ORTHOGONAL SYSTEMS IN FINITE GRAPHS

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ABSTRACT. To a finite graph there corresponds a free partially commutative group: with the given graph as commutation graph. In this paper we construct an orthogonality theory for graphs and their corresponding free partially commutative groups. The theory developed here provides tools for the study of the structure of partially commutative groups, their universal theory and automorphism groups. In particular the theory is applied in this paper to the centraliser lattice of such groups.

CONTENTS

Glossary of Notation.

Γ	—	a finite undirected graph with vertex set X
$\Gamma_1 \oplus \Gamma_2$	—	the join of graphs Γ_1 and Γ_2
$C_G(S)$	—	the centraliser of a a subset S of G
$\mathrm{C}(G)$	—	the set of centralisers of a group G
G or $G(\Gamma)$	—	the (free) partially commutative group with underlying graph Γ
$\mathrm{lg}(w)$	—	the length of a geodesic word w' such that $w =_G w'$
$d(x, y)$	—	the distance from x to y, $x, y \in \Gamma$
$\mathcal{O}^Z(Y)$	—	the orthogonal complement of Y in Z, i.e. $\{u \in Z \mid d(u, y) \leq 1, \text{ for all } y \in Y\}$
$Y^\perp$	—	the orthogonal complement of Y in X, $\mathcal{O}^X(Y)$
$\mathrm{cl}^Z(Y)$	—	the closure of Y in Z with respect to $\mathcal{O}^Z(Y)$, i.e. $\mathrm{cl}^Z(Y) = \mathcal{O}^Z(\mathcal{O}^Z(Y))$
$\mathrm{cl}(Y)$	—	the closure of Y in X, i.e. $\mathrm{cl}(Y) = Y^{\perp\perp}$
$L(\Gamma)$ or L	—	the lattice of closed sets of Γ
$\overline{X}$	—	the set $X \cup \{t\}$
$\overline{\Gamma}$	—	the graph $(\overline{X}, E(\Gamma) \cup E_t)$, $E_t = \{(t, x) \mid x \in J_t\}$, $J_t \subseteq X$
$\overline{L}$	—	the lattice of closed sets, $L(\overline{\Gamma})$, of $\overline{\Gamma}$
L_t	—	$\{Y \subseteq X \mid Y = C \cap J_t, \text{ where } C \in L\}$
$\tilde{L}$	—	the set $L \cup L_t$
$h(L)$	—	the height of a lattice L
$Y \sim_\perp Z$	—	$Y, Z \subseteq X$ are $\perp$-equivalent in X, that is $Y^\perp = Z^\perp$
$\mathrm{acl}(S)$	—	the Abelian closure of a simplex S, that is the union of $T \subseteq X$ such that $S \sim_\perp T$
$Y \sim_o Z$	—	subsets $Y, Z \subseteq X$ are o-equivalent, i.e. $Y^\perp \smallsetminus Y = Z^\perp \smallsetminus Z$
$\mathrm{fcl}(A)$	—	the free-closure of a free co-simplex A, that is the union of all free co-simplexes B such that $A \sim_o B$
$[x]_\perp$	—	the $\perp$-equivalence class of x, that is $\{y \in X \mid x \sim_\perp y\}$
$[x]_o$	—	the o-equivalence class of x, that is $\{y \in X \mid x \sim_o y\}$
$x \sim y$	—	$x, y \in X$ are equivalent, i.e. either $x \sim_\perp y$ or $x \sim_o y$
$[x]$	—	the equivalence class of x with respect to $\sim$
Γ^{c}	—	the compression of the graph Γ

Introduction

This paper is a continuation of a series of papers [4, 5] where the authors develop the theory of free partially commutative groups.

Free partially commutative groups arise in many branches of mathematics and computer science and consequently are known by a variety of names: *semifree groups, graph groups, right-angled Artin groups, trace groups, locally free groups.* We refer the reader to [2], [9] and references there for a survey of these groups, which we shall refer to here as partially commutative groups.

The analysis of proofs of results on partially commutative groups shows that these rely heavily upon two main ideas: divisibility and orthogonality. The divisibility theory of partially commutative groups has been formalised in [9] and is a convenient tool for solving major algorithmic problems. The idea of considering orthogonal complements of subsets of vertices of the underlying graph of a partially commutative implicitly occurs in many papers, see for instance, [14, 12] and also [11] pp. 650-651. In this paper we formalise this idea and establish the main results of orthogonality theory for graphs.

Definition. *Let $G(\Gamma)$ be the partially commutative group with underlying graph $\Gamma = (X, E)$. For a vertex $x \in X$ we define $x^\perp$ to be the set of all vertices of Γ connected with x. For a subset $Y \subseteq X$ we define*

$$Y^\perp = \bigcap_{y \in Y} y^\perp.$$

Let $L(\Gamma)$ be the set of all subsets Z of X of the form $Y^\perp$ for some $Y \subseteq X$. We call $L(\Gamma)$ the lattice of closed sets of Γ.

The importance of the lattice of closed sets $L(\Gamma)$ for the theory of partially commutative groups is a consequence of the the fact that the lattice $L(\Gamma)$ is isomorphic to the lattice of parabolic centralisers (see Section 1) of $G(\Gamma)$ which, in turn, is crucial for study of the group $G(\Gamma)$ itself and its automorphism group $\mathrm{Aut}(G(\Gamma))$.

The main problem that we consider in this paper is how the lattice of closed set behaves when one joins a vertex v to the graph Γ to form a new graph $\bar{\Gamma}$. Naturally this depends on which vertices of Γ are joined to v. In particular, we prove that the lattices $L = L(\Gamma)$ and $\overline{L} = \overline{L(\Gamma)}$ are isomorphic if and only if v is joined to the orthogonal complement of a simplex $S \subset X$; see Theorem 2.38.

Moreover, we prove that the height $h(\overline{L})$ of the extended lattice $\overline{L}$ is $h(\overline{L}) = h(L) + m$, where $m = 0, 1$ or 2, see Theorem 2.28.

In Sections 2.7 and 2.10 we introduce operations of free and Abelian inflation and deflation on graphs and prove that the lattice of closed sets

L behaves nicely under these operations. We then introduce the notion of compression of a graph Γ which plays an important role in the study of partially commutative groups and prove that the lattices of closed sets for the graph Γ and its compression are closely related. The compression of a graph allows us to give a decomposition of the automorphism group of the graph as a semi-direct product of the automorphism group of the compression with a direct sum of symmetric groups.

The results of the current paper play a key role in two papers of authors which are currently under preparation: one on the structure of lattices of centralisers of a given partially commutative group G, the other on the structure of the automorphism group $\mathrm{Aut}(G)$, [7, 8].

A major part of our research on partially commutative groups, [4, 5, 6, 7, 8] was carried out while the second and the third authors were visiting the University of Newcastle Upon Tyne, thanks to the support of the EPSRC grants EP/D065275/1 and GR/S61900/01.

1. Preliminaries

1.1. **Graphs.** Graph will mean undirected, finite graph throughout this paper. If x and y are vertices of a graph then we define the *distance* $d(x, y)$ from x to y to be the minimum of the lengths of all paths from x to y in Γ. A subgraph S of a graph Γ is called a *full* subgraph if vertices a and b of S are joined by an edge of S whenever they are joined by an edge of Γ.

Let Γ be a graph with $V(\Gamma) = X$. A subset Y of X is called a *simplex* if the full subgraph of Γ with vertices Y is isomorphic to a complete graph. A maximal simplex is called a *clique*. A subset Y of X is called a *free co-simplex* if the full subgraph of Γ with vertices Y is isomorphic to the null graph. The reason why the word "free" is necessary here will become apparent later (see Section 2.10).

Let Γ_i be a graph with vertex set X_i, for $i = 1, 2$. The *join* $\Gamma_1 \oplus \Gamma_2$ of Γ_1 and Γ_2 is the graph with vertex set the disjoint union $X_1 \sqcup X_2$ and edge set consisting of all the edges of Γ_i, for $i = 1$ and 2 and an edge joining x_1 to x_2 for all $x_1 \in X_1$ and $x_2 \in X_2$.

1.2. **Lattices.** Let P be a partially ordered set with order relation $\leq$. Then P is said to be a *lattice* if every pair of elements of P has a unique infimum and a unique supremum. We usually write $s \wedge t$ and $s \vee t$ for the infimum and supremum, respectively, of s and t.

A lattice is said to be *bounded* if it has both a minimum and a maximum element. An *ascending chain* in a lattice is a sequence of elements $a_0, a_1, \ldots$ such that $a_i < a_{i+1}$. The *length* of a finite chain $a_0 < \cdots < a_k$ is said to be k. *Descending chains* are defined analogously. A lattice may be bounded

and have infinite ascending or descending chains (or both). The *height* of a lattice L is defined to be the maximum of the lengths of all chains in L, if it exists, and ∞ otherwise.

A *homomorphism of partially ordered sets* is a map from one partially ordered set to another which preserves the order relation. If P and Q are lattices then a homomorphism of partially ordered sets $f : P \to Q$ is called a *homomorphism of lattices* if $f(s \vee t) = f(s) \vee f(t)$ and $s \wedge t = f(s) \wedge f(t)$, for all $s, t \in P$. For further details on lattices we refer the reader to [1].

1.3. **Centraliser Lattices.** If S is a subset of a group G then the centraliser of S in G is $C_G(S) = \{g \in G : gs = sg, \text{ for all } s \in S\}$. We write $C(S)$ instead of $C_G(S)$ when the meaning is clear. Let $\mathrm{C}(G)$ denote the set of centralisers of a group G. The relation of inclusion then defines a partial order '$\leq$' on $\mathrm{C}(G)$. We define the infimum of a pair of elements of $\mathrm{C}(G)$ in the obvious way as:

$$C(M_1) \wedge C(M_2) = C(M_1) \cap C(M_2) = C(M_1 \cup M_2).$$

Moreover the supremum $C(M_1) \vee C(M_2)$ of elements $C(M_1)$ and $C(M_2)$ of $\mathrm{C}(G)$ may be defined to be the intersection of all centralisers containing $C(M_1)$ and $C(M_2)$. Then $C(M_1) \vee C(M_2)$ is minimal among centralisers containing $C(M_1)$ and $C(M_2)$. These definitions make $\mathrm{C}(G)$ into a lattice, called the *centraliser lattice* of G. This lattice is bounded as it has a greatest element, $G = C(1)$, and a least element, $Z(G)$, the centre of G. Lattices of centralisers have been extensively studied; a brief survey of results can be found in [4].

The *centraliser dimension* of a group G is defined to be the height of the centraliser lattice of G and is denoted $\mathrm{cdim}(G)$. Centralisers have the properties that, for all subsets S and T of G, if $S \subseteq T$ then $C(S) \geq C(T)$ and $C(C(C(S))) = C(S)$. Therefore if $C_1 < C_2 < \cdots$ is an ascending chain then $\cdots > C(C_2) > C(C_1)$ is a descending chain and both chains are either infinite or of the same length. Thus $\mathrm{cdim}(G)$ is the maximum of the lengths of descending chains of centralisers in G, if such a maximum exists, and is infinite otherwise.

1.4. **Partially Commutative Groups.** Let Γ be a finite, undirected, simple graph. Let $X = V(\Gamma)$ be the set of vertices of Γ and let $F(X)$ be the free group on X. For elements g, h of a group we denote the commutator $g^{-1}h^{-1}gh$ of g and h by $[g, h]$. Let

$R = \{[x_i, x_j] \in F(X) \mid x_i, x_j \in X$ and there is an edge from x_i to x_j in $\Gamma\}$.

We define the *partially commutative group with (commutation) graph* Γ to be the group $G(\Gamma)$ with presentation $\langle X \mid R \rangle$. (Note that these are the

groups which are called finitely generated free partially commutative groups in [3].)

Let Γ be a simple graph, $G = G(\Gamma)$ and let $w \in G$. Denote by $\lg(w)$ the length of a *geodesic* word in $X \cup X^{-1}$ that represents the element $w \in G$: that is a word of minimal length amongst those representing w. We say that $w \in G$ is *cyclically minimal* if and only if

$$\lg(g^{-1}wg) \geq \lg(w)$$

for every $g \in G$.

The centraliser dimension of partially commutative groups is finite because all partially commutative groups are linear [10] and all linear groups have finite centraliser dimension, [13]. In [5] it is shown that the centraliser dimension of a partially commutative group is easy to calculate and depends only on the centralisers of subsets of X. If $Y \subseteq X$ then we call $C(Y)$ a *canonical parabolic* centraliser. It is not hard to prove that the intersection of two canonical parabolic centralisers is again a canonical parabolic centraliser and, as shown in [5], the supremum, in $C(G)$, of two canonical parabolic centralisers is also a canonical parabolic centraliser. Hence the set $C(X; G)$ of canonical parabolic centralisers forms a sublattice of $C(G)$. In [5, Theorem 3.3] it is shown that the centraliser dimension of G is equal to the height of the lattice $C(X; G)$. In [6] we give a short proof of this fact using the methods developed in this paper and give a characterisation of centralisers of arbitrary subsets of a partially commutative group. Moreover in [7, 8] we use these tools to give a description of the automorphism group of a partially commutative group.

2. THE LATTICE OF CLOSED SUBSETS OF A GRAPH

2.1. **Orthogonal Systems, Closure and Closed Sets.** As before let Γ be a finite, undirected, simple graph, with vertices X, and let $G = G(\Gamma)$ be the partially commutative group defined by Γ. Given vertices x, y in the same connected component of Γ we define the *distance* $d(x, y)$ from x to y to be the minimum of the lengths of paths from x to y. If x and y are in distinct connected components then we define $d(x, y) = \infty$.

Let Y and Z be subsets of X. We define the *orthogonal complement* of Y in Z to be

$$\mathcal{O}^Z(Y) = \{u \in Z | d(u, y) \leq 1, \text{ for all } y \in Y\}.$$

By convention we set $\mathcal{O}^Z(\emptyset) = Z$. If $Z = X$ we call $\mathcal{O}^X(Y)$ the orthogonal complement of Y, and if no ambiguity arises then we shall sometimes write $Y^\perp$ instead of $\mathcal{O}^X(Y)$. Also, if every vertex of Z is either in Y or is joined by an edge of Γ to every vertex of Y then we write $[Y, Z] = 1$. Thus $[Y, Z] = 1$ if and only if $Z \subseteq \mathcal{O}^X(Y)$ if and only if every element of Y commutes with

every element of Z in the group G. For future reference we record some of the basic properties of orthogonal complements in the next lemma.

Lemma 2.1. *Let Y, Y_1, Y_2 and Z be subsets of X.*

(1) *If $Y \subseteq Z$ then $Y \subseteq \mathcal{O}^Z(\mathcal{O}^Z(Y))$.*
(2) *If $Y \subseteq Z$ then $\mathcal{O}^Z(Y) = \mathcal{O}^Z(\mathcal{O}^Z(\mathcal{O}^Z(Y)))$.*
(3) *If $Y_1 \subseteq Y_2$ then $\mathcal{O}^Z(Y_2) \subseteq \mathcal{O}^Z(Y_1)$.*
(4) *$\mathcal{O}^Z(Y_1 \cap Y_2) \supseteq \mathcal{O}^Z(Y_1) \cup \mathcal{O}^Z(Y_2)$.*
(5) *$\mathcal{O}^Z(Y_1 \cup Y_2) = \mathcal{O}^Z(Y_1) \cap \mathcal{O}^Z(Y_2)$.*
(6) *Y is a simplex if and only if $Y \subseteq Y^\perp$.*
(7) *Y is a clique if and only if $Y = Y^\perp$.*

In particular from 1 and 2 we have $Y \subseteq Y^{\perp\perp}$ and $Y^\perp = Y^{\perp\perp\perp}$, where we write $Y^{\perp\perp}$ for $(Y^\perp)^\perp$.

Proof. If $y \in Y \subseteq Z$ then, for all $u \in \mathcal{O}^Z(Y)$, we have $d(u, y) \leq 1$. Hence $y \in \mathcal{O}^Z(\mathcal{O}^Z(Y))$ and 1 follows. Statement 3 follows directly from the definition of orthogonal complement. Statement 2 follows from 1 and 3. Statement 4 follows from 3. To see 5 suppose first that $Z = X$. It follows from 3 that $\mathcal{O}^X(Y_1 \cup Y_2) \subseteq \mathcal{O}^X(Y_1) \cap \mathcal{O}^X(Y_2)$. From 4 and 1 we have $\mathcal{O}^X(\mathcal{O}^X(Y_1) \cap \mathcal{O}^X(Y_2)) \supseteq Y_1 \cup Y_2$. Hence, from 1 and 3, $\mathcal{O}^X(Y_1) \cap \mathcal{O}^X(Y_2) \subseteq \mathcal{O}^X(\mathcal{O}^X(\mathcal{O}^X(Y_1) \cap \mathcal{O}^X(Y_2))) \subseteq \mathcal{O}^X(Y_1 \cup Y_2)$, so 5 holds in this case. In general, $\mathcal{O}^Z(Y_1 \cup Y_2) = \mathcal{O}^X(Y_1 \cup Y_2) \cap Z = \mathcal{O}^X(Y_1) \cap \mathcal{O}^X(Y_2) \cap Z = \mathcal{O}^Z(Y_1) \cap \mathcal{O}^Z(Y_2)$, as required. If Y is a simplex and $y \in Y$ then $d(y, z) = 1$, for all $z \in Y$, $z \neq y$. Hence $Y \subset Y^\perp$. Conversely, if $Y \subseteq Y^\perp$ and $y, z \in Y$ then $d(y, z) \leq 1$, so Y is a simplex. Therefore 6 holds. If Y is a clique and $x \in Y^\perp \backslash Y$ then $Y \cup \{x\}$ is a simplex, contrary to maximality of Y. Hence, using 6, $Y = Y^\perp$. Conversely, if $Y = Y^\perp$, then Y is a simplex and, by a similar argument, there is no simplex strictly containing Y. Hence 7 holds. $\square$

Example 2.2.   (1) In general the inclusions of Lemma 2.1 are strict. For instance, take Γ to be the graph of Figure 2.1, let $Y_1 = \{a, c\}$ and $Y_2 = \{b, c, d\}$. Then $Y_1^\perp = \{b\}$, $Y_2^\perp = \{c\}$ and $(Y_1 \cap Y_2)^\perp = \{b, c, d\}$: so $(Y_1 \cap Y_2)^\perp \neq Y_1^\perp \cup Y_2^\perp$. Moreover $Y_1^{\perp\perp} = \{a, b, c\} \neq Y_1$.
(2) The subgroup $G(X^\perp)$ is the centre of the group $G = G(\Gamma)$.
(3) If $X = X_1 \sqcup X_2$ is a disjoint union of X_1 and X_2 and Γ is the direct sum of graphs $\Gamma(X_1)$ and $\Gamma(X_2)$ then $G = G(X_1) \times G(X_2)$. If $\mathcal{O}^{X_1}(X_1) = \mathcal{O}^{X_2}(X_2) = \emptyset$ then the groups $G(X_i)$, $i = 1, 2$ have trivial centre. In this case $\mathcal{O}^X(X_1) = X_2$ and $\mathcal{O}^X(X_2) = X_1$.

The connection between orthogonal complements and centralisers is made explicit in the following lemma.

Lemma 2.3. *Let $G = G(\Gamma)$ and $Y \subseteq X$. Then $C_G(Y) = G(Y^\perp)$.*
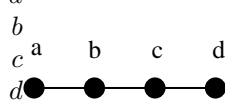
PSfrag replacements

a

b

c

d

FIGURE 2.1. A path graph

Proof. If $x \in X$ then $C_G(x) \supseteq G(x^\perp)$. From [9, Lemma 2.4] we also have $C_G(x) \subseteq G(x^\perp)$. Hence $C_G(Y) = \cap_{y \in Y} C_G(y) = \cap_{y \in Y} G(y^\perp) = G(\cap_{y \in Y} y^\perp) = G(Y^\perp)$. □

For subsets Y and Z of X we define the *closure* of Y in Z to be $\text{cl}^Z(Y) = \mathcal{O}^Z \mathcal{O}^Z(Y)$. When $Z = X$ we write $\text{cl}(Y)$ for $\text{cl}^X(Y)$. The closure operator in Γ satisfies the following properties.

Lemma 2.4. *Let Y, Y_1, Y_2 and Z be subsets of X.*

(1) $Y \subseteq \text{cl}(Y)$.

(2) $\text{cl}(Y^\perp) = Y^\perp$.

(3) $\text{cl}(\text{cl}(Y)) = \text{cl}(Y)$.

(4) *If $Y_1 \subseteq Y_2$ then $\text{cl}(Y_1) \subseteq \text{cl}(Y_2)$.*

(5) $\text{cl}(Y_1 \cap Y_2) \subseteq \text{cl}(Y_1) \cap \text{cl}(Y_2)$ *and* $\text{cl}(Y_1) \cup \text{cl}(Y_2) \subseteq \text{cl}(Y_1 \cup Y_2)$.

(6) *If $Z = \text{cl}(Y)$ then $Z = U^\perp$, for some $U \subset X$, and then $\text{cl}(U) = Z^\perp = Y^\perp$.*

(7) *If $\text{cl}(Y_1) = \text{cl}(Y_2)$ then $Y_1^\perp = Y_2^\perp$.*

(8) *Y is a simplex if and only if $\text{cl}(Y)$ is a simplex if and only if $\text{cl}(Y) \subseteq Y^\perp$.*

(9) *If $Y_1 \subseteq Y_2$ then $\text{cl}(\text{cl}(Y_1) \cap Y_2) = \text{cl}(Y_1)$.*

(10) $\text{cl}(\text{cl}(Y_1) \cup \text{cl}(Y_2)) = \text{cl}(Y_1 \cup Y_2)$.

(11) $\text{cl}(\text{cl}(Y_1) \cap Y_2) \cap Y_2 = \text{cl}(Y_1) \cap Y_2$.

Proof. Statements 1 and 2 are restatements of Lemma 2.1, 1 and 2, respectively. To see 3 apply the operator $\mathcal{O}^X$ to both sides of 2. Statement 4 is a consequence of Lemma 2.1.3. Statement 5 follows from 4. If $Z = \text{cl}(Y)$ then $Z = U^\perp$, where $U = Y^\perp$. If $Z = U^\perp$ then $\text{cl}(U) = U^{\perp\perp} = Z^\perp = (\text{cl}(Y))^\perp = Y^\perp$, using Lemma 2.1.2. Hence 6 holds. To see 7 apply the operator $\mathcal{O}^X$ to both $\text{cl}(Y_1)$ and $\text{cl}(Y_2)$ and use Lemma 2.1.2. For 8, if $\text{cl}(Y)$ is a simplex then $\text{cl}(Y) \subseteq \text{cl}(Y)^\perp$, from Lemma 2.1.6, so from 1 and Lemma 2.1.2 $\text{cl}(Y) \subseteq Y^\perp$. If $\text{cl}(Y) \subseteq Y^\perp$ then, from 1 and Lemma 2.1.6, $Y \subseteq Y^\perp$ so Y is a simplex, and $\text{cl}(Y) \subseteq \text{cl}(Y^\perp) = \text{cl}(Y)^\perp$, so $\text{cl}(Y)$ is a simplex. $Y^{\perp\perp} \subseteq (Y^{\perp\perp})^\perp = Y^\perp$ so $Y \subseteq Y^{\perp\perp} \subseteq Y^\perp$; and Y is a simplex. In the setting of 9 we have, from 1, $Y_1 \subseteq \text{cl}(Y_1) \cap Y_2$, so $\text{cl}(Y_1) \subseteq \text{cl}(\text{cl}(Y_1) \cap Y_2)$. On the other hand $\text{cl}(Y_1) \cap Y_2 \subseteq \text{cl}(Y_1)$ so, from 3 and 4, $\text{cl}(\text{cl}(Y_1) \cap Y_2) \subseteq \text{cl}(Y_1)$. To see 10 use the second part of 5 and then 3 to obtain $\text{cl}(\text{cl}(Y_1) \cup \text{cl}(Y_2)) \subseteq \text{cl}(Y_1 \cup Y_2)$. For the opposite

inclusion use 1 to obtain $Y_1 \cup Y_2 \subseteq \mathrm{cl}(Y_1) \cup \mathrm{cl}(Y_2)$ and then 4 implies that $\mathrm{cl}(Y_1 \cup Y_2) \subseteq \mathrm{cl}(\mathrm{cl}(Y_1) \cup \mathrm{cl}(Y_2))$, as required. For 11 first note that 1 implies that $\mathrm{cl}(Y_1) \cap Y_2 \subseteq \mathrm{cl}(\mathrm{cl}(Y_1) \cap Y_2) \cap Y_2$. Also $\mathrm{cl}(Y_1) \cap Y_2 \subseteq \mathrm{cl}(Y_1)$ so 4 and 3 imply that $\mathrm{cl}(\mathrm{cl}(Y_1) \cap Y_2) \subseteq \mathrm{cl}(Y_1)$. On intersection with Y_2 this gives the inclusion required to complete the proof. $\qquad\square$

Example 2.5. (1) If $x \in X$ and $Y = \mathrm{cl}(x) = x^{\perp\perp}$ then Y is a simplex.
   (2) In terms of the group G the subset Y of X is a simplex if and only if $G(Y)$ is Abelian. As $C_G(Z) = G(Z^{\perp})$, for any subset Z of X, Lemma 2.1.6 states that $G(Y)$ is Abelian if and only if $G(Y) \subseteq C_G(Y)$. The content of Lemma 2.4.8 is that $G(Y)$ is Abelian if and only if $C_G^2(Y)$ is Abelian if and only if $C_G^2(Y) \subseteq C_G(Y)$.

Definition 2.6. *A subset Y of X is called* closed *(with respect to Γ) if $Y = \mathrm{cl}(Y)$. Denote by $L(\Gamma)$ the set of all closed subsets of X.*

We list some basic properties of $L(\Gamma)$.

Lemma 2.7. *Let Y be a subset of X. The following hold.*
   (1) $\mathrm{cl}(Y) \in L(\Gamma)$.
   (2) X *is the unique maximal element of* $L(\Gamma)$.
   (3) Y *is closed if and only if* $Y = \mathcal{O}^X(U)$, *for some* $U \in L(\Gamma)$.
   (4) $\mathcal{O}^X(X)$ *is the unique minimal element of* $L(\Gamma)$.
   (5) *If* $Y_1, Y_2 \in L(\Gamma)$ *then* $Y_1 \cap Y_2 \in L(\Gamma)$.

Proof. (1) This follows from Lemma 2.4.3.
   (2) This is clear, given the previous statement and the fact that $X \subseteq \mathrm{cl}(X)$.
   (3) It follows, from Lemma 2.4, 2 and 6, that $Y \in L(\Gamma)$ if and only if $Y = \mathcal{O}^X(U)$, for some subset U of X. If Y is closed and $Y = \mathcal{O}^X(U)$ then $Y = \mathrm{cl}(Y) = \mathcal{O}^X(\mathrm{cl}(U))$ and, as $\mathrm{cl}(U)$ is closed, the result follows.
   (4) From the previous statement it follows that $\mathcal{O}^X(X) \in L(\Gamma)$. If $Y \in L(\Gamma)$ then $Y = \mathcal{O}^X(U)$, for some $U \subseteq X$. From Lemma 2.1 then $\mathcal{O}^X(X) \subseteq \mathcal{O}^X(U) = Y$, as required.
   (5) From Lemma 2.4, 1 and 5, we have

$$Y_1 \cap Y_2 \subseteq \mathrm{cl}(Y_1 \cap Y_2) \subseteq \mathrm{cl}(Y_1) \cap \mathrm{cl}(Y_2) = Y_1 \cap Y_2,$$

   the last equality holding by definition of closed set. Therefore $Y_1 \cap Y_2 = \mathrm{cl}(Y_1 \cap Y_2)$.
$\qquad\square$

The relation $Y_1 \subseteq Y_2$ defines a partial order on the set $L(\Gamma)$. As the closure operator cl is inclusion preserving and maps arbitrary subsets of X into closed sets we can make $L(\Gamma)$ into a lattice by defining the the infimum

$Y_1 \wedge Y_2$ of Y_1 and Y_2 by $Y_1 \wedge Y_2 = \mathrm{cl}(Y_1 \cap Y_2) = Y_1 \cap Y_2$ and the supremum $Y_1 \vee Y_2 = \mathrm{cl}(Y_1 \cup Y_2)$.

Proposition 2.8. *The set $L(\Gamma)$ with operations $\wedge$ and $\vee$ above is a complete lattice.*

Proof. As we have seen $L(\Gamma)$ is a lattice. From Lemma 2.7 it has maximum element X and minimum element $\mathcal{O}^X(X)$, so is complete. □

Proposition 2.9. *The operator $\mathcal{O}^X$ maps $L(\Gamma)$ to itself and is a lattice anti-automorphism.*

Proof. If $Y \in L(\Gamma)$ then, from Lemma 2.7, $\mathcal{O}^X(Y) \in L(\Gamma)$; so $\mathcal{O}^X$ maps $L(\Gamma)$ to itself. From Lemma 2.1 $\mathcal{O}^X$ is inclusion reversing. Moreover, for $Y \in L(\Gamma)$ we have $\mathcal{O}^X(\mathcal{O}^X(Y)) = Y$; so the restriction of $\mathcal{O}^X$ to $L(\Gamma)$ is a bijection. Hence this restriction is a lattice anti-automorphism. □

If $Z \subseteq X$ and Γ_Z is the full subgraph of Γ with vertex set Z then, by abuse of notation, we write $L(Z)$ for $L(\Gamma_Z)$. As long as it is clear that Γ is fixed this should cause no confusion. We have $\mathcal{O}^Z(Y) = \mathcal{O}^X(Y) \cap Z$ so $L(Z)$ consists of subsets Y of Z such that $Y = \mathrm{cl}^Z(Y) = \mathcal{O}^X(\mathcal{O}^X(Y) \cap Z) \cap Z$.

2.2. **Disconnected Graphs and Joins of Graphs.** Now suppose that X is a disjoint union $X = X_1 \sqcup X_2$, where X_1 and X_2 are non-empty, and $\Gamma = \Gamma(X_1) \sqcup \Gamma(X_2)$ (that is no edge of Γ joins a vertex of X_1 to a vertex of X_2). Write Γ_i for $\Gamma(X_i)$, $i = 1, 2$. We wish to describe $L(\Gamma)$ in terms of the lattices $L(\Gamma_i)$. First of all we note the following lemma.

Lemma 2.10. *With the hypotheses above, if U is a non-empty subset of X_i then $\mathcal{O}^{X_i}(U) = \mathcal{O}^X(U)$.*

Proof. By definition $\mathcal{O}^{X_i}(U) \subseteq \mathcal{O}^X(U)$. We have $\mathcal{O}^X(U) = \{x \in X | d(u, x) \leq 1, \forall u \in U\}$. If $x \notin X_i$ then, as $U \neq \emptyset$, there is some $u \in U$ such that $d(x, u) = \infty$. Hence $x \in \mathcal{O}^X(U)$ implies $x \in X_i$, so $x \in \mathcal{O}^{X_i}(U)$. □

The relationship between $L(\Gamma)$ and the $L(\Gamma_i)$ is specified by the following proposition.

Proposition 2.11. *Let $\Gamma = \Gamma_1 \sqcup \Gamma_2$, as above.*

    (1) $\emptyset \in L(\Gamma)$.
    (2) A non-empty set Y is in $L(\Gamma) \backslash \{X, X_1, X_2\}$ if and only if Y is in precisely one of $L(\Gamma_i) \backslash \{X_i\}$, $i = 1$ or 2.
    (3) If $\mathcal{O}^{X_i}(X_i) = \emptyset$ then $\emptyset \in CS(\Gamma_i)$ and $X_i \notin CS(\Gamma)$.
    (4) If $\mathcal{O}^{X_i}(X_i) \neq \emptyset$ then $\emptyset \notin CS(\Gamma_i)$ and $X_i \in CS(\Gamma)$.

Proof.   (1) As X_i is non-empty it follows that $\emptyset = \mathcal{O}^X(X)$, so $\emptyset \in L(\Gamma)$.

(2) Let Y be a non-empty element of $L(\Gamma)\backslash\{X, X_1, X_2\}$. Then $Y = \mathcal{O}^X(U)$, for some subset U of X. If $U \cap X_i \neq \emptyset$, for $i = 1$ and 2, then $\mathcal{O}^X(U) = \emptyset$. Hence $U \subseteq X_i$, for $i = 1$ or 2. If $U = \emptyset$ then $Y = X$, so $U \neq \emptyset$ and, from Lemma 2.10, $Y = \mathcal{O}^{X_i}(U)$ so is in $L(\Gamma_i)$. Note that in this case $Y \subseteq X_i$ and is non-empty; so cannot be in $L(\Gamma_j)$, $j \neq i$. Conversely if Y is a non-empty element of $L(\Gamma_i)\backslash\{X_i\}$ then $Y = \mathcal{O}^{X_i}(U)$, for some $U \subseteq X_i$. As $Y \neq X_i$ we have $U \neq \emptyset$ and so, from Lemma 2.10 again, $Y \in CS(\Gamma)$.

(3) From Lemma 2.7, $\emptyset \in L(\Gamma_i)$. From Lemma 2.10 we have $\emptyset = \mathcal{O}^{X_i}(X_i) = \mathcal{O}^X(X_i)$. If $X_i \in L(\Gamma)$ then $X_i = \mathcal{O}^X(U)$, for some $U \in L(\Gamma)$. Hence $\emptyset = \mathcal{O}^X(X_i) = U$ which implies $X_i = \mathcal{O}^X(U) = X$, a contradiction.

(4) As $\mathcal{O}^{X_i}(X_i)$ is the minimal element of $L(\Gamma_i)$, in this case $\emptyset \notin L(\Gamma_i)$. We have $X_i = \mathcal{O}^{X_i}(U)$, for some $U \in L(\Gamma_i)$, so $U \neq \emptyset$ and $U \subseteq X_i$. That $X_i \in L(\Gamma)$ now follows from Lemma 2.10.

$\square$

Let $L = L(\Gamma)$, $L_i = L(\Gamma_i)$ and $L_i' = L(\Gamma_i)\backslash\{X_i\}$. Then Figure 2.2 illustrates the composition of $L(\Gamma)$ in terms of the $L(\Gamma_i)$. Now suppose
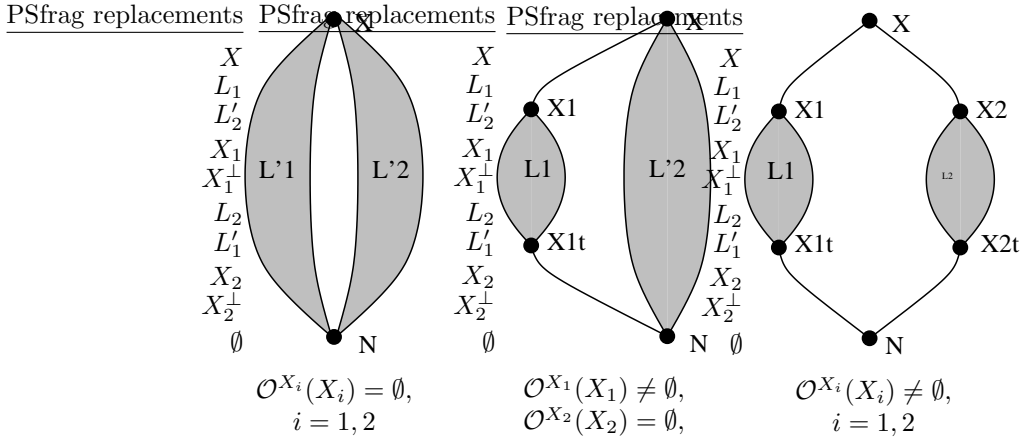
FIGURE 2.2. The lattice L of closed sets in a disconnected graph

that Γ has connected components $\Gamma_1, \ldots, \Gamma_m$, where $V(\Gamma_i) = X_i$. Assume that $\mathcal{O}^{X_i}(X_i) \neq \emptyset$, for $i = 1, \ldots, r$ and that $\mathcal{O}^{X_i}(X_i) = \emptyset$, for $i > r$. A straightforward induction using Proposition 2.11 shows that the lattice $L(\Gamma)$ takes the form shown in Figure 2.3: where we use the obvious extension of the notation introduced above for the lattices $L(\Gamma_i)$. We may often therefore

PSfrag replacements
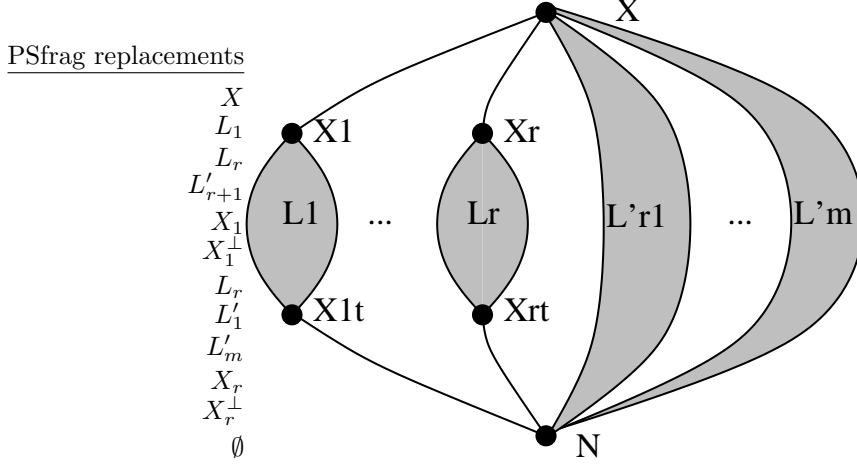
X
L_1
L_r
L'_{r+1}
X_1
$X_1^\perp$
L_r
L'_1
L'_m
X_r
$X_r^\perp$
$\emptyset$

FIGURE 2.3. The lattice L of the graph with connected components $\Gamma_1, \ldots, \Gamma_m$.

reduce to the study of $L(\Gamma)$ where Γ is a connected graph.

Now suppose that $X^\perp \neq \emptyset$ and set $X^* = X \backslash X^\perp$. Let $\Gamma(X^*) = \Gamma^*$ the full subgraph of Γ with vertex set X^*.

Proposition 2.12. *The set $\mathcal{O}^{X^*}(X^*) = \emptyset$ and the lattice $L(\Gamma)$ is isomorphic to the lattice $L(\Gamma^*)$.*

Proof. From the definitions it follows that $\mathcal{O}^X(X^*) = \mathcal{O}^X(X)$. Therefore $\mathcal{O}^{X^*}(X^*) = \mathcal{O}^X(X^*) \cap X^* = \mathcal{O}^X(X) \cap X^* = \emptyset$. If $Y = \mathcal{O}^X(U)$, where $U \in L(\Gamma)$ then $Y \backslash X^\perp = \mathcal{O}^X(U) \backslash \mathcal{O}^X(X) = \mathcal{O}^{X^*}(U \backslash \mathcal{O}^X(X))$. Hence the map $\phi : Y \to Y \backslash X^\perp$ maps $L(\Gamma)$ to $L(\Gamma^*)$.

Clearly ϕ is inclusion preserving. To see that ϕ is surjective, note that if $V \subseteq X^*$ then $\mathcal{O}^{X^*}(V) = C \backslash X^\perp$, where $C = \mathcal{O}^X(V \backslash \mathcal{O}^X(X))$. Therefore ϕ is a surjective homomorphism of partially ordered sets. Since $Y \in L(\Gamma)$ implies $X^\perp \subseteq Y$ it follows that ϕ is also injective; so ϕ is an isomorphism of lattices. $\square$

The set $\mathcal{O}^X(X)$ is called the *kernel* of the graph Γ. Given the proposition above we may restrict to the study of lattices with the trivial kernel.

Now suppose that $\Gamma = \Gamma(X_1) \oplus \Gamma(X_2)$, for some partition $X = X_1 \cup X_2$ of X (see Section 1.1). Let $\Gamma_i = \Gamma(X_i)$ and let $G_i = G(\Gamma_i)$, $i = 1, 2$; so $G = G_1 \times G_2$.

Proposition 2.13. *In the above notation, if $\Gamma = \Gamma_1 \oplus \Gamma_2$ then $L(\Gamma) = L(\Gamma_1) \times L(\Gamma_2)$.*

In this case the study of the lattice $L(\Gamma)$ reduces to the study of $L(\Gamma_1)$ and $L(\Gamma_2)$.

2.3. **Adjoining Vertices.** We now consider the effect on the lattice of closed sets of the addition to Γ, or removal from Γ, of a vertex. In particular we shall see how the heights of these lattices are related and how to make restrictions on the way in which the new vertex is added to obtain isomorphism of the two lattices.

We shall see below that if we adjoin a single vertex to Γ then the height of the lattice of closed sets of the new graph is equal to $h(L(\Gamma)) + k$, where $k = 0, 1$ or 2.

As usual Γ is a graph with $V(\Gamma) = X$ and edges $E(\Gamma)$. Let t be an element not in X and define $\overline{X} = X \cup \{t\}$. Let J_t be a subset of X. Define $\overline{\Gamma}$ to be the graph with vertices $\overline{X}$ and edges $E(\Gamma) \cup E_t$, where E_t is the set $E_t = \{(t, x) | x \in J_t\}$. Let $L = L(\Gamma)$ and $\overline{L} = L(\overline{\Gamma})$.

In order to understand how L and $\overline{L}$ are related we introduce a lattice intermediate between L and $\overline{L}$. This will help us to give a simple description of the structure of the lattice $\overline{L}$ in terms of the lattice L. Let

$$L_t = \{Y \subseteq X | Y = C \cap J_t, \text{ where } C \in L\}.$$

Now define the set of subsets $\tilde{L}$ of X to be $\tilde{L} = L \cup L_t$. We shall see that $\tilde{L}$ is a lattice which embeds in the lattice $\overline{L}$. Note that if $Y \in L_t$ then $Y = C \cap J_t$, for some $C \in L$, so

$$Y = Y \cap J_t \subseteq \mathrm{cl}(Y) \cap J_t \subseteq \mathrm{cl}(C) \cap J_t = C \cap J_t = Y.$$

Hence $Y = \mathrm{cl}(Y) \cap J_t$ and it follows that $\mathrm{cl}(Y)$ is the minimal element of L which intersects with J_t to give Y, for all $Y \in L_t$. Setting $Z = \mathrm{cl}(Y)$ this gives $Z = \mathrm{cl}(Y) = \mathrm{cl}(\mathrm{cl}(Y) \cap J_t) = \mathrm{cl}(Z \cap J_t)$. Also if $Y \in L_t \setminus L$ then $\mathrm{cl}(Y) \neq Y = \mathrm{cl}(Y) \cap J_t$, so $Z = \mathrm{cl}(Y) \nsubseteq J_t$. Conversely, given $Z \in L$ such that $Z \nsubseteq J_t$ and $Z = \mathrm{cl}(Z \cap J_t)$ then $\mathrm{cl}(Z \cap J_t) \neq Z \cap J_t$, so $Z \cap J_t \in L_t \setminus L$. Therefore

$$(2.1) \qquad L_t \setminus L = \{Y = Z \cap J_t | Z \in L, Z \nsubseteq J_t \text{ and } Z = \mathrm{cl}(Z \cap J_t)\}.$$

We define a closure operation $\mathrm{icl}^X = \mathrm{icl}$ on subsets of X by

$$\mathrm{icl}(U) = \begin{cases} \mathrm{cl}^X(U), & \text{if } U \nsubseteq J_t \\ \mathrm{cl}^X(U) \cap J_t, & \text{if } U \subseteq J_t \end{cases},$$

for $U \subseteq X$. Then $\mathrm{icl}(U) \in \tilde{L}$, for all $U \subseteq X$.

Now assume that Y_1 and Y_2 are in $\tilde{L}$ and $Y_1 \subseteq Y_2$. If $Y_1 \nsubseteq J_t$ then $\mathrm{icl}(Y_1) = \mathrm{cl}^X(Y_1)$ and $\mathrm{icl}(Y_2) = \mathrm{cl}^X(Y_2)$ so $\mathrm{icl}(Y_1) \subseteq \mathrm{icl}(Y_2)$. If $Y_1 \subseteq J_t$ then $\mathrm{icl}(Y_1) = \mathrm{cl}^X(Y_1) \cap J_t \subseteq \mathrm{cl}^X(Y_2) \cap J_t \subseteq \mathrm{icl}(Y_2)$. Therefore icl is an inclusion preserving map from subsets of X to $\tilde{L}$. It also follows from the definition

and 2.1 that $\mathrm{icl}(U) = U$, for all $U \in \tilde{L}$, so $\tilde{L}$ is a retract of X. We may therefore make $\tilde{L}$ into a lattice by defining

$$Y_1 \wedge Y_2 = \mathrm{icl}(Y_1 \cap Y_2) \text{ and } Y_1 \vee Y_2 = \mathrm{icl}(Y_1 \cup Y_2),$$

for $Y_1, Y_2 \in \tilde{L}$.

Lemma 2.14. *If $U, V \in \tilde{L}$ then $U \wedge V = U \cap V$ and*

$$U \vee V = \left\{ \begin{array}{ll} \mathrm{cl}(U \cup V) \cap J_t, & \text{if } U \cup V \subseteq J_t \\ \mathrm{cl}(U \cup V), & \text{otherwise} \end{array} \right. .$$

Proof. The expression for $U \vee V$ is merely a restatement of the definitions. If $U \in L$ then $\mathrm{icl}(U) = \mathrm{cl}^X(U)$. Therefore, for U and V in L we have (in the lattice $\tilde{L}$) $U \wedge V = U \cap V$. If either U or V belongs to L_t then $U \cap V \subseteq J_t$ so

$$U \wedge V = \mathrm{cl}(U \cap V) \cap J_t \subseteq \mathrm{cl}(U) \cap \mathrm{cl}(V) \cap J_t = U \cap V \subseteq \mathrm{cl}(U \cap V) \cap J_t$$

and the Lemma follows. $\square$

Definition 2.15. *Define $\tilde{\beta}$ to be the inclusion map of L into $\tilde{L}$ and $\tilde{\gamma}$ to be the map from $\tilde{L}$ to L given by $\tilde{\gamma}(Y) = \mathrm{cl}^X(Y)$, for $Y \in \tilde{L}$.*

Lemma 2.16. *The maps $\tilde{\beta}$ and $\tilde{\gamma}$ are homomorphisms of partially ordered sets and $\tilde{\gamma}\tilde{\beta} = \mathrm{id}_L$. We have $\tilde{\beta}(Y \wedge Z) = \tilde{\beta}(Y) \wedge \tilde{\beta}(Z)$, for all $Y, Z \in L$, and $\tilde{\gamma}(U \vee V) = \tilde{\gamma}(U) \vee \tilde{\gamma}(V)$, for all $U, V \in \tilde{L}$. If $U, V \in \tilde{L}$ such that $U \neq V$ and $\tilde{\gamma}(U) = \tilde{\gamma}(V)$ then (after interchanging U and V if necessary) $U \in L \backslash L_t$ and $V \in L_t \backslash L$ and $U = \mathrm{cl}^X(V)$.*

Proof. The first statement is a direct consequence of the definitions, as is the fact that $\tilde{\beta}$ respects the lattice infimum operation. For all $U, V \in \tilde{L}$ we have

$$\tilde{\gamma}(U) \vee \tilde{\gamma}(V) = \mathrm{cl}(\mathrm{cl}(U) \cup \mathrm{cl}(V)) = \mathrm{cl}(U \cup V),$$

from Lemma 2.4.10. If $U \cup V \not\subseteq J_t$ then $\tilde{\gamma}(U \vee V) = \mathrm{cl}\,\mathrm{cl}(U \cup V) = \mathrm{cl}(U \cup V)$. On the other hand, if $U \cup V \subseteq J_t$ then $\tilde{\gamma}(U \vee V) = \mathrm{cl}(\mathrm{cl}(U \cup V) \cap J_t) = \mathrm{cl}(U \cup V)$, using Lemma 2.4.9. Hence $\tilde{\gamma}(U) \vee \tilde{\gamma}(V) = \tilde{\gamma}(U \vee V)$, for all $U, V \in \tilde{L}$.

Let $U, V \in \tilde{L}$. If $U, V \in L$ then $\tilde{\gamma}(U) = \tilde{\gamma}(V)$ implies $U = V$. If $U, V \in L_t$ then $U = \mathrm{cl}(U) \cap J_t$ and $V = \mathrm{cl}(V) \cap J_t$ and $\mathrm{cl}(U) = \mathrm{cl}(\mathrm{cl}(U) \cap J_t) = \tilde{\gamma}(U)$ and similarly $\mathrm{cl}(V) = \tilde{\gamma}(V)$. Therefore $\tilde{\gamma}(U) = \tilde{\gamma}(V)$ implies that $U = \mathrm{cl}(U) \cap J_t = \mathrm{cl}(V) \cap J_t = V$. Therefore, if $U \neq V$ and $\tilde{\gamma}(U) = \tilde{\gamma}(V)$ then one of U, V is in $L \backslash L_t$ and the other in $L_t \backslash L$. Assume then that $U \in L \backslash L_t$ and $V \in L_t \backslash L$. In this case $U = \tilde{\gamma}(U) = \tilde{\gamma}(V) = \mathrm{cl}(\mathrm{cl}(V) \cap J_t) = \mathrm{cl}(V)$. $\square$

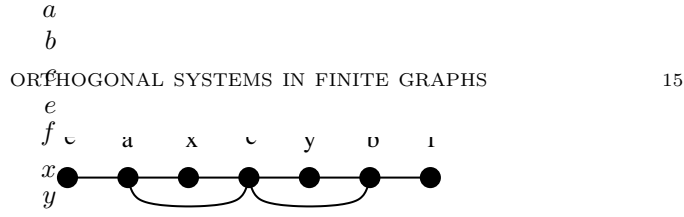In general $\tilde{\beta}$ does not preserve supremums and $\tilde{\gamma}$ does not preserve infimums.

FIGURE 2.4. Example 2.17

Example 2.17. In the graph of Figure 2.4 the sets $B = \{b\}$ and $C = \{c\}$ are closed. The supremum $B \vee C = \mathrm{cl}(B \cup C) = \{b, c, y\}$ and setting $J_t = \{b, c\}$ we have $\tilde{\beta}(B \vee C) = \{b, c, y\}$ and $\tilde{\beta}(B) \vee \tilde{\beta}(C) = \mathrm{cl}(\{b, c\}) \cap \{b, c\} = \{b, c\}$. In the same graph $\mathrm{cl}(x) = \{a, x, c\}$ and $\mathrm{cl}(y) = \{b, y, c\}$. Set $J_t = \{x, y\}$ and then $U = \mathrm{cl}(x) \cap J_t = \{x\}$ and $V = \mathrm{cl}(y) \cap J_t = \{y\}$ are both elements of L_t. Now $U \wedge V = \emptyset$ so $\tilde{\gamma}(U \wedge V) = \mathrm{cl}(\emptyset) = \emptyset$. However $\tilde{\gamma}(U) \wedge \tilde{\gamma}(V) = \mathrm{cl}(x) \cap \mathrm{cl}(y) = \{c\}$.

Next we show that the lattice $\tilde{L}$ is embedded, as a partially ordered set, in $\overline{L}$.

Definition 2.18. *Let $\overline{\beta} : \tilde{L} \to \overline{L}$ and $\overline{\gamma} : \overline{L} \to \tilde{L}$ be the maps given by $\overline{\beta}(Y) = \mathrm{cl}^{\overline{X}}(Y)$, for $Y \in \tilde{L}$ and $\overline{\gamma}(Z) = \mathrm{icl}(Z \backslash \{t\})$, for $Z \in \overline{L}$.*

Lemma 2.19. *The maps $\overline{\beta}$ and $\overline{\gamma}$ are homomorphisms of partially ordered sets and $\overline{\gamma}\overline{\beta} = \mathrm{id}_{\tilde{L}}$; so $\overline{\beta}$ is injective and $\overline{\gamma}$ is surjective. We have $\overline{\gamma}(Z) = Z \backslash \{t\}$, for all $Z \in \overline{L}$, and*

$$(2.2) \qquad \overline{\beta}(Y) = \begin{cases} Y & \text{if } \mathcal{O}^X(Y) \nsubseteq J_t, \\ Y \cup \{t\} & \text{if } \mathcal{O}^X(Y) \subseteq J_t \end{cases} ,$$

for all $Y \in \tilde{L}$.

If Z_1 and Z_2 are elements of $\overline{L}$ such that $Z_1 \neq Z_2$ then $\overline{\gamma}(Z_1) = \overline{\gamma}(Z_2)$ if and only if (after interchanging Z_1 and Z_2 if necessary) $t \in Z_1$ and $Z_2 = Z_1 \backslash \{t\} \in \overline{L}$.

Proof. Since the closure operations in $\tilde{L}$ and $\overline{L}$ preserve inclusion of sets it follows from the definitions that $\overline{\beta}$ and $\overline{\gamma}$ are homomorphisms of partially ordered sets.

Now let $U \in \tilde{L}$. If $U \nsubseteq J_t$ then $\mathcal{O}^{\overline{X}}(U) = \mathcal{O}^X(U)$. On the other hand if $U \subseteq J_t$ then $\mathcal{O}^{\overline{X}}(U) = \mathcal{O}^X(U) \cup \{t\}$. Therefore, if $U \nsubseteq J_t$ then

$$\overline{\beta}(U) = \mathcal{O}^{\overline{X}}\mathcal{O}^X(U) = \begin{cases} \mathcal{O}^X\mathcal{O}^X(U), & \text{if } \mathcal{O}^X(U) \nsubseteq J_t \\ \mathcal{O}^X\mathcal{O}^X(U) \cup \{t\}, & \text{if } \mathcal{O}^X(U) \subseteq J_t \end{cases}$$

and (2.2) holds as $U \nsubseteq J_t$ implies that $U \in L$. If $U \subseteq J_t$ then

$$\overline{\beta}(U) = \mathcal{O}^{\overline{X}}(\mathcal{O}^X(U) \cup \{t\}) = \mathcal{O}^{\overline{X}}(\mathcal{O}^X(U)) \cap (J_t \cup \{t\})$$

so

$$\overline{\beta}(U) = \begin{cases} \mathcal{O}^X \mathcal{O}^X(U) \cap J_t, & \text{if } \mathcal{O}^X(U) \nsubseteq J_t \\ (\mathcal{O}^X \mathcal{O}^X(U) \cap J_t) \cup \{t\}, & \text{if } \mathcal{O}^X(U) \subseteq J_t \end{cases}.$$

In this case, as $U \subseteq J_t$ we have $\mathcal{O}^X \mathcal{O}^X(U) \cap J_t = \mathrm{cl}^X(U) \cap J_t = U$. Thus, in all cases, (2.2) holds.

Now suppose that $Z \in \overline{L}$ and let $Y \in \overline{L}$ such that $Z = \mathcal{O}^{\overline{X}}(Y)$. If $t \in Y$ then $Z = \mathcal{O}^{\overline{X}}(Y) \subseteq \mathcal{O}^{\overline{X}}(t) = J_t \cup \{t\}$. Conversely if $Z \subseteq J_t \cup \{t\}$ then $t \in Y = \mathcal{O}^{\overline{X}}(Z)$. Hence $Z \backslash \{t\} \subseteq J_t$ if and only if $t \in Y$. Similarly $Y \backslash \{t\} \subseteq J_t$ if and only if $t \in Z$. To show that $\overline{\gamma}(Z) = Z \backslash \{t\}$ we consider various cases.

(1) Suppose that $t \in Z$ and that $t \notin Y$. Then $Y \subseteq J_t$ and $Z = \mathcal{O}^{\overline{X}}(Y) = \mathcal{O}^X(Y) \cup \{t\}$. Therefore $Z \backslash \{t\} = \mathcal{O}^X(Y) \in L$ and, since $Z \backslash \{t\} \nsubseteq J_t$, it follows that $\overline{\gamma}(Z) = \mathrm{icl}(Z \backslash \{t\}) = Z \backslash \{t\}$.

(2) Assume that $t \in Z$ and $t \in Y$. Then $Y \subseteq J_t \cup \{t\}$ and

$$\begin{aligned} Z = \mathcal{O}^{\overline{X}}(Y) &= \mathcal{O}^{\overline{X}}((Y \backslash \{t\}) \cup \{t\}) \\ &= \mathcal{O}^{\overline{X}}(Y \backslash \{t\}) \cap \mathcal{O}^{\overline{X}}(t) \\ &= (\mathcal{O}^X(Y \backslash \{t\}) \cup \{t\}) \cap (J_t \cup \{t\}) \\ &= (\mathcal{O}^X(Y \backslash \{t\}) \cap J_t) \cup \{t\}. \end{aligned}$$

Therefore $Z \backslash \{t\} = \mathcal{O}^X(Y \backslash \{t\}) \cap J_t$ and, since $Z \backslash \{t\} \subseteq J_t$, we have, using Lemma 2.4.11, $\mathrm{icl}(Z \backslash \{t\}) = \mathrm{cl}(\mathcal{O}^X(Y \backslash \{t\}) \cap J_t) \cap J_t = \mathcal{O}^X(Y \backslash \{t\}) \cap J_t$. Therefore $\overline{\gamma}(Z) = \mathrm{icl}(Z \backslash \{t\}) = Z \backslash \{t\}$.

(3) Assume that $t \notin Z$ and $t \notin Y$. In this case $Z = \mathcal{O}^X(Y) \in L$ and, since $Z \nsubseteq J_t$, it follows that $\overline{\gamma}(Z) = \mathrm{cl}(Z) = Z = Z \backslash \{t\}$.

(4) Assume that $t \notin Z$ and $t \in Y$. Since $t \notin Z$ this means that $Z \subseteq J_t$ and $\overline{\gamma}(Z) = \mathrm{cl}(Z) \cap J_t$. Now

$$\begin{aligned} Z = \mathcal{O}^{\overline{X}}(Y) &= \mathcal{O}^{\overline{X}}(Y \backslash \{t\}) \cap (J_t \cup \{t\}) \\ &= \mathcal{O}^X(Y \backslash \{t\}) \cap J_t \in L_t, \end{aligned}$$

as $Y \backslash \{t\} \nsubseteq J_t$. Hence $Z = \mathrm{cl}(Z) \cap J_t$ and so $\overline{\gamma}(Z) = Z = Z \backslash \{t\}$.

Thus $\overline{\gamma}(Z) = Z \backslash \{t\}$, for all $Z \in \overline{L}$.

Now suppose that $Z_1, Z_2 \in \overline{L}$ such that $Z_1 \neq Z_2$. Suppose that $\overline{\gamma}(Z_1) = \overline{\gamma}(Z_2)$. As $\gamma(Z_i) = Z_i \backslash \{t\}$ we must have, after interchanging Z_1 and Z_2 if necessary, $Z_1 = Z_2 \cup \{t\}$; so $t \in Z_1 \in \overline{L}$ and $Z_1 \backslash \{t\} \in \overline{L}$. $\qquad \square$

Definition 2.20. *Let $\beta : L \to \overline{L}$ be the map given by $\beta(Y) = \mathrm{cl}^{\overline{X}}(Y)$, for $Y \in L$. Let $\gamma : \overline{L} \to L$ be the map given by $\gamma(Z) = \mathrm{cl}^X(X \cap Z)$, for $Z \in \overline{L}$.*

Corollary 2.21. *We have $\beta = \overline{\beta}\tilde{\beta}$ and $\gamma = \tilde{\gamma}\overline{\gamma}$. The maps β and γ are homomorphisms of partially ordered sets. For $Y \in L$*

$$\beta(Y) = \begin{cases} Y & \text{if } \mathcal{O}^X(Y) \nsubseteq J_t, \\ Y \cup \{t\} & \text{if } \mathcal{O}^X(Y) \subseteq J_t \end{cases}.$$

Moreover $\gamma\beta = \mathrm{id}_L$, β is injective and γ is surjective.

2.4. **The height of the extended lattice.** In this section we determine the possible differences in height between the lattices L and $\overline{L}$. By a *strong ascending chain* in a partially ordered set L is meant a sequence $C_0, C_1 \ldots$ of elements of L such that $C_i < C_{i+1}$, for all $i \geq 0$. *Strong descending chains* are defined analogously, replacing $<$ by $>$. The *length* of a finite strong chain $C_0, \ldots, C_d$ is d. If $C_0, C_1 \ldots$ is a sequence of elements of L such that $C_i \leq C_{i+1}$, for all $i \geq 0$, then we call $C_0, C_1 \ldots$ a *weak* ascending chain. Weak descending chains are defined analogously. The *length* of a weak chain $\mathcal{C}$ is the maximum of the lengths of strong chains obtained by taking subsequences of $\mathcal{C}$. We shall from now on use chain to mean either weak or strong chain, if the meaning is clear. We denote the length of a chain $\mathcal{C}$ by $l(\mathcal{C})$. Let L and L' be partially ordered sets and let $\phi : L \to L'$ be a homomorphism or anti-homomorphism of partially ordered sets. If $\mathcal{C}$ is a chain $C_0, \ldots, C_d$ in L then we denote by $\phi(\mathcal{C})$ the chain $\phi(C_0), \ldots, \phi(C_d)$, in L'. Clearly the length of $\mathcal{C}$ is greater than or equal to the length of $\phi(\mathcal{C})$.

Definition 2.22. *The* height $h(L)$ *of a lattice L is the length of its maximal chain, if this exists, and is infinite otherwise.*

The following is a corollary of Lemmas 2.16 and 2.19

Corollary 2.23. $h(L) \leq h(\tilde{L}) \leq h(\overline{L})$.

Proof. If $\mathcal{C}$ is a maximal chain in L then $\tilde{\beta}(\mathcal{C})$ is a chain in $\tilde{L}$. As $\tilde{\beta}$ is injective $\tilde{\beta}(\mathcal{C})$ has the same length as $\mathcal{C}$ and the result follows. The second inequality follows similarly. $\square$

Example 2.24. Let Γ be the graph of Figure 2.1 and let $J_t = \{a, c\}$. Then L consists of X, the orthogonal complements (in X) of a, b, c and d, and also $\{b, c\} = \mathcal{O}^X\{b, c\}$, $\{b\} = \mathcal{O}^X\{a, c\}$, $\{c\} = \mathcal{O}^X\{b, d\}$ and $\emptyset$. Therefore $h(L) = 4$. $\tilde{L}$ contains in addition the set J_t and the set $\{a\} = J_t \cap \mathcal{O}^X(a)$. It follows that $h(\tilde{L}) = 4$ as well. Finally, the maximal proper subsets of $\overline{L}$ are the orthogonal complements (in $\overline{X}$) of a, b, c and t (as $\mathcal{O}^{\overline{X}}(d) \subseteq \mathcal{O}^{\overline{X}}(c)$). The only one of these sets with 4 elements is $\mathcal{O}^{\overline{X}}(c)$. However, the intersection of $\mathcal{O}^{\overline{X}}(c)$ with any other proper maximal subset has at most 2 elements. Hence $\overline{L}$ can have height at most 4. As $h(\tilde{L}) = 4$ it now follows that $h(\overline{L}) = h(\tilde{L}) = h(L) = 4$.

Example 2.25. Let $\overline{\Gamma}$ be the graph of Figure 2.1 and Γ be the graph obtained by removing vertex c. Then, with $t = c$ we have $X = \{a, b, d\}$ and $J_t = \{b, d\}$. In this case L consists of the sets X, $\mathcal{O}^X(a)$, $\mathcal{O}^X(d)$ and $\emptyset$, so $h(L) = 2$. L_t contains in addition the sets J_t and $\mathcal{O}^X(a) \cap J_t = \{b\}$. Thus $h(\tilde{L}) = 3$. Moreover, from the previous example $h(\overline{L}) = 4$. (The *semibraid* group on n generators is the partially commutative group G_n with presentation

$$\langle x_1, \ldots, x_n | [x_i, x_j] = 1, \text{ if } |i - j| \geq 2 \rangle.$$

The graphs of this example are those of G_3 and G_4, see [5] for further details)

In fact these two examples illustrate the two extremes in differences of height between L and $\tilde{L}$ and between $\tilde{L}$ and $\overline{L}$: as the following propositions show.

Proposition 2.26. $h(\tilde{L}) = h(L) + m$, *where* $m = 0$ *or* 1.

Proof. Let

$$\mathcal{C} = Z_0 < \cdots < Z_k$$

be a strictly ascending chain in $\tilde{L}$, with $k = h(\tilde{L})$. Then $\tilde{\gamma}(\mathcal{C})$ is an ascending chain in L. If $Z_i \in L$ for all i then $\tilde{\gamma}(\mathcal{C}) = \mathcal{C}$, so Lemma 2.23 implies that $h(\tilde{L}) = h(L)$. Assume then that $Z_i \notin L$, for some i, and let r be the smallest integer such that $Z_r \in L$, for all $i \geq r$. Then $Z_i \subseteq J_t$, so $Z_i \in L_t$, for all $i \leq r - 1$. Using Lemma 2.16, $\tilde{\gamma}(Z_r) < \cdots < \tilde{\gamma}(Z_k)$ and $\tilde{\gamma}(Z_0) < \cdots < \tilde{\gamma}(Z_{r-1})$ are strictly ascending chains in L. The length of $\tilde{\gamma}(\mathcal{C})$ is therefore at least $k - 1 = h(\tilde{L}) - 1$; so $h(L) \geq h(\tilde{L}) - 1$, and the lemma follows from Lemma 2.23. $\square$

Proposition 2.27. $h(\overline{L}) = h(\tilde{L}) + m$, *where* $m = 0$ *or* 1.

Proof. Let $\overline{\mathcal{C}} = Z_0 < \cdots < Z_k$ be a strictly ascending chain in $\overline{L}$. As $\overline{\gamma}$ is inclusion preserving the sequence $\overline{\gamma}(\overline{\mathcal{C}})$ is ascending. Let r be the least integer such that $t \in Z_i$ for $r \geq i$. Then, from Lemma 2.19,

$$\overline{\gamma}(Z_0) < \cdots < \overline{\gamma}(Z_{r-1}) \leq \overline{\gamma}(Z_r) < \overline{\gamma}(Z_{r+1}) < \cdots < \overline{\gamma}(Z_k),$$

so $\overline{\gamma}(\overline{\mathcal{C}})$ has length at least $k - 1$. $\square$

Theorem 2.28. $h(\overline{L}) = h(L) + m$, *where* $m = 0, 1$ *or* 2.

The next two examples show that a difference of one between the heights of L and $\overline{L}$ may occur and may be due either to a difference in height between L and $\tilde{L}$ or between $\tilde{L}$ and $\overline{L}$.
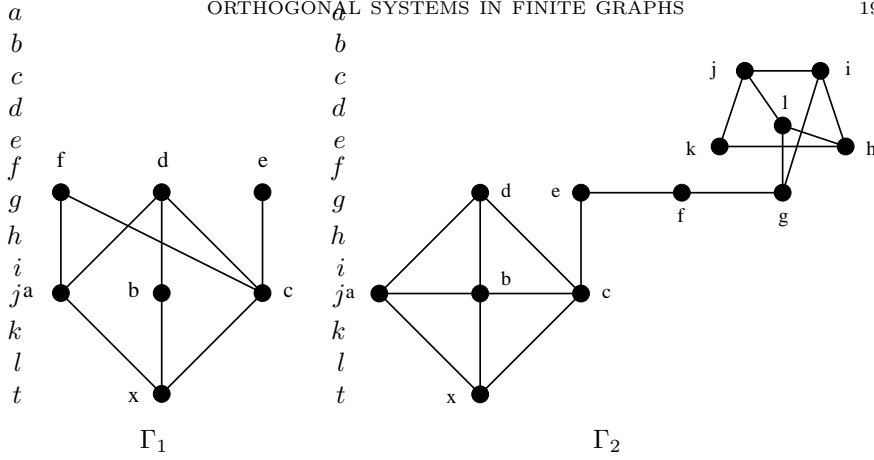
FIGURE 2.5. Examples 2.29 and 2.30

Example 2.29. Let Γ be the graph obtained by removing vertex t from the graph $\overline{\Gamma} = \Gamma_1$ of Figure 2.5 and let $J_t = \{a, b, c\}$. Then $h(L) = 4$ and $h(\overline{L}) = 5$. In this case the height of the lattice $\tilde{L}$ is 5, with a maximal chain

$$X > \mathcal{O}^X(d) > J_t > \mathcal{O}^X(f) \cap J_t > \mathcal{O}^X(f) \cap J_t \cap \mathcal{O}^X(a) > \emptyset.$$

Example 2.30. Let Γ be the graph obtained by removing vertex t from the graph $\overline{\Gamma} = \Gamma_2$ of Figure 2.5 and again let $J_t = \{a, b, c\}$. Then $h(L) = 5$ and $h(\tilde{L}) = 5$. In this case the maximal chains in the lattice L involve only the vertices g, h, i, j, k, l and the sets of L_t involve only vertices a, b, c. Therefore the lattice $\tilde{L}$ has some new chains of length 5 but none of length 6. However computation shows (see [5]) that $h(\overline{L}) = 6$.

2.5. **The structure of the extended lattice.** Next we use the results of the Section 2.3 to describe the lattice $\overline{L}$ in terms of the lattice L. We make the following definition. Suppose that L is a lattice which is a subset of a lattice L' and that the partial ordering in L is the restriction of the partial ordering in L'. Assume that L contains a subset S such that there is an isomorphism of partially ordered sets, ρ, from S to $L' \backslash L$. Then we say that L' is obtained from L by *doubling S along ρ*.

Recall from Section 2.3 that if $Z \in \tilde{L}$ and $Z \nsubseteq J_t$ then $Z \in L$. This, together with (2.1), prompts the following definition.

Definition 2.31. *Let*

$$R = \{Z \in \tilde{L} | Z \nsubseteq J_t \text{ and } \text{cl}(Z \cap J_t) = Z\}$$

and let ρ be the map from R to $\tilde{L}$ given by $\rho(Z) = Z \cap J_t$.

If $Z \in R$ then $Z \in L$ and $Z \notin L_t$, as $Z \nsubseteq J_t$. Furthermore, from (2.1), $\rho(Z) \in L_t \backslash L = \tilde{L} \backslash L$.

Proposition 2.32. $\tilde{L}$ *is obtained from* $L \subseteq \tilde{L}$ *by doubling* R *along* ρ.

Proof. As ρ clearly preserves inclusion it suffices to show that ρ is a bijection. If $\rho(Y) = \rho(Z)$, with $Y, Z \in R$ then $Z = \text{cl}(Z \cap J_t) = \text{cl}(Y \cap J_t) = Y$, so ρ is injective. From (2.1) if follows that ρ is also surjective. $\qquad\square$

The lattice $\overline{L}$ is obtained from $\tilde{L}$ by a doubling on an appropriate subset of $\tilde{L}$. To see this we use the following strengthening of the final part of Lemma 2.19. We remark that condition (2.3) of the lemma can be expressed more succinctly in terms of complements by noting that

(1) $\mathcal{O}^X(\mathcal{O}^X(Y) \cap J_t) = \mathcal{O}^X(\mathcal{O}^{J_t}(Y))$ and
(2) if $Y \subseteq J_t$ then $(\mathcal{O}^X(\mathcal{O}^X(Y) \cap J_t)) \cap J_t = \text{cl}^{J_t}(Y) \in L(J_t)$.

Lemma 2.33. *Let* $Y \subset X$. *Then* Y *and* $Y \cup \{t\}$ *belong to* $\overline{L}$ *if and only if* $\mathcal{O}^X(Y) \nsubseteq J_t$ *and*

$$(2.3) \qquad Y = \begin{cases} \mathcal{O}^X(\mathcal{O}^X(Y) \cap J_t), & \text{if } Y \nsubseteq J_t \\ \mathcal{O}^X(\mathcal{O}^X(Y) \cap J_t) \cap J_t, & \text{if } Y \subseteq J_t \end{cases}.$$

Proof. Suppose that $\mathcal{O}^X(Y) \nsubseteq J_t$. If $Y \nsubseteq J_t$ then

$$\text{cl}^{\overline{X}}(Y \cup \{t\}) = \mathcal{O}^{\overline{X}}(\mathcal{O}^X(Y) \cap J_t)$$
$$(2.4) \qquad\qquad = \mathcal{O}^X(\mathcal{O}^X(Y) \cap J_t) \cup \{t\}.$$

If, on the other hand, $Y \subseteq J_t$ then $\mathcal{O}^{\overline{X}}(Y \cup \{t\}) = (\mathcal{O}^X(Y) \cap J_t) \cup \{t\}$ so

$$\text{cl}^{\overline{X}}(Y \cup \{t\}) = \mathcal{O}^{\overline{X}}\left((\mathcal{O}^X(Y) \cap J_t) \cup \{t\}\right)$$
$$= \left(\mathcal{O}^X(\mathcal{O}^X(Y) \cap J_t) \cup \{t\}\right) \cap (J_t \cup \{t\})$$
$$(2.5) \qquad\qquad = \left(\mathcal{O}^X(\mathcal{O}^X(Y) \cap J_t) \cap J_t\right) \cup \{t\}.$$

In both cases, if in addition (2.3) holds then $\text{cl}^{\overline{X}}(Y \cup \{t\}) = Y \cup \{t\}$ and $Y \cup \{t\} \in \overline{L}$.

Now, given that $\mathcal{O}^X(Y) \nsubseteq J_t$ and (2.3) holds, choose $x \in \mathcal{O}^X(Y)$ such that $x \notin J_t$. Then $\mathcal{O}^{\overline{X}}(x) = \mathcal{O}^X(x) \supseteq \text{cl}^X(Y) \supseteq Y$ and $t \notin \mathcal{O}^{\overline{X}}(x)$. From the above $Y \cup \{t\} \in \overline{L}$, so $Y \cup \{t\} = \mathcal{O}^{\overline{X}}(Z)$, for some $Z \in \overline{L}$. Then $\mathcal{O}^{\overline{X}}(Z \cup \{x\}) = \mathcal{O}^{\overline{X}}(Z) \cap \mathcal{O}^{\overline{X}}(x) = Y$; and $Y \in \overline{L}$.

Conversely suppose that Y and $Y \cup \{t\}$ belong to $\overline{L}$. In this case if $\mathcal{O}^X(Y) \subseteq J_t$ then $\mathcal{O}^{\overline{X}}(Y) \subseteq \mathcal{O}^X(Y) \cup \{t\}$ so $\text{cl}^{\overline{X}}(Y) \supseteq \mathcal{O}^{\overline{X}}(\mathcal{O}^X(Y)) \cap (J_t \cup \{t\})$. Thus $t \in \text{cl}^{\overline{X}}(Y)$ and $Y \notin \overline{L}$, a contradiction. Thus $\mathcal{O}^X(Y) \nsubseteq J_t$. If $Y \nsubseteq J_t$ then, from (2.4), $Y = \text{cl}^{\overline{X}}(Y \cup \{t\}) \backslash \{t\} = \mathcal{O}^X(\mathcal{O}^X(Y) \cap J_t)$. If, on

the other hand, $Y \subseteq J_t$ then (2.5) implies that $Y = (\mathcal{O}^X(\mathcal{O}^X(Y) \cap J_t)) \cap J_t$, as claimed. $\qquad \square$

The lemma prompts the following definition.

Definition 2.34. *Let*

$$S_1 = \{Y \subset X | Y \nsubseteq J_t, \mathcal{O}^X(Y) \nsubseteq J_t, \text{ and } Y = \mathcal{O}^X(\mathcal{O}^X(Y) \cap J_t)\}$$

and

$$S_2 = \{Y \subset X | Y \subseteq J_t, \mathcal{O}^X(Y) \nsubseteq J_t, \text{ and } Y = (\mathcal{O}^X(\mathcal{O}^X(Y) \cap J_t)) \cap J_t\}.$$

Let $S = S_1 \cup S_2$ and let $T = \{Y \cup \{t\} | Y \in S\}$. Let σ be the map from S to T given by $\sigma(Y) = Y \cup \{t\}$.

From Lemma 2.33 it follows that $S \cup T \subseteq \overline{L}$ and by definition $S \subseteq \tilde{L}$. Moreover, from Lemma 2.19, $\overline{\beta}(Y) = Y$, for all $Y \in S$, so $S = \overline{\beta}(S) \subseteq \overline{\beta}(\tilde{L}) \subseteq \overline{L}$.

Proposition 2.35. *The lattice $\overline{L}$ is obtained from $\overline{\beta}(\tilde{L}) \subseteq \overline{L}$ by doubling S along σ.*

Proof. Using Lemma 2.19, if $Y \in S$ and $Y \cup \{t\} = \overline{\beta}(U)$, for some element $U \in \tilde{L}$, then $t \in \overline{\beta}(U)$ implies that $\mathcal{O}^X(U) \in J_t$. However $Y \cup \{t\} = \overline{\beta}(U) = U \cup \{t\}$ so $U = Y$ and $\mathcal{O}^X(Y) \nsubseteq J_t$, a contradiction. Hence no element of T belongs to the image of $\overline{\beta}$. If $Z \in \overline{L}$ and Z is not in the image of $\overline{\beta}$ then, from Lemma 2.19 again, $\overline{\gamma}(Z) = Z \backslash \{t\} \in \tilde{L}$ and so $\overline{\beta}(Z \backslash \{t\}) \neq Z$. Thus either $t \notin Z$ and $\overline{\beta}(Z \backslash \{t\}) = \overline{\beta}(Z) = Z \cup \{t\}$ or $t \in Z$ and $\overline{\beta}(Z \backslash \{t\}) = Z \backslash \{t\}$. In the former case $Z \in \overline{L}$ and $\overline{\beta}(Z) = Z \cup \{t\} \in \overline{L}$ so $Z \in S$ and $Z \cup \{t\} \in T \cap \text{Im}(\overline{\beta})$, a contradiction. Hence $\overline{\beta}(Z \backslash \{t\}) = Z \backslash \{t\} \in \overline{L}$ and $t \in Z$. It follows from Lemma 2.33 that $Z \backslash \{t\} \in S$ so $Z \in T$. That is, $T = \overline{L} \backslash \overline{\beta}(\tilde{L})$. As σ is an inclusion preserving bijection the result follows. $\qquad \square$

2.6. **Extension along the complement of a simplex.** In those cases where γ is injective it follows, from Corollary 2.21, that γ is a bijection and so an isomorphism of lattices. We now consider under which conditions this may occur. Let $V = \mathcal{O}^{\overline{X}}(t) = J_t \cup \{t\} \in \overline{L}$. If γ is injective then $V = \beta\gamma(V) = \text{cl}^X(J_t) \cup \{t\}$, so $J_t = \text{cl}^X(J_t) \in L$. Therefore $J_t \in L$ is a necessary condition for γ to be injective. We shall show, in Section 2.8, that if J_t is closed then $h(L) = h(\overline{L})$; but we shall also see in Lemma 2.37 that a further condition is required to ensure that γ is injective. First however we establish a simple form for γ when J_t is closed.

Lemma 2.36. *If $J_t \in L$ then $\tilde{L} = L$ and $\gamma(Z) = Z \backslash \{t\}$, for all $Z \in \overline{L}$. Moreover, in the notation of* Definition 2.34, *$S_1 = \emptyset$ so $\overline{L}$ is obtained from $\beta(L) \subseteq \overline{L}$ by doubling S_2 along σ.*

Proof. If $J_t \in L$ then L_t is a subset of L, so $\tilde{L} = L$, as claimed. In this case $\gamma = \overline{\gamma}$ and $\beta = \overline{\beta}$, so the first statement of the Lemma follows from Lemma 2.19. If $Y \in S_1$ then $Y \in L$ and $Y = \mathcal{O}^X(W)$, where $W = \mathcal{O}^X(Y) \cap J_t \in L$. However this means $\mathcal{O}^X(Y) = W \subseteq J_t$, a contradiction. $\square$

Lemma 2.37. *The map γ is an isomorphism of lattices if and only if $J_t = \mathcal{O}^X(S)$, where S is a simplex of Γ.*

Proof. First assume that $J_t = \mathcal{O}^X(A)$, where $A \subseteq X$ is a simplex. In this case, in the notation of Definition 2.34, $Y \in S_2$ implies $Y \in L(J_t)$, so $Y = \mathcal{O}^{J_t}(W)$, for some $W \subseteq J_t$. Now $W \subseteq J_t = \mathcal{O}^X(A) = \mathcal{O}^{J_t}(A)$ which implies $\mathcal{O}^{J_t}(\mathcal{O}^{J_t}(A)) \subseteq \mathcal{O}^{J_t}(W) = Y$. As A is a simplex $A \subseteq J_t$ so $A \subseteq \mathcal{O}^{J_t}(\mathcal{O}^{J_t}((A))$ and thus $\mathcal{O}^X(Y) \subseteq J_t$, contrary to the definition of S_2. Therefore $S_1 = S_2 = \emptyset$ and from Lemma 2.36 $L = \overline{L}$.

On the other hand suppose that $J_t = \mathcal{O}^X(N)$, where N is not a simplex. Then, from Lemma 2.1.6, there is $s \in N$ such that $s \notin J_t$. Therefore $t \notin \mathcal{O}^{\overline{X}}(N)$ and we have $J_t = \mathcal{O}^{\overline{X}}(N) \in \overline{L}$. Hence $\gamma(J_t) = J_t = \gamma(J_t \cup \{t\})$ and γ is not injective. From the remarks at the beginning of the Section it follows that if J_t is not the orthogonal complement of a simplex in X then γ is not injective. (It is not difficult to see that in this case $J_t \in S_2$.) $\square$

As a consequence of this lemma we obtain the following theorem.

Theorem 2.38. *The lattices L and $\overline{L}$ are isomorphic if and only if $J_t = \mathcal{O}^X(S)$, where $S \subset X$ is a simplex, in which case γ is an isomorphism.*

Proof. From Lemma 2.37, if $J_t = \mathcal{O}^X(S)$, where $S \subset X$ is a simplex, then the lattices are isomorphic and γ is an isomorphism. Now suppose that J_t is not of this form. The map $\beta : L \to \overline{L}$ is injective so $|L| \leq |\overline{L}|$. If $|L| = |\overline{L}|$ then, as $\gamma\beta = \mathrm{id}_L$, it follows that γ is also injective, contrary to Lemma 2.37. Thus $|L| < |\overline{L}|$ and the lattices are not isomorphic. $\square$

2.7. **Abelian Inflation and Deflation.** In this section we consider further the case where the set J_t defined above is the orthogonal complement of a simplex, as in the previous section. First we introduce some equivalence classes on subsets of vertices Γ. We say that two subsets S and T of X are $\perp$-*equivalent* in X and write $S \sim_\perp T$ if and only if $S^\perp = T^\perp$; that is $\mathcal{O}^X(S) = \mathcal{O}^X(T)$.

Lemma 2.39. *Let S and T be subsets of X.*

(1) $S \sim_\perp T$ if and only if $T \subseteq \mathrm{cl}^X(S)$ and $S \subseteq \mathrm{cl}^X(T)$.
(2) If $S \sim_\perp T$ and $Y \in L(\Gamma)$ then $S \subseteq Y$ implies that $T \subseteq Y$.
(3) If S is a simplex and $S \sim_\perp T$ then T is a simplex. In particular, in this case, $G(\Gamma')$ is an Abelian group, where Γ' denotes the full subgraph of Γ on $S \cup T$.

Proof. To see the first statement note that, using Lemma 2.1, $S \sim_\perp T$ if and only if $\mathrm{cl}^X(S) = \mathrm{cl}^X(T)$. It follows that $S \sim_\perp T$ implies that $S \subseteq \mathrm{cl}^X(T)$ and $T \subseteq \mathrm{cl}^X(S)$. Conversely if $S \subseteq \mathrm{cl}^X(T)$ then $S^\perp \supseteq T^{\perp\perp\perp} = T^\perp$. Similarly if $T \subseteq \mathrm{cl}^X(S)$ then $T^\perp \supseteq S^\perp$ and the result follows. To prove the second statement note that by Lemma 2.4, $S \subseteq Y$ and Y closed implies $\mathrm{cl}^X(S) \subseteq Y$. Thus $T \subseteq \mathrm{cl}^X(T) = \mathrm{cl}^X(S) \subseteq Y$. For the third statement we have $S \subseteq \mathcal{O}^X(S) = \mathcal{O}^X(T)$, since S is a simplex, and so $T \subseteq \mathcal{O}^X(S) = \mathcal{O}^X(T)$. Hence T is a simplex and the result follows. $\square$

In the light of Lemma 2.39.3 we define the *Abelian closure* $\mathrm{acl}(S)$ of a simplex S to be the union of subsets T of X such that $S \sim_\perp T$. Then $S \subseteq \mathrm{acl}(S)$ and it is easy to see then that $\mathrm{acl}(S)$ is the unique maximal simplex such that $S \sim_\perp \mathrm{acl}(S)$.

Now let Δ be a graph with vertices V. Let S be a simplex of Δ and $y \in V$ with $y \notin S$ and suppose that $S \sim_\perp \{y\}$ in Δ: that is $\mathcal{O}^V(S) = \mathcal{O}^V(y)$. Let $\Delta_y = \Delta \backslash \{y\}$. Then Δ_y is called an *elementary Abelian deflation* of Δ and Δ is called an *elementary Abelian inflation* of Δ_y. In this case the subgroup of Δ_y generated by S is a free Abelian group of rank $|S|$ and the subgroup of Δ generated by $S \cup \{y\}$ is free Abelian of rank $|S| + 1$.

If a graph Ω can be obtained from a graph Γ by finitely many elementary Abelian inflations then Ω is called an *Abelian inflation* of Γ and Γ is called an *Abelian deflation* of Ω. The same terminology carries over to the respective partially commutative groups.

Proposition 2.40. *If Δ is an Abelian inflation of Γ then $L(\Delta) \simeq L(\Gamma)$.*

Proof. It suffices to prove the result in the case where Δ is an elementary Abelian inflation of Γ. Suppose then that $\Gamma = \Delta_t$, for some vertex t of Δ. To be more explicit let $V(\Delta) = \overline{X}$, assume that $t \in \overline{X}$, $S \subseteq \overline{X}$ is a simplex, $t \notin S$ and $S \sim_\perp \{t\}$ in Δ. Let $X = V(\Gamma)$. Then, as $\Gamma = \Delta_t$ we have $\overline{X} = X \cup \{t\}$ and $\mathcal{O}^{\overline{X}}(t) = \mathcal{O}^{\overline{X}}(S)$. Let $J_t = \mathcal{O}^{\overline{X}}(t) \backslash \{t\}$. Then, as $S \subseteq X$, we have $\mathcal{O}^X(S) = J_t \in L(\Gamma)$. As Δ is obtained from Γ by adding the vertex t which is joined to precisely those vertices in $J_t = \mathcal{O}^X(S)$, and S is a simplex, it follows from Theorem 2.38 that $L(\Gamma) \simeq L(\Delta)$, as claimed. $\square$

2.8. **Extension along a closed set.** We saw in Section 2.6 that if J_t is a closed set then, in the notation of Definition 2.34, $\overline{L}$ is obtained from $\beta(L)$ by doubling S_2 along σ. In this section we shall show that if J_t is closed then $h(\overline{L}) = h(L)$. If $J_t = \mathcal{O}^X(S)$ where S is a simplex then $\overline{\Gamma}$ is an Abelian inflation of Γ, so this follows from Proposition 2.40. Therefore we assume that $A \subseteq X$, such that A is not a simplex, and $J_t = \mathcal{O}^X(A) \in L$. As A is not a simplex the set $A' = A \backslash J_t$ is non-empty. Fix $a \in A'$.

Now let $Y \in \overline{L}$ with $t \in Y$. Then $Y = \mathcal{O}^{\overline{X}}(Z)$, where $Z \subseteq J_t \cup \{t\}$. There are two possibilities. Either

(1) $Z \subseteq J_t$, in which case $A \cup \{t\} \subseteq \mathcal{O}^{\overline{X}}(Z) = Y$; or
(2) $Z \nsubseteq J_t$, in which case $Z = W \cup \{t\}$, where $W \subseteq J_t$, so $a \notin Y$.

In the latter case

$$
\begin{aligned}
Y &= \mathcal{O}^{\overline{X}}(W \cup \{t\}) \\
&= (\mathcal{O}^X(W) \cup \{t\}) \cap (J_t \cup \{t\}) \\
&= (\mathcal{O}^X(W) \cap J_t) \cup \{t\}
\end{aligned}
$$

whereas

$$
\begin{aligned}
\mathcal{O}^{\overline{X}}(W \cup \{a\}) &= \mathcal{O}^{\overline{X}}(W) \cap \mathcal{O}^{\overline{X}}(a) \\
&= (\mathcal{O}^X(W) \cup \{t\}) \cap \mathcal{O}^X(a) \\
&= \mathcal{O}^X(W) \cap \mathcal{O}^X(a).
\end{aligned}
$$

This prompts us to define a map $\alpha : \overline{L} \to \overline{L}$ by

$$
\alpha(Y) = \begin{cases} \mathcal{O}^{\overline{X}}(W \cup \{a\}) \text{ if } t \in Y, a \notin Y \\ Y \text{ otherwise} \end{cases} .
$$

Note that

$$
(2.6) \qquad t \notin \alpha(Y) \text{ and } Y \backslash \{t\} \cup \{a\} \subseteq \alpha(Y), \text{ if } t \in Y \text{ and } a \notin Y
$$

and that

$$
(2.7) \qquad \text{either } t \notin \alpha(Y) \text{ or } A \cup \{t\} \subseteq \alpha(Y) \text{ for all } Y \in \overline{L}.
$$

Now let $\overline{\mathcal{C}} = Z_1 < \cdots < Z_k$ be a strong ascending chain in $\overline{L}$. Let $\alpha(\overline{\mathcal{C}}) = \alpha(Z_1) \leq \cdots \leq \alpha(Z_k)$.

Lemma 2.41. $\alpha(\overline{\mathcal{C}})$ *is a strong ascending chain in* $\overline{L}$.

Proof. Define $r = r(\overline{\mathcal{C}})$ to be the smallest integer such that $t \in Z_r$. If no such r exists then $\alpha(\overline{\mathcal{C}}) = \overline{\mathcal{C}}$ and there is nothing to prove. Suppose then that $1 \leq r \leq k$. Let s be the smallest integer such that $A \cup \{t\} \subseteq Z_s$ (and set $s = k + 1$ if $A \cup \{t\} \nsubseteq Z_k$). Then $r \leq s \leq k + 1$. For i such that $1 \leq i \leq r - 1$ or $s \leq i \leq k$ we have $\alpha(Z_i) = Z_i$. Therefore we need only check that $\alpha(Z_i) < \alpha(Z_{i+1})$ for i such that $r - 1 \leq i \leq s$. If $r = s$ then also $\alpha(Z_r) = Z_r$ and so $\alpha(\overline{\mathcal{C}}) = \overline{\mathcal{C}}$ and the Lemma holds.

Assume then that $r < s$. In this case $a \notin Z_r$ and so $a \notin Z_{r-1}$. Therefore $a \notin \alpha(Z_{r-1}) = Z_{r-1}$ but $a \in \alpha(Z_r)$. As $t \notin Z_{r-1}$ we have

$$
Z_{r-1} \leq Z_r \backslash \{t\} \leq \alpha(Z_r),
$$

so $\alpha(Z_{r-1}) < \alpha(Z_r)$.

To see that $\alpha(Z_{s-1}) < \alpha(Z_s)$ write $Z_s = \mathcal{O}^{\overline{X}}(Y_s)$, where $Y_s \subseteq J_t$ and $Z_{s-1} = \mathcal{O}^{\overline{X}}(W_{s-1} \cup \{t\})$, where $W_{s-1} \subseteq J_t$. As $Z_{s-1} < Z_s$ we have $W_{s-1} \cup \{t\} \geq Y_s$ and, as $t \notin Y_s$, $W_{s-1} \geq Y_s$; so $\mathcal{O}^X(W_{s-1}) \leq \mathcal{O}^X(Y_s)$. Therefore

$$\alpha(Z_{s-1}) = \mathcal{O}^X(W_{s-1}) \cap \mathcal{O}^X(a) \leq \mathcal{O}^X(Y_s) < \mathcal{O}^X(Y_s) \cup \{t\} = Z_s = \alpha(Z_s).$$

It remains to check that $\alpha(Z_i) < \alpha(Z_{i+1})$, where $r \leq i \leq s - 2$. Given such i we have, for $j = i$ and $j = i + 1$,

$$Z_j = \mathcal{O}^{\overline{X}}(W_j \cup \{t\}) = (\mathcal{O}^X(W_j) \cap J_t) \cup \{t\},$$

where $W_j \subseteq J_t$. As $Z_i < Z_{i+1}$ we have $W_i > W_{i+1}$ so $\mathcal{O}^X(W_i) \leq \mathcal{O}^X(W_{i+1})$. Therefore

$$\alpha(Z_i) = \mathcal{O}^X(W_i) \cap \mathcal{O}^X(a) \leq \mathcal{O}^X(W_{i+1}) \cap \mathcal{O}^X(a) = \alpha(Z_{i+1}).$$

Moreover, as $Z_i < Z_{i+1}$ there is $x \in \mathcal{O}^X(W_{i+1}) \cap J_t$ such that $x \notin \mathcal{O}^X(W_i) \cap J_t$. Hence $x \notin \mathcal{O}^X(W_i)$ and therefore $x \notin \alpha(Z_i)$. However $J_t \subseteq \mathcal{O}^X(a)$ so $x \in \mathcal{O}^X(W_{i+1}) \cap J_t$ implies $x \in \alpha(Z_{i+1})$. Thus $\alpha(Z_i) < \alpha(Z_{i+1})$. $\qquad\square$

Given a chain $\overline{\mathcal{C}} = Z_1 < \cdots < Z_k$ in $\overline{L}$ define $\gamma(\overline{\mathcal{C}})$ to be the chain $\gamma(Z_1) \leq \cdots \leq \gamma(Z_k)$.

Lemma 2.42. *If $\overline{\mathcal{C}}$ is a strictly ascending chain in $\overline{L}$ such that Z_i satisfies (2.7), for $i = 1, \ldots, k$, then $\gamma(\overline{\mathcal{C}})$ is a strictly ascending chain in L.*

Proof. As before define $r = r(\overline{\mathcal{C}})$ to be the smallest integer such that $t \in Z_r$. As J_t is closed we have $\gamma(Z) = Z \backslash \{t\}$, for all $Z \in \overline{L}$. Therefore it suffices to show that $\gamma(Z_{r-1}) < \gamma(Z_r)$. We have $A \subseteq \gamma(Z_r)$, by (2.7). If $A \subseteq Z_{r-1} = \mathcal{O}^{\overline{X}}(Y_{r-1})$ then $\mathcal{O}^{\overline{X}}(t) = J_t \cup \{t\} = \mathcal{O}^{\overline{X}}(A) \supseteq Y_{r-1}$. In this case $t \in \mathrm{cl}^{\overline{X}}(t) \subseteq Z_{r-1}$, contrary to the definition of r. Hence $A \nsubseteq Z_{r-1}$ and so $\gamma(Z_{r-1}) < \gamma(Z_r)$. $\qquad\square$

Now, given any strictly ascending chain $\overline{\mathcal{C}}$ in $\overline{L}$ we may, according to Lemma 2.41, construct a strictly ascending chain $\mathcal{C} = \alpha(\overline{\mathcal{C}})$, satisfying (2.7); as in the hypothesis of Lemma 2.42. Applying γ to $\mathcal{C}$ we obtain a strictly ascending chain $\gamma(\mathcal{C})$ in L of the same length as $\overline{\mathcal{C}}$. Therefore we have the following proposition.

Proposition 2.43. *If J_t is closed then $h(L) = h(\overline{L})$.*

2.9. **Extension along the complement of a co-simplex.** A subset $A \subseteq X$ is called a *co-simplex* if $A \cap \mathcal{O}^X(A) = \emptyset$. In this section we consider the case $J_t = \mathcal{O}^X(A)$ where A is a co-simplex. In this case if $Y \in L$ and $Y \subseteq J_t$ then $\mathcal{O}^X(Y) \supseteq \mathcal{O}^X(J_t) \supseteq A$. As $A \cap J_t = \emptyset$ we have $\mathcal{O}^X(Y) \nsubseteq J_t$, for all such Y. Therefore, if A is a co-simplex,

$$S_2 = \{Y \in L | Y \subseteq J_t, Y \in L(J_t)\} = L \cap L(J_t) = L(J_t),$$

as $L(J_t) \subseteq L$. Therefore $\overline{L}$ is obtained from L by doubling $L(J_t)$ along σ.

It is easy to find examples showing that in general there may be elements of L which are subsets of J_t but do not belong to $L(J_t)$. This motivates the following definition.

Definition 2.44. *A closed subset $J \in L$ is* realisable *if $L(J) = \{Y \in L | Y \subseteq J\}$.*

Lemma 2.45. *An element $J \in L$ is realisable if and only if, for all $s \in X\backslash J$ there exists $W \subseteq J$ such that $\mathcal{O}^X(s) \cap J = \mathcal{O}^X(W) \cap J$.*

Proof. Let $J = \mathcal{O}^X(A)$, where $A \in L$. Suppose that J is realisable and that $s \in X\backslash J$. Then $Y = \mathcal{O}^X(s) \cap J \in L$ and $Y \subseteq J$; so $Y \in L(J)$. Hence $Y = \mathcal{O}^X(W) \cap J$, where $W = \mathcal{O}^X(U) \cap J$, for some $U \subseteq J$, so $W \subseteq J$ as required.

Now suppose that J satisfies the condition of the Lemma. Let $Y \in L$ such that $Y \subseteq J$. Then $Y = \mathcal{O}^X(Z)$, for some $Z \in L$. Let $Z_1 = Z \cap J$ and $Z_2 = Z\backslash Z_1$. Fix $z \in Z_2$. By hypothesis there exists $W_z \subseteq J$ such that $\mathcal{O}^X(z) \cap J = \mathcal{O}^X(W_z) \cap J \in L(J)$. Therefore

$$\mathcal{O}^X(Z_2) \cap J = \bigcap_{z \in Z_2} (\mathcal{O}^X(z) \cap J) \in L(J).$$

As $Z_1 \subseteq J$ it is also true that $\mathcal{O}^X(Z_1) \cap J \in L(J)$. We have $Y \subseteq J$ so $Y = \mathcal{O}^X(Z) \cap J = (\mathcal{O}^X(Z_1) \cap J) \cap (\mathcal{O}^X(Z_2) \cap J) \in L(J)$, as required. $\square$

We now have the following proposition.

Proposition 2.46. *Let A be a co-simplex such that $\mathcal{O}^X(A)$ is realisable. If $J_t = \mathcal{O}^X(A)$ then $\overline{L}$ is obtained from L by doubling $S_2 = \{Y \in L | Y \subseteq J_t\}$ along σ.*

2.10. **Free Inflation and Deflation.** We now define another relation on the subsets of X, similar to that of Section 2.7: but giving rise to free groups instead of free Abelian groups. If J_t is the orthogonal complement of a co-simplex A then $\mathcal{O}^{\overline{X}}(A) = J_t$, since $A \cap J_t = \emptyset$, so $\mathcal{O}^{\overline{X}}(A)\backslash A = \mathcal{O}^{\overline{X}}(t)\backslash \{t\}$. This suggests the following definition. If Y and Z are subsets of X we say that Y and Z are *o-equivalent* and write $Y \sim_o Z$ if

$$(2.8) \qquad\qquad \mathcal{O}^X(Y)\backslash Y = \mathcal{O}^X(Z)\backslash Z.$$

Note that if Y is a co-simplex then $Y \sim_o Z$ implies that $\mathcal{O}^X(Y, Z) = \mathcal{O}^X(Y)$ and that $G(\mathcal{O}^X(Y)) = C(Y \cup Z)$ — the centraliser of Y and Z in $G(\Gamma)$.

We call a co-simplex A such that the full subgraph on A is the null graph a *free* co-simplex. If A is a free co-simplex and B is either a free co-simplex or a single vertex then $A \sim_o B$ implies that the subgroup of G generated

by A and B is a free group. We define the *free-closure* $\mathrm{fcl}(A)$ of a free co-simplex A to be the union of all free co-simplexes B such that $A \sim_o B$. It is easy to see that $A \subseteq \mathrm{fcl}(A)$ and that $\mathrm{fcl}(A)$ is the unique maximal free co-simplex such that $A \sim_o \mathrm{fcl}(A)$.

If J_t is the orthogonal complement of a free co-simplex then we say that $\overline{\overline{\Gamma}}$ is an *elementary free inflation* of Γ and that Γ is an *elementary free deflation* of $\overline{\Gamma}$. We say that Δ is a *free inflation* of Γ and Γ is a *free deflation* of Δ if Δ is obtained from Γ by a finite sequence of elementary free inflations.

2.11. **The Compression of a Graph.** We now use the ideas of Sections 2.7 and 2.10 to define an equivalence relation on the vertices of a finite graph Γ; which will give a decomposition of the automorphism group of Γ. We build this equivalence relation up out of the restrictions to singleton sets of the relations defined in Sections 2.7 and 2.10.

The restriction of the relation of $\perp$-equivalence to one-element subsets of X gives and equivalence relation $\sim_\perp$ on X such that $x \sim_\perp y$ if and only if $x^\perp = y^\perp$. Denote the equivalence class of x under $\sim_\perp$ by $[x]_\perp$.

Similarly, restricting the relation of o-equivalence to one-element subsets gives an equivalence relation $\sim_o$ on X such that $x \sim_o y$ if and only if $x^\perp \backslash \{x\} = y^\perp \backslash \{y\}$. Denote the equivalence class of x under $\sim_o$ by $[x]_o$.

Lemma 2.47.

> (1) $[x]_\perp$ *is a simplex, for all* $x \in X$.
> (2) $[x]_\perp \cap [x]_o = \{x\}$, *for all* $x \in X$.
> (3) *If* $|[x]_\perp| \geq 2$ *then* $|[x]_o| = 1$.
> (4) *If* $|[x]_o| \geq 2$ *then* $[x]_o$ *is a free co-simplex and* $|[x]_\perp| = 1$.

Proof. If $x \sim_\perp y$ then, as $x \in x^\perp$ we have $x \in y^\perp$, so $[x]_\perp$ is a simplex. If $z \in [x]_\perp \cap [x]_o$ then $x \in x^\perp = z^\perp$ but $x \notin x^\perp \backslash \{x\} = z^\perp \backslash \{z\}$: so it must be that $x = z$. If $|[x]_\perp| \geq 2$ then suppose that $y \neq x$ and $y \sim_\perp x$. If $z \neq x$ and $z \sim_o x$ then $z \neq y$, as $[x]_\perp \cap [x]_o = \{x\}$ from the above. Thus $y \in y^\perp = x^\perp$ implies $y \in x^\perp \backslash \{x\} = z^\perp \backslash \{z\}$, so $z \in y^\perp = x^\perp$, contradicting $z \sim_o x$. A similar argument shows that if $|[x]_o| \geq 2$ then $|[x]_\perp| = 1$. If $y \neq x$ and $y \sim_o x$ then $y \notin x^\perp$, as otherwise $x^\perp \backslash \{x\} \neq y^\perp \backslash \{y\}$. Hence $[x]_o$ is a free co-simplex if $|[x]_o| \geq 2$. $\qquad\square$

Now define a relation $\sim$ on X by $x \sim y$ if and only if either $x \sim_\perp y$ or $x \sim_o y$. From Lemma 2.47 $\sim$ is an equivalence relation and we denote the equivalence class of x under $\sim$ by $[x]$. Define subsets M_1, $M_\perp$ and M_o of X

by

$$M_1 = \{x \in X : [x] = [x]_o = [x]_\perp = \{x\}\},$$
$$M_\perp = \{x \in X : |[x]_\perp| \geq 2\} \text{ and}$$
$$M_o = \{x \in X : |[x]_o| \geq 2\}.$$

From Lemma 2.47 it follows that X is the disjoint union $X = M_1 \sqcup M_\perp \sqcup M_o$.

We use the equivalence $\sim$ to define a quotient graph of Γ.

Definition 2.48. *The* compression *of the graph* Γ *is the graph* Γ^c *with vertices* $X^c = \{[v] : v \in X\}$ *and an edge joining* $[u]$ *to* $[v]$ *if and only if* (u', v') *is an edge of* Γ *for all* $u' \in [u]$ *and* $v' \in [v]$.

Note that although Γ has no loops it may be that there are loops in Γ^c (if there are vertices of Γ such that $[x]_\perp$ has more than two elements). If Γ and Γ' are graphs without multiple edges, and there is a map $f : V(\Gamma) \to V(\Gamma')$ then we say that f induces a graph homomorphism $f : \Gamma \to \Gamma'$ if $(f(u), f(v)) \in E(\Gamma)$ for all $(u, v) \in \Gamma$.

Proposition 2.49. *The map* $\mathsf{c} : X \to X^c$ *given by* $\mathsf{c}(x) = [x]$, *for* $x \in X$, *induces a surjective graph homomorphism* $\mathsf{c} : \Gamma \to \Gamma^c$.

Proof. The map $\mathsf{c} : X \to X^c$ is surjective by definition. If c maps edges of Γ to edges of Γ^c then, since neither graph has multiple edges, the induced map is a surjective graph homomorphism. Therefore it suffices to show that if (u, v) is an edge of Γ then $([u], [v])$ is an edge of Γ^c.

Suppose then that $u, v \in X$, $u \neq v$ and (u, v) is an edge of Γ. If $[u] = [v]$ and $|[u]| = 1$ or $[u] = [u]_o$ then there are no edges of Γ joining elements of $[u]$ to each other. Therefore if $[u] = [v]$ we may assume that $[u] = [v] = [u]_\perp$. In this case $[u]$ is a simplex, with more than one element since $u \neq v$, and so there is a loop e in Γ^c from $[u]$ to itself. Thus (u, v) maps to e, as required.

Now suppose that $[u] \neq [v]$. If $|[u]| = |[v]| = 1$ then $([u], [v])$ is clearly an edge of Γ^c. Suppose then that $|[v]| \geq 2$ and that $z \in [v]$, $z \neq v$. Then $(u, v) \in E(\Gamma)$ implies $u \in v^\perp$. As either $z \sim_\perp v$ or $z \sim_o v$ and $z \neq v$ it follows that $z \in u^\perp$. If $|[u]| = 1$ this implies that $([u], [v]) \in E(\Gamma^c)$. If $|[u]| \geq 2$ then let $w \in [u]$, $w \neq u$. Then $w \neq z$ (as $[u] \neq [v]$) and $z \in u^\perp$ implies $z \in w^\perp$. Hence $(w, z) \in E(\Gamma)$ and it follows that $([u], [v])$ is an edge of Γ^c. $\qquad\square$

As usual we extend c to a map from subsets of X to subsets of X^c by setting $\mathsf{c}(Y) = \cup_{y \in Y}\{\mathsf{c}(y)\}$, for $Y \subseteq X$. If $[y] \in X^c$ then $[y]^\perp = \{[u] \in X^c : d([u], [y]) \leq 1\} = \{[u] \in X^c : d(u, y) \leq 1\}$, by definition of Γ^c, so for all $y \in X$,

$$\mathsf{c}(y)^\perp = \{[u] \in X^c : u \in y^\perp\} = \bigcup_{u \in y^\perp} \{\mathsf{c}(u)\} = \mathsf{c}(y^\perp).$$

Now suppose that $Z = \{z_1, \ldots, z_n\} \subseteq X$. Then $\mathsf{c}(Z)^\perp = (\cup_{i=1}^n \mathsf{c}(z_i))^\perp = \cap_{i=1}^n \mathsf{c}(z_i)^\perp = \cap_{i=1}^n \mathsf{c}(z_i^\perp)$. Clearly $\cap_{i=1}^n \mathsf{c}(z_i^\perp) \supseteq \mathsf{c}(\cap_{i=1}^n z_i^\perp)$. On the other hand, if $[u] \in \cap_{i=1}^n \mathsf{c}(z_i^\perp)$ then $[u] \in [z_i]^\perp$, so $d([u], [z_i]) \leq 1$ and so $d(u, z_i) \leq 1$, for $i = 1, \ldots, n$. Therefore $u \in \cap_{i=1}^n z_i^\perp$ from which it follows that $[u] \in \mathsf{c}(\cap_{i=1}^n z_i^\perp)$. Hence $\mathsf{c}(Z)^\perp = \mathsf{c}(\cap_{i=1}^n z_i^\perp) = \mathsf{c}(Z^\perp)$.

Now restricting the map c to closed sets we see that if $Y \in L(\Gamma)$ then $Y = Z^\perp$, for some $Z \subseteq X$ so $\mathsf{c}(Y) = \mathsf{c}(Z^\perp) = \mathsf{c}(Z)^\perp \in L(\Gamma^\mathsf{c})$. Hence c induces a map from $L(\Gamma)$ to $L(\Gamma^\mathsf{c})$, which we denote by c_L. Let L denote the lattice $L(\Gamma)$ and L^c the lattice $L(\Gamma^\mathsf{c})$.

Proposition 2.50. *The map* $\mathsf{c}_L : L \to L^c$ *is a lattice epimorphism which preserves the unary relation* $\perp$*: that is* $\mathsf{c}_L(Y^\perp) = \mathsf{c}_L(Y)^\perp$*, for all* $Y \in L$.

Proof. As c is a surjective map it follows that every subset of X^c is the image of a subset of X. If W is a closed subset of X^c then $W = V^\perp$ for some subset V of X^c. Choose $Y \subseteq X$ such that $\mathsf{c}(Y) = V$. As we have seen above we have $\mathsf{c}(Y^\perp) = V^\perp = W$. As $Y^\perp \in L$ we have $\mathsf{c}_L(Y^\perp) = W$, so c_L is a surjective map. It therefore suffices to show that c_L is a lattice homomorphism. If $S, T \in L$ then $S = U^\perp$ and $T = V^\perp$, for some $U, V \in L$. Then $S \wedge T = S \cap T$ and
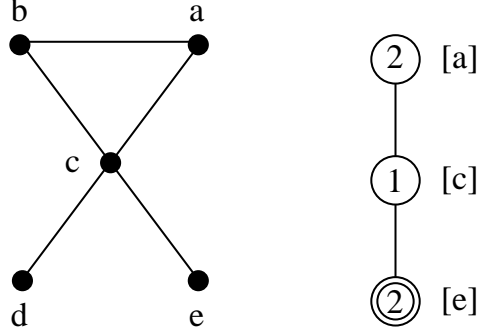
$$\begin{aligned}
\mathsf{c}_L(S) \wedge \mathsf{c}_L(T) &= \mathsf{c}_L(U^\perp) \cap \mathsf{c}_L(V^\perp) \\
&= \mathsf{c}_L(U)^\perp \cap \mathsf{c}_L(V)^\perp \\
&= (\mathsf{c}_L(U) \cup \mathsf{c}_L(V))^\perp \\
&= (\mathsf{c}_L(U \cup V))^\perp \\
&= \mathsf{c}_L(U^\perp \cap V^\perp) \\
&= \mathsf{c}_L(S \wedge T).
\end{aligned}$$

Moreover

$$\begin{aligned}
\mathsf{c}_L(S \vee T) &= \mathsf{c}_L((S \cup T)^{\perp\perp}) \\
&= (\mathsf{c}_L(S \cup T))^{\perp\perp} \\
&= (\mathsf{c}_L(S) \cup \mathsf{c}_L(T))^{\perp\perp} \\
&= \mathsf{c}_L(S) \vee \mathsf{c}_L(T).
\end{aligned}$$

Hence c_L is a lattice homomorphism as claimed. $\square$

We make Γ^c into a labelled graph as follows. For $x \in X$ define $\mu(x) = |[x]|$ and $\nu(x) = 1$, if $x \in M_1$, $\nu(x) = \perp$, if $x \in M_\perp$ and $\nu(x) = o$, if $x \in M_o$. Define a labelling function $l : X^c \to \mathbb{N} \times \{1, \perp, o\}$ by $l([y]) = (\mu(y), \nu(y))$, for all $y \in X^\mathsf{c}$.

FIGURE 2.6. A graph Γ and its compression Γ^c

Example 2.51. In drawing the compressed graph vertices with labels of the form $(1,1)$ or $(r,\perp)$ are represented as single circles containing the integer 1 or r, respectively, and vertices with labels of the form (r,o) are represented as two concentric circles containing the integer r, as in Figure 2.6.

Let $\mathrm{Aut}(\Gamma^c)$ denote the group of automorphisms of Γ^c as a labelled graph: that is $\phi \in \mathrm{Aut}(\Gamma^c)$ if and only if ϕ is an automorphism of the graph Γ^c such that $l(\phi([v])) = l([v])$, for all $[v] \in X^c$. Let $\mathrm{Aut}(\Gamma)$ denote the group of graph automorphisms of Γ and let $\phi \in \mathrm{Aut}(\Gamma)$. Since ϕ is an automorphism we have $\phi(u^\perp) = \phi(u)^\perp$, for all $u \in X$. It follows that if $u, v \in X$ and $[u] = [v]$ then $[\phi(u)] = [\phi(v)]$. Applying ϕ^{-1} to the latter equality we see that $[u] = [v]$ if and only if $[\phi(u)] = [\phi(v)]$. Since c and ϕ are graph homomorphisms it follows that $\phi_\mathsf{c} = \mathsf{c} \circ \phi$ is an automorphism of Γ^c as a labelled graph: that is $\phi_\mathsf{c} \in \mathrm{Aut}(\Gamma^c)$. Denote by $\mathrm{Aut}(\mathsf{c})$ the map which takes $\phi \in \mathrm{Aut}(\Gamma)$ to $\phi_\mathsf{c} \in \mathrm{Aut}(\Gamma^c)$. For $[v] \in X^c$ let $S_{\mu(v)}$ denote the symmetric group of degree $\mu(v)$.

Proposition 2.52. *The map* $\mathrm{Aut}(\mathsf{c})$ *is an epimorphism from* $\mathrm{Aut}(\Gamma)$ *to* $\mathrm{Aut}(\Gamma^c)$. *There is a split short exact sequence*

$$(2.9) \qquad 1 \to \prod_{[v] \in X^c} S_{\mu(v)} \to \mathrm{Aut}(\Gamma) \xrightarrow{\mathrm{Aut}(\mathsf{c})} \mathrm{Aut}(\Gamma^c) \to 1.$$

Proof. We have seen that $\mathrm{Aut}(\mathsf{c})$ is a map from $\mathrm{Aut}(\Gamma)$ to $\mathrm{Aut}(\Gamma^c)$. If $\phi, \phi' \in \mathrm{Aut}(\Gamma)$ then $(\phi \circ \phi')_\mathsf{c}([v]) = [\phi \circ \phi'(v)] = \phi_\mathsf{c}([\phi'(v)]) = \phi_\mathsf{c} \circ \phi'_\mathsf{c}([v])$, for all $[v] \in X^c$. Hence $\mathrm{Aut}(\mathsf{c})$ is a homomorphism.

Let $[v] \in X^c$ and consider the subgraph $\Gamma([v])$ of Γ. If $\phi \in \mathrm{Aut}(\Gamma([v]))$ then we may extend ϕ to Γ by setting $\phi(u) = u$, for all $u \notin [v]$. Hence we may regard $\mathrm{Aut}(\Gamma([v]))$ as a subgroup of $\mathrm{Aut}(\Gamma)$. If $u, v \in X$ and

$[u] \neq [v]$ then $\phi \circ \phi' = \phi' \circ \phi$, for all $\phi \in \mathrm{Aut}(\Gamma([u]))$ and $\phi' \in \mathrm{Aut}(\Gamma([v]))$. Moreover, as $[u] \cap [v] = \emptyset$ we have $\mathrm{Aut}(\Gamma([u])) \cap \mathrm{Aut}(\Gamma([v])) = 1$. Therefore $\mathrm{Aut}(\Gamma)$ contains the subgroup $A = \prod_{[v] \in X^c} \mathrm{Aut}(\Gamma([v]))$. If $\phi \in A$ then $\phi(v) \in [v]$, for all $v \in [v]$ and for all $[v] \in X^{\mathsf{c}}$. Therefore $\phi \in \ker(\mathrm{Aut}(\mathsf{c}))$ and so $A \subseteq \ker(\mathrm{Aut}(\mathsf{c}))$. Conversely if $\phi \in \ker(\mathrm{Aut}(\mathsf{c}))$ then $[\phi(v)] = [v]$ so $\phi(v) \in [v]$, for all $v \in X$. Hence if $\phi \in \ker(\mathrm{Aut}(\mathsf{c}))$ then $\phi|_{[v]} \in \mathrm{Aut}(\Gamma([v]))$ and so $\phi \in A$. Therefore $A = \ker(\mathrm{Aut}(\mathsf{c}))$. For all $[v] \in X^c$ the graph $\Gamma([v])$ is either a simplex or a free co-simplex so $\mathrm{Aut}(\Gamma([v]))$ is isomorphic to the symmetric group $S_{\mu(v)}$ of degree $\mu(v)$. Therefore $\prod_{[v] \in X^c} S_{\mu(v)} \cong A$. To show that the sequence (2.9) is exact it remains only to show that $\mathrm{Aut}(\mathsf{c})$ is surjective. However we shall first construct an embedding $\iota : \mathrm{Aut}(\Gamma^{\mathsf{c}}) \to \mathrm{Aut}(\Gamma)$.

Fix a transversal $V = \{v_1, \ldots, v_n\}$ for the map $\mathsf{c} : \Gamma \to \Gamma^c$. For i such that $1 \leq i \leq n$ choose an ordering $(v_{i,1}, \ldots v_{i,\mu(v_i)})$ of the class $[v_i]$, with $v_i = v_{i,1}$. Then $X = \sqcup_{i=1}^n \sqcup_{j=1}^{\mu(v_i)} \{v_{i,j}\}$. For i, k such that $1 \leq i \leq k \leq n$ and $l(v_i) = l(v_k)$, define a map $\tau_{i,k} : [v_i] \to [v_k]$ by $\tau_{i,k}(v_{i,j}) = v_{k,j}$, $j = 1, \ldots, \mu(v_i)$. Note that, as $l(v_i) = l(v_k)$ the map $\tau_{i,k}$ is a graph isomorphism from $\Gamma([v_i])$ to $\Gamma([v_k])$. If $\tau_{i,k}$ is defined and $i < k$ we define $\tau_{k,i} = \tau_{i,k}^{-1}$. Furthermore if $\tau_{i,k}$ and $\tau_{k,l}$ are both defined then so is $\tau_{i,l}$ and by construction $\tau_{i,l} = \tau_{k,l} \circ \tau_{i,k}$.

Now let $\phi_{\mathsf{c}} \in \mathrm{Aut}(\Gamma^{\mathsf{c}})$ and define a map ϕ of X to itself as follows. Let $v \in X$. Then $[v] = [v_i]$, so $v = v_{i,j}$, for unique i and j. There is a unique k such that $\phi_{\mathsf{c}}([v_i]) = [v_k]$ and as $l(\phi_c(v_i)) = l(\phi_c(v_i))$ the map $\tau_{i,k}$ is defined. Set $\phi(v) = \tau_{i,k}(v_{i,j}) = v_{k,j}$. As all the $\tau_{i,k}$ are isomorphisms and as ϕ_c is a graph automorphism it follows that ϕ is a graph automorphism. Thus $\iota : \phi_c \to \phi$ is a map from $\mathrm{Aut}(\Gamma^c)$ to $\mathrm{Aut}(\Gamma)$. That ι is an injective homomorphism follows directly from the definition.

If $\phi_c \in \mathrm{Aut}(\Gamma^{\mathsf{c}})$ and $[v] \in X^{\mathsf{c}}$ then $\mathrm{Aut}(\mathsf{c}) \circ \iota(\phi_c)$ maps $[v]$ to $[\iota\phi_c(v)] = \phi_c([v])$, so $\mathrm{Aut}(\mathsf{c}) \circ \iota$ is the identity on $\mathrm{Aut}(\Gamma^c)$. This implies that $\mathrm{Aut}(\mathsf{c})$ is surjective; so the sequence (2.9) is exact. Furthermore ι is a transversal for $\mathrm{Aut}(\mathsf{c})$ and so (2.9) splits, as claimed. $\qquad \square$

The compression Γ^c of Γ gives rise to a natural decomposition of $G(\Gamma)$ which we now describe; using the following generalisation of a partially commutative group. Let Γ be a graph and to each vertex of Γ associate a group G_v. Let $F = *_{v \in V(\Gamma)} G_v$ and let N be the normal subgroup of F generated by all elements of the form $[g_u, g_v]$, where $g_u \in G_u$, $g_v \in G_v$ and u and v are joined by an edge of Γ. The group $G = F/N$ is called a *partially commutative product of groups*. If all the vertex groups G_v are infinite cyclic groups then G is a partially commutative group. In the case in question take Γ^c to be the underlying graph and associate the the partially commutative

group with commutation graph $\Gamma([v])$ to the vertex $[v]$. The vertex groups are all then free Abelian groups or free groups.

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