{t\to 1} \frac{\det(\Phi(\frac{\partial}{\partial x} w^{-m}xw^m y^{-1}))}{(t-1)\det(\Phi(y-1))} \\ &= \frac{\tau_0}{s+s^{-1}-2}\lim_{t\to 1}\frac{\det(\mathbf{1}+Z_{-m})}{(t-1)^2}. \end{aligned} \tag{50}$$

Here the matrix Z_{-m} is given by $(\mathbf{1}-tX)S_m(W)(YX^{-1}-t^{-1}YX^{-1}Y^{-1})$. The right hand side of Equation (50) is given by

$$\begin{aligned} \frac{\tau_0}{s+s^{-1}-2}\Big[&-\mathrm{tr}\,(X^{-1}S_m(W)W) + 3\sigma_2(WX^{-1}S_m(W)) + \sigma_2(YS_m(W)) \\ &- \mathrm{tr}\,(X^{-1}S_m(W)W)\mathrm{tr}\,((\mathbf{1}+X^{-1}Y)S_m(W)W) \\ &+ \mathrm{tr}\,(X^{-1}S_m(W)W(I+X^{-1}Y)S_m(W)W) \\ &-(s+s^{-1}-2)^2\cdot mt_m^2\Big]. \end{aligned} \tag{51}$$

Each term in Equation (51) can be computed similarly as in Claims 19, 20 and 21 which give the second item of Theorem 13.

5.2. **Examples.** As an illustration of our main Theorem 13, we explicitly compute $\mathbb{T}_\lambda^K$ on the character variety $X_\lambda^{\mathrm{irr}}(\Pi(K))$ for the following four examples: the trefoil knot $3_1 = J(2,2)$, $5_2 = J(2,4)$, the figure eight knot $4_1 = J(2,-2)$, and $6_1 = J(2,-4)$ respectively.

(1) For the trefoil knot $3_1 = J(2,2)$, the Riley polynomial is given by

$$\phi_{J(2,2)}(s,u) = -1 + s + s^{-1} - u.$$

The computation of the twisted Reidemeister torsion for $J(2,2)$ is expressed as follows.

$$\begin{aligned}\mathbb{T}^{J(2,2)}_{\lambda}(\rho_{\sqrt{s},u}) &= \frac{\tau_0}{s+s^{-1}-2}\left(-3(s+s^{-1}-2)\right)\\ &= -3\tau_0.\end{aligned}$$

This coincides with the inverse of the result [6, Subsection 6.1] (see Remark 11).

(2) For $5_2 = J(2,4)$, the Riley polynomial is given by

$$\begin{aligned}\phi_{J(2,4)}(s,u) = -3 + 2(s+s^{-1}) + \left(-4 + 3(s+s^{-1}) - (s+s^{-1})^2\right)u\\ + \left(-3 + 2(s+s^{-1})\right)u^2 - u^3.\end{aligned}$$

The twisted Reidemeister torsion for $5_2 = J(2,4)$ is expressed as follows.

$$\begin{aligned}\mathbb{T}^{J(2,4)}_{\lambda}(\rho_{\sqrt{s},u}) &= \frac{\tau_0}{s+s^{-1}-2}[-2 + 21(s+s^{-1}) - 10(s+s^{-1})^2\\ &\quad + \{-2 + 15(s+s^{-1}) - 17(s+s^{-1})^2 + 5(s+s^{-1})^3\}u\\ &\quad + \{6 + 7(s+s^{-1}) - 5(s+s^{-1})^2\}u^2]\\ &= \tau_0\left(-10(s+s^{-1}) + 1 + \left(5(s+s^{-1})^2 - 7(s+s^{-1}) + 1\right)u\right.\\ &\qquad \left. + \left(-5(s+s^{-1}) - 3\right)u^2\right).\end{aligned}$$

(3) For the figure eight knot $4_1 = J(2,-2)$, the Riley polynomial is given by

$$\phi_{J(2,-2)}(s,u) = (3 - s - s^{-1})(u+1) + u^2.$$

The computation of the twisted Reidemeister torsion for $J(2,-2)$ is expressed as follows.

$$\begin{aligned}\mathbb{T}^{J(2,-2)}_{\lambda}(\rho_{\sqrt{s},u}) &= \frac{\tau_0}{s+s^{-1}-2}\left(-2(s+s^{-1})+1\right)\left(s+s^{-1}-2\right)\\ &= \tau_0\left(-2(s+s^{-1})+1\right).\end{aligned}$$

This coincides with the inverse of the result [6, Subsection 6.3] (see Remark 11) in which the torsion is expressed as $\pm\sqrt{17+4I_\lambda}$.

Since the longitude λ is equal to $[y,x^{-1}][x,y^{-1}]$, one has

$$I_\lambda = -2 - (s+s^{-1}) + s^2 + s^{-2}.$$

Thus, up to sign, we have

$$\sqrt{17+4I_\lambda} = 2(s+s^{-1}) - 1.$$

(4) For $6_1 = J(2,-4)$, the Riley polynomial is given by

$$\phi_{J(2,-4)}(s,u) = 5 - 2(s+s^{-1}) + \Big(12 + (s+s^{-1})^2 - 7(s+s^{-1})\Big)u + \Big(11 + (s+s^{-1})^2 - 6(s+s^{-1})\Big)u^2 + \Big(5 - 2(s+s^{-1})\Big)u^3 + u^4.$$

The twisted Reidemeister torsion for $J(2,-4)$ is expressed as follows.

$$\begin{aligned}
&\mathbb{T}_\lambda^{J(2,-4)}(\rho_{\sqrt{s},u}) \\
&= \frac{\tau_0}{s+s^{-1}-2}[-14 - 13(s+s^{-1}) + 26(s+s^{-1})^2 - 8(s+s^{-1})^3 \\
&\quad + \{-8 - 34(s+s^{-1}) + 49(s+s^{-1})^2 - 23(s+s^{-1})^3 + 4(s+s^{-1})^4\}u \\
&\quad + \{-2 - 31(s+s^{-1}) + 32(s+s^{-1})^2 - 8(s+s^{-1})^3\}u^2 \\
&\quad + \{2 - 9(s+s^{-1}) + 4(s+s^{-1})^2\}u^3] \\
&= \tau_0\Big[\Big(-8(s+s^{-1})^2 + 10(s+s^{-1}) + 7\Big) \\
&\qquad + \Big(4(s+s^{-1})^3 - 15(s+s^{-1})^2 + 19(s+s^{-1}) + 4\Big)u \\
&\qquad + \Big(-8(s+s^{-1})^2 + 16(s+s^{-1}) + 1\Big)u^2 + \Big(4(s+s^{-1}) - 1\Big)u^3\Big].
\end{aligned}$$

5.3. **Twisted Reidemeister torsion at the holonomy representation.** In this section we consider the twisted Reidemeister torsion for hyperbolic twist knots at holonomy representations. Formulas of the twisted Reidemeister torsion associated to twist knots are complicated. But we see here that formulas for the twisted Reidemeister torsion at holonomy representations are simpler.

Every twist knots except the trefoil knot are hyperbolic. It is well known that an exterior of a hyperbolic knot admits at most a complete hyperbolic structure and this hyperbolic structure determines the holonomy representation of the knot group (see Section 3.3). With Fact 12 in mind we know that such lifts are λ-regular representations.

Remark 17. If we substitute $s = 1$ into the Riley polynomial $\phi_{J(2,2m)}(s,u)$ given in Equation (33), then

$$\phi_{J(2,2m)}(1,u) = (1-u)t_m - t_{m-1} = (1-u)\frac{\xi_+^m - \xi_-^m}{\xi_+ - \xi_-} - \frac{\xi_+^{m-1} - \xi_-^{m-1}}{\xi_+ - \xi_-}.$$

The $SL_2(\mathbb{C})$-representations which lifts the holonomy representation correspond to roots of Riley's equation $\phi_{J(2,2m)}(1,u) = 0$. We let ρ_u denote such representations.

We are now ready to give some closed formulas for the twisted Reidemeister torsion of twist knots at the holonomy representation.

Theorem 22. *Let $m > 0$ and u denote one of the two complex conjugate roots of Riley's equation $\phi_{J(2,2m)}(1,u) = 0$ (resp. $\phi_{J(2,-2m)}(1,u) = 0$) corresponding to holonomy representations, then*

(1) *the twisted Reidemeister torsion* $\mathbb{T}_\lambda^{J(2,2m)}(\rho_u)$ *satisfies the following closed formulas:*

$$\begin{aligned}\mathbb{T}_\lambda^{J(2,2m)}(\rho_u) = \frac{-\tau_0}{u^2+4}\Big[&\Big(4+m(u^2-4u+8)\Big)t_m(\xi_+^m+\xi_-^m)\\ &+\Big(t_m(\xi_+^{m-1}+\xi_-^{m-1})-1\Big)(u^2-4)m\\ &+(-5u^2-8u+4)t_m^2\Big],\end{aligned}$$

(2) *similarly the Reidemeister torsion* $\mathbb{T}_\lambda^{J(2,-2m)}(\rho_u)$ *satisfies the following closed formula:*

$$\begin{aligned}\mathbb{T}_\lambda^{J(2,-2m)}(\rho_u) = \frac{-\tau_0}{u^2+4}\Big[&\Big(-4+m(u^2-4u+8)\Big)t_m(\xi_+^m+\xi_-^m)\\ &+\Big(t_m(\xi_+^{m+1}+\xi_-^{m+1})+1\Big)(u^2-4)m\\ &+(-5u^2-8u+4)t_m^2\Big].\end{aligned}$$

Remark 18. Combining results of Theorem 22 with Equation (12), the twisted Reidemeister torsion of a twist knot at the holonomy is expressed in terms of the cusp shape of the knot.

Proof. First we make the computations in the case of $J(2,2m)$, where $m>0$. If we substitute $s=1$ in Equation (41), then we obtain:

$$\begin{aligned}\mathbb{T}_\lambda^{J(2,2m)}(\rho) &= -\tau_0 \lim_{t\to 1}\frac{\det(\mathbf{1}+Z_m)}{(t-1)^4}\\ &= \frac{-\tau_0}{24}\,\frac{d^4}{dt^4}\det(\mathbf{1}+Z_m)\Big|_{t=1}.\end{aligned}$$

Next, using the splitting of $\det(\mathbf{1}+Z_m)$ given in Equation (42), we get:

$$\mathbb{T}_\lambda^{J(2,2m)}(\rho) = \frac{-\tau_0}{24}\,\frac{d^4}{dt^4}\left[1+\sigma_1(Z_m)+\sigma_2(Z_m)+\sigma_3(Z_m)\right]\Big|_{t=1}. \tag{52}$$

It follows from Equation (44) that $(t-1)^6$ divides $\sigma_3(Z_m)$ in the case of $s=1$. Hence the term $\frac{d^4}{dt^4}\sigma_3(Z_m)\Big|_{t=1}$ in Equation (52) vanishes. By degrees of t in $\sigma_1(Z_m)$ and $\sigma_2(Z_m)$, we obtain the following equation from direct computations of the above differentials:

$$\begin{aligned}\mathbb{T}_\lambda^{J(2,2m)}(\rho) = -\tau_0[&\mathrm{tr}\,(X^{-1}S_m(W^{-1}))+5\sigma_2(X^{-1}S_m(W^{-1}))\\ &-\mathrm{tr}\,(X^{-1}S_m(W^{-1}))\mathrm{tr}\,((\mathbf{1}+X^{-1}Y)S_m(W^{-1}))\\ &+\mathrm{tr}\,(X^{-1}S_m(W^{-1})(\mathbf{1}+X^{-1}Y)S_m(W^{-1}))].\end{aligned} \tag{53}$$

Note that $\xi_+ + \xi_- = u^2 + 2$ and $(\xi_+ - \xi_-)^2 = u^2(u^2 + 4)$. By Claims 19, 20 and 21 we have

$$\operatorname{tr}(X^{-1}S_m(W^{-1})) = \frac{1}{u^2+4}\left[4(\xi_+^m + \xi_-^m)t_m - (u^2-4)m\right]$$
$$\sigma_2(X^{-1}S_m(W^{-1})) = \frac{1}{u^2+4}\left[4m(\xi_+^m + \xi_-^m)t_m - (u^2-4)t_m^2\right]$$

and

$$-\operatorname{tr}(X^{-1}S_m(W^{-1}))\operatorname{tr}((\mathbf{1} + X^{-1}Y)S_m(W^{-1})) + \operatorname{tr}(X^{-1}S_m(W^{-1})(\mathbf{1} + X^{-1}Y)S_m(W^{-1}))$$
$$= -\frac{1}{u^2+4}\left[8(u+2)t_m^2 - m(u+2)(u-6)(\xi_+^m+\xi_-^m)t_m - m(u^2-4)(\xi_+^{m-1}+\xi_-^{m-1})t_m\right].$$

If we substitute these results into Equation (53), then we obtain the wanted formula for $J(2, 2m)$.

Similarly, in the case of twist knots $J(2, -2m)$, $m > 0$, from computations of differentials we have

$$\begin{aligned}\mathbb{T}_\lambda^{J(2,-2m)}(\rho_u) = -\tau_0[&-\operatorname{tr}(X^{-1}S_m(W)W) + 5\sigma_2(X^{-1}S_m(W)W) \\ &- \operatorname{tr}(X^{-1}S_m(W)W)\operatorname{tr}((\mathbf{1} + X^{-1}Y)S_m(W)W) \\ &+ \operatorname{tr}(X^{-1}S_m(W)W(\mathbf{1} + X^{-1}Y)S_m(W)W)].\end{aligned} \tag{54}$$

It follows from Claims 19, 20 and 21 that

$$\operatorname{tr}(X^{-1}S_m(W)W) = \frac{1}{u^2+4}\left[4(\xi_+^m + \xi_-^m)t_m - (u^2-4)m\right]$$
$$\sigma_2(X^{-1}S_m(W)W) = \frac{1}{u^2+4}\left[4m(\xi_+^m + \xi_-^m)t_m - (u^2-4)t_m^2\right]$$

and

$$-\operatorname{tr}(X^{-1}S_m(W)W)\operatorname{tr}((\mathbf{1}+X^{-1}Y)S_m(W)W)+\operatorname{tr}(X^{-1}S_m(W)W(\mathbf{1}+X^{-1}YS_m(W)W))$$
$$= \frac{1}{u^2+4}\left[8(u+2)t_m^2 - m(u+2)(u-6)(\xi_+^m+\xi_-^m)t_m - m(u^2-4)(\xi_+^{m+1}+\xi_-^{m+1})t_m\right].$$

If we substitute these results into Equation (54) , then we obtain the wanted formula for $J(2, -2m)$. □

5.4. **Program list for Maxima.** We give a program list in order to compute the twisted Reidemeister torsion for a given twist knot. This program works on the free computer algebra system *Maxima* [19]. The function $R(m)$ in the list computes the Riley polynomial of $J(2, 2m)$. The function $T(m)$ computes the twisted Reidemeister torsion for $J(2, 2m)$. It gives a polynomial of s and u such that the top degree of u is lower than that in the Riley polynomial $\phi_{J(2,2m)}(s, u)$. Here we use Expressions (49) & (51) and the following remark for computing the twisted Reidemeister torsion.

Remark 19. It follows from Equation (33) that the highest degree term of u in $\phi_{J(2,2m)}(s,u)$ is equal to $-u^{2m-1}$ (resp. $u^{2|m|}$) if $m > 0$ (resp. $m < 0$).

Program list

```
load("nchrpl");/*We need this package for using mattrace*/
R(m):=block(/*function for calculating the Riley polynomial of J(2,2m)*/
        [/*w is the matrix of w=[y,x^{-1}]*/
         w:matrix([1-s*u,1/s-u-1],[-u+s*u*(u+1),(-u)/s+(u+1)^2]),
         p],
        w:w^^m,
        p:w[1,1]+(1-s)*w[1,2],
        p:expand(p),
        return(p)
      );
T(m):=if integerp(m) then
        if m=0 then "J(2,0) is unknot." else
          block(
            [/*matrix for adjoint action of x*/
             X:matrix([s,-2,(-1)/s],[0,1,1/s],[0,0,1/s]),
             /*matrix for adjoint action of y*/
             Y:matrix([s,0,0],[s*u,1,0],[(-s)*u^2,(-2)*u,1/s]),
             IX,/*inverse matrix of X*/
             IY,/*inverse matrix of Y*/
             S:ident(3),/*marix for series of W or W^{-1}*/
             AS:ident(3),/*adjoint matrix of S*/
             W,/*matrix W=[Y,IX] */
             IW,/*matrix [IX,Y]*/
             d:1,/*the highest degree of u in the numerator
                    of R-torsion*/
             k:1,/*the highest degree of u in the Riley poly*/
             p:0,
             r:R(m),/*the Riley poly*/
             r1 /*a polynomial removed the top term of u
                  from the Riley polynomial*/
            ],
            IX:invert(X),
            IY:invert(Y),
            W:Y.IX.IY.X,
            IW:IX.Y.X.IY,
            /*calculating the numerator of R-torsion*/
            if m>0 then
              block(
                /*calculation of S*/
                for i:1 thru m-1 do(S:ident(3)+S.IW),
                AS:adjoint(S),
```

```
    /*the numerator of R-torsion*/
    p:p+mattrace(IX.S),
    p:p+3*mattrace(X.AS)+mattrace(IY.W.AS),
    p:p-mattrace(IX.S)*mattrace((ident(3)+IX.Y).S),
    p:p+mattrace(IX.S.(ident(3)+IX.Y).S),
    p:p+(2-s+(-1)/s)^2*determinant(S),
    /*The top term of u in the Riley poly r
       is given by -u^(2m-1).
      We use the relation u^(2m-1) = r + u^(2m-1) later*/
    k:2*m-1,/*the highets degree of u in the Riley poly*/
    r1:r+u^(2*m-1)
  )
else
  block(
    /*calculation of S*/
    for i:1 thru -m-1 do(S:ident(3)+S.W),
    AS:adjoint(S),
    p:p-mattrace(IX.S.W),
    p:p+3*mattrace(X.IW.AS)+mattrace(IY.AS),
    p:p-mattrace(IX.S.W)*mattrace((ident(3)+IX.Y).S.W),
    p:p+mattrace(IX.S.W.(ident(3)+IX.Y).S.W),
    p:p-(2-s+(-1)/s)^2*determinant(S),
    /*The top term of u in the Riley poly r
       is given by u^(2|m|).
      We use the relation u^(2|m|) = -r + u^(2|m|) later*/
    k:2*(-m),/*the highets degree of u in the Riley poly*/
    r1:-r+u^(2*(-m))
  ),
p:expand(p),
/* simplify by using r (decreasing the degrees of u)*/
/* set the degree of u in p*/
d:hipow(p,u),
/*decreasing the degrees of u*/
for j:1 while d >= k do(
  p:subst(r1*u^(d-k),u^d,p),
  p:expand(p),
  d:hipow(p,u)
),
p:factor(p),
/*multiplying p
   by the denominator of twisted Alexander*/
p:expand(p*(s/(s^2-2*s+1))),
p:factorout(p,s),
r:factorout(r,s),
print("The Riley polynomial of J(2,",2*m,"):",r),
```

```
        print("The Reidemeister torsion for J(2,",2*m,"):"),
        return(p)
      )
  else print(m,"is not an integer.");
```

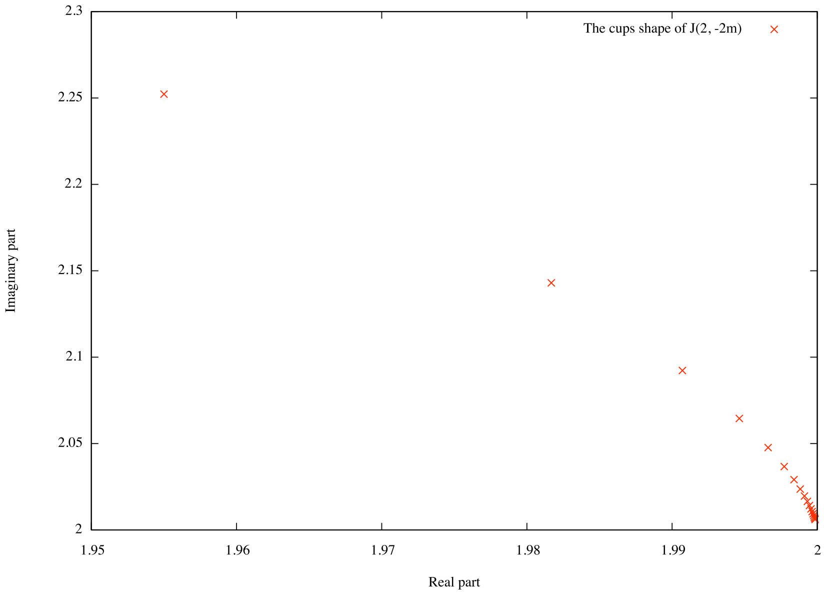

FIGURE 3. Graph of the cusp shape of $J(2,-2m)$.

5.5. **A remark on the asymptotic behavior of the twisted Reidemeister torsion at holonomy.** We close this paper with some remarks on the behavior of the cusp shape and of the twisted Reidemeister torsion at the holonomy for twist knots.

Remark 20 (Behavior of the cusp shape). In Notes [21, p. 5.63], Thurston explains that the sequence of exterior of the following knots $(J(2,-2m))_{m\geqslant 1}$ converges to the exterior of the Whitehead link on Figure 2 (link 5^2_1 in Rolfsen's table [18]). Note that, if the number of crossings m increases to infinity, then the cusp shape of the twist knot $J(2,\pm 2m)$ converges to $2+2i$, which is the common value of the cusp shapes of the Whitehead link, see the graph on Figure 3. This result is a consequence of Dehn's hyperbolic surgery Theorem.

The graph on Figure 4 gives the behavior of the sequence of the absolute value of $\mathbb{T}^{J(2,-2m)}_{\lambda}(\rho_0)$ with respect to the number of crossings $\sharp J(2,-2m) = 2+2m$ of the knot. The order of growing can be deduced by a "surgery argument" using

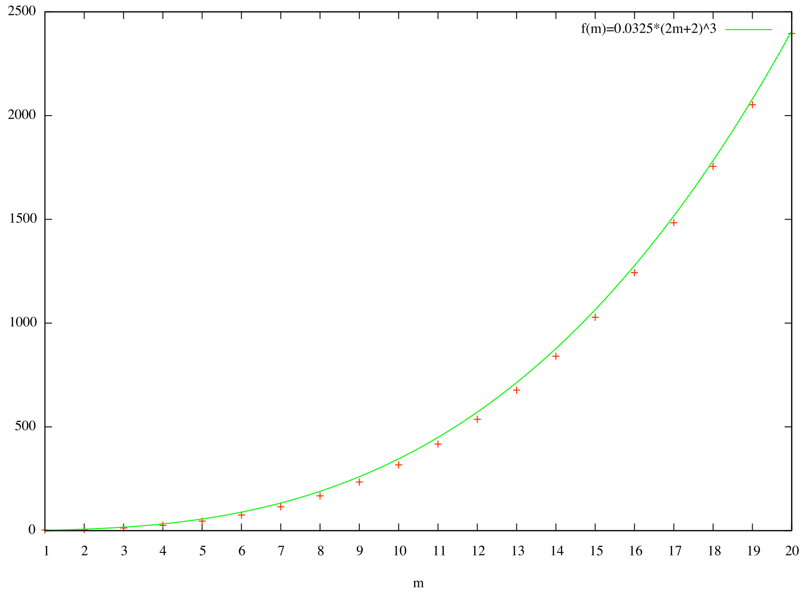

FIGURE 4. Graph of $|\mathbb{T}^{J(2,-2m)}_{\lambda}(\rho_0)|$ and $f(m) = C(\sharp J(2,-2m))^3$.

Item (5) of Remark 2 and the surgery formula for the Reidemeister torsion [16, Theorem 4.1].

Proposition 23. *The sequence* $\left(|\mathbb{T}^{J(2,-2m)}_{\lambda}(\rho_0)|\right)_{m\geqslant 1}$ *has the same behavior as the sequence* $\left(C(\sharp J(2,-2m))^3\right)_{m\geqslant 1}$, *for some constant* C.

Ideas of the Proof. Item (5) of Remark 2 gives us that $E_{J(2,-2m)} = \mathcal{W}(1/m)$ is obtained by a surgery of slope $1/m$ on the trivial component of the Whitehead link $\mathcal{W}$. Let V denote the glued solid torus and γ its core. Using [16, Theorem 4.1 (iii) and Proposition 2.25] we have, up to sign:

$$\mathbb{T}^{J(2,-2m)}_{\lambda}(\rho_0) = \mathbb{T}^{\mathcal{W}}_{(\lambda,\mu'\lambda'^{-m})}(\rho_0) \cdot \mathrm{TOR}(V; \mathfrak{sl}_2(\mathbb{C})_{\rho_0}, \gamma) \tag{55}$$

where $\mathbb{T}^{\mathcal{W}}_{(\lambda,\mu'\lambda'^{-m})}(\rho_0)$ stands for the $(Ad \circ \rho_0)$-twisted Reidemeister torsion of the Whitehead link exterior computed with respect to the bases of the twisted homology groups determined by the two curves λ, and $\mu'\lambda'^{-m}$ (see [16, Theorem 4.1]), and $\mathrm{TOR}(V; \mathfrak{sl}_2(\mathbb{C})_{\rho_0}, \gamma)$ stands for the $(Ad \circ \rho_0)$-twisted Reidemeister torsion the solid torus V computed with respect to its core γ. Here ρ_0 denotes the holonomy representation of the Whitehead link exterior.

To obtain the behavior of $\mathbb{T}_{\lambda}^{J(2,-2m)}(\rho_0)$ we estimate the two terms on the right–hand side of Equation (55):

(1) Using [16, Proof of Theorem 4.17], it is easy to see that $\mathrm{TOR}(V; \mathfrak{sl}_2(\mathbb{C})_{\rho_0}, \gamma)$ goes as $\frac{1}{4\pi^2}m^2$ when m goes to infinity.

(2) Using [16, Theorem 4.1 (ii)], one can prove that

$$\mathbb{T}^{\mathcal{W}}_{(\lambda, \mu'\lambda'^{-m})}(\rho_0) = \mathbb{T}^{\mathcal{W}}_{(\lambda,\lambda')}(\rho_0) \cdot \left(\frac{1}{\mathfrak{c}(\lambda,\mu)} - m\right),$$

where $\mathfrak{c} = \mathfrak{c}(\lambda,\mu)$ denotes the cusp shape of $J(2,-2m)$ (computed with respect to the usual meridian/longitude system). Thus, $\mathbb{T}^{\mathcal{W}}_{(\lambda, \mu'\lambda'^{-m})}(\rho_0)$ goes as $\mathbb{T}^{\mathcal{W}}_{(\lambda,\lambda')}(\rho_0) \cdot m$ when m goes to infinity. One can also prove that, at the holonomy, we have

$$\mathbb{T}^{\mathcal{W}}_{(\lambda,\lambda')}(\rho_0) = 8(1+i).$$

As a result, $\mathbb{T}_{\lambda}^{J(2,-2m)}(\rho_0)$ goes as $C \cdot m^3$ for some constant C. □

Appendix: Tables

In this appendix, except in the case of the trefoil knot (the only twist knot which is not hyperbolic), u denotes the root of Riley's equation $\phi_K(1,u) = 0$ corresponding to the discrete and faithful representation of the complete structure.

Remark 21. As explained in Section 3.4, one can express the root of Riley's equation $\phi_K(1,u) = 0$ corresponding to the holonomy representation using the cusp shape $\mathfrak{c}$. Equation (12) implies that the Reidemeister torsion at the holonomy representation for twist knots can be expressed as a rational function evaluated at the cusp shape. Such a formula is interesting because it expresses the twisted Reidemeister torsion at the holonomy in terms of a hyperbolic–geometric quantity.

Tables 1 and 2 gives the twisted Reidemeister torsion for twist knots at the holonomy (except in the case of the trefoil) with respect to the corresponding root of Riley's equation and to knot exterior's cusp shape.

Moreover we can know the approximate value of the cusp shape for the twist knot complement $E_{J(2,2m)}$ by using *SnapPea* [25], which is a program for creating and studying hyperbolic three–dimensional manifolds by Jeffrey Weeks. We give the lists containing the approximate values of cusp shapes for twist knot complements and the results by substituting them into the formulas of the twisted Reidemeister torsion in Tables 3 and 4.

Twisted torsion for $J(2,2m)$ $(1 \le m \le 10)$	
m	Torsion at the holonomy $\mathbb{T}^{J(2,2m)}_{\lambda}(\rho_u)$ (divided by a sign $-\tau_0$)
	Result by substituting $u = 4/(\mathfrak{c}-2)$ into $\mathbb{T}^{J(2,2m)}_{\lambda}(\rho_u)$, where $\mathfrak{c}$ is the cusp shape
1	3
	3
2	$13u^2 - 7u + 19$
	$(19\mathfrak{c}^2 - 104\mathfrak{c} + 340)/(\mathfrak{c}-2)^2$
3	$26u^4 - 17u^3 + 98u^2 - 45u + 55$
	$(55\mathfrak{c}^4 - 620\mathfrak{c}^3 + 3968\mathfrak{c}^2 - 11280\mathfrak{c} + 17424)/(\mathfrak{c}-2)^4$
4	$46u^6 - 34u^5 + 263u^4 - 157u^3 + 402u^2 - 159u + 118$
	$2(59\mathfrak{c}^6 - 1026\mathfrak{c}^5 + 9936\mathfrak{c}^4 - 52912\mathfrak{c}^3 + 180592\mathfrak{c}^2 - 352032\mathfrak{c} + 369280)/(\mathfrak{c}-2)^6$
5	$69u^8 - 54u^7 + 540u^6 - 366u^5 + 1360u^4 - 733u^3 + 1186u^2 - 411u + 215$
	$(215\mathfrak{c}^8 - 5084\mathfrak{c}^7 + 66072\mathfrak{c}^6 - 509040\mathfrak{c}^5 + 2656960\mathfrak{c}^4 - 9378624\mathfrak{c}^3 + 22613632\mathfrak{c}^2 - 33723648\mathfrak{c} + 26688768)/(\mathfrak{c}-2)^8$
6	$99u^{10} - 81u^9 + 971u^8 - 710u^7 + 3400u^6 - 2123u^5 + 5052u^4 - 2469u^3 + 2875u^2 - 884u + 353$
	$(353\mathfrak{c}^{10} - 10596\mathfrak{c}^9 + 173188\mathfrak{c}^8 - 1742080\mathfrak{c}^7 + 12219808\mathfrak{c}^6 - 61550208\mathfrak{c}^5 + 228030592\mathfrak{c}^4 - 612284416\mathfrak{c}^3 + 1160955136\mathfrak{c}^2 - 1411093504\mathfrak{c} + 903214080)/(\mathfrak{c}-2)^{10}$
7	$132u^{12} - 111u^{11} + 1566u^{10} - 1203u^9 + 7057u^8 - 4810u^7 + 14996u^6 - 8647u^5 + 15044u^4 - 6710u^3 + 6076u^2 - 1678u + 539$
	$(539\mathfrak{c}^{12} - 19648\mathfrak{c}^{11} + 387176\mathfrak{c}^{10} - 4799040\mathfrak{c}^9 + 42208784\mathfrak{c}^8 - 274741248\mathfrak{c}^7 + 1363062528\mathfrak{c}^6 - 5187840000\mathfrak{c}^5 + 15118560512\mathfrak{c}^4 - 33001455616\mathfrak{c}^3 + 51856091136\mathfrak{c}^2 - 53202206720\mathfrak{c} + 28544299008)/(\mathfrak{c}-2)^{12}$
8	$172u^{14} - 148u^{13} + 2383u^{12} - 1899u^{11} + 13098u^{10} - 9475u^9 + 36258u^8 - 23106u^7 + 52884u^6 - 28275u^5 + 38518u^4 - 15774u^3 + 11636u^2 - 2914u + 780$
	$4(195\mathfrak{c}^{14} - 8374\mathfrak{c}^{13} + 193288\mathfrak{c}^{12} - 2846448\mathfrak{c}^{11} + 30095568\mathfrak{c}^{10} - 239812000\mathfrak{c}^9 + 1487434752\mathfrak{c}^8 - 7287857664\mathfrak{c}^7 + 28404952320\mathfrak{c}^6 - 87772645888\mathfrak{c}^5 + 212579837952\mathfrak{c}^4 - 393068802048\mathfrak{c}^3 + 529782681600\mathfrak{c}^2 - 471281852416\mathfrak{c} + 218188021760)/(\mathfrak{c}-2)^{14}$
9	$215u^{16} - 188u^{15} + 3416u^{14} - 2796u^{13} + 22210u^{12} - 16767u^{11} + 76022u^{10} - 51847u^9 + 146639u^8 - 87602u^7 + 157972u^6 - 78647u^5 + 87864u^4 - 33238u^3 + 20652u^2 - 4730u + 1083$
	$(1083\mathfrak{c}^{16} - 53576\mathfrak{c}^{15} + 1417872\mathfrak{c}^{14} - 24177568\mathfrak{c}^{13} + 298484224\mathfrak{c}^{12} - 2810875520\mathfrak{c}^{11} + 20889506048\mathfrak{c}^{10} - 124832580096\mathfrak{c}^9 + 606632721920\mathfrak{c}^8 - 2406783375360\mathfrak{c}^7 + 7784342106112\mathfrak{c}^6 - 20354210914304\mathfrak{c}^5 + 42341634637824\mathfrak{c}^4 - 68068269064192\mathfrak{c}^3 + 80435317243904\mathfrak{c}^2 - 63142437978112\mathfrak{c} + 25688198283264)/(\mathfrak{c}-2)^{16}$
10	$265u^{18} - 235u^{17} + 4739u^{16} - 3964u^{15} + 35520u^{14} - 27711u^{13} + 144776u^{12} - 103759u^{11} + 348155u^{10} - 224404u^9 + 501055u^8 - 281458u^7 + 417368u^6 - 194245u^5 + 183500u^4 - 64454u^3 + 34537u^2 - 7285u + 1455$
	$(1455\mathfrak{c}^{18} - 81520\mathfrak{c}^{17} + 2433812\mathfrak{c}^{16} - 47158400\mathfrak{c}^{15} + 665730240\mathfrak{c}^{14} - 7230947840\mathfrak{c}^{13} + 62585931008\mathfrak{c}^{12} - 440831379456\mathfrak{c}^{11} + 2561522930176\mathfrak{c}^{10} - 12371911213056\mathfrak{c}^9 + 49827680770048\mathfrak{c}^8 - 167134091640832\mathfrak{c}^7 + 464297966682112\mathfrak{c}^6 - 1056316689612800\mathfrak{c}^5 + 1931794260754432\mathfrak{c}^4 - 2753296051208192\mathfrak{c}^3 + 2901909811167232\mathfrak{c}^2 - 2040504620417024\mathfrak{c} + 741196988416000)/(\mathfrak{c}-2)^{18}$

TABLE 1. Table for the sequence of knots $J(2,2m)$ $(1 \le m \le 10)$

	Twisted torsion for $J(2,-2m)$ $(1 \leq m \leq 10)$
m	Torsion at the holonomy $\mathbb{T}^{J(2,-2m)}_{\lambda}(\rho_u)$ (divided by a sign τ_0)
	Result by substituting $u = 4/(\mathfrak{c}-2)$ into $\mathbb{T}^{J(2,-2m)}_{\lambda}(\rho_u)$, where $\mathfrak{c}$ is the cusp shape
1	-3
	-3
2	$7u^3 + u^2 + 14u - 5$
	$-(5\mathfrak{c}^3 - 86\mathfrak{c}^2 + 268\mathfrak{c} - 680)/(\mathfrak{c}-2)^3$
3	$17u^5 + 8u^4 + 79u^3 + 26u^2 + 73u + 1$
	$(\mathfrak{c}^5 + 282\mathfrak{c}^4 - 1880\mathfrak{c}^3 + 9488\mathfrak{c}^2 - 22448\mathfrak{c} + 34848)/(\mathfrak{c}-2)^5$
4	$34u^7 + 22u^6 + 225u^5 + 119u^4 + 439u^3 + 162u^2 + 229u + 22$
	$2(11\mathfrak{c}^7 + 304\mathfrak{c}^6 - 3276\mathfrak{c}^5 + 25488\mathfrak{c}^4 - 112432\mathfrak{c}^3 + 359808\mathfrak{c}^2 - 661824\mathfrak{c} + 738560)/(\mathfrak{c}-2)^7$
5	$54u^9 + 39u^8 + 474u^7 + 300u^6 + 1411u^5 + 730u^4 + 1619u^3 + 586u^2 + 551u + 65$
	$(65\mathfrak{c}^9 + 1034\mathfrak{c}^8 - 16528\mathfrak{c}^7 + 175520\mathfrak{c}^6 - 1125280\mathfrak{c}^5 + 5374144\mathfrak{c}^4 - 17783040\mathfrak{c}^3 + 42336768\mathfrak{c}^2 - 62848768\mathfrak{c} + 53377536)/(\mathfrak{c}-2)^9$
6	$81u^{11} + 63u^{10} + 872u^9 + 611u^8 + 3462u^7 + 2104u^6 + 6167u^5 + 3036u^4 + 4714u^3 + 1615u^2 + 1129u + 137$
	$(137\mathfrak{c}^{11} + 1502\mathfrak{c}^{10} - 34340\mathfrak{c}^9 + 468616\mathfrak{c}^8 - 3940960\mathfrak{c}^7 + 25007808\mathfrak{c}^6 - 117308544\mathfrak{c}^5 + 420442368\mathfrak{c}^4 - 1109472000\mathfrak{c}^3 + 2123257344\mathfrak{c}^2 - 2628703232\mathfrak{c} + 1806428160)/(\mathfrak{c}-2)^{11}$
7	$111u^{13} + 90u^{12} + 1425u^{11} + 1062u^{10} + 7105u^9 + 4747u^8 + 17286u^7 + 9956u^6 + 21045u^5 + 9752u^4 + 11610u^3 + 3724u^2 + 2070u + 245$
	$(245\mathfrak{c}^{13} + 1910\mathfrak{c}^{12} - 62696\mathfrak{c}^{11} + 1057552\mathfrak{c}^{10} - 11025808\mathfrak{c}^9 + 87196704\mathfrak{c}^8 - 526180096\mathfrak{c}^7 + 2499806720\mathfrak{c}^6 - 9272200448\mathfrak{c}^5 + 26825366016\mathfrak{c}^4 - 58773907456\mathfrak{c}^3 + 94191718400\mathfrak{c}^2 - 99494326272\mathfrak{c} + 57088598016)/(\mathfrak{c}-2)^{13}$
8	$148u^{15} + 124u^{14} + 2195u^{13} + 1711u^{12} + 13125u^{11} + 9354u^{10} + 40453u^9 + 25698u^8 + 68070u^7 + 37044u^6 + 60745u^5 + 26422u^4 + 25394u^3 + 7604u^2 + 3502u + 396$
	$4(99\mathfrak{c}^{15} + 532\mathfrak{c}^{14} - 26060\mathfrak{c}^{13} + 529856\mathfrak{c}^{12} - 6606160\mathfrak{c}^{11} + 62595520\mathfrak{c}^{10} - 460836032\mathfrak{c}^9 + 2719805952\mathfrak{c}^8 - 12910096128\mathfrak{c}^7 + 49528327168\mathfrak{c}^6 - 152285103104\mathfrak{c}^5 + 370984124416\mathfrak{c}^4 - 695663718400\mathfrak{c}^3 + 961422573568\mathfrak{c}^2 - 884915453952\mathfrak{c} + 436376043520)/(\mathfrak{c}-2)^{15}$
9	$188u^{17} + 161u^{16} + 3172u^{15} + 2552u^{14} + 22171u^{13} + 16540u^{12} + 82961u^{11} + 56366u^{10} + 179181u^9 + 108029u^8 + 224190u^7 + 115204u^6 + 154037u^5 + 62916u^4 + 50650u^3 + 14172u^2 + 5570u + 597$
	$(597\mathfrak{c}^{17} + 1982\mathfrak{c}^{16} - 161440\mathfrak{c}^{15} + 3885760\mathfrak{c}^{14} - 56503104\mathfrak{c}^{13} + 624108160\mathfrak{c}^{12} - 5417133568\mathfrak{c}^{11} + 38142084096\mathfrak{c}^{10} - 219869856256\mathfrak{c}^9 + 1045959103488\mathfrak{c}^8 - 4105465389056\mathfrak{c}^7 + 13257704030208\mathfrak{c}^6 - 34864687169536\mathfrak{c}^5 + 73473552449536\mathfrak{c}^4 - 120432383098880\mathfrak{c}^3 + 146315203575808\mathfrak{c}^2 - 119078046597120\mathfrak{c} + 51376396566528)/(\mathfrak{c}-2)^{17}$
10	$235u^{19} + 205u^{18} + 4434u^{17} + 3659u^{16} + 35404u^{15} + 27360u^{14} + 155687u^{13} + 111176u^{12} + 410964u^{11} + 266255u^{10} + 665399u^9 + 380935u^8 + 647346u^7 + 314408u^6 + 353985u^5 + 135980u^4 + 94041u^3 + 24637u^2 + 8440u + 855$
	$(855\mathfrak{c}^{19} + 1270\mathfrak{c}^{18} - 236348\mathfrak{c}^{17} + 6649256\mathfrak{c}^{16} - 110706240\mathfrak{c}^{15} + 1397436800\mathfrak{c}^{14} - 13965196032\mathfrak{c}^{13} + 114155938304\mathfrak{c}^{12} - 773068443136\mathfrak{c}^{11} + 4380077954048\mathfrak{c}^{10} - 20842143467520\mathfrak{c}^9 + 83401615028224\mathfrak{c}^8 - 279834167951360\mathfrak{c}^7 + 782314999349248\mathfrak{c}^6 - 1800640172326912\mathfrak{c}^5 + 3349441682210816\mathfrak{c}^4 - 4881403696185344\mathfrak{c}^3 + 5296224268058624\mathfrak{c}^2 - 3862939033141248\mathfrak{c} + 1482393976832000)/(\mathfrak{c}-2)^{19}$

TABLE 2. Table for the sequence of knots $J(2,-2m)$ $(1 \leq m \leq 10)$

Table of cusp shape and twisted torsion for $J(2, 2m)$ $(1 \leq m \leq 10)$	
m	The cusp shape of $E_{J(2,2m)}$ by SnapPea
	Result by substituting the cusp shape into $\mathbb{T}_{\lambda}^{J(2,2m)}(\rho_u)$ (divided by a sign $-\tau_0$)
1	6
	3
2	$2.490244668 + 2.979447066\,i$
	$-4.11623 + 1.84036\,i$
3	$2.08126429145 + 2.36227823937\,i$
	$-7.90122 + 4.10883\,i$
4	$2.0276856933 + 2.1860003244\,i$
	$-15.7856 + 9.8702\,i$
5	$2.012780611 + 2.113453657\,i$
	$-28.639 + 19.945\,i$
6	$2.006968456 + 2.076533648\,i$
	$-47.61 + 35.51\,i$
7	$2.0042238896 + 2.0551565883\,i$
	$-73.9 + 58\,i$
8	$2.0027560835 + 2.0416569961\,i$
	$-108.71 + 87.94\,i$
9	$2.001898908 + 2.032581856\,i$
	$-153.25 + 127.23\,i$
10	$2.0013643244 + 2.0261854785\,i$
	$-208.74 + 176.85\,i$

TABLE 3. Approximate values of cusp shape and torsion for $J(2, 2m)$

Table of cusp shape and twisted torsion for $J(2,-2m)\,(1 \leq m \leq 10)$	
m	The cusp shape of $E_{J(2,-2m)}$ by SnapPea
	Result by substituting the cusp shape into $\mathbb{T}_{\lambda}^{J(2,-2m)}(\rho_u)$ (divided by a sign τ_0)
1	$2\sqrt{3}\,i$
	-3
2	$1.8267382783 + 2.5647986322\,i$
	$-3.56727 + 4.42520\,i$
3	$1.9550035735 + 2.2522368192\,i$
	$-7.65836 + 10.2328\,i$
4	$1.9816823033 + 2.1429951300\,i$
	$-15.613 + 20.31\,i$
5	$1.9907131276 + 2.0922630798\,i$
	$-28.493 + 35.873\,i$
6	$1.9946330273 + 2.0645297541\,i$
	$-47.48 + 58.13\,i$
7	$1.9966145588 + 2.0476951383\,i$
	$-73.77 + 88.3\,i$
8	$1.9977257728 + 2.0367007634\,i$
	$-108.586 + 127.59\,i$
9	$1.9983978648 + 2.0291212618\,i$
	$-153.13 + 177.2\,i$
10	$1.9988285125 + 2.0236732778\,i$
	$-208.6 + 238.4\,i$

TABLE 4. Approximate values of cusp shape and torsion for $J(2,-2m)$

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