

# NORM INEQUALITIES IN MULTIDIMENSIONAL LORENTZ SPACES

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ABSTRACT. In this paper we obtain necessary and sufficient conditions for double trigonometric series to belong to generalized Lorentz spaces, not symmetric in general. Estimates for the norms are given in terms of coefficients.

## 1. INTRODUCTION

Let  $f(x, y)$  be a measurable function on  $[0, 2\pi]^2$ . We define the rearrangement of  $f(x, y)$  with respect to  $y$  to be  $f_y^*(x, t_2) = \left(f(x, y)\right)_y^*(x, t_2)$ , i.e.,  $f_y^*(x, t_2)$  is a nonincreasing function on  $t_2$  and the functions  $f_y^*(x, t_2)$  and  $|f(x, y)|$  are equimeasurable as functions of one variable for almost all  $x$ . We define the rearrangement of  $f_y^*(x, t_2)$  with respect to  $x$  to be  $f_{yx}^*(t_1, t_2)$ . Therefore  $f_{yx}^*(t_1, t_2)$  is nonincreasing on  $t_1$  and  $t_2$  and equimeasurable with  $|f(x, y)|$ .

According to [BKPS]-[BPS], if  $0 < \alpha < \infty$ , we say that a measurable function  $f(x, y)$ , which is  $2\pi$ -periodic on each variable, belongs to the two-dimensional weighted Lorentz space  $\Lambda_2^\alpha(\omega)$ , if

$$\|f\|_{\Lambda_2^\alpha(\omega)} = \left( \int_0^{2\pi} \int_0^{2\pi} w(t_1, t_2) \left( f_{yx}^*(t_1, t_2) \right)^\alpha dt_1 dt_2 \right)^{\frac{1}{\alpha}} < \infty,$$

where the weighted function  $w \in W$ , i.e.,  $w(t_1, t_2)$  is an a.e. positive measurable function on  $[0, 2\pi]^2$ .

Similarly, we say that  $f(x, y)$  belongs to  $\bar{\Lambda}_2^\alpha(\omega)$  provided that

$$\|f\|_{\bar{\Lambda}_2^\alpha(\omega)} = \left( \int_0^{2\pi} \int_0^{2\pi} w(t_1, t_2) \left( f_{xy}^*(t_1, t_2) \right)^\alpha dt_1 dt_2 \right)^{\frac{1}{\alpha}} < \infty.$$

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We consider the following series

$$(1.1) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} \cos mx \cos ny,$$

$$(1.2) \quad \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a_{mn} \cos mx \sin ny,$$

$$(1.3) \quad \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_{mn} \sin mx \cos ny,$$

$$(1.4) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} \sin mx \sin ny,$$

where the coefficients  $a_{mn}$  are real numbers and  $\cos 0x := \cos 0y := \frac{1}{2}$ . We also suppose that the sequence  $\{a_{mn}\}$  satisfies the condition

$$(1.5) \quad a_{mn} \rightarrow 0 \quad \text{as} \quad m + n \rightarrow \infty.$$

For integers  $k_1$  and  $k_2$  we define

$$\Delta_{k_1 k_2} a_{mn} = \sum_{i=0}^{k_1} (-1)^i C_{k_1}^i \sum_{j=0}^{k_2} (-1)^j C_{k_2}^j a_{m+i, n+j} \quad (m, n \geq 0),$$

where  $C_k^l = \frac{k(k-1)\dots(k-l+1)}{l!}$  for  $l \geq 1$  and  $C_k^l = 1$  for  $l = 0$ .

We note that if  $\{a_{mn}\}$  satisfies (1.5) and  $\Delta_{k_1 k_2} a_{mn} \geq 0$  for integers  $k_1, k_2 \geq 1$ , then  $\Delta_{s_1 s_2} a_{mn} \geq 0$  for all  $0 \leq s_1 \leq k_1$  and  $0 \leq s_2 \leq k_2$ .

By  $C, C_i$  we denote positive constants that may be different on different occasions. Also,  $F \asymp G$  means that there exist constants  $C_1$  and  $C_2$  such that  $C_1 F \leq G \leq C_2 F$ .

Now we recall the definition of convergence of double series in the Pringsheim's sense. The partial sums of the series  $\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} c_{\mu\nu}$  are defined to be

$S_{mn} = \sum_{\nu=0}^n \sum_{\mu=0}^m c_{\mu\nu}$ . If there exists a number  $S$  such that for any  $\varepsilon > 0$  there exist integers  $k$  and  $l$  such that  $|S_{mn} - S| < \varepsilon$  for all  $n > k$  and  $m > l$ , then series  $\sum_{\mu, \nu} c_{\mu\nu}$  is said to converge to  $S$ , in the sense of Pringsheim.

It is well-known (see e.g. [H]) that if a sequence  $\{a_{mn}\}$  satisfies (1.5) and  $\Delta_{k_1 k_2} a_{mn} \geq 0$  for any integers  $k_1$  and  $k_2$ , then series (1.1)–(1.4) converge in Pringsheim's sense except, possibly, on a set of measure zero. We define by  $f_1(x, y)$ ,  $f_2(x, y)$ ,  $f_3(x, y)$  and  $f_4(x, y)$  the sums of series (1.1)–(1.4), respectively.

The aim of the paper is to study the necessary and sufficient conditions for the functions  $f_1(x, y)$ ,  $f_2(x, y)$ ,  $f_3(x, y)$  and  $f_4(x, y)$  to belong to the Lorentz spaces, not symmetric in general, in terms of coefficients  $\{a_{mn}\}$ .

This problem has a long history starting from the well-known theorem by Hardy and Littlewood (see [HL], [Z, V.2, XII, §6]): *A necessary and sufficient condition that the function*

$$f(x) \sim \sum_n (a_n \cos nx + b_n \sin nx), \quad a_n, b_n \downarrow,$$

*should belong to  $L^p$ ,  $1 < p < \infty$ , is that*

$$\sum_n (a_n^p + b_n^p) n^{p-2} < \infty.$$

This result was generalized by Sagher for the Lorentz spaces (see [S1] and [S2]): *Assume that  $f(x)$  is either the Fourier sine or Fourier cosine series associated with  $\{a_n\}$ . If  $a_n \downarrow$ , then for  $1 < p < \infty$  and  $0 < q \leq \infty$ ,*

$$\|f(x)\|_{L^{p,q}} \asymp \|\{a_n\}\|_{l^{p',q}}$$

*where  $L^{p,q}$  and  $l^{p',q}$  are continuous and discrete Lorentz spaces [BeSh].*

For non-weighted multidimensional  $L^p$ -spaces, where  $1 < p < \infty$ , the Hardy-Littlewood-type result was obtained by Moricz [Mo]. The cases of different Lebesgue and Lorentz spaces were also investigated in [BoSa], [G], [Mi], [N], [VD], among others.

In section 2 we present our main results for the spaces  $\Lambda_2^\alpha(\omega)$  and  $\bar{\Lambda}_2^\alpha(\omega)$  in the case of  $\alpha \in (0, \infty)$ . First (Theorem 2.1), we find the estimate of  $\|f_i\|_{\Lambda_2^\alpha(\omega)}$  from above. Next (Theorems 2.2 and 2.3), we give the necessary conditions for  $f_i$  to belong to  $\Lambda_2^\alpha(\omega)$ , depending on the monotonicity properties of the weight  $\omega$ . Sections 3 and 4 contain the preliminary results and proofs, respectively.

Finally, we remark that our results, in particular, provide criteria for  $f_1, f_2, f_3$  and  $f_4$  to belong to classical Lorentz space  $L^{p,q}$ ,  $0 < p, q < \infty$ , and to the Lorentz-Zygmund space  $L^{p,q}\varphi(L)$  (here  $\varphi$  is a slowly varying function; see [BeSh],[Mi]).

## 2. MAIN RESULTS

For convenience purposes we set

$$w_{mn} := \int_{\pi/(n+1)}^{\pi/\bar{n}} \int_{\pi/(m+1)}^{\pi/\bar{m}} w(x, y) dx dy,$$

where  $\bar{n} := \begin{cases} \frac{1}{2} & \text{if } n = 0, \\ n & \text{if } n \in \mathbf{N}. \end{cases}$  Then our results are read as follows.

**Theorem 2.1.** *Let  $0 < \alpha < \infty$ ,  $w \in W$ , and let  $w$  satisfy the following condition: for all  $\delta_1, \delta_2 \in (0, 2\pi)$*

$$(2.1) \quad \int_0^{\delta_1} \int_0^{\delta_2} w(t_1, t_2) dt_1 dt_2 \leq C \int_{\frac{\delta_1}{2}}^{\delta_1} \int_{\frac{\delta_2}{2}}^{\delta_2} w(t_1, t_2) dt_1 dt_2,$$

where  $C$  is independent of  $\delta_1, \delta_2$ . Suppose the sequence  $\{a_{mn}\}$  satisfies condition (1.5) and  $\Delta_{k_1 k_2} a_{mn} \geq 0$  for all  $m, n \in \mathbf{N}$  and appropriate  $k_1, k_2$  appearing in (A)-(D).

(A). If  $k_1 = 2$  and  $k_2 = 2$ , then

(2.2)

$$\|f_1(x, y)\|_{\Lambda_2^\alpha(\omega)} \leq C_1 \left( \sum_{n=0}^{\infty} (n+1)^{2\alpha} \sum_{m=0}^{\infty} (m+1)^{2\alpha} w_{mn} (\Delta_{11} a_{mn})^\alpha \right)^{\frac{1}{\alpha}};$$

(B). If  $k_1 = 2$  and  $k_2 = 1$ , then

(2.3)

$$\|f_2(x, y)\|_{\Lambda_2^\alpha(\omega)} \leq C_2 \left( \sum_{n=1}^{\infty} (n+1)^\alpha \sum_{m=0}^{\infty} (m+1)^{2\alpha} w_{mn} (\Delta_{10} a_{mn})^\alpha \right)^{\frac{1}{\alpha}};$$

(C). If  $k_1 = 1$  and  $k_2 = 2$ , then

(2.4)

$$\|f_3(x, y)\|_{\Lambda_2^\alpha(\omega)} \leq C_3 \left( \sum_{n=0}^{\infty} (n+1)^{2\alpha} \sum_{m=1}^{\infty} (m+1)^\alpha w_{mn} (\Delta_{01} a_{mn})^\alpha \right)^{\frac{1}{\alpha}};$$

(D). If  $k_1 = 1$  and  $k_2 = 1$ , then

(2.5)

$$\|f_4(x, y)\|_{\Lambda_2^\alpha(\omega)} \leq C_4 \left( \sum_{n=1}^{\infty} (n+1)^\alpha \sum_{m=1}^{\infty} (m+1)^\alpha w_{mn} a_{mn}^\alpha \right)^{\frac{1}{\alpha}};$$

where  $C_1, C_2, C_3, C_4$  are independent of  $\{a_{mn}\}$ .

**Remark 2.1.** For any  $\gamma_1, \gamma_2 > 0$  and for any slowly varying functions  $\varphi_1(x), \varphi_2(y)$  on  $(0, \infty)$ ,  $w(t_1, t_2) = t_1^{\gamma_1-1} \varphi_1(1/t_1) t_2^{\gamma_2-1} \varphi_2(1/t_2)$  satisfies (2.1).

**Theorem 2.2.** *Let  $0 < \alpha < \infty$ ,  $w \in W$ , and let  $w$  be nonincreasing with respect to  $t_1$  for almost all  $t_2$  and nonincreasing with respect to  $t_2$  for almost all  $t_1$ . Suppose the sequence  $\{a_{mn}\}$  satisfies condition (1.5) and  $\Delta_{k_1 k_2} a_{mn} \geq 0$  for all  $m, n \in \mathbf{N}$  and appropriate  $k_1, k_2$  appearing in (A)-(D).*

(A). *If  $k_1 = 2$  and  $k_2 = 2$ , then*

(2.6)

$$\|f_1(x, y)\|_{\Lambda_2^\alpha(\omega)} \geq C_1 \left( \sum_{n=0}^{\infty} (n+1)^{2\alpha} \sum_{m=0}^{\infty} (m+1)^{2\alpha} w_{mn} (\Delta_{11} a_{mn})^\alpha \right)^{\frac{1}{\alpha}};$$

(B). *If  $k_1 = 2$  and  $k_2 = 1$ , then*

(2.7)

$$\|f_2(x, y)\|_{\Lambda_2^\alpha(\omega)} \geq C_2 \left( \sum_{n=1}^{\infty} (n+1)^\alpha \sum_{m=0}^{\infty} (m+1)^{2\alpha} w_{mn} (\Delta_{10} a_{mn})^\alpha \right)^{\frac{1}{\alpha}};$$

(C). *If  $k_1 = 1$  and  $k_2 = 2$ , then*

(2.8)

$$\|f_3(x, y)\|_{\Lambda_2^\alpha(\omega)} \geq C_3 \left( \sum_{n=0}^{\infty} (n+1)^{2\alpha} \sum_{m=1}^{\infty} (m+1)^\alpha w_{mn} (\Delta_{01} a_{mn})^\alpha \right)^{\frac{1}{\alpha}};$$

(D). *If  $k_1 = 1$  and  $k_2 = 1$ , then*

(2.9)

$$\|f_4(x, y)\|_{\Lambda_2^\alpha(\omega)} \geq C_4 \left( \sum_{n=1}^{\infty} (n+1)^\alpha \sum_{m=1}^{\infty} (m+1)^\alpha w_{mn} a_{mn}^\alpha \right)^{\frac{1}{\alpha}};$$

where  $C_1, C_2, C_3, C_4$  are independent of  $\{a_{mn}\}$ .

**Theorem 2.3.** *Let  $1 \leq \alpha < \infty$ ,  $w \in W$ , and let  $w$  satisfy the following two conditions:*

- (1): *for almost all  $t_2 \in [0, 2\pi]$   $w(t_1, t_2)$  is nondecreasing and satisfies  $\Delta_2$ -condition with respect to  $t_1$ , i.e.,  $w(2t_1, t_2) \leq Cw(t_1, t_2)$ ;*
- (2): *for almost all  $t_1 \in [0, 2\pi]$ ,  $w(t_1, t_2)$  is nondecreasing and satisfies  $\Delta_2$ -condition with respect to  $t_2$ , i.e.,  $w(t_1, 2t_2) \leq Cw(t_1, t_2)$ .*

*Suppose the sequence  $\{a_{mn}\}$  satisfies condition (1.5) and  $\Delta_{k_1 k_2} a_{mn} \geq 0$  for all  $m, n \in \mathbf{N}$  and appropriate  $k_1, k_2$  appearing in (A)-(D).*

(A). *If  $k_1 = 2$  and  $k_2 = 2$ , then inequality (2.6) is satisfied;*

(B). *If  $k_1 = 2$  and  $k_2 = 1$ , then inequality (2.7) is satisfied;*

(C). *If  $k_1 = 1$  and  $k_2 = 2$ , then inequality (2.8) is satisfied;*

(D). If  $k_1 = 1$  and  $k_2 = 1$ , then inequality (2.9) is satisfied; where  $C_1, C_2, C_3, C_4$  are independent of  $\{a_{mn}\}$ .

**Corollary 2.1.** *Under the conditions of Theorem 2.1 and Theorem 2.2 (or Theorem 2.3), we have*

$$(2.10) \quad \|f_j(x, y)\|_{\Lambda_2^p(\omega)} \asymp \|f_j(x, y)\|_{\bar{\Lambda}_2^p(\omega)}, \quad j = 1, 2, 3, 4.$$

We note that, in general, (2.10) does not hold (see [BPS] and [B]). We also remark that in [BPS, §3] the authors showed that if  $\|\cdot\|_{\Lambda_2^p(\omega)}$  is a rearrangement invariant norm (see [BeSh, Ch.2]), then  $\omega$  is a constant and  $\Lambda_2^p(\omega) \equiv L^p([0, 2\pi]^2)$ . The following corollary follows from Theorems 2.1-2.3 (see also [VD]).

**Corollary 2.2.** *Let  $0 < p < \infty$ . Suppose the sequence  $\{a_{mn}\}$  satisfies condition (1.5) and  $\Delta_{k_1 k_2} a_{mn} \geq 0$  for all  $m, n \in \mathbf{N}$  and appropriate  $k_1, k_2$  appearing in (A)-(D).*

(A). *If  $k_1 = 2$  and  $k_2 = 2$ , then*

$$\|f_1(x, y)\|_{L^p([0, 2\pi]^2)} \asymp \left( \sum_{n=0}^{\infty} (n+1)^{2p-2} \sum_{m=0}^{\infty} (m+1)^{2p-2} (\Delta_{11} a_{mn})^p \right)^{\frac{1}{p}};$$

(B). *If  $k_1 = 2$  and  $k_2 = 1$ , then*

$$\|f_2(x, y)\|_{L^p([0, 2\pi]^2)} \asymp \left( \sum_{n=1}^{\infty} (n+1)^{p-2} \sum_{m=0}^{\infty} (m+1)^{2p-2} (\Delta_{10} a_{mn})^p \right)^{\frac{1}{p}};$$

(C). *If  $k_1 = 1$  and  $k_2 = 2$ , then*

$$\|f_3(x, y)\|_{L^p([0, 2\pi]^2)} \asymp \left( \sum_{n=0}^{\infty} (n+1)^{2p-2} \sum_{m=1}^{\infty} (m+1)^{p-2} (\Delta_{01} a_{mn})^p \right)^{\frac{1}{p}};$$

(D). *If  $k_1 = 1$  and  $k_2 = 1$ , then*

$$\|f_4(x, y)\|_{L^p([0, 2\pi]^2)} \asymp \left( \sum_{n=1}^{\infty} (n+1)^{p-2} \sum_{m=1}^{\infty} (m+1)^{p-2} a_{mn}^p \right)^{\frac{1}{p}}.$$

This result is the two-dimensional version of the classical Hardy-Littlewood theorem. Note also that if  $1 < p < \infty$  and  $k_1 = k_2 = 1$ , then (compare with [Mo])

$$\|f_j(x, y)\|_{L^p([0, 2\pi]^2)} \asymp \left( \sum_{n=1}^{\infty} (n+1)^{p-2} \sum_{m=1}^{\infty} (m+1)^{p-2} a_{mn}^p \right)^{\frac{1}{p}}, \quad j = 1, 2, 3, 4.$$

## 3. LEMMAS

Let

$$\begin{aligned} B_0^1(x) &= \frac{1}{2}; \quad B_n^1(x) = \frac{1}{2} + \cos x + \cdots + \cos nx \quad \text{for } n \geq 1; \\ B_n^k(x) &= \sum_{m=0}^n B_m^{k-1}(x) \quad \text{for } k = 2, 3, \dots \quad \text{and } n = 0, 1, 2, \dots; \\ \overline{B}_n^1(x) &= \sin x + \cdots + \sin nx \quad \text{for } n = 1, 2, \dots; \\ \overline{B}_n^k(x) &= \sum_{m=1}^n \overline{B}_m^{k-1}(x) \quad \text{for } k = 2, 3, \dots \quad \text{and } n = 1, 2, \dots. \end{aligned}$$

We consider series

$$(3.1) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{k_1 k_2} a_{mn} B_m^{k_1}(x) B_n^{k_2}(y);$$

$$(3.2) \quad \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \Delta_{k_1 k_2} a_{mn} B_m^{k_1}(x) \overline{B}_n^{k_2}(y);$$

$$(3.3) \quad \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \Delta_{k_1 k_2} a_{mn} \overline{B}_m^{k_1}(x) B_n^{k_2}(y);$$

$$(3.4) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Delta_{k_1 k_2} a_{mn} \overline{B}_m^{k_1}(x) \overline{B}_n^{k_2}(y).$$

**Lemma 3.1.** *Suppose the sequence  $\{a_{mn}\}$  satisfies condition (1.5) and  $\Delta_{k_1 k_2} a_{mn} \geq 0$  for integers  $k_1$  and  $k_2$ . Then any series (3.1)-(3.4) converges in Pringsheim's sense except possibly the set of measure zero to  $f_1(x, y)$ ,  $f_2(x, y)$ ,  $f_3(x, y)$  and  $f_4(x, y)$ , respectively.*

The proof follows easily from the Abel transform (see also [VD]).

**Lemma 3.2.** ([L]) *Let  $a_n > 0, b_n \geq 0$  ( $n = 1, 2, \dots$ );  $1 \leq p < \infty$ . Then*

$$(A). \quad \sum_{k=1}^{\infty} a_k \left( \sum_{m=1}^k b_m \right)^p \leq p^p \sum_{m=1}^{\infty} a_m^{1-p} \left( b_m \sum_{n=m}^{\infty} a_n \right)^p;$$

$$(B). \quad \sum_{k=1}^{\infty} a_k \left( \sum_{m=k}^{\infty} b_m \right)^p \leq p^p \sum_{m=1}^{\infty} a_m^{1-p} \left( b_m \sum_{n=1}^m a_n \right)^p.$$

**Lemma 3.3.** ([KPS, Ch. II, §2]) *Let  $f$  and  $g$  be measurable functions on  $[0, 2\pi]$ . Then*

$$\begin{aligned} (f+g)^*(t_1+t_2) &\leq f^*(t_1) + g^*(t_2), \\ (fg)^*(t_1+t_2) &\leq f^*(t_1)g^*(t_2). \end{aligned}$$

**Lemma 3.4.** ([KPS, Ch. II, §2]) *Let  $f$  and  $g$  be measurable functions on  $[0, 2\pi]$ . Suppose  $|f(t)| \leq |g(t)|$ ; then  $f_1^*(t) \leq f_2^*(t)$ .*

**Lemma 3.5.** ([BeSh, Ch. 2, §2]) *Let  $f$  be a measurable function on  $[0, 2\pi]$ . Then for all  $0 < \alpha < \infty$*

$$(f^*(t))^\alpha = (|f|^\alpha)^*(t).$$

**Lemma 3.6.** ([KPS]) *Let  $h(t)$  be an increasing non-negative function. Then*

$$\int_0^\infty (f+g)^*(t)h(t) dt \geq \int_0^\infty f^*(t)h(t) dt + \int_0^\infty g^*(t)h(t) dt.$$

**Lemma 3.7.** ([B]) (A). *Suppose  $|f(x, y)| \leq |g(x, y)|$ ; then  $f_{yx}^*(t_1, t_2) \leq g_{yx}^*(t_1, t_2)$ ;*

$$(B). \quad (f+g)_{yx}^*(s_1+s_2, t_1+t_2) \leq f_{yx}^*(s_1, t_1) + g_{yx}^*(s_2, t_2).$$

**Lemma 3.8.** ([KPS, Ch. II, §2]) *Let  $f(t)$  and  $g(t)$  be non-negative summable functions on  $[0, 2\pi]$  such that*

$$\int_0^x f(t) dt \leq \int_0^x g(t) dt$$

*Suppose  $h(t)$  is nonincreasing non-negative function on  $(0, 2\pi)$ ; then*

$$\int_0^{2\pi} f(t)h(t) dt \leq \int_0^{2\pi} g(t)h(t) dt.$$

**Lemma 3.9.** ([B]) *Let  $f^*(t_1, t_2) = f_{yx}^*(t_1, t_2)$ ,  $\sigma > 0$  and let*

$$\begin{aligned} \lambda_f(\sigma) &= \mu\left\{(x, y) \in \Omega_1 \times \Omega_2 : |f(x, y)| > \sigma\right\}, \\ \lambda_{f^*}(\sigma) &= \mu\left\{(s, t) \in [0, \infty) \times [0, \infty) : f^*(s, t) > \sigma\right\}. \end{aligned}$$

*Then  $\lambda_f(\sigma) = \lambda_{f^*}(\sigma)$ .*

**Lemma 3.10.** *Let a measurable set  $E \subset [0, a] \times [0, b]$ , where  $0 < a, b < \infty$ . Suppose for some  $k \in (0, 1)$ ,*

$$\mu E > k\mu([0, a] \times [0, b]) = kab.$$

*Then*

$$\begin{aligned} U &:= \left\{(s, t) \in \left[0, \frac{ka}{2}\right] \times \left[0, \frac{kb}{2}\right]\right\} \\ &\subset \left\{(s, t) \in [0, \infty) \times [0, \infty) : \left(\chi_E(x, y)\right)_{yx}^*(s, t) > \frac{1}{2}\right\} =: T. \end{aligned}$$



*Proof.* From the nonincrease of  $\left(\chi_E(x, y)\right)_{yx}^*(s, t)$  it is clear that the boundary of  $\left\{(s, t) : \left(\chi_E(x, y)\right)_{yx}^*(s, t) > \frac{1}{2}\right\}$  is the nondecreasing function  $t = d(s)$ . Let us assume that  $U \subseteq T$ . Therefore,  $\mu(T) \leq \frac{kab}{2} + \frac{kab}{2} - \frac{k^2ab}{4} < kab$ . By Lemma 3.9, we have  $\mu E = \mu T$  and hence  $\mu E < kab$  which contradicts our assumption. Thus,  $U \subset T$ , which completes the proof.

#### 4. PROOFS

*Proof of Theorem 2.1.* By Lemma 3.1, the following identity holds a.e.

$$F(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{l_1 l_2} a_{mn} \tilde{B}_m^{l_1}(x) \tilde{B}_n^{l_2}(y), \quad \text{where in the case of}$$

(A):  $F(x, y) = f_1(x, y)$ ,  $l_1 = l_2 = 2$ ,  $\tilde{B}_m^{l_1}(x) = B_m^2(x)$ ,  $\tilde{B}_n^{l_2}(y) = B_n^2(y)$ ;  
 (B):  $F(x, y) = f_2(x, y)$ ,  $l_1 = 2, l_2 = 1$ ,  $\tilde{B}_m^{l_1}(x) = B_m^2(x)$ ,  $\tilde{B}_n^{l_2}(y) = \bar{B}_n^1(y)$ ,  
 and  $\Delta_{21} a_{m0} = \Delta_{10} a_{m0} = 0$ ,  $m = 0, 1, 2, \dots$ ;  
 (C):  $F(x, y) = f_3(x, y)$ ,  $l_1 = 1, l_2 = 2$ ,  $\tilde{B}_m^{l_1}(x) = \bar{B}_m^1(x)$ ,  $\tilde{B}_n^{l_2}(y) = B_n^2(y)$ ,  
 and  $\Delta_{12} a_{0n} = \Delta_{01} a_{0n} = 0$ ,  $n = 0, 1, 2, \dots$ ;  
 (D):  $F(x, y) = f_4(x, y)$ ,  $l_1 = l_2 = 1$ ,  $\tilde{B}_m^{l_1}(x) = \bar{B}_m^1(x)$ ,  $\tilde{B}_n^{l_2}(y) = \bar{B}_n^1(y)$ ,  
 and  $\Delta_{11} a_{m0} = a_{m0} = 0$ ,  $m = 0, 1, 2, \dots$ ,  $\Delta_{11} a_{0n} = a_{0n} = 0$ ,  $n = 0, 1, 2, \dots$ .

We estimate  $\|F(x, y)\|_{\Lambda_2^p(\omega)}$  from above. Using Lemmas 3.3 and 3.4, we obtain

$$\begin{aligned} & \|F(x, y)\|_{\Lambda_2^p(\omega)}^\alpha = \\ &= \int_0^{2\pi} \int_0^{2\pi} \omega(t_1, t_2) \left( F(x, y) \left[ \chi_{(0, \pi]}(x) \chi_{(0, \pi]}(y) + \chi_{(0, \pi]}(x) \chi_{(\pi, 2\pi]}(y) \right. \right. \\ &+ \left. \left. \chi_{(\pi, 2\pi]}(x) \chi_{(0, \pi]}(y) + \chi_{(\pi, 2\pi]}(x) \chi_{(\pi, 2\pi]}(y) \right]_{yx}^*(t_1, t_2) \right)^\alpha dt_1 dt_2 \\ &\leq C \int_0^{\pi/2} \int_0^{\pi/2} \omega(4t_1, 4t_2) \left( \left[ F(x, y) \chi_{(0, \pi]}(x) \chi_{(0, \pi]}(y) \right]_{yx}^*(t_1, t_2) \right)^\alpha dt_1 dt_2 \\ &\leq C (I_1 + I_2 + I_3 + I_4), \end{aligned}$$

where

$$I_1 = \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2) \\ \left( \left[ \sum_{\nu=0}^{2^{n+1}} \sum_{\mu=0}^{2^{m+1}} \Delta_{l_1 l_2} a_{\mu\nu} \tilde{B}_{\mu}^{l_1}(x) \tilde{B}_{\nu}^{l_2}(y) \chi_{(0,\pi]}(x) \chi_{(0,\pi]}(y) \right]_{yx}^* (t_1, t_2) \right)^{\alpha} dt_1 dt_2;$$

$$I_2 = \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2) \\ \left( \left[ \sum_{\nu=2^{n+1}+1}^{\infty} \sum_{\mu=0}^{2^{m+1}} \Delta_{l_1 l_2} a_{\mu\nu} \tilde{B}_{\mu}^{l_1}(x) \tilde{B}_{\nu}^{l_2}(y) \chi_{(0,\pi]}(x) \chi_{(0,\pi]}(y) \right]_{yx}^* (t_1, t_2) \right)^{\alpha} dt_1 dt_2;$$

$$I_3 = \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2) \\ \left( \left[ \sum_{\nu=0}^{2^{n+1}} \sum_{\mu=2^{m+1}+1}^{\infty} \Delta_{l_1 l_2} a_{\mu\nu} \tilde{B}_{\mu}^{l_1}(x) \tilde{B}_{\nu}^{l_2}(y) \chi_{(0,\pi]}(x) \chi_{(0,\pi]}(y) \right]_{yx}^* (t_1, t_2) \right)^{\alpha} dt_1 dt_2;$$

$$I_4 = \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2) \\ \left( \left[ \sum_{\nu=2^{n+1}+1}^{\infty} \sum_{\mu=2^{m+1}+1}^{\infty} \Delta_{l_1 l_2} a_{\mu\nu} \tilde{B}_{\mu}^{l_1}(x) \tilde{B}_{\nu}^{l_2}(y) \chi_{(0,\pi]}(x) \chi_{(0,\pi]}(y) \right]_{yx}^* (t_1, t_2) \right)^{\alpha} dt_1 dt_2.$$

First we estimate  $I_1$ . Since  $|\tilde{B}_\mu^{l_1}(x)| \leq C_1(\mu+1)^{l_1}$ ,  $|\tilde{B}_\nu^{l_2}(y)| \leq C_2(\nu+1)^{l_2}$ , where  $C_1, C_2$  are independent of  $x, y$  and  $\mu, \nu$ , we have

$$I_1 \leq \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2) \cdot \left( \sum_{\nu=0}^{2^{n+1}} \sum_{\mu=0}^{2^{m+1}} \Delta_{l_1 l_2} a_{\mu\nu} (\mu+1)^{l_1} (\nu+1)^{l_2} \right)^\alpha dt_1 dt_2.$$

Since  $\Delta_{l_1-1 l_2} a_{mn} \geq 0$ ,  $\Delta_{l_1-1 l_2-1} a_{mn} \geq 0$  and  $\Delta_{l_1 l_2} a_{mn} = \Delta_{l_1-1 l_2} a_{mn} - \Delta_{l_1-1 l_2} a_{m+1 n}$ ,  $\Delta_{l_1-1 l_2} a_{mn} = \Delta_{l_1-1 l_2-1} a_{mn} - \Delta_{l_1-1 l_2-1} a_{m n+1}$ , then

(4.1)

$$\begin{aligned} \sum_{n=0}^N \sum_{m=0}^M \Delta_{l_1 l_2} a_{mn} (m+1)^{l_1} (n+1)^{l_2} &\leq \\ &\leq C \sum_{n=0}^N \sum_{m=0}^M \Delta_{l_1-1 l_2-1} a_{mn} (m+1)^{l_1-1} (n+1)^{l_2-1}, \end{aligned}$$

where  $C = C(l_1, l_2)$ . Using this estimate, we get

$$I_1 \leq C \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2) dt_1 dt_2 \cdot \left( \sum_{\nu=0}^n \sum_{\mu=0}^m \Delta_{l_1-1 l_2-1} a_{2^\mu-1, 2^\nu-1} 2^{\mu l_1} 2^{\nu l_2} \right)^\alpha.$$

Now if  $\alpha \geq 1$ , we use Lemma 3.2 and if  $\alpha < 1$ , we apply Jensen's inequality. Then using (2.1), we write

$$\begin{aligned} I_1 &\leq \\ &\leq C \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2) dt_1 dt_2 \left( \Delta_{l_1-1 l_2-1} a_{2^m-1, 2^n-1} 2^{m l_1} 2^{n l_2} \right)^\alpha \\ &\leq C \sum_{n=0}^{\infty} (n+1)^{2\alpha l_1} \sum_{m=0}^{\infty} (m+1)^{2\alpha l_2} w_{mn} (\Delta_{l_1-1 l_2-1} a_{mn})^\alpha =: CB(\alpha, l_1, l_2). \end{aligned}$$

Now we estimate  $I_2$ . We note that for  $y \in (0, \pi)$ ,  $|\tilde{B}_n^{l_2}(y)| \leq C y^{-l_2}$  ( $l_2 = 1, 2$ ), where  $C$  is independent of  $n$  and  $y$ . Then by Lemma

3.4, we have

$$\begin{aligned}
I_2 &\leq \\
&\leq C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2) \\
&\times \left( \left[ \sum_{\nu=2^{n+1}+1}^{\infty} \sum_{\mu=0}^{2^{m+1}} \Delta_{l_1 l_2} a_{\mu\nu} (\mu+1)^{l_1} y^{-l_2} \chi_{(0,\pi]}(x) \chi_{(0,\pi]}(y) \right]_{yx}^* (t_1, t_2) \right)^{\alpha} dt_1 dt_2 \\
&\leq C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2) dt_1 dt_2 \cdot \\
&\quad \cdot \left( 2^{nl_2} \sum_{\nu=2^{n+1}+1}^{\infty} \sum_{\mu=0}^{2^{m+1}} \Delta_{l_1 l_2} a_{\mu\nu} (\mu+1)^{l_1} \right)^{\alpha} \\
&\leq C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2) dt_1 dt_2 \cdot \\
&\quad \cdot \left( 2^{nl_2} \sum_{\mu=0}^{2^{m+1}} 2^{\mu l_1} \Delta_{l_1-1, l_2-1} a_{\mu, 2^{n+1}+1} \right)^{\alpha}.
\end{aligned}$$

To verify the last inequality, we use  $\sum_{\nu=2^{n+1}+1}^{\infty} \Delta_{l_1 l_2} a_{\mu\nu} = \Delta_{l_1 l_2-1} a_{\mu, 2^{n+1}+1}$  and (4.1).

Finally, using Lemma 3.2 for  $\alpha \geq 1$ , and Jensen's inequality for  $\alpha < 1$ , we arrive at  $I_2 \leq CB(\alpha, l_1, l_2)$ .

In a similar way to the estimates of  $I_1$  and  $I_2$ , one can also obtain the inequality  $I_3 + I_4 \leq CB(\alpha, l_1, l_2)$ , i.e.,  $I \leq CB(\alpha, l_1, l_2)$ . Then this gives us inequality (2.2) for  $l_1 = l_2 = 2$ , inequality (2.3) for  $l_1 = 2, l_2 = 1$ , inequality (2.4) for  $l_1 = 1, l_2 = 2$ , and inequality (2.5) for  $l_1 = l_2 = 1$ . QED

*Proof of Theorem 2.2.* By Lemma 3.1, the following identities hold a.e.

$$\begin{aligned}
\text{(A).} \quad \varphi_1(x, y) = f_1(x, y) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} \cos mx \cos ny \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{22} b_{mn} \frac{\sin^2 \frac{(m+1)x}{2}}{4 \sin^2 \frac{x}{2}} \frac{\sin^2 \frac{(n+1)y}{2}}{4 \sin^2 \frac{y}{2}},
\end{aligned}$$

where  $b_{mn} = a_{mn}$ ,

(B).

$$\begin{aligned}\varphi_2(x, y) &= \frac{f_2(x, y) + f_2(x, \pi - y)}{2} \\ &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a_{m, 2n-1} \cos mx \sin(2n-1)y \\ &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \Delta_{21} b_{mn} \frac{\sin^2 \frac{(m+1)x}{2}}{4 \sin^2 \frac{x}{2}} \frac{\sin^2 ny}{\sin y},\end{aligned}$$

where  $b_{mn} = a_{m, 2n-1}$ ,

(C).

$$\begin{aligned}\varphi_3(x, y) &= \frac{f_3(x, y) + f_3(\pi - x, y)}{2} \\ &= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_{2m-1, n} \sin(2m-1)x \cos ny \\ &= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \Delta_{12} b_{mn} \frac{\sin^2 mx}{\sin x} \frac{\sin^2 \frac{(n+1)y}{2}}{4 \sin^2 \frac{y}{2}},\end{aligned}$$

where  $b_{mn} = a_{2m-1, n}$ ,

$$\begin{aligned}\text{(D). } \varphi_4(x, y) &= \frac{f_4(x, y) + f_4(\pi - x, y) + f_4(x, \pi - y) + f_4(\pi - x, \pi - y)}{2} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{2m-1, 2n-1} \sin(2m-1)x \sin(2n-1)y \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Delta_{11} b_{mn} \frac{\sin^2 mx}{\sin x} \frac{\sin^2 ny}{\sin y},\end{aligned}$$

where  $b_{mn} = a_{2m-1, 2n-1}$ .

Thus, we have a.e.

$$\Phi(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{l_1 l_2} b_{mn} \hat{B}_m^{l_1}(x) \hat{B}_n^{l_2}(y), \quad \text{where}$$

- (A):  $\Phi = \varphi_1, l_1 = 2, l_2 = 2, \hat{B}_m^{l_1}(x) = B_m^2(x), \hat{B}_n^{l_2}(y) = B_n^2(y);$   
 (B):  $\Phi = \varphi_2, l_1 = 2, l_2 = 1, \hat{B}_m^{l_1}(x) = B_m^2(x), \hat{B}_n^{l_2}(y) = \frac{\sin^2 ny}{\sin y};$   
 and  $\Delta_{21} b_{m0} = \Delta_{10} b_{m0} = b_{m0} = 0, \quad m = 0, 1, 2, \dots;$   
 (C):  $\Phi = \varphi_3, l_1 = 1, l_2 = 2, \hat{B}_m^{l_1}(x) = \frac{\sin^2 mx}{\sin x}, \hat{B}_n^{l_2}(y) = B_n^2(y);$   
 and  $\Delta_{12} b_{0n} = \Delta_{01} b_{0n} = b_{0n} = 0, \quad n = 0, 1, 2, \dots;$

$$\begin{aligned}
(\mathbf{D}): \quad & \Phi = \varphi_4, \quad l_1 = 1, l_2 = 1, \quad \widehat{B}_m^{l_1}(x) = \frac{\sin^2 mx}{\sin x}, \quad \widehat{B}_n^{l_2}(y) = \\
& \frac{\sin^2 ny}{\sin y} \\
& \text{and } \Delta_{11}b_{m0} = b_{m0} = 0, \quad m = 0, 1, 2, \dots, \quad \Delta_{11}b_{0n} = b_{0n} = 0, \quad n = \\
& 0, 1, 2, \dots.
\end{aligned}$$

Using Lemmas 3.5 and 3.8, we obtain

$$\int_0^{2\pi} \int_0^{2\pi} w(t_1, t_2) \left( F_{yx}^*(t_1, t_2) \right)^\alpha dt_1 dt_2 \geq \int_0^{2\pi} \int_0^{2\pi} w(x, y) \left| F(x, y) \right|^\alpha dx dy$$

because of

$$\left( F_{yx}^*(t_1, t_2) \right)^\alpha = \left( \left( \left( F(x, y) \right)_y^* \right)_x^\alpha \right)^*(t_1, t_2) = \left( \left| F(x, y) \right|^\alpha \right)_{yx}^*(t_1, t_2).$$

We also note the following:

$$\begin{aligned}
& \int_0^\pi \int_0^\pi w\left(\frac{t_1}{2}, \frac{t_2}{2}\right) w\left(\frac{t_1}{2}, \frac{t_2}{2}\right) \left[ \left( f(x, y) + f(\pi - x, y) \right)_{yx}^* \right]^\alpha(t_1, t_2) dt_1 dt_2 \\
& \leq C \left( \int_0^\pi \int_0^\pi w\left(\frac{t_1}{2}, \frac{t_2}{2}\right) \left( |f(x, y)|_{yx}^* \left(\frac{t_1}{2}, \frac{t_2}{2}\right) \right)^\alpha dt_1 dt_2 \right. \\
& \quad \left. + \int_0^\pi \int_0^\pi w\left(\frac{t_1}{2}, \frac{t_2}{2}\right) \left( |f(\pi - x, y)|_{yx}^* \left(\frac{t_1}{2}, \frac{t_2}{2}\right) \right)^\alpha dt_1 dt_2 \right) \\
& \leq C \int_0^{2\pi} \int_0^{2\pi} w(t_1, t_2) \left( f_{yx}^*(t_1, t_2) \right)^\alpha dt_1 dt_2.
\end{aligned}$$

Using this, let us estimate

$$I := \int_0^\pi \int_0^\pi w\left(\frac{x}{2^k}, \frac{y}{2^k}\right) \left| \Phi(x, y) \right|^\alpha dx dy,$$

$$\text{where } k := \begin{cases} 0 & \text{if } \Phi = \varphi_1, \\ 1 & \text{if } \Phi = \varphi_2 \text{ (or } \varphi_4), \\ 2 & \text{if } \Phi = \varphi_3. \end{cases}$$

We have

$$\begin{aligned} I &= \sum_{s_2=0}^{\infty} \int_{\pi/2^{s_2}}^{\pi/2^{s_2+1}} \sum_{s_1=0}^{\infty} \int_{\pi/2^{s_1}}^{\pi/2^{s_1+1}} w\left(\frac{x}{2^k}, \frac{y}{2^k}\right) |\Phi(x, y)|^\alpha dx dy, \\ &\geq C \sum_{s_2=0}^{\infty} \sum_{s_1=0}^{\infty} w\left(\frac{\pi}{2^{k+s_1}}, \frac{y}{2^{k+s_2}}\right) \int_{\pi/2^{s_2}}^{\pi/2^{s_2+1}} \int_{\pi/2^{s_1}}^{\pi/2^{s_1+1}} |\Phi(x, y)|^\alpha dx dy \\ &=: C \sum_{s_2=0}^{\infty} \sum_{s_1=0}^{\infty} w\left(\frac{\pi}{2^{k+s_1}}, \frac{y}{2^{k+s_2}}\right) A_{s_1 s_2} \\ &\geq C \sum_{s_2=j}^{\infty} \sum_{s_1=i}^{\infty} w\left(\frac{\pi}{2^{k+s_1}}, \frac{\pi}{2^{k+s_2}}\right) A_{s_1 s_2}, \end{aligned}$$

where  $i = j = 0$  for  $\varphi_1$ ,  $i = 0, j = 1$  for  $\varphi_2$ ,  $i = 1, j = 0$  for  $\varphi_3$ , and  $i = j = 1$  for  $\varphi_4$ .

Using the following estimate (see [VD])

$$A_{s_1 s_2} \geq C 2^{s_1(l_1\alpha-1)+s_2(l_2\alpha-1)} \left( \Delta_{l_1-1 l_2-1} b_{[2^{s_1-1}][2^{s_2-1}]} \right)^\alpha,$$

we have

$$I \geq C \sum_{s_2=j}^{\infty} \sum_{s_1=i}^{\infty} w\left(\frac{\pi}{2^{k+s_1}}, \frac{y}{2^{k+s_2}}\right) 2^{s_1(l_1\alpha-1)+s_2(l_2\alpha-1)} \left( \Delta_{l_1-1 l_2-1} b_{[2^{s_1-1}][2^{s_2-1}]} \right)^\alpha.$$

Thus, we get for  $i = j = 0$

$$\|f_1(x, y)\|_{\Lambda_2^s(\omega)} \geq C_1 \left( \sum_{n=0}^{\infty} (n+1)^{2\alpha} \sum_{m=0}^{\infty} (m+1)^{2\alpha} w_{mn} (\Delta_{11} a_{mn})^\alpha \right)^{\frac{1}{\alpha}}.$$

In a similar way, one can also obtain inequalities (2.7)–(2.9). QED

*Proof of Theorem 2.3.* We only consider in detail the case (A). By Lemma 3.1, one has

$$f_1(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{22} a_{mn} B_m^2(x) B_n^2(y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{11} b_{mn} B_m^2(x) B_n^2(y),$$

$$\text{where } b_{mn} = \Delta_{11} a_{mn}, \quad B_k^2(z) = \frac{\sin^2 \frac{(k+1)z}{2}}{4 \sin^2 \frac{z}{2}}.$$

Let  $x \in [\frac{\pi}{2^{m+1}}, \frac{\pi}{2^m}]$ ,  $y \in [\frac{\pi}{2^{n+1}}, \frac{\pi}{2^n}]$ ,  $I_{mn} = [\frac{\pi}{2^{m+1}}, \frac{\pi}{2^m}] \times [\frac{\pi}{2^{n+1}}, \frac{\pi}{2^n}]$ , and let

$$\Psi_{mn}(x, y) = \sum_{\nu=[2^{n-1}]}^{\infty} \sum_{\mu=[2^{m-1}]}^{\infty} \Delta_{11} b_{\mu\nu} B_{\mu}^2(x) B_{\nu}^2(y).$$

Therefore,

$$f_1(x, y) \geq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \chi_{I_{mn}}(x, y) \Psi_{mn}(x, y).$$

Then by Lemmas 3.5, 3.6, 3.7, and Jensen's inequality, we have

$$\begin{aligned} \|f_1(x, y)\|_{\Lambda_2^{\alpha}(\omega)}^{\alpha} &= \\ &= \int_0^{2\pi} \int_0^{2\pi} \omega(t_1, t_2) \left( \left( [f_1(x, y)]_{yx}^* \right)^* (t_1, t_2) \right)^{\alpha} dt_1 dt_2 \\ &\geq C \int_0^{2\pi} \int_0^{2\pi} \omega(t_1, t_2) \left( \left( \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \chi_{I_{mn}}(x, y) \Psi_{mn}(x, y) \right]_{yx}^* \right)^* (t_1, t_2) \right)^{\alpha} dt_1 dt_2 \\ &= C \int_0^{2\pi} \int_0^{2\pi} \omega(t_1, t_2) \left( \left[ \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \chi_{I_{mn}}(x, y) \Psi_{mn}(x, y) \right)^{\alpha} \right]_{yx}^* \right)^* (t_1, t_2) dt_1 dt_2 \\ &\geq C \int_0^{2\pi} \int_0^{2\pi} \omega(t_1, t_2) \left( \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \chi_{I_{mn}}(x, y) \Psi_{mn}(x, y) \right)^{\alpha} \right]_{yx}^* \right)^* (t_1, t_2) dt_1 dt_2 \\ &\geq C \int_0^{2\pi} \int_0^{2\pi} \omega(t_1, t_2) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \left[ \chi_{I_{mn}}(x, y) \Psi_{mn}(x, y) \right]_{yx}^* \right)^* (t_1, t_2)^{\alpha} dt_1 dt_2. \end{aligned}$$

We will use the following fact (see [VD]): there exists a set  $J_{mn} \subset I_{mn}$  such that

$$J_{mn} = \{(x, y) \in I_{mn} : \Psi_{mn}(x, y) \geq C_1 2^{2(n+m)} \Delta_{11} b_{[2^{m-1}][2^{n-1}]}\}$$



and  $\mu(J_{mn}) \geq C_2 \mu(I_{mn})$ , where  $C_1, C_2$  do not depend on  $n, m, \{b_{mn}\}$ . Hence, we have

$$\begin{aligned} & \|f_1(x, y)\|_{\Lambda_2^\alpha(\omega)}^\alpha \geq \\ & C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \omega(t_1, t_2) \left( \left( \chi_{I_{mn}}(x, y) 2^{2(n+m)} \Delta_{11} b_{[2^{m-1}][2^{n-1}]} \right)_{yx}^* (t_1, t_2) \right)^\alpha dt_1 dt_2 \\ & = C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \omega(t_1, t_2) \left( \chi_T(t_1, t_2) 2^{2(n+m)} \Delta_{11} b_{[2^{m-1}][2^{n-1}]} \right)^\alpha dt_1 dt_2, \end{aligned}$$

where

$$T = \left\{ (t_1, t_2) : (\chi_{I_{mn}}(x, y))_{yx}^* (t_1, t_2) > \frac{1}{2} \right\}.$$

By Lemma 3.10, we write

$$\begin{aligned} & \|f_1(x, y)\|_{\Lambda_2^\alpha(\omega)}^\alpha \geq \\ & \geq C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( 2^{2(n+m)} \Delta_{11} b_{[2^{m-1}][2^{n-1}]} \right)^\alpha \int_{C_2 \pi 2^{-(n+2)}}^{C_2 \pi 2^{-(n+1)}} \int_{C_2 \pi 2^{-(m+2)}}^{C_2 \pi 2^{-(m+1)}} \omega(t_1, t_2) dt_1 dt_2 \\ & \geq C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( 2^{2(n+m)} \Delta_{11} b_{[2^{m-1}][2^{n-1}]} \right)^\alpha \omega\left(\frac{\pi}{2^m}, \frac{\pi}{2^n}\right) \\ & \geq C \sum_{n=0}^{\infty} (n+1)^{2\alpha} \sum_{m=0}^{\infty} (m+1)^{2\alpha} w_{mn} (\Delta_{11} a_{mn})^\alpha. \end{aligned}$$

In a similar way, one can also obtain inequalities (2.7)–(2.9). QED

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