

CATALAN'S INTERVALS AND REALIZERS OF TRIANGULATIONS

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ABSTRACT. The Stanley lattice, Tamari lattice and Kreweras lattice are three remarkable orders defined on the set of Catalan objects of a given size. These lattices are ordered by inclusion: the Stanley lattice is an extension of the Tamari lattice which is an extension of the Kreweras lattice. The Stanley order can be defined on the set of Dyck paths of size n as the relation of *being above*. Hence, intervals in the Stanley lattice are pairs of non-crossing Dyck paths. In a former article, the second author defined a bijection Φ between pairs of non-crossing Dyck paths and the realizers of triangulations (or Schnyder woods). We give a simpler description of the bijection Φ . Then, we study the restriction of Φ to Tamari's and Kreweras' intervals. We prove that Φ induces a bijection between Tamari intervals and minimal realizers. This gives a bijection between Tamari intervals and triangulations. We also prove that Φ induces a bijection between Kreweras intervals and the (unique) realizers of stack triangulations. Thus, Φ induces a bijection between Kreweras intervals and stack triangulations which are known to be in bijection with ternary trees.

1. INTRODUCTION

A *Dyck path* is a lattice path made of $+1$ and -1 steps that starts from 0, remains non-negative and ends at 0. It is often convenient to represent a Dyck path by a sequence of North-East and South-East steps as is done in Figure 1 (a). The set \mathbf{D}_n of Dyck paths of length $2n$ can be ordered by the relation $P \leq_S Q$ if P stays below Q . This partial order is in fact a distributive lattice on \mathbf{D}_n known as the *Stanley lattice*. The Hasse diagram of the Stanley lattice on \mathbf{D}_3 is represented in Figure 2 (a).

It is well known that the Dyck paths of length $2n$ are counted by the n^{th} *Catalan number* $C_n = \frac{1}{n+1} \binom{2n}{n}$. The Catalan sequence is a pervasive guest in enumerative combinatorics. Indeed, beside Dyck paths, this sequence enumerates the binary trees, the plane trees, the non-crossing partitions and over 60 other fundamental combinatorial structures [17, Ex. 6.19].

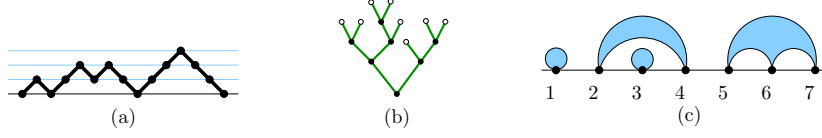


FIGURE 1. (a) A Dyck path. (b) A binary tree. (c) A non-crossing partition.

These different incarnations of the Catalan family gave rise to several lattices beside Stanley's. The *Tamari lattice* appears naturally in the study of binary trees where the covering relation corresponds to right rotation. This lattice is actively studied due to its link with the associahedron (Stasheff polytope). Indeed, the Hasse diagram of the Tamari lattice is the 1-skeleton of the associahedron. The *Kreweras lattice* appears naturally in the setting of non-crossing partitions. In the seminal paper [9], Kreweras proved that the refinement order on non-crossing partitions defines a lattice. Kreweras lattice appears to support a great deal of mathematics that reach far beyond enumerative combinatorics [10, 16]. Using suitable bijection between Dyck paths, binary trees, non-crossing partitions and plane trees, the three *Catalan lattices* can be defined on the set of plane trees of size n in such way that the Stanley lattice \mathcal{L}_n^S is an extension of the Tamari lattice \mathcal{L}_n^T which in turn is an extension of the Kreweras lattice \mathcal{L}_n^K (see [8, Ex. 7.2.1.6 - 26, 27 and 28]). In this paper, we shall find convenient to embed the three Catalan lattices on the set \mathbf{D}_n of Dyck paths. The Hasse diagram of the Catalan lattices on \mathbf{D}_3 is represented in Figure 2.

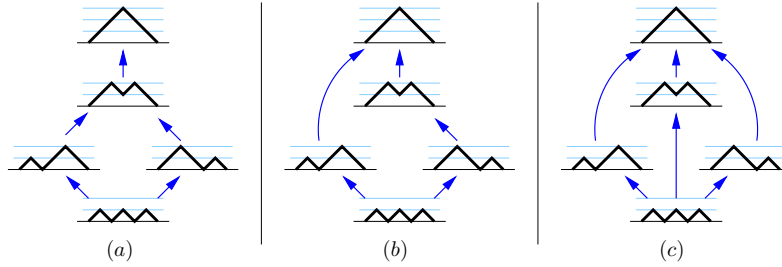


FIGURE 2. Hasse diagrams of the Catalan lattices on the set \mathbf{D}_3 of Dyck paths: (a) Stanley lattice, (b) Tamari lattice, (c) Kreweras lattice.

There are closed formulas for the number of *intervals* (i.e. pairs of comparable elements) in each of the Catalan lattices. The intervals of the Stanley lattice are the pairs of non-crossing Dyck paths and the number $|\mathcal{L}_n^S|$ of such pairs can be calculated using the lattice path determinant formula of Lindström-Gessel-Viennot [6]. It is shown in [4] that

$$(1) \quad |\mathcal{L}_n^S| = C_{n+2}C_n - C_{n+1}^2 = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}.$$

The intervals of the Tamari lattice were recently enumerated by Chapoton [3] using a generating function approach. It was proved that the number of intervals in the Tamari lattice is

$$(2) \quad |\mathcal{L}_n^T| = \frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

Chapoton also noticed that (2) is the number of triangulations (i.e. maximal planar graphs) and asked for an explanation. The number $|\mathcal{L}_n^K|$ of intervals of the Kreweras Lattice has an even simpler formula. In [9], Kreweras proved by a recursive method that

$$(3) \quad |\mathcal{L}_n^K| = \frac{1}{2n+1} \binom{3n}{n}.$$

This is also the number of ternary trees and a bijection was exhibited in [5].

In [1], the second author defined a bijection Φ between the pairs of non-crossing Dyck paths (equivalently, Stanley's intervals) and the *realizers* (or *Schnyder woods*) of triangulations. The main purpose of this article is to study the restriction of the bijection Φ to the Tamari intervals and to the Kreweras intervals. We first give an alternative, simpler, description of the bijection Φ . Then, we prove that the bijection Φ induces a bijection between the intervals of the Tamari lattice and the realizers which are *minimal*. Since every triangulation has a unique *minimal* realizer, we obtain a bijection between Tamari intervals and triangulations. As a corollary, we obtain a bijective proof of Formula (2) thereby answering the question of Chapoton. Turning to the Kreweras lattice, we prove that the mapping Φ induces a bijection between Kreweras intervals and the realizers which are both *minimal* and *maximal*. We then characterize the triangulations having a realizer which is both minimal and maximal and prove that these triangulations are in bijection with ternary trees. This gives a new bijective proof of Formula (3).

The outline of this paper is as follows. In Section 2, we review our notations about Dyck paths and characterize the covering relations for the

Stanley, Tamari and Kreweras lattices in terms of Dyck paths. In Section 3, we recall the definitions about triangulations and realizers. We then give an alternative description of the bijection Φ defined in [1] between pairs of non-crossing Dyck paths and the realizers. In Section 4, we study the restriction of Φ to the Tamari intervals. Lastly, in Section 5 we study the restriction of Φ to the Kreweras intervals.

2. CATALAN LATTICES

Dyck paths. A *Dyck path* is a lattice path made of steps $N = +1$ and $S = -1$ that starts from 0, remains non-negative and ends at 0. A Dyck path is said to be *prime* if it remains positive between its start and end. The *size* of a path is half its length and the set of Dyck paths of size n is denoted by \mathbf{D}_n .

Let P be a Dyck path of size n . Since P begins by an N step and has n N steps, it can be written as $P = NS^{\alpha_1}NS^{\alpha_2}\dots NS^{\alpha_n}$. We call i^{th} *descent* the subsequence S^{α_i} of P . For $i = 0, 1, \dots, n$ we call i^{th} *exceedence* and denote by $e_i(P)$ the height of the path P after the i^{th} descent, that is, $e_i(P) = i - \sum_{j \leq i} \alpha_j$. For instance, the Dyck path represented in Figure 3 (a) is $P = NS^1NS^0NS^1NS^2NS^0NS^0NS^3$ and $e_0(P) = 0$, $e_1(P) = 0$, $e_2(P) = 1$, $e_3(P) = 1$, $e_4(P) = 0$, $e_5(P) = 1$, $e_6(P) = 2$ and $e_7(P) = 0$. If P, Q are two Dyck paths of size n , we denote $\delta_i(P, Q) = e_i(Q) - e_i(P)$ and $\Delta(P, Q) = \sum_{i=1}^n \delta_i(P, Q)$. For instance, if P and Q are respectively the lower and upper paths in Figure 3 (b), the values $\delta_i(P, Q)$ are zero except for $\delta_1(P, Q) = 1$, $\delta_4(P, Q) = 2$ and $\delta_5(P, Q) = 1$.

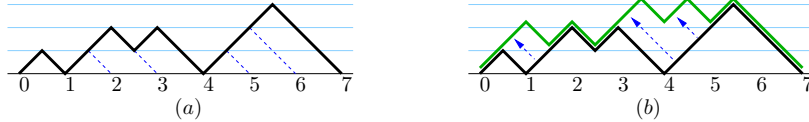


FIGURE 3. (a) Exceedence of a Dyck path. (b) Differences between two Dyck paths.

For $0 \leq i \leq j \leq n$, we write $i \underline{\leq} j$ (resp. $i \underline{\leq} j$) if $e_i(P) \geq e_j(P)$ and $e_i(P) \leq e_k(P)$ (resp. $e_i(P) < e_k(P)$) for all $i < k < j$. In other words, $i \underline{\leq} j$ (resp. $i \underline{\leq} j$) means that the subpath $NS^{\alpha_{i+1}}NS^{\alpha_{i+2}}\dots NS^{\alpha_j}$ is a Dyck path (resp. prime Dyck path) followed by $e_i(P) - e_j(P)$ S steps. For instance, for the Dyck path P of Figure 3 (a), we have $0 \underline{\leq} 4$, $1 \underline{\leq} 4$ and $2 \underline{\leq} 4$ (and many other relations).

We will now define the Stanley, Tamari and Kreweras lattices in terms of Dyck paths. More precisely, we will characterize the covering relation of each lattice in terms of Dyck paths and show that our definitions respects the known hierarchy between the three lattices (the Stanley lattice is a refinement of the Tamari lattice which is refinement of the Kreweras Lattice; see [8, Ex. 7.2.1.6 - 26, 27 and 28]).

Stanley lattice. Let $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ be two Dyck paths of size n . We denote by $P \leq_S Q$ if the path P stays below the path Q . Equivalently, $e_i(P) \leq e_i(Q)$ for all $1 \leq i \leq n$. The relation \leq_S defines the *Stanley lattice* \mathcal{L}_n^S on the set \mathbf{D}_n . Clearly the path P is covered by the path Q in the Stanley lattice if Q is obtained from P by replacing a subpath SN by NS . Equivalently, there is an index $1 \leq i \leq n$ such that $\beta_i = \alpha_i - 1$, $\beta_{i+1} = \alpha_{i+1} + 1$ and $\beta_k = \alpha_k$ for all $k \neq i, i+1$. The covering relation of the Stanley lattice is represented in Figure 4 (a) and the Hasse Diagram of \mathcal{L}_3^S is represented in Figure 2 (a).



FIGURE 4. Covering relations in (a) Stanley lattice, (b) Tamari lattice.

Tamari lattice. The Tamari lattice has a simple interpretation in terms of binary trees. The set of binary trees can be defined recursively by the following grammar. A binary tree B is either a leaf denoted by \circ or is an ordered pair of binary trees, denoted $B = (B_1, B_2)$. It is often convenient to draw a binary tree by representing the leaf by a white vertex and the tree $B = (B_1, B_2)$ by a black vertex at the bottom joined to the subtrees B_1 (on the left) and B_2 (on the right). The tree $(((\circ, \circ), ((\circ, \circ), \circ)), (\circ, (\circ, \circ)))$ is represented in Figure 5.

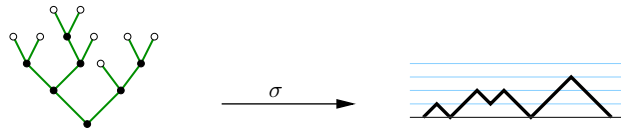


FIGURE 5. The binary tree $(((\circ, \circ), ((\circ, \circ), \circ)), (\circ, (\circ, \circ)))$ and its image by the bijection σ .

The set \mathbf{B}_n of binary trees with n nodes has cardinality $C_n = \frac{1}{n+1} \binom{2n}{n}$ and there are well known bijections between the set \mathbf{B}_n and the set \mathbf{D}_n . We

call σ the bijection defined as follows: the image of the binary tree reduced to a leaf is the empty word and the image of the binary tree $B = (B_1, B_2)$ is the Dyck path $\sigma(B) = \sigma(B_1)N\sigma(B_2)S$. An example is given in Figure 5.

In [7], Tamari defined a partial order on the set \mathbf{B}_n of binary trees and proved to be a lattice. The covering relation for the Tamari lattice is defined as follows: a binary tree B containing a subtree of type $X = ((B_1, B_2), B_3)$ is covered by the binary tree B' obtained from B by replacing X by $(B_1, (B_2, B_3))$. The Hasse diagram of the Tamari lattice on the set of binary trees with 4 nodes is represented in Figure 6 (left).

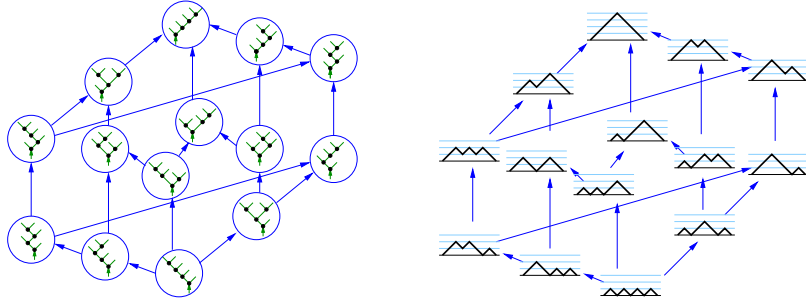


FIGURE 6. Hasse diagram of the Tamari lattice \mathcal{L}_4^T .

The bijection σ allows to transfer the Tamari lattice to the set of \mathbf{D}_n Dyck paths. We denote by \mathcal{L}_n^T the image of the Tamari lattice on \mathbf{D}_n and denote by $P \leq_T Q$ if the path P is less than or equal to the path Q for this order. The Hasse diagram of \mathcal{L}_4^T is represented in Figure 6 (right). The following proposition expresses the covering relation of the Tamari lattice \mathcal{L}_n^T in terms of Dyck paths. This covering relation is illustrated in Figure 4 (b).

Proposition 2.1. *Let $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ be two Dyck paths. The path P is covered by the path Q in the Tamari lattice \mathcal{L}_n^T if Q is obtained from P by swapping an S step and the prime Dyck subpath following it, that is, there are indices $1 \leq i < j \leq n$ with $\alpha_i > 0$ and $i \underline{P} j$ such that $\beta_i = \alpha_i - 1$, $\beta_j = \alpha_j + 1$ and $\beta_k = \alpha_k$ for all $k \neq i, j$.*

Corollary 2.2. *The Stanley lattice \mathcal{L}_n^S is a refinement of the Tamari lattice \mathcal{L}_n^T . That is, for any pair of Dyck paths P, Q , $P \leq_T Q$ implies $P \leq_S Q$.*

Proof of Proposition 2.1: Let B be a binary tree and let $P = \sigma(B)$.

• We use the well known fact that there is a one-to-one correspondence between the subtrees of B and the Dyck subpaths of P which are either a

prefix of P or are preceded by an N step. (This classical property is easily shown by induction on the size of P .)

- If the binary tree B' is obtained from B by replacing a subtree $X = ((B_1, B_2), B_3)$ by $X' = (B_1, (B_2, B_3))$, then the Dyck path $Q = \sigma(B')$ is obtained from P by replacing a subpath $\sigma(X) = \sigma(B_1)N\sigma(B_2)SN\sigma(B_3)S$ by $\sigma(X') = \sigma(B_1)N\sigma(B_2)N\sigma(B_3)SS$; hence by swapping an S step and the prime Dyck subpath following it.

- Suppose conversely that the Dyck path Q is obtained from P by swapping an S step with a prime Dyck subpath NP_3S following it. Then, there are two Dyck paths P_1 and P_2 (possibly empty) such that $W = P_1NP_2SNP_3S$ is a Dyck subpath of P which is either a prefix of P or is preceded by an N step. Hence, the binary tree B contains the subtree $X = \sigma^{-1}(W) = ((B_1, B_2), B_3)$, where $B_i = \sigma^{-1}(P_i)$, $i = 1, 2, 3$. Moreover, the binary tree $B' = \sigma^{-1}(Q)$ is obtained from B by replacing the subtree $X = ((B_1, B_2), B_3)$ by $X' = (B_1, (B_2, B_3)) = \sigma^{-1}(P_1NP_2NP_3SS)$. \square

Kreweras lattice. A partition of $\{1, \dots, n\}$ is *non-crossing* if whenever four elements $1 \leq i < j < k < l \leq n$ are such that i, k are in the same class and j, l are in the same class, then the two classes coincide. The non-crossing partition whose classes are $\{1\}$, $\{2, 4\}$, $\{3\}$, and $\{5, 6, 7\}$ is represented in Figure 7. In this figure, each class is represented by a connected cell incident to the integers it contains.

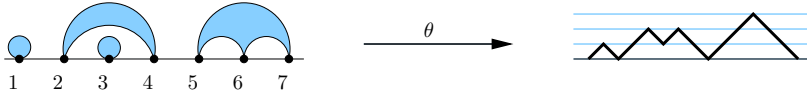


FIGURE 7. A non-crossing partition and its image by the bijection θ .

The set \mathbf{NC}_n of non-crossing partition on $\{1, \dots, n\}$ has cardinality $C_n = \frac{1}{n+1} \binom{2n}{n}$ and there are well known bijections between non-crossing partitions and Dyck paths. We consider the bijection θ defined as follows. The image of a non-crossing partition π of size n by the mapping θ is the Dyck path $\theta(\pi) = NS^{\alpha_1}NS^{\alpha_2} \dots NS^{\alpha_n}$, where α_i is the size of the class containing i if i is maximal in its class and $\alpha_i = 0$ otherwise. An example is given in Figure 7.

In [9], Kreweras showed that the partial order of refinement defines a lattice on the set \mathbf{NC}_n of non-crossing partitions. The covering relation of this lattice corresponds to the merging of two parts when this operation does

not break the *non-crossing condition*. The Hasse diagram of the Kreweras lattice on the set \mathbf{NC}_4 is represented in Figure 8 (left).

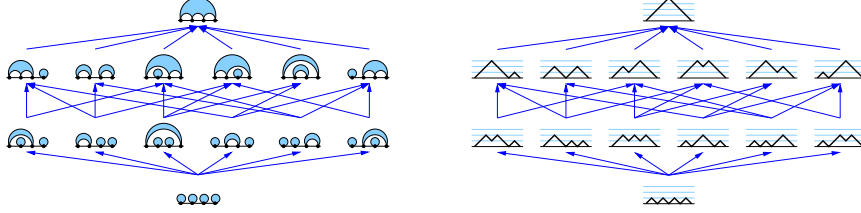


FIGURE 8. Hasse diagram of the Kreweras lattice \mathcal{L}_4^K .

The bijection θ allows to transfer the Kreweras lattice on the set \mathbf{D}_n of Dyck paths. We denote by \mathcal{L}_n^K the lattice structure obtained on \mathbf{D}_n and denote by $P \leq_K Q$ if the path P is less than or equal to the path Q for this order. The Hasse diagram of \mathcal{L}_4^K is represented in Figure 8 (right). The following proposition expresses the covering relation of the Kreweras lattice \mathcal{L}_n^K in terms of Dyck paths. This covering relation is represented in Figure 9.

Proposition 2.3. *Let $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ be two Dyck paths of size n . The path P is covered by the path Q in the Kreweras lattice \mathcal{L}_n^K if Q is obtained from P by swapping a (non-empty) descent with a Dyck subpath following it, that is, there are indices $1 \leq i < j \leq n$ with $\alpha_i > 0$ and $i \underline{P} j$ such that $\beta_i = 0$, $\beta_j = \alpha_i + \alpha_j$ and $\beta_k = \alpha_k$ for all $k \neq i, j$.*

Corollary 2.4. *The Tamari lattice \mathcal{L}_n^T is a refinement of the Kreweras lattice \mathcal{L}_n^K . That is, for any pair P, Q of Dyck paths, $P \leq_K Q$ implies $P \leq_T Q$.*



FIGURE 9. Two examples of covering relation in the Kreweras lattice.

Proposition 2.3 is a immediate consequence of the following lemma.

Lemma 2.5. *Let π be a non-crossing partition and let $P = \theta(\pi)$. Let c and c' be two classes of π with the convention that $i = \max(c) < j = \max(c')$. Then, the classes c and c' can be merged without breaking the non-crossing condition if and only if $i \underline{P} j$.*

Proof: For any index $k = 1, \dots, n$, we denote by c_k the class of π containing k . Observe that the classes c and c' can be merged without breaking the non-crossing condition if and only if there are no integers r, s with $c_r = c_s$ such that $r < i < s < j$ or $i < r < j < s$. Observe also from the definition of the mapping θ that for all index $l = 1, \dots, n$, the exceedence $e_l(P)$ is equal to the number of indices $k \leq l$ such that $\max(c_k) > l$.

- We suppose that $i \underline{P} j$ and we want to prove that merging the classes c and c' does not break the non-crossing condition. We first prove that there are no integers r, s such that $i < r < j < s$ and $c_r = c_s$. Suppose the contrary. In this case, there is no integer $k \leq r$ such that $r < \max(c_k) \leq j$ (otherwise, $c_k = c_r = c_s$ by the non-crossing condition, hence $\max(c_k) \geq \max(c_s) > j$). Thus, $\{k \leq r / \max(c_k) > r\} = \{k \leq r / \max(c_k) > j\} \subsetneq \{k \leq j / \max(c_k) > j\}$. This implies $e_r(P) < e_j(P)$ and contradicts the assumption $i \underline{P} j$. It remains to prove that there are no integers r, s such that $r < i < s < j$ and $c_r = c_s$. Suppose the contrary and let $s' = \max(c_r)$. The case where $s' \geq j$ has been treated in the preceding point so we can assume that $s' < j$. In this case, there is no integer k such that $i < k \leq s'$ and $\max(c_k) > s'$ (otherwise, $c_k = c_r = c_{s'}$ by the non-crossing condition, hence $\max(c_k) = \max(c_r) = s'$). Thus, $\{k \leq i / \max(c_k) > i\} \subsetneq \{k \leq i / \max(c_k) > s'\} = \{k \leq s' / \max(c_k) > s'\}$. This implies $e_i(P) < e_{s'}(P)$ and contradicts the assumption $i \underline{P} j$.

- We suppose now that merging the classes c and c' does not break the non-crossing partition and we want to prove that $i \underline{P} j$. Observe that there is no integer k such that $i < k \leq j$ and $\max(c_k) > j$ (otherwise, merging the classes c and c' would break the non-crossing condition). Thus, $\{k \leq j / \max(c_k) > j\} = \{k \leq i / \max(c_k) > j\} \subseteq \{k \leq i / \max(c_k) > i\}$. This implies $e_j(P) \leq e_i(P)$. It remains to prove that there is no index s such that $i < s < j$ and $e_s(P) < e_i(P)$. Suppose the contrary and consider the minimal such s . Observe that s is maximal in its class, otherwise $e_{s-1}(P) = e_s(P) - 1 < e_i(P)$ contradicts the minimality of s . Observe also that $i < r = \min(c_s)$ otherwise merging the classes c and c' would break the non-crossing condition. By the non-crossing condition, there is no integer $k < r$ such that $r \leq \max(c_k) \leq s$. Thus, $\{k \leq r - 1 / \max(c_k) > r - 1\} = \{k \leq r - 1 / \max(c_k) > s\} \subseteq \{k \leq s / \max(c_k) > s\}$. This implies $e_{r-1}(P) \leq e_s(P) < e_i(P)$ and contradicts the minimality of s . \square

3. A BIJECTION BETWEEN STANLEY INTERVALS AND REALIZERS

In this section, we recall some definitions about triangulations and realizers. Then, we define a bijection between pairs of non-crossing Dyck paths

and realizers.

3.1. Triangulations and realizers.

Maps. A *planar map*, or *map* for short, is an embedding of a connected finite planar graph in the sphere considered up to continuous deformation. In this paper, maps have no loop nor multiple edge. The *faces* are the connected components of the complement of the graph. By removing the midpoint of an edge we get two *half-edges*, that is, one dimensional cells incident to one vertex. Two consecutive half-edges around a vertex define a *corner*. If an edge is oriented we call *tail* (resp. *head*) the half-edge incident to the origin (resp. end).

A *rooted map* is a map together with a special half-edge which is not part of a complete edge and is called the *root*. (Equivalently, a rooting is defined by the choice of a corner.) The root is incident to one vertex called *root-vertex* and one face (containing it) called the *root-face*. When drawing maps in the plane the root is represented by an arrow pointing on the root-vertex and the root-face is the infinite one. See Figure 10 for an example. The vertices and edges incident to the root-face are called *external* while the others are called *internal*. From now on, *maps are rooted* without further notice.

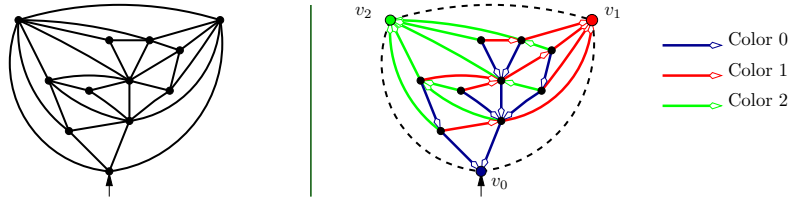


FIGURE 10. A rooted triangulation (left) and one of its realizers (right).

Triangulations. A *triangulation* is a map in which any face has degree 3 (has 3 corners). A triangulation has *size* n if it has n internal vertices. The incidence relation between faces and edges together with Euler formula show that a triangulation of size n has $3n$ internal edges and $2n + 1$ internal triangles.

In one of its famous *census* paper, Tutte proved by a generating function approach that the number of triangulations of size n is $t_n = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$

[18]. A bijective proof of this result was given in [12].

Realizers. We now recall the notion of *realizer* (or *Schnyder wood*) defined by Schnyder [14, 15]. Given an edge coloring of a map, we shall call *i-edge* (resp. *i-tail*, *i-head*) an edge (resp. tail, head) of color i .

Definition 3.1 ([14]). Let M be a triangulation and let U be the set of its internal vertices. Let v_0 be the root-vertex and let v_1, v_2 be the other external vertices with the convention that v_0, v_1, v_2 appear in counterclockwise order around the root-face.

A *realizer* of M is a coloring of the internal edges in three colors $\{0, 1, 2\}$ such that:

- (1) *Tree condition*: for $i = 0, 1, 2$, the i -edges form a tree T_i with vertex set $U \cup \{v_i\}$. The vertex v_i is considered to be the root-vertex of T_i and the i -edges are oriented toward v_i .
- (2) *Schnyder condition*: in clockwise order around any internal vertex there is: one 0-tail, some 1-heads, one 2-tail, some 0-heads, one 1-tail, some 2-heads. This situation is represented in Figure 11.

We denote by $R = (T_0, T_1, T_2)$ this realizer.

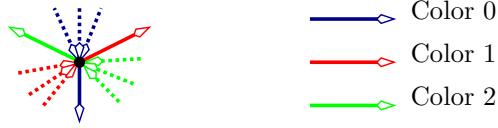


FIGURE 11. Edges coloration and orientation around a vertex in a realizer (Schnyder condition).

A realizer is represented in Figure 10 (right). Let $R = (T_0, T_1, T_2)$ be a realizer. We denote by $\overline{T_0}$ the tree made of T_0 together with the edge (v_0, v_1) . For any internal vertex u , we denote by $\mathbf{p}_i(u)$ the parent of u in the tree T_i . A *cw-triangle* (resp. *ccw-triangle*) is a triple of vertices (u, v, w) such that $\mathbf{p}_0(u) = v, \mathbf{p}_2(v) = w$ and $\mathbf{p}_1(w) = u$ (resp. $\mathbf{p}_0(u) = v, \mathbf{p}_1(v) = w$ and $\mathbf{p}_2(w) = u$). A realizer is called *minimal* (resp. *maximal*) if it has no *cw-triangle* (resp. *ccw-triangle*). It was proved in [11, 13] that every triangulation has a unique minimal (resp. maximal) realizer. (The appellations *minimal* and *maximal* refer to a classical lattice which is defined on the set of realizers of any given triangulation [11, 13].)

3.2. A bijection between pairs of non-crossing Dyck paths and realizers.

In this subsection, we give an alternative (and simpler) description of the bijection defined in [1] between realizers and pairs of non-crossing Dyck paths.

We first recall a classical bijection between plane trees and Dyck paths. A *plane tree* is a rooted map whose underlying graph is a tree. Let T be a plane tree. We *make the tour* of the tree T by following its border in clockwise direction starting and ending at the root (see Figure 14 (a)). We denote by $\omega(T)$ the word obtained by making the tour of the tree T and writing N the first time we follow an edge and S the second time we follow this edge. For instance, $w(T) = NNSNNSNNSNNSNNSSS$ for the tree in Figure 14 (a). It is well known that the mapping ω is a bijection between plane trees with n edges and Dyck paths of size n [8].

Let T be a plane tree. Consider the order in which the vertices are encountered while making the tour of T . This defines the *clockwise order around T* (or *preorder*). For the tree in Figure 14 (a), the clockwise order is $v_0 < u_0 < u_1 < \dots < u_8$. The tour of the tree also defines an order on the set of corners around each vertex v . We shall talk about the *first* (resp. *last*) *corner of v around T* .

We are now ready to define a mapping Ψ which associates an ordered pair of Dyck paths to each realizer.

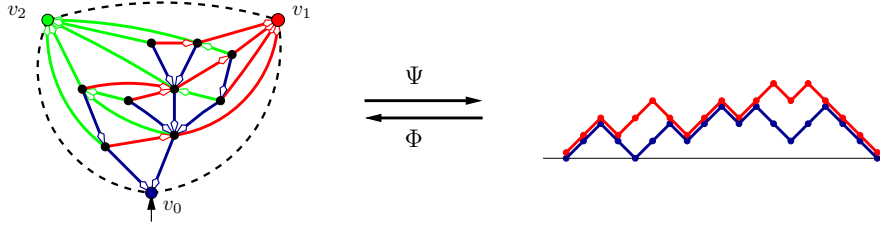
Definition 3.2. Let M be a rooted triangulation of size n and let $R = (T_0, T_1, T_2)$ be a realizer of M . Let u_0, u_1, \dots, u_{n-1} be the internal vertices of M in clockwise order around T_0 . Let $\beta_i, i = 1, \dots, n-1$ be the number of 1-heads incident to u_i and let β_n be the number of 1-heads incident to v_1 . Then $\Psi(R) = (P, Q)$, where $P = \omega^{-1}(T_0)$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$.

The image of a realizer by the mapping Ψ is represented in Figure 12.

Theorem 3.3. *The mapping Ψ is a bijection between realizers of size n and pairs of non-crossing Dyck paths of size n .*

The rest of this section is devoted to the proof of Theorem 3.3. We first prove that the image of a realizer is indeed a pair of non-crossing Dyck paths.

Proposition 3.4. *Let $R = (T_0, T_1, T_2)$ be a realizer of size n and let $(P, Q) = \Psi(R)$. Then, P and Q are both Dyck paths and moreover the path P stays below the path Q .*

FIGURE 12. The bijections Ψ and Φ .

Proposition 3.4 is closely related to the Lemma 3.6 below which, in turn, relies on the following technical lemma.

Lemma 3.5. *Let M be a map in which every face has degree three. We consider an orientation of the internal edges of M such that every internal vertex has outdegree 3 (i.e. is incident to exactly 3 tails). Let C be a simple cycle made of c edges. By the Jordan Lemma, the cycle C separates the sphere into two connected regions. We call inside the region not containing the root. Then, the number of tails incident with C and lying strictly inside C is $c - 3$.*

Proof: Let v (resp. f , e) be the number of vertices (resp. faces, edges) lying strictly inside C . Note that the edges strictly inside C are internal hence are oriented. The number i of tails incident with C and lying strictly inside C satisfies $e = 3v + i$. Moreover, the incidence relation between edges and faces implies $3f = 2e + c$ and the Euler relation implies $(f+1) + (v+c) = (e+m)+2$. Solving for i gives $i = c - 3$. \square

Lemma 3.6. *Let $R = (T_0, T_1, T_2)$ be a realizer. Then, for any 1-edge e the tail of e is encountered before its head around the tree $\overline{T_0}$.*

Proof of Lemma 3.6: Suppose a 1-edge e breaks this rule and consider the cycle C made of e and the 0-path joining its endpoints. Using the Schnyder condition it is easy to show that the number of tails incident with C and lying strictly inside C is equal to the number of edges of C (the different possibilities are represented in Figure 13). This contradicts Lemma 3.5. \square

Lemma 3.7. *Let $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ be a Dyck path and let $T = \omega^{-1}(P)$. Let v_0 be the root-vertex of the tree T and let u_0, u_1, \dots, u_{n-1} be its other vertices in clockwise order around T . Then, the word obtained by making the tour of T and writing S^{β_i} when arriving at the first corner of u_i and N when arriving at the last corner of u_i is $W = S^{\beta_0} N^{\alpha_1} S^{\beta_1} \dots S^{\beta_{n-1}} N^{\alpha_n}$.*

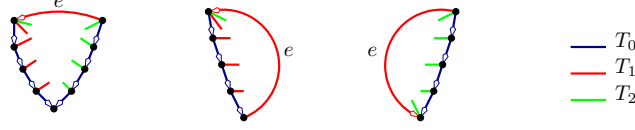


FIGURE 13. Case analysis for a 1-edge e whose head appears before its tail around the tree \overline{T}_0 .

Proof: We consider the word \mathcal{W} obtained by making the tour of T and writing \mathbf{NS}^{β_i} when arriving at the first corner of u_i and \mathbf{NS} when arriving at the last corner of u_i for $i = 0, \dots, n-1$. By definition of the mapping ω , the restriction of \mathcal{W} to the letters N, S is $W = \omega(T) = P = NS^{\alpha_1} \dots NS^{\alpha_n}$. Therefore, $\mathcal{W} = \mathbf{NS}^{\beta_0} (\mathbf{NS})^{\alpha_1} \mathbf{NS}^{\beta_1} (\mathbf{NS})^{\alpha_2} \dots \mathbf{NS}^{\beta_{n-1}} (\mathbf{NS})^{\alpha_n}$. Hence, the restriction of \mathcal{W} to the letters \mathbf{N}, \mathbf{S} is $\mathbf{W} = \mathbf{S}^{\beta_0} \mathbf{N}^{\alpha_1} \mathbf{S}^{\beta_1} \mathbf{N}^{\alpha_2} \dots \mathbf{S}^{\beta_{n-1}} \mathbf{N}^{\alpha_n}$.

□

Proof of Proposition 3.4: We denote $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$.

- The mapping ω is known to be a bijection between trees and Dyck paths, hence $P = \omega(T)$ is a Dyck path.
- We want to prove that Q is a Dyck path staying above P . Consider the word \mathbf{W} obtained by making the tour of \overline{T}_0 and writing \mathbf{N} (resp. \mathbf{S}) when we encounter a 1-tail (resp. 1-head). By Lemma 3.7, the word \mathbf{W} is $\mathbf{S}^{\beta_0} \mathbf{N}^{\alpha_1} \mathbf{S}^{\beta_1} \mathbf{N}^{\alpha_2} \dots \mathbf{S}^{\beta_{n-1}} \mathbf{N}^{\alpha_n} \mathbf{S}^{\beta_n}$. By Lemma 3.6, the word \mathbf{W} is a Dyck path. In particular, $\mathbf{S}^{\beta_0} = 0$ and $\sum_{i=1}^n \beta_i = \sum_{i=1}^n \alpha_i = n$, hence the path Q returns to the origin. Moreover, for all $i = 1, \dots, n$, $\delta_i(P, Q) = \sum_{j=1}^i \alpha_j - \beta_j \geq 0$. Thus, the path Q stays above P . In particular, Q is a Dyck path. □

In order to prove Theorem 3.3, we shall now define a mapping Φ from pairs of non-crossing Dyck paths to realizers and prove it to be the inverse of Ψ . We first define *prerealizers*.

Definition 3.8. Let M be a map. Let v_0 be the root-vertex, let v_1 be another external vertex and let U be the set of the other vertices. A *prerealizer* of M is a coloring of the edges in two colors $\{0, 1\}$ such that:

- (1) *Tree condition:* for $i = 0, 1$, the i -edges form a tree T_i with vertex set $U \cup \{v_i\}$. The vertex v_i is considered to be the root-vertex of T_i and the i -edges are oriented toward v_i .
- (2) *Corner condition:* in clockwise order around any vertex $u \in U$ there is: one 0-tail, some 1-heads, some 0-heads, one 1-tail.

- (3) *Order condition*: for any 1-edge e , the tail of e is encountered before its head around the tree $\overline{T_0}$, where $\overline{T_0}$ is the tree obtained from T_0 by adding the edge (v_0, v_1) at the right of the root.

We denote by $PR = (T_0, T_1)$ this prerealizer.

An example of prerealizer is given in Figure 14 (c).

Lemma 3.9. *Let $PR = (T_0, T_1)$ be a prerealizer. Then, there exists a unique tree T_2 such that $R = (T_0, T_1, T_2)$ is a realizer.*

In order to prove Lemma 3.9, we need to study the sequences of corner around the faces of prerealizers. If h and h' are two consecutive half-edges in clockwise order around a vertex u we denote by $c = (h, h')$ the corner delimited by h and h' . For $0 \leq i, j \leq 2$, we call (h_i, h_j) -corner (resp. (h_i, t_j) -corner, (t_i, h_j) -corner, (t_i, t_j) -corner) a corner $c = (h, h')$ where h and h' are respectively an i -head (resp. i -head, i -tail, i -tail) and a j -head (resp. j -tail, j -head, j -tail).

Proof of Lemma 3.9: Let $PR = (T_0, T_1)$ be a prerealizer and let $N = T_0 \cup T_1$ be the underlying map. Let v_0 (resp. v_1) be the root-vertex of T_0 (resp. T_1) and let U be the set of vertices distinct from v_0, v_1 . Let $\overline{T_0}$ (resp. \overline{N}) be the tree (resp. map) obtained from T_0 (resp. N) by adding the edge (v_0, v_1) at the right of the root. We first prove that there is at most one tree T_2 such that $R = (T_0, T_1, T_2)$ is a realizer.

- Let f be an internal face of \overline{N} and let c_1, c_2, \dots, c_k be the corners of f encountered in clockwise order around $\overline{T_0}$. Note that c_1, c_2, \dots, c_k also correspond to the clockwise order of the corners around the face f . We want to prove the following properties:
 - the corner c_1 is a (t_1, t_0) -corner,
 - the corner c_2 is either a (h_0, h_0) - or a (h_0, t_1) -corner,
 - the corners c_3, \dots, c_{k-1} are (h_1, h_0) -, (h_1, t_1) -, (t_0, h_0) - or (t_0, t_1) -corners,
 - the corner c_k is either a (h_1, h_1) - or a (t_0, h_1) -corner.

First note that by the *corner condition* of the prerealizers the possible corners are of type (h_0, h_0) , (h_0, t_1) , (h_1, h_0) , (h_1, h_1) , (h_1, t_1) , (t_0, h_0) , (t_0, h_1) , (t_0, t_1) and (t_1, t_0) . By the *order condition*, one enters a face for the first time (during a tour of T_0) when crossing a 1-tail. Hence, the first corner c_1 of f is a (t_1, t_0) -corner while the corners c_i , $i = 2, \dots, k$ are not (t_1, t_0) -corners. Since c_1 is a (t_1, t_0) -corner, the corner c_2 is either a (h_0, h_0) - or a (h_0, t_1) -corner. Similarly, since c_1 is a (t_1, t_0) -corner, the corner c_k is either a (h_1, h_1) - or a (t_0, h_1) -corner. Moreover, for $i = 2, \dots, k-1$, the corner c_i is not a (h_1, h_1) - nor a (t_0, h_1) -corner or c_{i+1} would be a

(t_1, t_0) -corner. Therefore, it is easily seen by induction on i that the corners c_i , $i = 3, \dots, k-1$ are either (h_1, h_0) -, (h_1, t_1) -, (t_0, h_0) - or (t_0, t_1) -corners.

- By a similar argument we prove that the corners of the external face of \bar{N} are (h_1, h_0) -, (h_1, t_1) -, (t_0, h_0) - or (t_0, t_1) -corners except for the corner incident to v_0 which is a (h_0, h_0) -corner and the corner incident to v_1 which is a (h_1, h_1) -corner.
- Let v_2 be an isolated vertex in the external face of N . If a tree T_2 with vertex set $U \cup \{v_2\}$ is such that $R = (T_0, T_1, T_2)$ is a realizer, then there is one 2-tail in each (h_1, h_0) -, (h_1, t_1) -, (t_0, h_0) - or (t_0, t_1) -corner of \bar{N} while the 2-heads are only incident to the (t_0, t_1) -corners and to the vertex v_2 . By the preceding points, there is exactly one (t_1, t_0) corner in each internal face and none in the external face. Moreover there is at most one way of connecting the 2-tails and the 2-heads in each face of \bar{N} . Thus, there is at most one tree T_2 such that $R = (T_0, T_1, T_2)$ is a realizer.

We now prove that there exists a tree T_2 such that $R = (T_0, T_1, T_2)$ is a realizer. Consider the colored map (T_0, T_1, T_2) obtained by

- adding an isolated vertex v_2 in the external face of \bar{N} .
- adding a 2-tail in each (h_1, h_0) -, (h_1, t_1) -, (t_0, h_0) - and (t_0, t_1) -corner of \bar{N} .
- joining each 2-tail in an internal face f (resp. the external face) to the unique (t_0, t_1) -corner of f (resp. to v_2).

We denote by $M = T_0 \cup T_1 \cup T_2 \cup \{(v_0, v_1), (v_0, v_2), (v_1, v_2)\}$ the underlying map.

- We first prove that the map $M = T_0 \cup T_1 \cup T_2 \cup \{(v_0, v_1), (v_0, v_2), (v_1, v_2)\}$ is a triangulation. Let f be an internal face. By a preceding point, f has exactly one (t_1, t_0) corner c and the (h_1, h_0) -, (h_1, t_1) -, (t_0, h_0) - or (t_0, t_1) -corners are precisely the ones that are not consecutive with c around f . Thus, the internal faces of N are triangulated (split into sub-faces of degree 3) by the 2-edges. Moreover, the only corners of the external face of \bar{N} which are not of type (h_1, h_0) , (h_1, t_1) , (t_0, h_0) or (t_0, t_1) are the (unique) corner around v_0 and the (unique) corner around v_1 . Hence the external face of \bar{N} is triangulated by the 2-edges together with the edges (v_0, v_2) and (v_1, v_2) . Thus, every face of M has degree 3. It only remains to prove that M has no multiple edge. Since the faces of M are of degree 3 and every internal vertex has outdegree 3, the hypothesis of Lemma 3.5 are satisfied. By this lemma, there can be no multiple edge (this would create a cycle of length 2 incident to -1 tails!). Thus, the map M has no multiple edge and is a triangulation.

- We now prove that the coloring $R = (T_0, T_1, T_2)$ is a realizer of M . By construction, R satisfies the *Schnyder-condition*. Hence it only remains to prove that T_2 is a tree. Suppose there is a cycle C of 2-edges. Since every vertex in C is incident to one 2-tail, the cycle C is directed. Therefore, the Schnyder condition proves that there are $c = |C|$ tails incident with C and lying strictly inside C . This contradicts Lemma 3.5. Thus, T_2 has no cycle. Since T_2 has $|U|$ edges and $|U| + 1$ vertices it is a tree.

We are now ready to define a mapping Φ from pairs of non-crossing Dyck paths to realizers. This mapping is illustrated by Figure 14. Consider a pair of Dyck paths $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ such that P stays below Q . The image of (P, Q) by the mapping Φ is the realizer $R = (T_0, T_1, T_2)$ obtained as follows.

Step 1. The tree T_0 is $\omega^{-1}(P)$. We denote by v_0 its root-vertex and by u_0, \dots, u_n the other vertices in clockwise order around T_0 . We denote by $\overline{T_0}$ the tree obtained from T_0 by adding a new vertex v_1 and an edge (v_0, v_1) at the right of the root.

Step 2. We glue a 1-tail in the last corner of each vertex $u_i, i = 0, \dots, n-1$ and we glue β_i 1-heads in the first corner of each vertex $u_i, i = 1, \dots, n-1$ (if u_i is a leaf we glue the 1-heads before the 1-tail in clockwise order around u_i). We also glue β_n 1-heads in the (unique) corner of v_1 . This operation is illustrated by Figure 14 (b).

Step 3. We consider the sequence of 1-tails and 1-heads around $\overline{T_0}$. Let W be the word obtained by making the tour of $\overline{T_0}$ and writing N (resp. S) when we cross a 1-tail (resp. 1-head). By Lemma 3.7, $W = N^{\alpha_1} S^{\beta_1} \dots N^{\alpha_n} S^{\beta_n}$. Since the path P stays below the path Q , we have $\delta_i(P, Q) = \sum_{j \leq i} \alpha_j - \beta_j \geq 0$ for all $i = 1, \dots, n$, hence W is a Dyck path. Thus, there exists a unique way of joining each 1-tail to a 1-head that appears after it around the tree $\overline{T_0}$ so that the 1-edges do not intersect (this statement is equivalent to the well-known fact that there is a unique way of matching parenthesis in a well parenthesized word); we denote by T_1 the set of 1-edges obtained in this way. This operation is illustrated in Figure 14 (c).

Step 4. The set T_1 of 1-edges is a tree directed toward v_1 ; see Lemma 3.10 below. Hence, by construction, $PR = (T_0, T_1)$ is a prerealizer. By Lemma 3.9, there is a unique tree T_2 such that $R = (T_0, T_1, T_2)$ is a realizer and we define $\Phi(P, Q) = R$.

In order to prove that *step 4* of the bijection Φ is well defined, we need the following lemma.

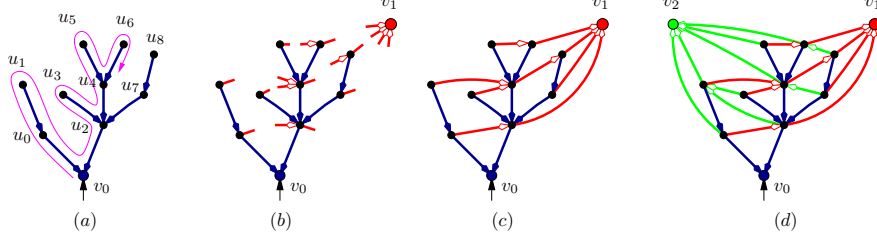


FIGURE 14. Steps of the mapping $\Phi : (P, Q) \mapsto (T_0, T_1, T_2)$. (a) Step 1: build the tree T_0 . (b) Step 2: add the 1-tails and 1-heads. (c) Step 3: join the 1-tails and 1-heads together. (d) Step 4: determine the third tree T_2 .

Lemma 3.10. *The set T_1 of 1-edges obtained at step 3 in the definition of Φ is a tree directed toward v_1 and spanning the vertices in $U_1 = \{u_0, \dots, u_{n-1}, v_1\}$.*

Proof: • Every vertex in U_1 is incident to an edge in T_1 since there is a 1-tail incident to each vertex u_i , $i = 1, \dots, n-1$ and at least one 1-head incident to v_1 since $\beta_n > 0$.

• We now prove that the tree T_1 has no cycle. Since every vertex in U_1 is incident to at most one 1-tail, any 1-cycle is directed. Moreover, if e is a 1-edge directed from u_i to u_j then $i < j$ since the last corner of u_i appears before the first corner of u_j around T_0 . Therefore, there is no directed cycle.

• Since T_1 is a set of n edges incident to $n+1$ vertices and having no cycle, it is a tree. Since the only sink is v_1 , the tree T_1 is directed toward v_1 (make an induction on the size of the oriented tree T_1 by removing a leaf). \square

The mapping Φ is well defined and the image of any pair of non-crossing Dyck paths is a realizer. Conversely, by Proposition 3.4, the image of any realizer by Ψ is a pair of non-crossing Dyck paths. It is clear from the definitions that $\Psi \circ \Phi$ (resp. $\Phi \circ \Psi$) is the identity mapping on pairs of non-crossing Dyck paths (resp. realizers). Thus, Φ and Ψ are inverse bijections between realizers of size n and pairs of non-crossing Dyck paths of size n . This concludes the proof of Theorem 3.3. \square

4. INTERVALS OF THE TAMARI LATTICE

In the previous section, we defined a bijection Φ between pairs of non-crossing Dyck paths and realizers. Recall that the pairs of non-crossing Dyck

paths correspond to the intervals of the Stanley lattice. In this section, we study the restriction of the bijection Φ to the intervals of the Tamari lattice.

Theorem 4.1. *The bijection Φ induces a bijection between the intervals of the Tamari lattice \mathcal{L}_n^T and minimal realizers of size n .*

Since every triangulation has a unique minimal realizer, Theorem 4.1 implies that the mapping Φ' which associates with a Tamari interval (P, Q) the triangulation underlying $\Phi(P, Q)$ is a bijection. This gives a bijective explanation to the relation between the number of Tamari intervals enumerated in [3] and the number of triangulations enumerated in [18, 12].

Corollary 4.2. *The number of intervals in the Tamari lattice \mathcal{L}_n^T is equal to the number $\frac{2(4n+1)!}{(n+1)!(3n+2)!}$ of triangulations of size n .*

The rest of this section is devoted to the proof of Theorem 4.1. We first recall a characterization of minimality given in [2] and illustrated in Figure 15.

Proposition 4.3 ([2]). *A realizer $R = (T_0, T_1, T_2)$ is minimal if and only if for any internal vertex u , the vertex $\mathbf{p}_0(\mathbf{p}_1(u))$ is an ancestor of u in the tree T_0 .*

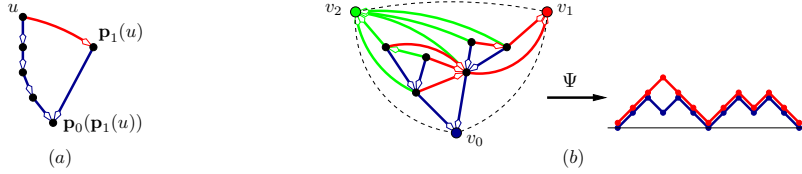


FIGURE 15. (a) Characterization of minimality: $\mathbf{p}_0(\mathbf{p}_1(u))$ is an ancestor of u in T_0 . (b) A minimal realizer and its image by Ψ .

Using Proposition 4.3, we obtain the following characterization of the pairs of non-crossing Dyck paths (P, Q) whose image by the bijection Φ is a minimal realizer.

Proposition 4.4. *Let (P, Q) be a pair of non-crossing Dyck paths and let $R = (T_0, T_1, T_2) = \Phi(P, Q)$. Let u_0, \dots, u_{n-1} be the non-root vertices of T_0 in clockwise order. Then, the realizer R is minimal if and only if $\delta_i(P, Q) \leq \delta_j(P, Q)$ whenever u_i is the parent of u_j in $T_0 = \omega^{-1}(P)$.*

In order to prove Proposition 4.4, we need to interpret the value of $\delta_i(P, Q)$ in terms of the realizer $R = \Phi(P, Q)$. Let u be an internal vertex of the triangulation underlying the realizer $R = (T_0, T_1, T_2)$. We say

that a 1-tail is *available at u* if this tail appears before the first corner of u in clockwise order around T_0 while the corresponding 1-head appears (strictly) after the first corner of u .

Lemma 4.5. *Let (P, Q) be a pair of non-crossing Dyck paths and let $R = (T_0, T_1, T_2) = \Phi(P, Q)$. Let u_0, \dots, u_{n-1} be the non-root vertices of T_0 in clockwise order. The number of 1-tails available at u_i is $\delta_i(P, Q)$.*

Proof of Lemma 4.5: We denote $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$. Let \mathcal{W} be the word obtained by making the tour of T_0 and writing \mathbf{NS}^{β_i} when arriving at the first corner of u_i and \mathbf{NS} when arriving at the last corner of u_i for $i = 0, \dots, n-1$ (with the convention that $\beta_0 = 0$). By definition of the mapping ω , the restriction of \mathcal{W} to the letters N, S is $\omega(T_0) = P = NS^{\alpha_1} \dots NS^{\alpha_n}$. Therefore, $\mathcal{W} = \mathbf{NS}^{\beta_0} (\mathbf{NS})^{\alpha_1} \mathbf{NS}^{\beta_1} (\mathbf{NS})^{\alpha_2} \dots \mathbf{NS}^{\beta_{n-1}} (\mathbf{NS})^{\alpha_n}$. The prefix of \mathcal{W} written after arriving at the first corner of u_i is $\mathbf{NS}^{\beta_0} (\mathbf{NS})^{\alpha_1} \mathbf{NS}^{\beta_1} \dots (\mathbf{NS})^{\alpha_i} \mathbf{NS}^{\beta_i}$. The sub-word $\mathbf{S}^{\beta_0} \mathbf{N}^{\alpha_1} \mathbf{S}^{\beta_1} \dots \mathbf{N}^{\alpha_i} \mathbf{S}^{\beta_i}$ corresponds to the sequence of 1-tails and 1-heads encountered so far (\mathbf{N} for a 1-tail, \mathbf{S} for a 1-head). Thus, the number of 1-tails available at u_i is $\sum_{j \leq i} \alpha_j - \beta_j = \delta_i(P, Q)$. \square

Proof of Proposition 4.4:

- We suppose that a vertex u_i is the parent of a vertex u_j in T_0 and that $\delta_i(P, Q) > \delta_j(P, Q)$, and we want to prove that the realizer $R = \Phi(P, Q)$ is not minimal. Since u_i is the parent of u_j we have $i < j$ and all the vertices u_r , $i < r \leq j$ are descendants of u_i . By Lemma 4.5, $\delta_i(P, Q) > \delta_j(P, Q)$ implies that there is a 1-tail t available at u_i which is not available at u_j , hence the corresponding 1-head is incident to a vertex u_l with $i < l \leq j$. Let u_k be the vertex incident to the 1-tail t . Since t is available at u_i , the vertex u_k is not a descendant of u_i . But $\mathbf{p}_0(\mathbf{p}_1(u_k)) = \mathbf{p}_0(u_l)$ is either u_i or a descendant of u_i in T_0 . Thus, the vertex u_k contradicts the minimality condition given by Proposition 4.3. Hence, the realizer R is not minimal.
- We suppose that the realizer R is not minimal and we want to prove that there exists a vertex u_i parent of a vertex u_j in T_0 such that $\delta_i(P, Q) > \delta_j(P, Q)$. By Proposition 4.3, there exists a vertex u such that $\mathbf{p}_0(\mathbf{p}_1(u))$ is not an ancestor of u in T_0 . In this case, the 1-tail t incident to u is available at $u_i = \mathbf{p}_0(\mathbf{p}_1(u))$ but not at $u_j = \mathbf{p}_1(u)$ (since t cannot appear between the first corner of u_i and the first corner of u_j around T_0 , otherwise u would be a descendant of u_i). Moreover, any 1-tail t' available at u_j appears before the 1-tail t around T_0 (otherwise, the 1-edge corresponding to t' would cross the 1-edge (u, u_j)). Hence, any 1-tail t' available at u_j is also available at u_i . Thus, there are more 1-tails available at u_i than at u_j . By Lemma 4.5, this implies $\delta_i(P, Q) > \delta_j(P, Q)$. \square

Proposition 4.6. . Let (P, Q) be a pair of non-crossing Dyck paths. Let $T = \omega^{-1}(P)$, let v_0 be the root-vertex of the tree T and let u_0, \dots, u_{n-1} be its other vertices in clockwise order. Then, $P \leq_T Q$ if and only if $\delta_i(P, Q) \leq \delta_j(P, Q)$ whenever u_i is the parent of u_j .

Propositions 4.4 and Propositions 4.6 clearly imply Theorem 4.1. Hence, it only remains to prove Proposition 4.6.

Proof: We denote $Q = NS^{\beta_1} \dots NS^{\beta_n}$.

• We suppose that $P \leq_T Q$ and want to prove that $\delta_k(P, Q) \leq \delta_l(P, Q)$ whenever u_k is the parent of u_l . We make an induction on $\Delta(P, Q)$. If $\Delta(P, Q) = 0$, then $P = Q$ and the property holds. If $\Delta(P, Q) > 0$ there is a path $Q' = NS^{\beta'_1} \dots NS^{\beta'_n}$ such that $P \leq_T Q'$ and Q' is covered by Q in the Tamari lattice. The three paths P, Q', Q are represented in Figure 16. By definition, there are two indices $1 \leq i < j \leq n$ such that $i \stackrel{Q'}{\leq} j$ and $\beta_i = \beta'_i + 1$, $\beta_j = \beta'_j - 1$ and $\beta_k = \beta'_k$ for all $k \neq i, j$. Thus, $\delta_k(P, Q) = \delta_k(P, Q') + 1$ if $i \leq k < j$ and $\delta_k(P, Q) = \delta_k(P, Q')$ otherwise. By the induction hypothesis we can assume that $\delta_k(P, Q') \leq \delta_l(P, Q')$ whenever u_k is the parent of u_l . Suppose there exists u_k parent of u_l such that $\delta_k(P, Q) > \delta_l(P, Q)$. Note that if u_k is the parent of u_l then $k < l$ and for all $k < r \leq l$, the vertex u_r is a proper descendant of u_k . Since $\delta_k(P, Q) > \delta_l(P, Q)$ and $\delta_k(P, Q') \leq \delta_l(P, Q')$ we have $k < j \leq l$, hence u_j is a proper descendant of u_k . Note that for all $r = 0, \dots, n-1$, $e_r(P) + 1$ is equal to the height of the vertex u_r in the tree T (i.e. the distance between v_0 and u_r). Thus, $e_k(P) < e_j(P)$. Moreover, by the induction hypothesis, $\delta_k(P, Q') \leq \delta_j(P, Q')$. Hence, $e_k(Q') = e_k(P) + \delta_k(P, Q') < e_j(Q') = e_j(P) + \delta_j(P, Q')$. But since $i \leq k < j$ this contradicts the hypothesis $i \stackrel{Q'}{\leq} j$. We reach a contradiction, hence $\delta_k(P, Q) \leq \delta_l(P, Q)$ whenever u_k is the parent of u_l .

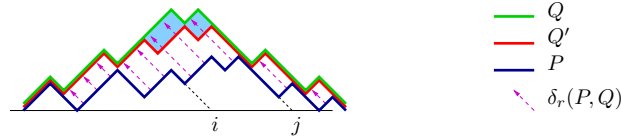


FIGURE 16. The Dyck paths $P \leq_T Q' \leq_T Q$.

• We suppose that $\delta_k(P, Q) \leq \delta_l(P, Q)$ whenever u_k is the parent of u_l and want to prove that $P \leq_T Q$. We make an induction on $\Delta(P, Q)$. If $\Delta(P, Q) = 0$, then $P = Q$ and the property holds. Suppose $\Delta(P, Q) > 0$ and let $\delta = \max\{\delta_k(P, Q), k = 0 \dots n\}$, let $e = \min\{e_k(P)/\delta_k(P, Q) = \delta\}$ and let $i = \max\{k/e_k(P) = e \text{ and } \delta_k(P, Q) = \delta\}$. Let j be the first index such that

$i < j \leq n$ and u_j is not a descendant of u_i ($j = n$ if u_{i+1}, \dots, u_{n-1} are all descendants of u_i). Let $Q' = NS^{\beta'_1} \dots NS^{\beta'_n}$ with $\beta'_i = \beta_i + 1$, $\beta'_j = \beta_j - 1$ and $\beta'_k = \beta_k$ for all $k \neq i, j$. The paths P , Q and Q' are represented in Figure 16. We want to prove that Q' is a Dyck path covered by Q in the Tamari lattice and $P \leq_T Q'$.

- We first prove that Q' is a Dyck path that stays above P . First note that $\delta_k(P, Q') = \delta_k(P, Q) - 1$ if $i \leq k < j$ and $\delta_k(P, Q') = \delta_k(P, Q)$ otherwise. If $\delta_k(P, Q') < 0$, then $i \leq k < j$, hence u_k is a descendant of u_i . Since the value of $\delta_r(P, Q)$ is weakly increasing along the branches of T , we have $\delta_k(P, Q) \geq \delta_i(P, Q) = \delta > 0$, hence $\delta_k(P, Q') \geq 0$. Thus for all $k = 0, \dots, n$, $\delta_k(P, Q') \geq 0$, that is, Q' stays above P .

- We now prove that $P \leq_T Q'$. Suppose there exist k, l , such that $\delta_k(P, Q') > \delta_l(P, Q')$ with u_k parent of u_l . Since $\delta_k(P, Q) \leq \delta_l(P, Q)$, we have $k < i \leq l < j$. Since a vertex u_r is a descendant of u_i if and only if $i < r < j$, the only possibility is $l = i$. Moreover, since u_k is the parent of u_i we have $e_k(P) < e_i(P) = e$, hence by the choice of e , $\delta_k(P, Q) < \delta = \delta_i(P, Q)$. Hence, $\delta_k(P, Q') = \delta_k(P, Q) \leq \delta_i(P, Q) - 1 = \delta_i(P, Q')$. We reach a contradiction. Thus $\delta_k(P, Q') \leq \delta_l(P, Q')$ whenever u_k is the parent of u_l . By the induction hypothesis, this implies $P \leq_T Q'$.

- It remains to prove that Q' is covered by Q in the Tamari lattice. It suffices to prove that $i \underline{Q'} j$. Recall that for all $r = 0, \dots, n-1$, $e_r(P) + 1$ is the height of the vertex u_r in the tree T . For all $i < r < j$, the vertex u_r is a descendant of u_i hence $e_r(P) > e_i(P)$. Moreover, since the value of $\delta_x(P, Q)$ is weakly increasing along the branches of T , $\delta_r(P, Q) \geq \delta_i(P, Q)$ for all $i < r < j$. Thus, for all $i < r < j$, $e_r(Q) = e_r(P) + \delta_r(P, Q) > e_i(Q) = e_i(P) + \delta_i(P, Q)$ and $e_r(Q') = e_r(Q) - 1 > e_i(Q') = e_i(Q) - 1$. It only remains to show that $e_j(Q') \leq e_i(Q')$. The vertex u_j is the first vertex not descendant of u_i around T , hence $e_j(P) \leq e_i(P)$. Moreover $\delta_j(P, Q) \leq \delta = \delta_i(P, Q)$. Furthermore, the equalities $e_i(P) = e_j(P)$ and $\delta_j(P) = \delta$ cannot hold simultaneously by the choice of i . Thus, $e_j(Q) = e_j(P) + \delta_j(P, Q) < e_i(Q) = e_i(P) + \delta_i(P, Q)$ and $e_j(Q') = e_j(Q) \leq e_i(Q') = e_i(Q) - 1$. \square

5. INTERVALS OF THE KREWERAS LATTICE

In this section, we study the restriction of the bijection Φ to the Kreweras intervals.

Theorem 5.1. *The mapping Φ induces a bijection between the intervals of the Kreweras lattice \mathcal{L}_n^K and realizers of size n which are both minimal and maximal.*

Before commenting on Theorem 5.1, we characterize the realizers which are both minimal and maximal. Recall that a triangulation is *stack* if it

is obtained from the map reduced to a triangle by recursively inserting a vertex of degree 3 in one of the (triangular) internal face. An example is given in Figure 17.

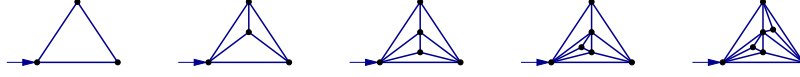


FIGURE 17. A stack triangulation is obtained by recursively inserting a vertex of degree 3.

Proposition 5.2. *A realizer R is both minimal and maximal if and only if the underlying triangulation M is stack. (In this case, R is the unique realizer of M .)*

The proof of Proposition 5.2 uses the following Lemma.

Lemma 5.3. *Let M be a triangulation and let $R = (T_0, T_1, T_2)$ be one of its realizers. Suppose that M has an internal vertex v of degree 3 and let M' be obtained from M by removing v (and the incident edges). Then, the restriction of the realizer R to the triangulation M' is a realizer.*

Proof: By Schnyder condition, the vertex v is incident to three tails and no head, hence it is a leaf in each of the trees T_1, T_2, T_3 . Thus, the *tree condition* is preserved by the deletion of v . Moreover, deleting v does not deprive any other vertex of an *i*-tail, hence the *Schnyder condition* is preserved by the deletion of v . \square

Proof of Proposition 5.2:

- We first prove that any realizer R of a stack triangulation M is minimal and maximal, that is, contains neither a cw- nor a ccw-triangle. We proceed by induction on the size of M . If M is reduced to the triangle, the property is obvious. Let M be a stack triangulation not reduced to the triangle. By definition, the triangulation M contains an internal vertex v of degree 3 such that the triangulation M' obtained from M by removing v is stack. By Lemma 5.3, the restriction of the realizer R to M' is a realizer. Hence, by the induction hypothesis, the triangulation M' contains neither a cw- nor a ccw-triangle. Thus, if C is either a cw- or a ccw-triangle of M , then C contains v . But this is impossible since v is incident to no head.
- We now prove that any realizer R of a non-stack triangulation M contains either a cw- or a ccw-triangle.
- We first prove that the property holds if M has no internal vertex of degree 3 nor *separating triangle* (a triangle which is not a face). It is known

that if R contains a directed cycle, then it contains either a cw- or ccw-triangle (proof omitted; see [11]). Thus it suffices to prove that R contains a directed cycle. Let u be the third vertex of the internal triangle incident to the edge (v_1, v_2) . The vertex u is such that $\mathbf{p}_1(u) = v_1$ and $\mathbf{p}_2(u) = v_2$ (see Figure 18). The vertex u has degree at least 4 and is not adjacent to v_0 (otherwise one of the triangles (v_0, v_1, u) or (v_0, v_2, u) contains some vertices, hence is separating). Thus, $u' = \mathbf{p}_0(u) \neq v_0$. Moreover, either $\mathbf{p}_1(u') \neq v_1$ or $\mathbf{p}_2(u') \neq v_2$, otherwise the triangle (v_1, v_2, u') is separating. Let us assume that $u'' = \mathbf{p}_1(u') \neq v_1$ (the other case is symmetrical). By Schnyder condition, the vertex u'' lies inside the cycle C made of the edges (v_0, v_1) , (v_1, u) and the 0-path from u to v_0 . By Schnyder condition, the 1-path from u'' to v_1 stays strictly inside C . Let C' be the cycle made of the edges (v_1, u) , (u, u') and the 1-path from u' to v_1 . By Schnyder condition, the 2-path from u'' to v_2 starts inside the cycle C' , hence cut this cycle. Let v be the first vertex of C' on the 2-path from u'' to v_2 . The vertex v is incident to a 2-head lying inside C' , hence by Schnyder condition $v = u$. Thus, the cycle made of the edges (u, u') , (u', u'') and the 2-path from u'' to u is directed.

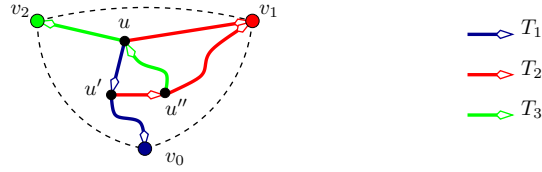


FIGURE 18. The vertices u , $u' = \mathbf{p}_0(u)$ and $u'' = \mathbf{p}_1(u')$.

- We now prove that the property holds for any non-stack triangulation M without internal vertex of degree 3. If M has no separating triangle then, by the preceding point, the realizer R contains either a cw- or ccw-triangle. Suppose now that M has a separating triangle Δ . We can choose Δ not containing any other separating triangle. In this case, the triangulation M' lying inside the triangle Δ has no separating triangle and is not stack (since no internal vertex has degree 3). Let t_0, t_1, t_2 be the vertices of the triangle Δ . By definition, there are some vertices lying inside the triangle Δ . By Lemma 3.5, there is no tail incident to Δ and lying inside Δ . Thus, for $i = 1, 2, 3$, the half-edges incident to the vertex t_i and lying inside Δ are heads. Moreover, the Schnyder condition implies that all the heads incident to t_i have the same color. Furthermore, for each color $i = 1, 2, 3$ there is an i -head incident to one of the vertices t_0, t_1, t_2 , otherwise the vertices inside Δ would not be connected to v_i by an i -path. Hence, we can assume without

loss of generality that for $i = 1, 2, 3$, the heads incident to t_i and lying inside Δ are of color i . Thus, the restriction R' of R to the triangulation lying inside Δ is a realizer. By the preceding point, the realizer R' contains either a cw- or ccw-triangle, hence so do R .

- We now prove that the property holds for any non-stack triangulation M . Let R be a realizer of a non-stack triangulation M . Let M' be the triangulation obtained from M by recursively deleting every internal vertex of degree 3. The triangulation M' is not stack and has no internal vertex of degree 3. Moreover, by Lemma 5.3, the restriction R' of the realizer R to the triangulation M' is a realizer. By the preceding point, the realizer R' contains either a cw- or ccw-triangle, hence so do R . \square

Given Theorem 5.1 and Proposition 5.2, the mapping Φ induces a bijection between the intervals of the Kreweras lattice and the stack triangulations. Stack triangulations are known to be in bijection with ternary trees (see for instance [19]), hence we obtain a new proof that the number of intervals in \mathcal{L}_n^K is $\frac{1}{2n+1} \binom{3n}{n}$. The rest of this section is devoted to the proof of Theorem 5.1. We first recall a characterization of the realizers which are both minimal and maximal. This characterization which is illustrated in Figure 19 follows immediately from the characterizations of minimality and of maximality given in [2].

Proposition 5.4 ([2]). *A realizer $R = (T_0, T_1, T_2)$ is both minimal and maximal if and only if for any internal vertex u , either $\mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$ or $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u)$.*

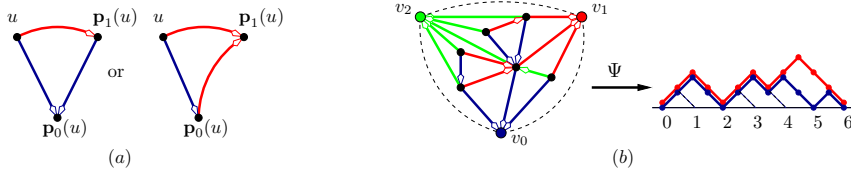


FIGURE 19. (a) Condition for a realizer to be both minimal and maximal: $\mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$ or $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u)$. (b) A minimal and maximal realizer and its image by Ψ .

Let $R = (T_0, T_1, T_2)$ be a realizer of a triangulation M and let u, u' be two vertices distinct from v_0 and v_2 . We say that there is a *1-obstruction* between u and u' if there is a 1-edge e such that the tail of e appears before the first corner of u while its head appears strictly between the first corner of u and the first corner of u' around the tree $\overline{T_0}$. This situation is

represented in Figure 20. Using Proposition 5.4, we obtain the following property satisfied by realizers which are both minimal and maximal.

Lemma 5.5. *Let $R = (T_0, T_1, T_2)$ be a minimal and maximal realizer and let $(P, Q) = \Psi(R)$. Let*

$v_0, u_0, u_1, \dots, u_n = v_1$ be the vertices of the tree $\overline{T_0}$ in clockwise order. Then, for all indices $0 \leq i < j \leq n$, the relation $i \underline{Q} j$ holds if and only if the three following properties are satisfied:

- (1) the vertex u_j is an ancestor of u_i in the tree T_1 ,*
- (2) either $\mathbf{p}_1(\mathbf{p}_0(u_i)) = u_j$ or $\mathbf{p}_0(u_i) = \mathbf{p}_0(u_j)$ (with the convention that $\mathbf{p}_0(u_n) = v_0$),*
- (3) there is no 1-obstruction between u_i and u_j .*

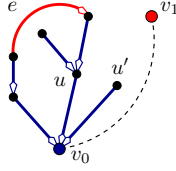


FIGURE 20. A 1-obstruction between the vertices u and u' .

The proof Lemma 5.5 of is based on the following result.

Lemma 5.6. *Let $R = (T_0, T_1, T_2)$ be a minimal and maximal realizer and let $(P, Q) = \Psi(R)$. Let*

$v_0, u_0, u_1, \dots, u_n = v_1$ be the vertices of the tree $\overline{T_0}$ in clockwise order. For all indices $0 \leq i < j \leq n$, the relation $i \underline{Q} j$ holds if and only if $\mathbf{p}_1(u_i) = u_j$. Moreover, in this case $e_j(Q) = e_i(Q)$ if and only if $\mathbf{p}_0(u_i) = \mathbf{p}_0(u_j)$ and there is no 1-edge whose head is incident to u_j and whose tail appears before the first corner of u_i .

Proof: Let $0 \leq i < j \leq n$ such that $\mathbf{p}_1(u_i) = u_j$.

- We first prove that for all index $r = i + 1, \dots, j - 1$, the inequality $e_k(Q) > e_i(Q)$ holds.

Let u_{i_1}, \dots, u_{i_s} be the vertices on the 0-path from $u_{i_0} = u_i$ to $u_{i_{s+1}} = \mathbf{p}_0(u_j)$ (that is, $\mathbf{p}_0(u_{i_k}) = u_{i_{k+1}}$ for all $k = 0, \dots, s$); see Figure 21. The characterization of minimal and maximal realizers given in Proposition 5.4 implies that $\mathbf{p}_1(u_{i_k}) = u_j$ for all $k = 1, \dots, s$. For all $k = 0, \dots, s$, we denote by r_k the index of the last descendant of u_{i_k} around $\overline{T_0}$ and we denote $r_{s+1} = j - 1$. Note that, for all $k = 0, \dots, s$, the vertices $u_{r_k+1}, \dots, u_{r_{k+1}}$ are descendants of $u_{i_{k+1}}$ in T_0 . Hence, for all $k = 0, \dots, s$ and all $l = r_k + 1, \dots, r_{k+1}$, the

inequality $e_l(P) > e_i(P) - k$ holds (since for any index h the value $e_h(P) + 1$ is the height of the vertex u_h in the tree T_0). By the minimality condition given by Proposition 4.3, none of the 1-tails available at u_i is matched to one of the vertices u_{i+1}, \dots, u_{r_0} (since these vertices are descendants of u_i). Moreover, none of these available 1-tails is matched to one of the vertices $u_{r_0+1}, \dots, u_{j-1}$ or there would be a crossing with the 1-edge (u_i, u_j) (see Figure 21). Hence, the 1-tails available at i are also available at all the vertices $u_{i+1}, \dots, u_j - 1$. Moreover, for all $k = 1, \dots, s$ and all $l = r_k + 1, \dots, r_{k+1}$ the k 1-tails incident to each of the vertices $u_{i_0}, \dots, u_{i_{k-1}}$ are available at the vertex u_l . Thus, given Lemma 4.5, for all $k = 0, \dots, s$, for all $l = r_k + 1, \dots, r_{k+1}$, $\delta_l(P, Q) \leq \delta_i(P, Q) + k$. Thus, for all $l = i+1, \dots, j-1$, the inequality $e_l(Q) = e_l(P) + \delta_l(P, Q) > e_i(Q) = e_i(P) + \delta_i(P, Q)$ holds.

• It only remains to prove that *the inequality $e_j(Q) \leq e_i(Q)$ holds and equality occurs if and only if $\mathbf{p}_0(u_i) = \mathbf{p}_0(u_j)$ and there is no 1-edge whose head is incident to u_j and whose tail appears before the first corner of u_i .*

- Since the realizer R is minimal the vertex $\mathbf{p}_0(u_j)$ is an ancestor of u_j in the tree T_0 (by Proposition 4.3). Hence, the inequality $e_j(P) \leq e_i(P)$ holds and equality occurs if and only if $\mathbf{p}_0(u_i) = \mathbf{p}_0(u_j)$. We now compare the values of $\delta_i(P, Q)$ and $\delta_j(P, Q)$ which are the number of tails available at u_i and at u_j respectively (by Lemma 4.5).

- We first prove that *any 1-tails available at u_j is also available at u_i .* No 1-tail available at u_j is incident to a vertex u_l with $r_0 < l < j$ or the corresponding 1-edge would cross the edge (u_i, u_j) (see Figure 21). Moreover, the characterization of minimal and maximal realizers given in Proposition 5.4 implies that no 1-tail available at u_j is incident to a vertex u_l with $i < l \leq r_0$ (since these vertices are descendants of u_i). Hence, any the 1-tail available at u_j is also available at u_i .

- We now prove *any 1-tail available at u_i is available at u_j except if the corresponding 1-head is incident to u_j .* Clearly, no 1-tail available at u_i is such that the corresponding 1-head is incident to a vertex u_l with $r_0 < l < j$ or the 1-edge under consideration would cross the edge (u_i, u_j) (see Figure 21). Since the realizer R is minimal, there is no 1-tail available at u_i and such that the corresponding 1-head is a vertex u_l with $i < l \leq r_0$ (since these vertices are descendants of u_i). Hence, if a 1-tail available at u_i is not available at u_j , then the corresponding 1-head is incident to u_j .

- Given Lemma 4.5, the preceding points imply that the inequality $\delta_i(P, Q) \leq \delta_j(P, Q)$ holds and equality occurs if and only if there is no 1-edge whose head is incident to u_j and whose tail appears before the first corner of u_i . Hence, $e_j(Q) = e_j(P) + \delta_j(P, Q) \leq e_i(Q) = e_i(P) + \delta_i(P, Q)$ and equality occurs if and only if $\mathbf{p}_0(u_i) = \mathbf{p}_0(u_j)$ and no index $k < i$ is such that $\mathbf{p}_1(u_k) = u_j$. \square

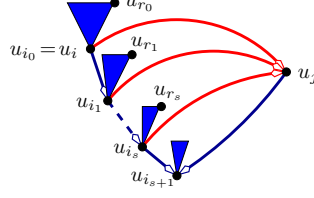


FIGURE 21. Notations for the proof of Lemma 5.6.

Proof of Lemma 5.5:

• We suppose that $i \underline{Q} j$ and we want to prove the properties (1), (2) and (3).

(1) Since $i \underline{Q} j$, there are indices $i_0 = i, i_1, \dots, i_{s+1} = j$ such that $i_0 \underline{Q} i_1 \underline{Q} \dots \underline{Q} i_{s+1}$ and $e_{i_0}(Q) = e_{i_1}(Q) = \dots = e_{i_s}(Q)$. Lemma 5.6 implies $\mathbf{p}_1(u_{i_k}) = u_{i_{k+1}}$ for all $k \leq s$. Hence u_j is a ancestor of u_i in the tree T_1 .

(2) Since $e_{i_k}(P) = e_i(P)$ for all $k \leq s$, Lemma 5.6 implies $\mathbf{p}_0(u_{i_k}) = \mathbf{p}_0(u_i)$ for all $k \leq s$. Moreover, $\mathbf{p}_1(u_{i_s}) = u_j$, thus Proposition 5.4 implies that either $\mathbf{p}_0(u_j) = \mathbf{p}_0(u_{i_s}) = \mathbf{p}_0(u_i)$ or $u_j = \mathbf{p}_1(\mathbf{p}_0(u_{i_s})) = \mathbf{p}_1(\mathbf{p}_0(u_i))$. This situation is represented in Figure 22.

(3) We want to prove that there is no 1-obstruction between u_i and u_j . We suppose that the tail of a 1-edge e appears before the first corner of u_i around T_0 and we want to prove that the corresponding 1-head h is not incident to a vertex u_k with $i < k < j$. Clearly, if the 1-head h is incident to a vertex u_k with $i < k < j$, then the vertex u_k is either one of the vertices $u_{i_0}, u_{i_2}, \dots, u_{i_s}$ or one of their descendants (otherwise, the edge e would cross one of the 1-edges $(u_{i_0}, u_{i_1}), \dots, (u_{i_s}, u_{i_{s+1}})$; see Figure 22). Since $e_{i_0}(Q) = e_{i_1}(Q) = \dots = e_{i_s}(Q)$, Lemma 5.6 implies that u_k is none of the vertices $u_{i_1}, u_{i_2}, \dots, u_{i_s}$. Moreover, since the realizer R is minimal, Proposition 4.3 implies that u_k is not a (proper) descendant of one of the vertices i_0, \dots, i_s . Thus, the 1-head h is not incident to a vertex u_k with $i < k < j$ and e is not creating a 1-obstruction.

• We suppose that the vertices u_i and u_j satisfy the properties (1), (2) and (3) and want to prove that $i \underline{Q} j$. Observe first that by property (1), there are indices $i_0 = i, i_1, \dots, i_{s+1} = j$ such that $\mathbf{p}_1(u_{i_k}) = u_{i_{k+1}}$.

- We first prove that, for all $k = 1, \dots, s$, $\mathbf{p}_0(u_{i_k}) = \mathbf{p}_0(u_i)$; this situation is represented in Figure 22.

Suppose the contrary and consider the first index $k \in \{1, \dots, s\}$ such that $\mathbf{p}_0(u_{i_k}) \neq \mathbf{p}_0(u_i)$. In this case, $u_{i_k} = \mathbf{p}_1(u_{i_{k-1}})$ and $\mathbf{p}_0(u_{i_k}) \neq \mathbf{p}_0(u_{i_{k-1}}) = \mathbf{p}_0(u_i)$. Since the realizer R is minimal and maximal, Proposition 5.4 implies

that $u_{i_k} = \mathbf{p}_1(\mathbf{p}_0(u_{i_{k-1}})) = \mathbf{p}_1(\mathbf{p}_0(u_i))$. Thus, the vertices $u_{i_{k+1}}, \dots, u_{i_{s+1}}$ are distinct from $\mathbf{p}_1(\mathbf{p}_0(u_i))$ and are ancestors of $\mathbf{p}_0(u_i)$ in the tree T_1 . In particular, $u_j = u_{i_{s+1}} \neq \mathbf{p}_1(\mathbf{p}_0(u_i))$, and $\mathbf{p}_0(u_j) \neq \mathbf{p}_0(u_i)$. This contradicts Property (2).

- We now prove that *for all index $k = 1, \dots, s$ there is no 1-edge e whose head is incident to u_{i_k} and whose tail appears before the first corner of u_{k-1}* . Suppose that such a 1-edge e exist. Observe that the 1-tail t of the edge e do not appear before the first corner of u_i otherwise the edge e creates a 1-obstruction between u_i and u_j . Hence, the 1-tail t is incident either to one of the vertices $u_{i_0}, \dots, u_{i_{k-2}}$ or to one of their descendants (otherwise, the edge e would cross one of the 1-edges $(u_{i_0}, u_{i_1}), \dots, (u_{i_{k-2}}, u_{i_{k-1}})$; see Figure 22). Moreover, the 1-tail t is not incident to the vertices $u_{i_0}, \dots, u_{i_{k-2}}$, otherwise e would create a cycle in the tree T_1 . Lastly, since the realizer R is minimal, the 1-tail t is not incident to a descendant of u_{i_l} , $l = 0, \dots, k-2$. Thus the 1-tail t does not appear before the first corner of u_{k-1} .

- By Lemma 5.6, the preceding points imply $i_k \stackrel{Q}{=} i_{k+1}$ and $e_{i_k}(Q) = e_i(Q)$ for all $k = 0 \dots s$. Thus, $i \stackrel{Q}{=} j$. \square

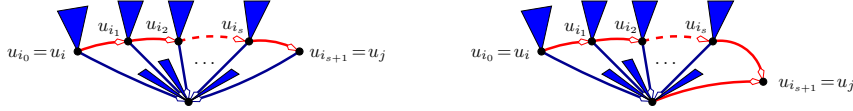


FIGURE 22. Notations for the proof of Lemma 5.5.

Proof of Theorem 5.1: Let $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ be two Dyck paths and let $R = (T_0, T_1, T_2) = \Phi(P, Q)$. Let $v_0, u_0, u_1, \dots, u_n = v_1$ be the vertices of the tree $\overline{T_0}$ in clockwise order.

• We suppose that $P \leq_K Q$ and we want to prove that the realizer R is minimal and maximal. We proceed by induction on $\Delta(P, Q)$.

- We first suppose that $\Delta(P, Q) = 0$, that is $P = Q$, and we want to prove that R is minimal and maximal. Let \mathcal{W} be the word obtained by making the tour of $\overline{T_0}$ and writing N (resp. S) when following an edge of $\overline{T_0}$ for the first (resp. second) time and writing \mathbf{N} (resp. \mathbf{S}) when crossing a 1-tail (resp. 1-head). By definition of the mapping ω , the restriction of \mathcal{W} to the letters N, S is $\omega(\overline{T_0}) = NS^{\alpha_1} \dots NS^{\alpha_n} NS$. Moreover, for all $i = 0, \dots, n$ there are α_i 1-heads incident to the first corner of u_i and one 1-head incident to its last corner. Thus, $\mathcal{W} = N(\mathbf{N}\mathbf{S})^{\alpha_1} \mathbf{N} \mathbf{S}^{\alpha_1} (\mathbf{N}\mathbf{S})^{\alpha_2} \dots \mathbf{N} \mathbf{S}^{\alpha_{n-1}} (\mathbf{N}\mathbf{S})^{\alpha_n} \mathbf{N} \mathbf{S}^{\alpha_n} S$. Between any letter \mathbf{N} of \mathcal{W} and the corresponding letter \mathbf{S} there is exactly one letter N . Thus, for any internal vertex u , the vertex $\mathbf{p}_1(u)$ is the first

vertex appearing after the last corner of u around $\overline{T_0}$ (that is, the first vertex which is not a descendant of u appearing after u around $\overline{T_0}$). By Proposition 5.4, this implies that R is minimal and maximal.

- We now suppose that $\Delta(P, Q) > 0$. In this case, there is a Dyck path $Q' = NS^{\beta'_1} \dots NS^{\beta'_n}$ covered by Q in the Kreweras lattice and such that $P \leq_K Q'$. Since Q' is covered by Q in the Kreweras lattice, there are indices $0 \leq i < j \leq n$ such that $i \underline{Q'} j$ and $\beta_i = 0$, $\beta_j = \beta'_i + \beta'_j$ and $\beta_k = \beta'_k$ for all $k \neq i, j$ (this situation is represented in Figure 23 (a)). By the induction hypothesis, the realizer $R' = (T'_0, T'_1, T'_2) = \Phi(P, Q')$ is both minimal and maximal. Moreover, by definition of the bijection Φ , the trees T_0 and T'_0 are the same. We use this fact to identify the vertices in the prerealizers $PR = (T_0, T_1)$ and $PR' = (T_0, T'_1)$ that we denote by $v_0, u_0, u_1, \dots, u_n = v_1$ in clockwise order around $\overline{T_0} = \overline{T'_0}$. We also denote by $\mathbf{p}'_1(u)$ the parent of any vertex u in T'_1 .

- We first prove that *for any vertex v , $\mathbf{p}'_1(v) = \mathbf{p}_1(v)$ except if $\mathbf{p}'_1(v) = u_i$ in which case $\mathbf{p}_1(v) = u_j$* . Since $i \underline{Q'} j$, Lemma 5.5 implies that there is no 1-obstruction between u_i and u_j in the realizer R' . Thus, the β'_i 1-heads incident to u_i can be unglued from the first corner of u_i and glued to the first corner of u_j without creating any crossing in the prerealizer $PR' = (T_0, T'_1)$ (the transfer of the β'_i 1-heads is represented in Figure 23 (b)). Let $PR'' = (T_0, T''_1)$ be the colored map obtained. Clearly, $PR'' = (T_0, T''_1)$ satisfies the *tree condition* (T''_1 is a tree), the *corner condition* (the 1-heads are in first corners, the 1-tails are in last corners) and the *order condition* (any 1-tail appears before the corresponding 1-head around $\overline{T_0}$), therefore PR'' is a prerealizer. Moreover, for all $i = 0, \dots, n$, there are β_i 1-heads incident to the vertex u_i . Thus, by definition of the mapping Φ , the prerealizer PR'' is equal to $PR = (T_0, T_1)$. Since the only difference between the prerealizers PR' and PR is that the 1-heads incident to u_i in PR' are incident to u_j in PR , the property holds.
- We now prove that *the realizer $R = (T_0, T_1, T_2)$ is minimal and maximal*. If the realizer R is not both minimal and maximal, there is a vertex u such that $\mathbf{p}_1(u) \neq \mathbf{p}_1(\mathbf{p}_0(u))$ and $\mathbf{p}_0(\mathbf{p}_1(u)) \neq \mathbf{p}_0(u)$. Since the realizer R' is both minimal and maximal, either $\mathbf{p}'_1(u) = \mathbf{p}'_1(\mathbf{p}_0(u))$ or $\mathbf{p}_0(\mathbf{p}'_1(u)) = \mathbf{p}_0(u)$. But $\mathbf{p}'_1(u) \neq \mathbf{p}'_1(\mathbf{p}_0(u))$, otherwise $\mathbf{p}_1(u) = \mathbf{p}_1(\mathbf{p}_0(u))$. Thus, $\mathbf{p}_0(\mathbf{p}'_1(u)) = \mathbf{p}_0(u)$ and $\mathbf{p}'_1(u) = u_i$. Hence, $\mathbf{p}_0(u_i) = \mathbf{p}_0(u)$ and $\mathbf{p}_1(u) = u_j$. Moreover, since $i \underline{Q'} j$, Lemma 5.5 implies that either $\mathbf{p}_0(u_i) = \mathbf{p}_0(u_j)$ or $\mathbf{p}'_1(\mathbf{p}_0(u_i)) = u_j$. But, if $\mathbf{p}_0(u_i) = \mathbf{p}_0(u_j)$, then $\mathbf{p}_0(u) = \mathbf{p}_0(u_i) = \mathbf{p}_0(u_j) = \mathbf{p}_0(\mathbf{p}_1(u))$ which is forbidden. And, if $\mathbf{p}'_1(\mathbf{p}_0(u_i)) = u_j$, then $\mathbf{p}_1(\mathbf{p}_0(u)) =$

$\mathbf{p}_1(\mathbf{p}_0(u_i)) = \mathbf{p}'_1(\mathbf{p}_0(u_i)) = u_j = \mathbf{p}_1(u)$ which is also forbidden. We reach a contradiction.

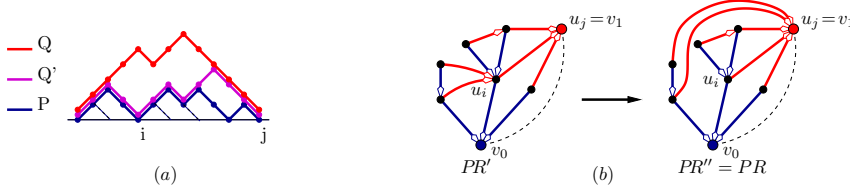


FIGURE 23. (a) The Dyck paths $P \leq_K Q' \leq_K Q$. (b) The prerealizer PR'' is obtained from $PR' = (T_0, T'_1)$ by moving β'_i 1-heads from the first corner of u_i to the first corner of u_j .

- We suppose that the realizer R is minimal and maximal and we want to prove that $P \leq_K Q$. We proceed by induction on $\Delta(P, Q)$. If $\Delta(P, Q) = 0$, then $P = Q$ and the property holds. We suppose now that $\Delta(P, Q) > 0$ and we denote by $v_0, u_0, u_1, \dots, u_n = v_1$ the vertices of the tree \overline{T}_0 in clockwise order.

- We first prove that *there are indices $0 \leq k < i < j \leq n$ such that $\mathbf{p}_0(u_k) = \mathbf{p}_0(u_i)$ and $\mathbf{p}_1(u_k) = u_j$* . We suppose that no such indices exist and we want to prove that $P = Q$. Let u be an internal vertex. If u has a sibling in \overline{T}_0 appearing after u around \overline{T}_0 , then $\mathbf{p}_1(u)$ is the first such sibling (since the indices i, j, k do not exist), else $\mathbf{p}_1(u) = \mathbf{p}_1(\mathbf{p}_0(u))$ (since the realizer R is minimal and maximal). Thus, for any vertex u , $\mathbf{p}_1(u)$ is the first vertex appearing after the last corner of u around \overline{T}_0 . Let \mathcal{W} be the word obtained by making the tour of \overline{T}_0 and writing N (resp. S) when following an edge of \overline{T}_0 for the first (resp. second) time and writing \mathbf{N} (resp. \mathbf{S}) when crossing a 1-tail (resp. 1-head). By definition of the mapping ω , the restriction of \mathcal{W} to the letters N, S is $\omega(\overline{T}_0) = NS^{\alpha_1} \dots NS^{\alpha_n} NS$. Moreover, for all $i = 0, \dots, n$ there are β_i 1-heads in the first corner of u_i and one 1-head in its last corner. Thus, $\mathcal{W} = N(\mathbf{N}\mathbf{S})^{\alpha_1} \mathbf{N}\mathbf{S}^{\beta_1} (\mathbf{N}\mathbf{S})^{\alpha_2} \dots \mathbf{N}\mathbf{S}^{\beta_{n-1}} (\mathbf{N}\mathbf{S})^{\alpha_n} \mathbf{N}\mathbf{S}^{\beta_n} \mathbf{S}$. Moreover, between any letter \mathbf{N} of \mathcal{W} and the corresponding letter \mathbf{S} there is exactly one letter N . Thus, $\beta_1 = \alpha_1, \dots, \beta_n = \alpha_n$, that is, $P = Q$.

- Let $k < i < j$ be as described in the preceding point with k maximal and i minimal with respect to k (i.e. u_i is the first sibling of u_k appearing after u_k around the tree \overline{T}_0). This situation is represented in Figure 24. Observe that no 1-head is incident to u_i in the prerealizer $PR = (T_0, T_1)$ (see Figure 24), hence $\beta_i = 0$. Let H be the set of 1-heads incident to u_j and such that the corresponding 1-tail is either incident to u_k or to one

of its descendants. One can unglue the 1-heads in H from the first corner of u_j and glue them to the first corner of u_i without creating any crossing (see Figure 24). Moreover, the resulting colored map PR' is easily seen to be a prerealizer that we denote by $PR' = (T_0, T'_1)$. Let R' be the realizer corresponding to the prerealizer PR' and let $Q' = NS^{\beta'_1} \dots NS^{\beta'_n}$ be the Dyck path such that $\Phi(P, Q') = R'$. By definition of Φ , we have $\beta'_i = |H|$, $\beta'_j = \beta_j - |H|$ and $\beta'_l = \beta_l$ for all $l \neq i, j$.

- We now prove that *the realizer $R' = \Phi(P, Q')$ is minimal and maximal*. By Proposition 5.4, we only need to prove that for every internal vertex u , either $\mathbf{p}_0(\mathbf{p}'_1(u)) = \mathbf{p}_0(u)$ or $\mathbf{p}'_1(\mathbf{p}_0(u)) = \mathbf{p}'_1(u)$, where $\mathbf{p}'_1(u)$ denotes the parent of u in the tree T'_1 . Suppose that there is a vertex u not satisfying this condition. Note first that $u \neq u_k$ since $\mathbf{p}_0(\mathbf{p}'_1(u_k)) = \mathbf{p}_0(u_k)$. Since the realizer R is minimal and maximal, either $\mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$ or $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u)$. Suppose first $\mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$. In this case, the vertex u is a descendant of u_k (otherwise, $\mathbf{p}_0(\mathbf{p}'_1(u)) = \mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$), and $\mathbf{p}'_1(u) = u_j$ (for the same reason). Therefore, $\mathbf{p}_0(u_j) = \mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$ implies that u_j is a descendant of u_k . This is impossible since u_j appears after u_i around $\overline{T_0}$. Suppose now that $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u)$. In this case, the vertex u is a descendant of u_k (otherwise, $\mathbf{p}'_1(\mathbf{p}_0(u)) = \mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u) = \mathbf{p}'_1(u)$), and $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u) = u_j$ (for the same reason). Thus $\mathbf{p}'_1(\mathbf{p}_0(u)) = \mathbf{p}'_1(u) = u_i$. We reach again a contradiction.

- We now prove that *the Dyck path Q' is covered by Q in the Kreweras lattice*. By definition of the covering relation in the Kreweras lattice \mathcal{L}^K , it suffices to prove that $i \underline{Q'} j$. Since the realizer R' is minimal and maximal, it suffices to prove that the conditions (1), (2) and (3) of Lemma 5.5 hold. Clearly, there is no 1-obstruction between the vertices u_i and u_j in the realizer R' (see Figure 24), hence condition (3) holds. Moreover, since the realizer R is minimal and maximal, either $\mathbf{p}_0(u_k) = \mathbf{p}_0(u_j)$ or $\mathbf{p}_1(\mathbf{p}_0(u_k)) = u_j$. Thus, either $\mathbf{p}_0(u_i) = \mathbf{p}_0(u_j)$ or $\mathbf{p}_1(\mathbf{p}_0(u_i)) = u_j$, hence condition (2) holds. Let $i = i_1, i_2, \dots, i_s$ be the indices of the siblings of u_k appearing between u_k and u_j in clockwise order around $\overline{T_0}$ (see Figure 24). By the choice of k , we get $\mathbf{p}_1(u_{i_r}) = u_{i_{r+1}}$ for all $r < s$. Moreover, since the realizer R is minimal and maximal, either $\mathbf{p}_0(u_k) = \mathbf{p}_0(u_j)$ or $\mathbf{p}_1(\mathbf{p}_0(u_k)) = u_j$. If either case, we get $\mathbf{p}_1(u_s) = u_j$. Thus, $\mathbf{p}'_1(u_{i_r}) = \mathbf{p}_1(u_{i_r}) = u_{i_{r+1}}$ for all $r < s$, and $\mathbf{p}'_1(u_s) = \mathbf{p}_1(u_s) = u_j$. Hence, u_j is an ancestor of u_i in the tree T'_1 , that is, condition (1) holds.

- The realizer $R' = \Phi(P, Q')$ is minimal and maximal, hence by the induction hypothesis $P \leq_K Q'$. Moreover, the path Q' is covered by Q in the Kreweras lattice. Thus, $P \leq_K Q$.

□

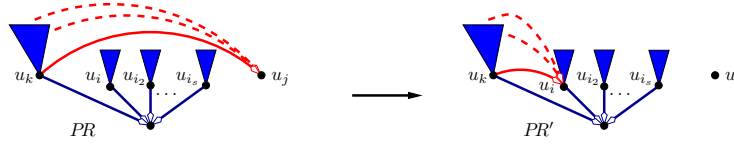


FIGURE 24. The vertices u_k, u_i, u_j in the prerealizer $PR = (T_0, T_1)$ and $PR' = (T_0, T'_1)$.

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