ON THE EXISTENCE AND UNIQUENESS OF LIMIT CYCLES IN LIÉNARD DIFFERENTIAL EQUATIONS ALLOWING DISCONTINUITIES

JAUME LLIBRE, ENRIQUE PONCE AND FRANCISCO TORRES

Abstract. In this paper we study the non–existence and the uniqueness of limit cycles for the Liénard differential system of the form $\ddot{x} - f(x)\dot{x} + g(x) = 0$ where the functions f and g satisfy $xf(x) > 0$ and $xg(x) > 0$ for $x \neq 0$ but they can be discontinuous at $x = 0$.

In particular our results allow first to prove the non–existence of limit cycles under suitable assumptions, and second to prove the existence and uniqueness of a limit cycle in a class of discontinuous Liénard systems which are relevant in engineering applications.

1. Introduction and statement of the main results

One of the main problems in the qualitative theory of planar differential equations is to know the existence of limit cycles and its number. This problem restricted to polynomial differential equations is the well known 16–th Hilbert's problem [7]. Since Hilbert's problem turned out a strongly difficult one Smale [12] particularized it to Liénard differential systems in his list of problems for the present century.

For Liénard systems there are many results about the non–existence, existence and uniqueness of limit cycles, see for instance [3, 5, 9, 13, 14]. In this paper we provide a new contribution to this subject which can be also applied to Liénard differential systems with some kind of discontinuities.

We consider for $x \in [a, b]$, where $-\infty < a < 0 < b < \infty$, the Liénard differential equation

$$x'' - f(x)x' + g(x) = 0, \tag{1}$$

1991 *Mathematics Subject Classification.* 58F21, 34C05, 58F14.

Key words and phrases. Liénard differential systems, uniqueness, non–existence, limit cycles, discontinuities.

* The first author is partially supported by a MEC/FEDER grant number MTM2005-06098-C02-01 and by a CICYT grant number 2005SGR 00550. The second and third authors are partially supported by a MEC/FEDER grant number MTM2006-00847. E. Ponce acknowledges the hospitality and support from Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain, from January to July 2007.

where the functions f and g are given by

$$f(x) = \begin{cases} f_1(x) & \text{if } x < 0, \\ f_2(x) & \text{if } x > 0, \end{cases} \qquad g(x) = \begin{cases} g_1(x) & \text{if } x < 0, \\ g_2(x) & \text{if } x > 0, \end{cases} \tag{2}$$

being f_1, g_1 continuously differentiable in $[a, 0]$, and f_2, g_2 continuously differentiable in $[0, b]$. Note that the functions f and g are not defined at $x = 0$ so that, if we eventually define $f(0)$ and $g(0)$, they are allowed to have a jump discontinuity at the origin.

By using the classical Liénard plane we can obtain the equivalent differential system

$$\begin{aligned} x' &= F(x) - y, \\ y' &= g(x), \end{aligned} \quad \text{where} \quad F(x) = \int_0^x f(s)ds, \tag{3}$$

and it is understood that $F(0) = 0$, while $g(0)$ is not defined by now.

This system has associated the vector field

$$\mathbf{X}(\mathbf{x}) = \begin{cases} \mathbf{X}_1(\mathbf{x}) & \text{if } x \leq 0, \\ \mathbf{X}_2(\mathbf{x}) & \text{if } x \geq 0, \end{cases} \quad \text{where} \quad \mathbf{X}_i(\mathbf{x}) = \begin{pmatrix} F(x) - y \\ g_i(x) \end{pmatrix}, \tag{4}$$

with $\mathbf{x} = (x, y)^T$ and standing $i = 1$ for $x \leq 0$, and $i = 2$ for $x \geq 0$. The ambiguity in the definition of $\mathbf{X}(\mathbf{x})$ on $x = 0$ will be clarified later on.

Since the system can be discontinuous we must adopt some criterion in order to define solutions starting at or passing through the allowed discontinuity line $x = 0$. Typically this is done by using the so called Filippov approach, see for instance [10]. However here only the vertical component of the vector field (4) could be discontinuous at the y-axis, while its horizontal component turns out to be continuous. In fact, we have $x' = -y$ on $x = 0$. Thus if we consider for instance orbits starting at points with $x < 0$, then these orbits are well defined whenever they do not touch the y-axis but they can arrive at this straight line (obviously only at points $(0, y)$ with $y \leq 0$) by extending $g(x)$ as if $g(0)$ were equal to $g_1(0)$. Now starting from the point $(0, y)$ with $y < 0$ we assume that $g(0) = g_2(0)$ and we continue the orbit inside $x > 0$ using system (3).

From the above paragraph and using the standard terminology of planar Filippov systems [10], the *crossing set* of the discontinuity line of system (3) includes the negative y-axis. Similar arguments for $x > 0$ imply that the crossing set is the y-axis without the origin. In [10] the origin is then called a *singular isolated sliding point*.

In short, except orbits arriving at the origin and assuming that the system is actually discontinuous, it is natural to allow concatenation of solutions in an obvious way so that the system has no sliding (Filippov) solutions. The only possible singular point may be the origin, where each vector field can

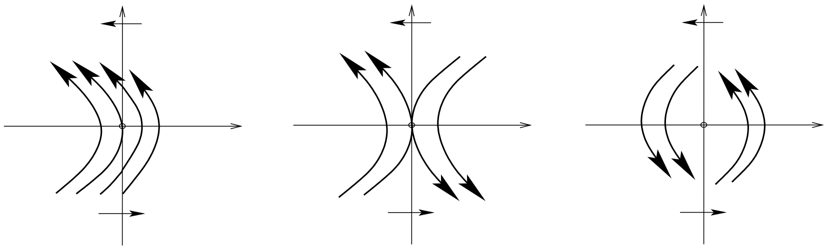

FIGURE 1. The three main cases for the local phase plane at the origin when it is not a boundary equilibrium point: regular point, pseudo-saddle and pseudo-focus.

either vanish or have a tangency with the y-axis. If at least one vector field vanishes at the origin we say that it is a *boundary equilibrium point.* If both vector fields are not zero at the origin we still can have a *pseudo-equilibrium point* when both vector fields are anti-collinear (i.e. $g_1(0)g_2(0) < 0$). Then it behaves as an equilibrium point that may be reached in finite time. Its stability and local phase portrait will be determined by studying its nearby orbits, see Figure 1.

Proposition 1. *For system* (3) *the following statements hold.*

(a) *If* $g_1(0)g_2(0) > 0$ *then the origin can be thought of a regular point.*
(b) *If* $g_1(0)g_2(0) = 0$ *then the origin is a boundary equilibrium point.*
(c) *If* $g_1(0)g_2(0) < 0$ *then the origin is a pseudo-equilibrium point, being of saddle type if* $g_1(0) > 0$ *and* $g_2(0) < 0$*, and of focus type if* $g_1(0) < 0$ *and* $g_2(0) > 0$.

Proposition 1 will be proved in Section 2.

From the point of view of practical engineering problems the most interesting case corresponds to the existence of a pseudo-equilibrium point or a proper equilibrium point of focus type at the origin, because then it is possible that the system behaves locally or even globally as an oscillator. Thus we will be mainly interested in possible periodic orbits. In this context, to assure that there is no more singular points the following hypothesis is assumed.

(H1) The function g satisfy $xg(x) > 0$ for $x \neq 0$.

We will require that the divergence of the vector field does not change its sign in each side of the discontinuity line, i.e.

(H2) The function f satisfy $xf(x) > 0$ for $x \neq 0$.

Under the last hypothesis we have positive divergence for $x > 0$ and negative divergence for $x < 0$. Then in order to have some periodic orbit surrounding the origin, there must be some balance between the x-positive and x-negative parts of the interior of the bounded region limited by the periodic orbit. This idea will be precisely stated below in Lemma 7 but in the same spirit of comparing the x-positive and x-negative half-planes and following [1], it will be useful to introduce some auxiliary functions as follows.

Under Hypothesis H2 and recalling the definition of F in (3), we define a variable $p = p(x) = F(x)$. As $p'(x) = f(x)$, then $p(x) \geq 0$ for all x, and $\operatorname{sgn}(p'(x)) = \operatorname{sgn}(x)$ for $x \neq 0$. We deduce that the function $p(x)$ has inverse functions both for $x \leq 0$ and for $x \geq 0$, namely the non-positive decreasing function

$$x_1 : [0, F(a)] \to [a, 0], \quad \text{such that } F(x_1(p)) = p, \tag{5}$$

and the non-negative increasing function

$$x_2 : [0, F(b)] \to [0, b], \quad \text{such that } F(x_2(p)) = p. \tag{6}$$

Hence for $x \neq 0$ we have that both systems (3) and (4) are equivalent to the two differential equations

$$\frac{dy(x_i(p))}{dp} = \frac{g(x_i(p))}{F(x_i(p)) - y}\,\frac{1}{f(x_i(p))} = \frac{1}{p-y}\,\frac{g(x_i(p))}{f(x_i(p))}, \tag{7}$$

where $i = 1, 2$, according to $x < 0$ or $x > 0$ respectively, and these new differential equations are both meaningful only for $p > 0$. Now by considering the functions

$$h_i(p) = \frac{g\left(x_i(p)\right)}{f\left(x_i(p)\right)}, \tag{8}$$

equations (7) can be written in the more compact form

$$\frac{dy\left(x_i(p)\right)}{dp} = \frac{h_i(p)}{p-y}. \tag{9}$$

Note that $h_i(p) > 0$ for $p > 0$ and $i = 1, 2$, and that the effect of considering equations (9) instead of the original systems (3) or (4) can be thought of as if the plane (x, y) had been folded into the half-plane (p, y) with $p > 0$.

When $h_1(p) = h_2(p)$ for p sufficiently small and the origin is a topological focus it is not difficult to show that we have indeed a center, see for instance Theorem 11.3 in [8]. We add a third hypothesis precluding such possibility. It is written in a dual way to facilitate the checking of its validity in the applications.

(H3) Assume that there exist the two limits

$$\lim_{x\to 0^-} \frac{g(x)}{f(x)} = \lim_{p\to 0^+} h_1(p) = l_1, \quad \lim_{x\to 0^+} \frac{g(x)}{f(x)} = \lim_{p\to 0^+} h_2(p) = l_2$$

satisfying

$$0 \leq l_2 \leq l_1 < \infty,$$

and if $l_2 = l_1$ then $h_2(p) < h_1(p)$ for $p > 0$ and sufficiently small (when $l_2 < l_1$ this last requirement is always fulfilled).

It is worth mentioning that this hypothesis implies that the origin is topologically an unstable focus when $l_2 > 0$, see Lemma 11. Next result states a necessary condition for the existence of periodic orbits under the above hypotheses.

Theorem 2. *Let f and g be the functions defined in* (2) *such that f_i and g_i are of class $\mathcal{C}^1$ in $[a, 0]$ and $[0, b]$ for $i = 1, 2$, respectively, where $-\infty < a < 0 < b < \infty$. Let F and h_i be the functions defined in* (3) *and* (8) *and assume that hypotheses H1-H3 are fulfilled. If system* (3) *has a periodic orbit contained in the band $a < x < b$, then the system*

$$F(x_1) = F(x_2), \qquad \frac{g(x_1)}{f(x_1)} = \frac{g(x_2)}{f(x_2)}, \tag{10}$$

has at least one solution $(x_1, x_2) = (s_1, s_2)$ with $a < s_1 < 0 < s_2 < b$, or equivalently there exists at least one solution $\hat{p} \in (0, F(a)) \cap (0, F(b))$ for the equation $h_1(p) = h_2(p)$.

Theorem 2 is proved in Section 2.

Now we give a result on uniqueness of limit cycles for Liénard equations where discontinuities are allowed at $x = 0$.

Theorem 3. *Under the same conditions of Theorem 2, assume that system* (10) *has exactly one solution $(x_1, x_2) = (s_1, s_2)$ with $a < s_1 < 0 < s_2 < b$, or equivalently there exists exactly one solution $\hat{p} \in (0, F(a)) \cap (0, F(b))$ for the equation $h_1(p) = h_2(p)$. The following statement holds.*

If the positive function

$$\alpha(x) = \frac{g(x)}{f(x)F(x)} \tag{11}$$

is increasing for $x \in (a, 0)$, or equivalently the positive function

$$\frac{h_1(p)}{p} \tag{12}$$

is decreasing for $p \in (0, F(a))$, then system (3) *has at most one periodic orbit contained in the band $a < x < b$, and if it exists has a negative characteristic exponent.*

Theorem 3 is proved in Section 2.

Although our main motivation is the case of discontinuous systems, it should be noted that the above results can be useful also for continuous differential equations. For instance we can state the following result.

Proposition 4. *The following Liénard system*

$$x' = \alpha x^2 + \beta x^3 + x^4 - y, \quad y' = x, \tag{13}$$

where $\beta > 0$ *and* $9\beta^2 - 32\alpha < 0$ *has no limit cycles in the plane.*

Proposition 4 is proved in Section 4.

We finish by considering an application of the above results to discontinuous piecewise linear differential systems. This class is increasingly used in engineering and applied sciences to model a large variety of technological devices and physical systems [2, 15]. Similar differential systems had been considered before in [6] but under the assumption of continuity for the corresponding vector field.

Theorem 5. *Consider the Liénard piecewise linear differential system*

$$\begin{cases} \dot{x} = & t_1 x - y, \\ \dot{y} = & d_1 x + a_1, \end{cases} \quad \text{if } x < 0, \qquad \begin{cases} \dot{x} = & t_2 x - y, \\ \dot{y} = & d_2 x + a_2, \end{cases} \quad \text{if } x \geq 0, \tag{14}$$

where it is assumed

$$t_1 < 0, \quad d_1 > 0, \quad a_1 < 0, \quad t_2 > 0, \quad d_2 > 0, \quad a_2 > 0.$$

Then the following statements hold.

(a) *If* $a_2/t_2 < a_1/t_1$ *then a necessary condition for the existence of periodic orbits is* $d_2/t_2^2 > d_1/t_1^2$. *If the system has periodic orbits, then it has a unique periodic orbit which is a stable limit cycle.*
(b) *If* $a_1/t_1 < a_2/t_2$ *then a necessary condition for the existence of periodic orbits is* $d_1/t_1^2 > d_2/t_2^2$. *If the system has periodic orbits, then it has a unique periodic orbit which is an unstable limit cycle.*
(c) *If* $a_2/t_2 = a_1/t_1$ *then either the system has no periodic orbits when* $d_1/t_1^2 \neq d_2/t_2^2$, *or it has a center at the origin when* $d_1/t_1^2 = d_2/t_2^2$.

Theorem 5 is proved in Section 5. Observe that statement (c) of Theorem 5 when $0 < a_2/t_2 = a_1/t_1$ and $d_1/t_1^2 = d_2/t_2^2$ says that the origin is a center even when the dynamics of the linear differential system in each half–plane could be of node type. This situation happens when

$$\frac{d_i}{t_i^2} \leq \frac{1}{4}$$

for $i = 1, 2$. When both dynamics are of focus type and we are under the assumptions of statements (a) and (b) of Theorem 5 the necessary condition for the existence of limit cycles is also sufficient, as stated in our last main result.

Theorem 6. *Under the assumptions of Theorem 5 and if*

$$\frac{d_i}{t_i^2} > \frac{1}{4}$$

for $i = 1, 2$, *then the following statements hold.*

(a) *If* $a_2/t_2 < a_1/t_1$ *then the system has periodic orbits if and only if* $d_2/t_2^2 > d_1/t_1^2$, *and in such case it has a unique periodic orbit which is a stable limit cycle.*

(b) *If* $a_1/t_1 < a_2/t_2$ *then the system has periodic orbits if and only if* $d_1/t_1^2 > d_2/t_2^2$, *and in such case it has a unique periodic orbit which is an unstable limit cycle.*

Theorem 6 is proved in Section 5.

2. On the origin and the periodic orbits

In this section we prove Proposition 1 and we give some preliminary results necessary for the proof of Theorem 2.

Proof of Proposition 1. The vector fields at the origin are $\mathbf{X}_1(0,0) = (0, g_1(0))$ and $\mathbf{X}_2(0,0) = (0, g_2(0))$, see (4). Then from (1) we have $x''(0) = -g_i(0)$ for $i = 1, 2$. Therefore if $g_1(0)$ and $g_2(0)$ are both positive or both negative then $\mathbf{X}_1(0,0)$ and $\mathbf{X}_2(0,0)$ are collinear and the orbits of both vector fields in a neighborhood of the origin have the same convexity. Consequently we can define the vector field at the origin in such a way the orbit through the origin has a quadratic tangency with the y-axis. This completes the proof of statement (a).

Statement (b) follows directly from the definitions.

If $g_1(0)g_2(0) < 0$ then the vector fields at the origin are anti-collinear and so the origin is a pseudo-equilibrium point. Assume $g_1(0) > 0$ and $g_2(0) < 0$. Then the vector field $\mathbf{X}_1$ has a *visible quadratic tangency*, that is, the orbit of $\mathbf{x}' = \mathbf{X}_1(\mathbf{x})$ through the origin is locally contained in $x \leq 0$ for backward and forward times. Similarly, the vector field $\mathbf{X}_2$ has also a visible quadratic tangency in $x \geq 0$, see Figure 1. Hence the origin is a topological saddle.

When $g_1(0) < 0$ and $g_2(0) > 0$ the vector fields $\mathbf{X}_1$ and $\mathbf{X}_2$ have *invisible quadratic tangencies.* That is, the unique point of the orbit of $\mathbf{x}' = \mathbf{X}_1(\mathbf{x})$ through the origin locally contained in $x \leq 0$ for backward and forward times is the origin itself, and similarly for $\mathbf{X}_2$; see Figure 1. Now the origin is a topological focus. This ends the proof of statement (c). □

Now we extend a necessary condition for the existence of periodic orbits fulfilled by smooth vector fields to the case of our discontinuous differential systems.

Lemma 7. *Consider the functions f and g defined as in* (2)*. If system* (3) *has a periodic orbit Γ and the interior of the bounded region limited by Γ includes the origin and it is denoted by Δ, then Γ crosses the y-axis in two points different from the origin, and the function f satisfies the condition*

$$\iint_{\Delta} f(x)dxdy = 0.$$

Proof. Since $x' = -y$ on $x = 0$ and the origin is in Δ, it follows that Γ intersects the y-axis in two points $M = (0, y_M)$ and $N = (0, y_N)$ with $y_M < 0 < y_N$.

We define Δ_1, Γ_1, and Δ_2 , Γ_2 to be the parts of Δ and Γ contained in $x < 0$ and $x > 0$ respectively. We denote by Λ the oriented segment on the y-axis from the point M to the point N while the same segment with the opposite orientation is denoted by $-\Lambda$. Then by applying the Green's Theorem we have

$$\begin{aligned}\iint_{\Delta} f(x)dxdy = & \iint_{\Delta_1} f(x)dxdy + \iint_{\Delta_2} f(x)dxdy \\ = & \int_{\Gamma_1} [F(x) - y]\, dy - g(x)dx + \int_{\Lambda} [F(x) - y]\, dy - g(x)dx + \\ & \int_{\Gamma_2} [F(x) - y]\, dy - g(x)dx + \int_{-\Lambda} [F(x) - y]\, dy - g(x)dx \\ = & \; 0 + \int_{y_M}^{y_N} (-y)\, dy + 0 + \int_{y_N}^{y_M} (-y)\, dy = 0,\end{aligned}$$

and the conclusion follows. □

3. Proof of Theorems 2 and 3

First we prove Theorem 2.

Proof of Theorem 2. We start by noticing that if system (3) has singular points they must be on the y-axis because $xg(x) > 0$ if $x \neq 0$. Also we have $g_1(0) \leq 0$ and $g_2(0) \geq 0$. Since $x' = -y$ when $x = 0$ the unique possible singular point is the origin, and from Proposition 1 it is a boundary equilibrium point or a pseudo-focus because $g_1(0)g_2(0) \leq 0$.

Assume that system (3) has a periodic orbit Γ contained in the band $a < x < b$. As a consequence of the Poincaré-Bendixson Theorem for phase portraits in the plane (see for instance [4]) the interior of the bounded region limited by Γ must contain a singular point. Such point in our system must be the origin.

Next some geometrical properties of the periodic orbit Γ will be established. First since $x' = -y$ on the y-axis, the orbit Γ is run in counterclockwise sense. Let $A = (x_A, y_A)$ and $B = (x_B, y_B)$ be the points on Γ

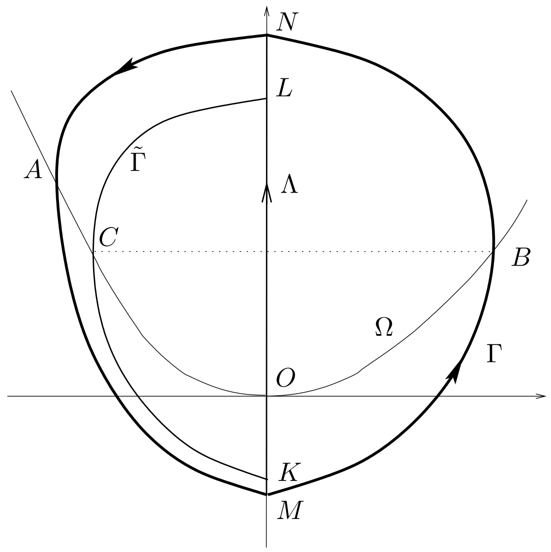

FIGURE 2. Notable points associated to a periodic orbit (thick line). For $x < 0$, it is sketched the orbit passing through $(x_{\rm C}, y_{\rm C})$, where $y_{\rm B} = y_{\rm C} = F(x_{\rm C})$. The line AOB is the graph of the curve $y = F(x)$.

for which the variable x assumes its minimum and maximum values, then $x_A < 0 < x_B$. Since for $x \neq 0$ we have

$$\frac{dx}{dy} = \frac{F(x) - y}{g(x)}, \tag{15}$$

and this derivative vanishes for $x = x_A$ and $x = x_B$, one obtains $y_A = F(x_A)$ and $y_B = F(x_B)$. Moreover this derivative only vanishes at the points A and B. Indeed when $dx/dy = 0$ the second derivative is given by

$$\frac{d^2x}{dy^2} = \left.\frac{\left(\frac{dF}{dx}\frac{dx}{dy} - 1\right) g(x) - [F(x) - y]\,\frac{dg}{dx}\frac{dx}{dy}}{g(x)^2}\right|_{\frac{dx}{dy}=0} = -\frac{1}{g(x)},$$

which has a definite sign, in fact the opposite sign to x. Then the derivative (15) vanishes only once for $x > 0$ and only once for $x < 0$, and so the points A and B are the unique points where the orbit Γ intersects the curve defined by the equation $y = F(x)$ denoted by Ω.

It follows that Γ intersects any straight line L defined by $x = q$ with $x_A < q < x_B$ in exactly two points (q, y_α) and (q, y_β) with $y_\alpha < F(q) < y_\beta$.

In particular for $q = 0$ such points are denoted by $M = (0, y_M)$ and $N = (0, y_N)$ with $y_M < 0 < y_N$. Moreover the path Γ can be described as the graph of $y = y_l(x)$ on the lower arc AMB and by the graph of $y = y_u(x)$ on the upper arc ANB. Clearly $y_l(x) < F(x) < y_u(x)$, that is Γ is below the curve Ω on the lower arc AMB, while it is over the curve Ω on the upper arc ANB, see Figure 2.

Differential equations (9) can be continuously extended to $x = 0$ by putting $h_i(0) = l_i$, so that they define the orbits of the following two differential systems, both defined for $p \geq 0$:

$$\begin{aligned} \frac{dp}{d\tau} &= p - y, \\ \frac{dy}{d\tau} &= h_i(p), \end{aligned} \tag{16}$$

for $i = 1, 2$. The arc MAN of the periodic orbit Γ can be parameterized as

$$\Gamma_1(p) = \begin{cases} y_l(x_1(p)) & \text{if } y_l(x_1(p)) \leq y_A, \\ y_u(x_1(p)) & \text{if } y_A \leq y_u(x_1(p)), \end{cases}$$

while the arc MBN of Γ can be parameterized as

$$\Gamma_2(p) = \begin{cases} y_l(x_2(p)) & \text{if } y_l(x_2(p)) \leq y_B, \\ y_u(x_2(p)) & \text{if } y_B \leq y_u(x_2(p)), \end{cases}$$

where

$$y_l(x_1(0)) = y_l(x_2(0)) = y_M < 0 \text{ and } y_u(x_1(0)) = y_u(x_2(0)) = y_N > 0. \tag{17}$$

Before proceeding further we state now some results from the theory of differential inequalities, providing their proof for sake of completeness.

Lemma 8. *Assume that the graphs of the two continuous functions $y_i : [c, d] \to \mathbb{R}$ are solution curves for the Lipschitz differential systems*

$$\begin{aligned} \frac{dp}{d\tau} &= p - y, \\ \frac{dy}{d\tau} &= \phi_i(p), \end{aligned}$$

for $i = 1, 2$, respectively. Assume also that the inequalities

$$p - y_1(p) > 0, \text{ and } p - y_2(p) > 0, \text{ for all } p \in (c, d),$$

and

$$0 < \phi_1(p) < \phi_2(p), \text{ for all } p \in (c, d),$$

are satisfied. The following statements hold.

(a) *If $y_1(c) \leq y_2(c)$ then $y_1(p) < y_2(p)$ for all $p \in (c, d]$.*
(b) *If $y_1(d) \geq y_2(d)$ then $y_1(p) > y_2(p)$ for all $p \in [c, d)$.*

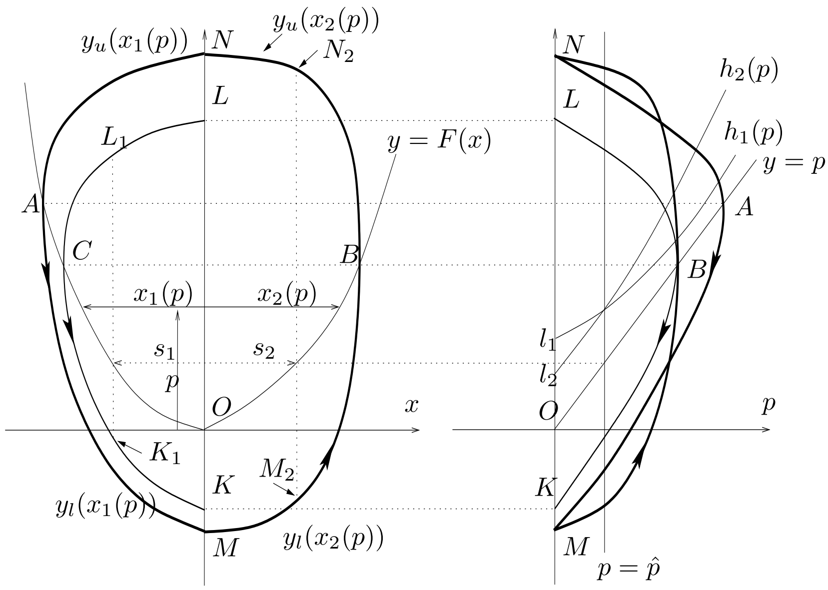

FIGURE 3. In the new coordinates (p, y) both semi-orbits are in the halfplane $p > 0$, and they must enclose the same area.

Proof. For all $p \in (c, d)$ such that $y_1(p) \le y_2(p)$ we have

$$\frac{dy_1}{dp} = \frac{\phi_1(p)}{p - y_1(p)} \le \frac{\phi_1(p)}{p - y_2(p)} < \frac{\phi_2(p)}{p - y_2(p)} = \frac{dy_2}{dp},$$

so that the function $y_2 - y_1$ is strictly increasing in (c, d) and the conclusion of statement (a) follows easily.

To show statement (b) suppose on the contrary that there exists $\bar{p} \in [c, d)$ such that $y_1(\bar{p}) \le y_2(\bar{p})$. Then by statement (a) we conclude that $y_1(p) < y_2(p)$ for all $p \in (\bar{p}, d]$, and in particular that $y_1(d) < y_2(d)$, which is a contradiction. □

Remark 9. *An analogous result for Lemma 8 is also true by reversing all the inequalities in the statements when* $p - y_1(p) < 0$ *and* $p - y_2(p) < 0$ *(i.e. when the orbits are in the region* $y > p$*) while* $0 < \phi_1(p) < \phi_2(p)$ *still holds for all* $p \in (c, d)$*.*

From the hypotheses we get that $0 \le h_2(0) \le h_1(0)$ and $h_2(p) < h_1(p)$ for $0 < p \ll 1$. Then from (9) and (17) and using statement (a) of Lemma

8 in the interval $[0,\bar{p}]$ with $\bar{p}$ sufficiently small, we obtain

$$y_l\left(x_2(p)\right) < y_l\left(x_1(p)\right) \quad \text{for } 0 < p \ll 1.$$

On the other hand by using Remark 9 we have analogously

$$y_u\left(x_1(p)\right) < y_u\left(x_2(p)\right) \quad \text{for } 0 < p \ll 1. \tag{18}$$

Next we will show that both paths $\Gamma_1(p)$ and $\Gamma_2(p)$ cross themselves at least at one point. It can be easily seen that the system

$$\left\{\begin{array}{rcl} \dfrac{dp}{d\tau} &=& -p-y, \\ \dfrac{dy}{d\tau} &=& -h_1(-p), \end{array}\right. \quad \text{if } p<0, \quad \left\{\begin{array}{rcl} \dfrac{dp}{d\tau} &=& p-y, \\ \dfrac{dy}{d\tau} &=& h_2(p), \end{array}\right. \quad \text{if } p>0, \tag{19}$$

has a counterclockwise periodic orbit $\hat{\Gamma}$ constituted by a path $\hat{\Gamma}_1$ which is the symmetrical one with respect to the y-axis of the path $\Gamma_1(p)$ along with the path $\Gamma_2(p)$. System (19) is the Liénard system

$$\begin{array}{rcl} \dfrac{dp}{d\tau} &=& |p|-y, \\ \dfrac{dy}{d\tau} &=& h(p), \end{array}$$

where

$$h(p) = \left\{\begin{array}{rl} h_2(p) & \text{if } p>0, \\ -h_1(-p) & \text{if } p<0. \end{array}\right.$$

Then by applying Lemma 7 we have

$$\iint_{\Delta} \operatorname{sgn}(x)\,dx\,dy = 0 = -S_1 + S_2, \tag{20}$$

where Δ is the interior of the region limited by $\hat{\Gamma}$, S_1 and S_2 are the areas of Δ on the left and on the right hand side of the line $x=0$ respectively. If the path $\Gamma_1(p)$ does not cut the path $\Gamma_2(p)$, then $S_1 \neq S_2$ and (20) cannot be fulfilled. So the path $\Gamma_1(p)$ must cut the path $\Gamma_2(p)$.

Assume now that $h_2(p) < h_1(p)$ for $0 < p \leq \min\{y_A, y_B\}$. Since $y_l\left(x_2(p)\right)$ and $y_l\left(x_1(p)\right)$ are solutions of the equations

$$\frac{dy}{dp} = \frac{h_2(p)}{p-y}, \quad \frac{dy}{dp} = \frac{h_1(p)}{p-y},$$

respectively, with $y_l\left(x_2(0)\right) = y_l\left(x_1(0)\right)$, then by statement (a) of Lemma 8 we must have $y_l\left(x_2(p)\right) < y_l\left(x_1(p)\right)$ for $0 < p \leq \min\{y_A, y_B\}$. Similarly, by using Remark 9 we must have $y_u\left(x_1(p)\right) < y_u\left(x_2(p)\right)$ for $0 < p \leq \min\{y_A, y_B\}$, and then the paths $\Gamma_1(p)$ and $\Gamma_2(p)$ do not cross themselves, which is a contradiction. Hence there must exist $\hat{p}$ such that $h_1(\hat{p}) = h_2(\hat{p})$, i.e. the system

$$\hat{p} = F(x_1) = F(x_2), \quad \frac{f(x_1)}{g(x_1)} = \frac{f(x_2)}{g(x_2)}$$

must have a solution $(x_1, x_2) = (s_1, s_2)$ with $x_A < s_1 < 0 < s_2 < x_B$, and Theorem 2 is proved. □

Proof of Theorem 3. We start by assuming again the existence of a periodic orbit Γ contained in the band $a < x < b$ with all the geometric properties already established in the proof of Theorem 2. Furthermore we assume that there is a unique value $\hat{p} < \min\{y_A, y_B\}$ such that $h_2(p) < h_1(p)$ for $0 < p < \hat{p}$, and $h_1(p) < h_2(p)$ for $p > \hat{p}$.

We claim first that $y_A > y_B$, as shown in Figure 2. Now we start the proof of the claim. By using statement (a) of Lemma 8 in the interval $[0, \hat{p}]$ it follows that

$$y_l\left(x_2(p)\right) < y_l\left(x_1(p)\right) \quad \text{for } 0 < p \leq \hat{p},$$

and analogously, by using Remark 9 for the upper part, we have

$$y_u\left(x_1(p)\right) < y_u\left(x_2(p)\right) \quad \text{for } 0 < p \leq \hat{p}.$$

From the above inequalities we see that when the paths start to separate from the y–axis the two arcs of path $\Gamma_2(p)$ are farther from the p–axis than the two arcs of path $\Gamma_1(p)$, see Figure 3. We already know from the proof of Theorem 2 that both paths intersect and now from the relative position of their beginning arcs at the y–axis we can assure that their crossing points must appear in an even number counting multiplicities. In fact due to the uniqueness of solutions of system (10), we conclude now that there exist a unique value $\delta_1 > \hat{p}$ such that

$$\begin{aligned} y_l\left(x_2(p)\right) &< y_l\left(x_1(p)\right) \quad \text{for } 0 < p < \delta_1, \\ y_l\left(x_2(p)\right) &> y_l\left(x_1(p)\right) \quad \text{for } \delta_1 < p < \min\{y_A, y_B\}. \end{aligned}$$

This lower crossing point at $p = \delta_1$ for $y_l\left(x_1(p)\right)$ and $y_l\left(x_2(p)\right)$ must be unique because resorting to Lemma 8(a) in the interval $[\delta_1, \min\{y_A, y_B\}]$ we have $y_l\left(x_1(p)\right) < y_l\left(x_2(p)\right)$ in such interval. Similarly, by using Remark 9 there is a unique value $\delta_2 > \hat{p}$ such that for the upper parts

$$\begin{aligned} y_u\left(x_1(p)\right) &< y_u\left(x_2(p)\right) \quad \text{for } 0 < p < \delta_2, \\ y_u\left(x_1(p)\right) &> y_u\left(x_2(p)\right) \quad \text{for } \delta_2 < p < \min\{y_A, y_B\}. \end{aligned}$$

Therefore, as these two crossing points are only possible when $y_A > y_B$, if system (3) has a periodic orbit then the condition $y_A > y_B$ holds and our first claim is proved.

We now claim that the characteristic exponent of a periodic orbit of system (3) is negative, that is the periodic orbit is a stable limit cycle. Hence the system has at most one periodic orbit because we cannot have two consecutive stable periodic orbits. This should complete the proof of Theorem 3. Now we prove this second claim.

Let $C = (x_C, y_C)$ be the point on the curve Ω for which $x_A < x_C < 0$ and $y_C = F(x_C) = F(x_B) = y_B > y_l(x_C)$, and let $\tilde{\Gamma}$ be the orbit of (3) passing through the point C. Then the orbit $\tilde{\Gamma}$ meets the y-axis in the points K and L (see Figure 3), where $y_M < y_K < 0 < y_L < y_N$. The orbit $\tilde{\Gamma}$ is given by the graph of $y = \tilde{y}_l(x)$ on the arc CK and by the graph of $y = \tilde{y}_u(x)$ on the arc LC. Since $y_M = y_l(0) < y_K = \tilde{y}_l(0)$, Lemma 8.a in the interval $[0, \hat{p}]$ implies

$$y_l(x_2(p)) < \tilde{y}_l(x_1(p)) \quad \text{for } 0 \le p \le \hat{p}. \tag{21}$$

The previous inequality can be extended to assure that

$$y_l(x_2(p)) < \tilde{y}_l(x_1(p)) \quad \text{for } \hat{p} \le p \le y_B. \tag{22}$$

by using statement (b) of Lemma 8 in the interval $[\hat{p}, y_B]$ because we know that $y_l\left(x_2(y_B)\right) = y_B = y_C = \tilde{y}_l\left(x_1(y_B)\right)$ and that $h_1(p) < h_2(p)$ for $p > \hat{p}$.

By using Remark 9 in an analogous way, we can show that

$$\tilde{y}_u(x_1(p)) < y_u(x_2(p)) \quad \text{for } 0 < p < y_B. \tag{23}$$

Next we compute the characteristic exponent ρ of the periodic orbit Γ, i.e.

$$\rho = \int_\Gamma f(x(t))dt,$$

where the line integral is described in the sense of the flow, that is counter-clockwise.

The periodic orbit $\Gamma = \{(x(t), y(t))\}$ intersects the line $x = s_2$ in the points M_2 and N_2, and the orbit $\tilde{\Gamma} = \{(\tilde{x}(t), \tilde{y}(t))\}$ intersects the line $x = s_1$ in the points K_1 and L_1, see Figure 3. We first compute the integral

$$I = \int_{MBN} f(x(t))dt + \int_{LCK} f(\tilde{x}(t))dt$$

along the arc MBN of the periodic orbit Γ and along the arc LCK of the path $\tilde{\Gamma}$.

To this end we compute the following integrals

$$I_1 = \int\limits_{\Gamma:MM_2} f(x(t))dt + \int\limits_{\tilde{\Gamma}:K_1K} f(\tilde{x}(t))dt =$$

$$\int\limits_0^{s_2} \frac{f(x)}{F(x) - y_l(x)} dx + \int\limits_{s_1}^{0} \frac{f(x)}{F(x) - \tilde{y}_l(x)} dx =$$

$$\int\limits_0^{\hat{p}} \frac{dp}{p - y_l\left(x_2(p)\right)} - \int\limits_0^{\hat{p}} \frac{dp}{p - \tilde{y}_l\left(x_1(p)\right)} =$$

$$\int\limits_0^{\hat{p}} \frac{\left[y_l\left(x_2(p)\right) - \tilde{y}_l\left(x_1(p)\right)\right] dp}{\left[p - y_l\left(x_2(p)\right)\right]\left[p - \tilde{y}_l\left(x_1(p)\right)\right]},$$

and from (21) we conclude that $I_1 < 0$. Now we consider

$$I_2 = \int\limits_{\Gamma:M_2B} f(x(t))dt + \int\limits_{\tilde{\Gamma}:CK_1} f(\tilde{x}(t))dt =$$

$$\int\limits_{s_2}^{x_B} \frac{f(x)}{F(x) - y_l(x)} dx + \int\limits_{x_C}^{s_1} \frac{f(x)}{F(x) - \tilde{y}_l(x)} dx =$$

$$\int\limits_{\hat{p}}^{y_B} \frac{dp}{p - y_l\left(x_2(p)\right)} - \int\limits_{\hat{p}}^{y_B} \frac{dp}{p - \tilde{y}_l\left(x_1(p)\right)} =$$

$$\lim_{\eta \to y_B} \int\limits_{\hat{p}}^{\eta} \frac{\left[y_l\left(x_2(p)\right) - \tilde{y}_l\left(x_1(p)\right)\right] dp}{\left[p - y_l\left(x_2(p)\right)\right]\left[p - \tilde{y}_l\left(x_1(p)\right)\right]},$$

and from (22) we conclude that $I_2 < 0$. We have

$$I_3 = \int\limits_{\Gamma:BN_2} f(x(t))dt + \int\limits_{\tilde{\Gamma}:L_1C} f(\tilde{x}(t))dt =$$

$$\int\limits_{x_B}^{s_2} \frac{f(x)}{F(x) - y_u(x)} dx + \int\limits_{s_1}^{x_C} \frac{f(x)}{F(x) - \tilde{y}_u(x)} dx =$$

$$\int\limits_{\hat{p}}^{y_B} \frac{dp}{y_u\left(x_2(p)\right) - p} - \int\limits_{\hat{p}}^{y_B} \frac{dp}{\tilde{y}_u\left(x_1(p)\right) - p} =$$

$$\lim_{\eta \to y_B} \int\limits_{\hat{p}}^{\eta} \frac{\left[\tilde{y}_u\left(x_1(p)\right) - y_u\left(x_2(p)\right)\right] dp}{\left[y_u\left(x_2(p)\right) - p\right]\left[\tilde{y}_u\left(x_1(p)\right) - p\right]},$$

and from (23) we conclude that $I_3 < 0$. We compute

$$I_4 = \int\limits_{\Gamma:N_2N} f(x(t))dt + \int\limits_{\tilde{\Gamma}:LL_1} f(\tilde{x}(t))dt =$$

$$\int\limits_{s_2}^{0} \frac{f(x)}{F(x) - y_u(x)}dx + \int\limits_{0}^{s_1} \frac{f(x)}{F(x) - \tilde{y}_u(x)}dx =$$

$$\int\limits_{0}^{\hat{p}} \frac{dp}{y_u\left(x_2(p)\right) - p} - \int\limits_{0}^{\hat{p}} \frac{dp}{\tilde{y}_u\left(x_1(p)\right) - p} =$$

$$\int\limits_{0}^{\hat{p}} \frac{\left[\tilde{y}_u\left(x_1(p)\right) - y_u\left(x_2(p)\right)\right] dp}{\left[y_u\left(x_2(p)\right) - p\right]\left[\tilde{y}_u\left(x_1(p)\right) - p\right]},$$

and from (23) we conclude that $I_4 < 0$. Hence $I = I_1 + I_2 + I_3 + I_4 < 0$ and

$$\rho = I + \int_{\Gamma:NAM} f(x(t))dt - \int_{\tilde{\Gamma}:LCK} f(\tilde{x}(t))dt.$$

We define

$$J_1 = \int_{\Gamma:NA} f(x(t))dt - \int_{\tilde{\Gamma}:LC} f(\tilde{x}(t))dt,$$

$$J_2 = \int_{\Gamma:AM} f(x(t))dt - \int_{\tilde{\Gamma}:CK} f(\tilde{x}(t))dt,$$

so that $\rho = I + J_1 + J_2$. Now we will show that $J_1 < 0$ and $J_2 < 0$.

We compute the integral

$$\begin{aligned} J_1 = & \int\limits_{0}^{y_A} \frac{dp}{p - y_u\left(x_1(p)\right)} - \int\limits_{0}^{y_B} \frac{dp}{p - \tilde{y}_u\left(x_1(p)\right)} \\ = & \int\limits_{0}^{y_A} \frac{dp}{p - y_u\left(x_1(p)\right)} - \int\limits_{0}^{y_A} \frac{dp}{p - \hat{y}_u\left(p\right)} \\ = & \int\limits_{0}^{y_A} \frac{y_u\left(x_1(p)\right) - \hat{y}_u\left(p\right)}{\left[p - y_u\left(x_1(p)\right)\right]\left[p - \hat{y}_u\left(p\right)\right]} dp \end{aligned}$$

where the function $\hat{y}_u\left(p\right)$ is given by

$$\hat{y}_u\left(p\right) = \mu\tilde{y}_u\left(x_1\left(\mu^{-1}p\right)\right) \quad \text{and} \quad \mu = \frac{y_A}{y_B} > 1.$$

Clearly the function $\hat{y}_u\left(p\right)$ is a solution of the differential equation

$$\frac{dy}{dp} = \frac{\hat{h}_1\left(p\right)}{p - y},$$

where $\hat{h}_1(p) = \mu h_1(\mu^{-1}p)$.

The function α defined in (11), which can be written for $x < 0$ as

$$\alpha(x_1(p)) = \frac{g(x_1(p))}{f(x_1(p))\, p} = \frac{h_1(p)}{p},$$

is an increasing function of x_1, and so a decreasing function of p. Then $h_1(\mu^{-1}p) > \mu^{-1}h_1(p)$, so $h_1(p) < \mu h_1(\mu^{-1}p) = \hat{h}_1(p)$ for $p \geq 0$. We recall that for $y = y_u(x_1(p))$ we knew that

$$\frac{dy}{dp} = \frac{h_1(p)}{p - y}.$$

Now from the equality

$$\hat{y}_u(y_A) = \frac{y_A}{y_B}\tilde{y}_u\left(x_1\left(\frac{y_B}{y_A}y_A\right)\right) = y_A = y_u(x_1(y_A)),$$

and using the statement (b) corresponding to Remark 9 for h_1 and $\hat{h}_1$ in the interval $[0, y_A]$ we get $y_u(x_1(p)) < \hat{y}_u(p)$ for $0 < p < y_A$, and consequently $J_1 < 0$.

Similarly we can show that $J_2 < 0$, and the proof is complete. □

4. Proof of Proposition 4

Hypotheses H1 clearly holds for system (13).

We see that $f(x) = x(2\alpha + 3\beta x + 4x^2)$. Obviously $\alpha > 0$ and since the discriminant of the quadratic factor is negative Hypotheses H2 holds. Regarding Hypothesis H3 we see that

$$\frac{g(x)}{f(x)} = \frac{1}{2\alpha + 3\beta x + 4x^2},$$

so that $l_1 = l_2 = 1/(2\alpha)$ but since $\beta > 0$ it is clear that $h_2(p) < h_1(p)$ for $p > 0$ and sufficiently small because $x_1(p) < 0$ but $x_2(p) > 0$.

Now we look for solutions $(x_1, x_2) = (s_1, s_2)$ with $s_1 < 0 < s_2$ of system

$$\alpha x_1^2 + \beta x_1^3 + x_1^4 = \alpha x_2^2 + \beta x_2^3 + x_2^4, \quad \frac{1}{2\alpha + 3\beta x_1 + 4x_1^2} = \frac{1}{2\alpha + 3\beta x_2 + 4x_2^2},$$

or equivalently of system

$$\alpha(x_2^2 - x_1^2) + \beta(x_2^3 - x_1^3) + x_2^4 - x_1^4 = 0, \quad 3\beta(x_2 - x_1) + 4(x_2^2 - x_1^2) = 0.$$

After removing the obvious factor $x_2 - x_1 > 0$ we get the system

$$\alpha(x_1+x_2)+\beta(x_1^2+x_1x_2+x_2^2)+x_1^3+x_1^2x_2+x_1x_2^2+x_2^3 = 0, \quad 3\beta+4(x_1+x_2) = 0.$$

Substituting the second equation in the first one multiplied by 4, we get

$$-3\alpha\beta+4\beta(x_1^2+x_1x_2+x_2^2)-3\beta(x_1^2-x_1x_2+x_2^2)-3\beta x_1x_2 = 0, \; 3\beta+4(x_1+x_2) = 0,$$

and new simplifications lead to the quadratic system

$$-3\alpha + x_1^2 + 4x_1x_2 + x_2^2 = 0, \qquad 3\beta + 4(x_1 + x_2) = 0.$$

A new substitution in the first equation multiplied by 16 gives the system

$$9\beta^2 - 48\alpha + 32x_1x_2 = 0, \qquad 3\beta + 4(x_1 + x_2) = 0.$$

So the solutions x_1 and x_2 must be roots of the quadratic

$$x^2 + \frac{3\beta}{4}x + \frac{48\alpha - 9\beta^2}{32} = 0,$$

which has real different solutions only if $9\beta^2 - 32\alpha > 0$, and this is contrary to the initial assumption. Then from Theorem 2 no limit cycles are possible.

5. Proof of Theorems 5 and 6

First we show Theorem 5.

Proof of Theorem 5. We first check the hypotheses H1–H3 in order to see that both Theorem 2 and Theorem 3 can be applied.

Hypotheses H1 and H2 are immediate.

We will use the functions h_i in checking Hypothesis H3, and noting that $x_i(p) = p/t_i$ for $i = 1, 2$, we have

$$h_i(p) = \frac{d_i}{t_i^2}p + \frac{a_i}{t_i},$$

for $i = 1, 2$. Now Hypothesis H3 is fulfilled whenever $l_2 = a_2/t_2 < l_1 = a_1/t_1$, or if $a_2/t_2 = a_1/t_1$ when $d_2/t_2^2 < d_1/t_1^2$. The equation $h_1(p) = h_2(p)$ becomes equivalent to

$$\left(\frac{d_1}{t_1^2} - \frac{d_2}{t_2^2}\right) p = \frac{a_2}{t_2} - \frac{a_1}{t_1}, \tag{24}$$

which has a unique positive solution only if

$$\left(\frac{d_1}{t_1^2} - \frac{d_2}{t_2^2}\right)\left(\frac{a_2}{t_2} - \frac{a_1}{t_1}\right) > 0.$$

Now statement (a) of Theorem 5 is a direct consequence of Theorems 2 and 3.

Statement (b) can be shown by using statement (a) applied to the system

$$\begin{cases} \dot{x} = & (-t_2)x - y, \\ \dot{y} = & d_2x + (-a_2), \end{cases} \text{if } x \le 0, \quad \begin{cases} \dot{x} = & (-t_1)x - y, \\ \dot{y} = & d_1x + (-a_1), \end{cases} \text{if } x > 0,$$

which corresponds to systems (14) after doing the change of variable $(x, \tau) \to (-x, -\tau)$.

Regarding statement (c), it is obvious that equation (24) has no solutions different from zero when $d_1/t_1^2 - d_2/t_2^2 \neq 0$ and the first assertion then comes

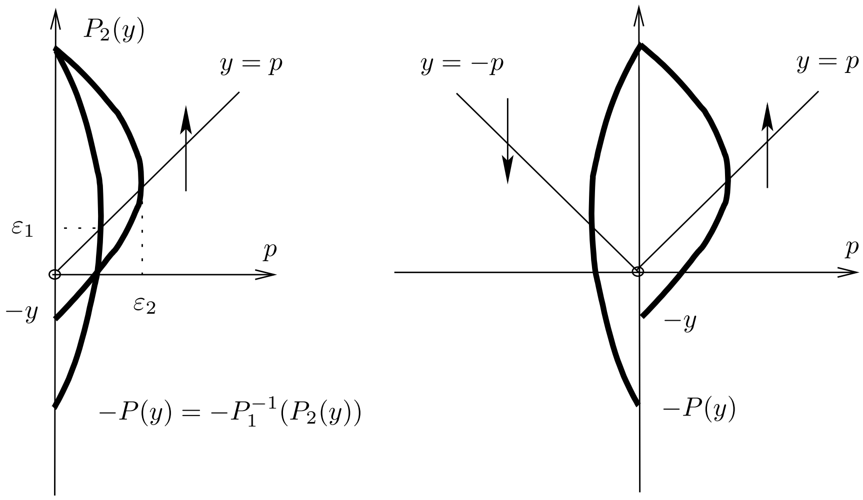

FIGURE 4. Building the Poincaré map near the origin by using the half–return maps P_1 and P_2.

from Theorem 2. In the remaining case we have $h_1(p) = h_2(p)$ for all p and the conclusion on having a center comes from the application of Theorem 11.3 in [8] to system (19). This completes the proof of Theorem 5. □

Starting from system (19) corresponding to system (14), when both dynamics are of focus type (stable for $x < 0$, unstable for $x > 0$) it is possible to define globally a Poincaré return map by introducing a transversal section to the flow. We select for that the negative y–axis and define $P : (0, \infty) \to (0, \infty)$ which maps the coordinate $y > 0$ of the point $(0, -y)$ into the vertical coordinate $P(y)$ of the point $(0, -P(y))$, where both points are the initial and the final point, respectively, of one orbit that gives counterclockwise a complete turn around the origin. For more details on the definition of this Poincaré map see the proof of Lemma 10. The explicit computation of this map P should solve the problem of determining the exact number of periodic orbits; however this is not possible in general. The following result will be needed later.

Lemma 10. *Under the assumptions of Theorem 6 the derivative of the Poincaré map P satisfy*

$$\lim_{y\to\infty} \frac{dP}{dy} = e^{\pi(\kappa_2-\kappa_1)},$$

where for $i = 1, 2$,

$$\kappa_i = \frac{1}{\sqrt{\frac{4d_i}{t_i^2} - 1}}.$$

Proof. To work in a more compact way we choose the folded plane, that is systems (16), namely

$$\begin{aligned} \frac{dp}{d\tau} &= p - y, \\ \frac{dy}{d\tau} &= \frac{d_i}{t_i^2} p + \frac{a_i}{t_i}, \end{aligned}$$

for $p \geq 0$, and we define

$$\omega_i = \sqrt{\frac{d_i}{t_i^2} - \frac{1}{4}}, \quad p_i^e = -\frac{a_i t_i}{d_i}$$

for $i = 1, 2$. Integrating both linear systems taking as initial point $(0, -y)$, we have

$$\begin{pmatrix} p_i(\tau) - p_i^e \\ y_i(\tau) - y_i^e \end{pmatrix} = \exp\left(\frac{\tau}{2}\right) C_i(\tau) \begin{pmatrix} 0 - p_i^e \\ -y - y_i^e \end{pmatrix}$$

where $y_i^e = p_i^e$, and

$$C_i(\tau) = \begin{pmatrix} \cos(\omega_i \tau) + \dfrac{\sin(\omega_i \tau)}{2\omega_i} & -\dfrac{\sin(\omega_i \tau)}{\omega_i} \\ \dfrac{d_i \sin(\omega_i \tau)}{t_i^2 \omega_i} & \cos(\omega_i \tau) - \dfrac{\sin(\omega_i \tau)}{2\omega_i} \end{pmatrix}.$$

After one half–turn around the origin following these solutions $(p_i(\tau), y_i(\tau))$, we will arrive up to the positive part of the y–axis for certain values τ_i such that $p_i(\tau_i) = 0$ with $0 < \omega_i \tau_i < \pi$, see Figure 4. The corresponding values of $y_i(\tau_i)$ allows to define the half–return maps

$$P_i : (0, \infty) \to (0, \infty) \text{ with } P_i(y) = y_i(\tau_i) \text{ and } p_i(\tau_i) = 0 \text{ with } 0 < \omega_i \tau_i < \pi,$$

for $i = 1, 2$. Now the return map $P(y)$ of system (19) corresponding to system (14) can be recovered by taking $P(y) = P_1^{-1}(P_2(y))$.

As it is shown in [6] for the continuous case, the study of such half–return Poincaré maps is not possible explicitly and must be done in a parametric way. Thus introducing the notation $\theta_i = \omega_i \tau_i$ and $\kappa_i = 1/(2\omega_i)$ for $i = 1, 2$, the map P_i is determined by the equation

$$e^{\kappa_i \theta_i} \begin{pmatrix} \cos\theta_i + \kappa_i \sin\theta_i & -2\kappa_i \sin\theta_i \\ \frac{1+\kappa_i^2}{2\kappa_i} \sin\theta_i & \cos\theta_i - \kappa_i \sin\theta_i \end{pmatrix} \begin{pmatrix} -p_i^e \\ -y - p_i^e \end{pmatrix} = \begin{pmatrix} -p_i^e \\ P_i(y) - p_i^e \end{pmatrix}$$

and it is parametrically described for each value of $\theta_i \in (0,\pi)$ as follows,

$$\begin{aligned} y = & \quad -p_i^e \frac{e^{-\kappa_i\theta_i} - \cos\theta_i + \kappa_i \sin\theta_i}{2\kappa_i \sin\theta_i}, \\ P_i(y) = & \quad -p_i^e \frac{e^{\kappa_i\theta_i} - \cos\theta_i - \kappa_i \sin\theta_i}{2\kappa_i \sin\theta_i}, \end{aligned}$$

for $i = 1, 2$. Hence a straightforward computation now shows that for the derivatives we also have the parametric representation

$$\frac{dP_i}{dy}(\theta_i) = \frac{1 - e^{\kappa_i\theta_i}(\cos\theta_i - \kappa_i \sin\theta_i)}{1 - e^{-\kappa_i\theta_i}(\cos\theta_i + \kappa_i \sin\theta_i)} = e^{2\kappa_i\theta_i} \frac{y}{P_i(y)},$$

so that

$$\lim_{y\to\infty} \frac{dP_i(y)}{dy} = \lim_{\theta_i\to\pi^-} \frac{1 - e^{\kappa_i\theta_i}(\cos\theta_i - \kappa_i \sin\theta_i)}{1 - e^{-\kappa_i\theta_i}(\cos\theta_i + \kappa_i \sin\theta_i)} = \frac{1 + e^{\kappa_i\pi}}{1 + e^{-\kappa_i\pi}} = e^{\kappa_i\pi}.$$

We can conclude by the chain rule and the inverse function theorem that

$$\lim_{y\to\infty} \frac{dP(y)}{dy} = \lim_{y\to\infty} \frac{dP_1^{-1}(P_2(y))}{dy} = \frac{1}{e^{\kappa_1\pi}} e^{\kappa_2\pi} = e^{\pi(\kappa_2-\kappa_1)},$$

and the lemma follows. □

This last result can be also obtained by resorting to the techniques followed in [11].

We finish by giving the proof of Theorem 6. For that we show first another technical result

Lemma 11. *Hypothesis H3 implies that the origin is an unstable topological focus if $l_2 > 0$.*

Proof. We will show that when $y > 0$ is sufficiently small the Poincaré map introduced in this section satisfies $P(y) > y$. Taking a point $(\varepsilon_i, \varepsilon_i)$ on the line $y = p$ sufficiently near the origin, we know that for the orbit passing through this point

$$\frac{dp_i}{dy} = 0, \quad \frac{d^2p_i}{dy^2} = -\frac{1}{h_i(\varepsilon_i)},$$

and then the corresponding orbits can be approximated by

$$p_i(y) = \varepsilon_i - \frac{1}{2h_i(\varepsilon_i)}(y - \varepsilon_i)^2 + \mathcal{O}(y - \varepsilon_i)^3,$$

which cuts the y–axis in the points

$$y_i^\pm \approx \pm\sqrt{2h_i(\varepsilon_i)\varepsilon_i} + \varepsilon_i \approx \pm\sqrt{2l_i\varepsilon_i} + \varepsilon_i.$$

Note that for ε_i sufficiently small we have that $y_i^- < 0$.

We choose ε_1 and ε_2 for the systems with $i = 1, 2$, in such a way that the two quadratic approximations for the orbits coincide in the positive y–axis, namely

$$\varepsilon_1 + \sqrt{2h_1(\varepsilon_1)\varepsilon_1} = \varepsilon_2 + \sqrt{2h_2(\varepsilon_2)\varepsilon_2}. \tag{25}$$

From Remark 9 and considering only the part of the orbits contained in the region $y > p$ we can assure that $\varepsilon_1 < \varepsilon_2$, see Figure 4. Now taking $y = -y_2^-$ we can make the approximation $P(y) \approx -y_1^-$ so that

$$P(y) - y \approx -(\varepsilon_1 - \sqrt{2h_1(\varepsilon_1)\varepsilon_1}) + \varepsilon_2 - \sqrt{2h_2(\varepsilon_2)\varepsilon_2} = 2(\varepsilon_2 - \varepsilon_1) > 0,$$

where we have taken into account the equality (25). This implies that the origin is an unstable topological focus, see Figure 4. □

Proof of Theorem 6. Reasoning like in the proof of Theorem 5 we must only show statement (a).

From Lemma 11 we know that the origin is unstable and in particular that for the Poincaré map P introduced in this section we have $P(y) > y$ for $y > 0$ and sufficiently small.

The assumptions assure that $d_2/t_2^2 > d_1/t_1^2$, and using that the function $1/\sqrt{4x-1}$ is decreasing for $x > 1/4$ we see that $\kappa_2 - \kappa_1 < 0$. Therefore from Lemma 10 we have

$$L = \lim_{y\to\infty} \frac{dP}{dy} = e^{\pi(\kappa_2-\kappa_1)} < 1.$$

We will now claim that there exists $y^* > 0$ with $P(y^*) < y^*$ so that from the intermediate value theorem we deduce the existence of a periodic orbit. Then the conclusion of the theorem follows from Theorem 3.

Effectively we can assure that there exists a certain value $\bar{y}$ such that for $y \geq \bar{y}$ we have

$$\frac{dP}{dy} < \frac{1+L}{2} = \bar{L} < 1.$$

If $P(\bar{y}) < \bar{y}$ we are done. Otherwise taking $y^* > \bar{y}$ and invoking the Mean Value Theorem we have

$$P(y^*) - P(\bar{y}) < \bar{L}(y^* - \bar{y}),$$

which implies that

$$P(y^*) - y^* < P(\bar{y}) - \bar{L}\bar{y} - (1 - \bar{L})y^*,$$

which is clearly negative if y^* is big enough and the claim is true. □

References

[1] W. A. Coppel, *Some Analytical Systems with at most one Limit Cycle*, Dynamics Reported, Vol. **2**, Edited by U. Kirchgraber & H.O.Walther, John Wiley Sons Ltd, 1989.

[2] S. Banerjee and G.C. Verghese, *Nonlinear Phenomena in Power Electronics*, IEEE Press, 2001.

[3] F. Dumortier and C. Li, *On the uniqueness of limit cycles surrounding one or more singularities for Liénard equations*, Nonlinearity **9** (1996), 1489–1500.

[4] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative theory of planar differential systems*, Universitext, Springer-Verlag, Berlin, 2006.

[5] F. Dumortier and C. Rousseau, *Cubic Liénard equations with linear damping*, Nonlinearity **3** (1990), 1015–1039.

[6] E. Freire, E. Ponce, F. Rodrigo and F. Torres, *Bifurcation sets of continuous piecewise linear systems with two zones*, Int. J. Bifurcation and Chaos **8** (1998), 2073–2097.

[7] D. Hilbert, *Mathematische Probleme*, Lecture, Second Internat. Congr. Math. (Paris, 1900), *Nachr. Ges. Wiss. G"ttingen Math. Phys. KL.* (1900), 253–297; English transl., *Bull. Amer. Math. Soc.* **8** (1902), 437–479.

[8] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations*, 3rd edition, Oxford University Press, Oxford UK, 1999.

[9] A.I. Khibnik, B. Krauskopf and C. Rousseau, *Global study of a family of cubic Liénard equations*, Nonlinearity **11** (1998), 1505–1519.

[10] Yu. A. Kuznetsov, S. Rinaldi and A. Gragnani, *One-Parameter Bifurcations in Planar Filippov Systems*, Int. J. Bifurcation and Chaos **13** (2003), 2157–2188.

[11] J. Llibre and E. Ponce, *Bifurcation of a periodic orbit from infinity in planar piecewise linear vector fields*, Nonlinear Analysis **36** (1999), 623–653.

[12] S. Smale, *Mathematical Problems for the Next Century*, Mathematical Intelligencer **20** (1998), 7–15.

[13] D. Xiao and Z. Zhang, *On the uniqueness and nonexistence of limit cycles for predator–prey systems*, Nonlinearity **16** (2003), 1185–1201.

[14] Zhang Zhifen, Ding Tongren, Huang Wenzao and Dong Zhenxi, *Qualitative Theory of Differential Equations*, Translations of Math. Monographs, Vol. **101**, Amer. Math. Soc, Providence, 1992.

[15] Z.T. Zhusubaliyev and E. Mosekilde, *Bifurcations and Chaos in Piecewise–Smooth Dynamical Systems*, World Scientific, Singapoore, 2003.

Jaume LLibre,
Departament de Matemàtiques, Universitat Autònoma de Barcelona,
08193 Bellaterra, Barcelona, Spain
E-mail:jllibre@mat.uab.cat

Enrique Ponce and Francisco Torres,
E.T.S. Ingenieros, Camino de los Descubrimientos, 41092 Sevilla, Spain,
E-mail:eponcem@us.es, ftorres@us.es