METRIC PROPERTIES OF THE BRAIDED THOMPSON'S GROUPS

JOSÉ BURILLO AND SEAN CLEARY

ABSTRACT. Braided Thompson's groups are finitely presented groups introduced by Brin and Dehornoy which contain the ordinary braid groups B_n, the finitary braid group B_∞ and Thompson's group F as subgroups. We describe some of the metric properties of braided Thompson's groups and give upper and lower bounds for word length in terms of the number of strands and the number of crossings in the diagrams used to represent elements.

INTRODUCTION

Thompson's groups form interesting examples of a range of unusual group-theoretic phenomena. Their metric properties are understood to varying extents of completeness. A natural notion of "size" of an element in these groups is the size in terms of the number of nodes in the smallest tree pair diagram which can represent that element. Fordham [9] developed an effective method for computing the word length of elements of Thompson's group F exactly from tree pair representations, which has led to good understanding of both fine-scale and large-scale geometry of the Cayley graph of F. Burillo, Cleary, Stein and Taback [6] give estimates for the word length in Thompson's group T, showing that the word length is quasi-isometric to the number of nodes in the smallest representative of a group element. This has led to some understanding of the large-scale geometry of T. The metric properties of Thompson's group V are less well-understood. Birget [1] gives lower and upper bounds in terms of the number of nodes of minimal representatives– the lower bound is linear in the number of nodes, and the upper bound is proportional to $n \log n$, where n is the number of nodes in the minimal representative. This upper bound is sharp in the sense that the fraction of elements which have lengths close to the $n \log n$ bound

converges quickly to 1 as n increases. So there is a gap between the lower and upper bounds which cannot be bridged merely by looking at the size of representative diagrams. Here, we consider the metric properties of the braided Thompson's groups BV and $\widehat{BV}$. These groups have a great deal in common with Thompson's groups V and can be regarded as extensions of V. Furthermore, they have some properties in common with the finitely generated braid groups, whose metric properties with respect to the standard generators are well-understood in terms of the number of crossings in a minimal diagram.

Below, we give upper and lower bounds for the word lengths of elements of braided Thompson's groups BV and $\widehat{BV}$ with respect to their standard generating sets. We give bounds based on the number of nodes and the number of crossings, and we give some sharper bounds based on the number of crossings of a single pair of strands. We give examples of families of elements where the number of crossings is significantly larger than the length, and use these examples to show that the bounds on word length are optimal in terms of order of growth.

1. Background on braided Thompson's groups

Thompson's group V is constructed as a group of piecewise linear maps of the interval $[0, 1]$, not necessarily continuous, but which are linear in open intervals bounded by dyadic rationals. Cannon, Floyd and Parry [7] give an excellent introduction with complete details. Elements in the group V can be seen as pairs of rooted binary trees, with a permutation defining how leaves are mapped to each other. The related group $\widehat{V}$ is a subgroup and a supergroup of V described in Brin [4] which has some aspects which make presentation and analysis more straightforward.

The standard "Artinification" construction that replaces permutations by braids is used to construct the braided versions BV and $\widehat{BV}$. See Brin [4] and Dehornoy [8] for details of the constructions.

We construct a braided tree pair diagram representative in "tree-braid-tree" form of an element of BV as a pair of rooted binary trees, each with n nodes, and a braid with $n + 1$ strands, thought of as joining the leaves of both trees via the given braiding, with the second tree positioned upside down. See Figure 1 for an example of an element of BV in tree-braid-tree form.

An element in BV is an equivalence class of diagrams, where the equivalence is given by replacing two corresponding leaves of the two trees by nodes, and splitting the corresponding strand into two parallel strands. Using this operation and its inverse operation of collapsing parallel strands
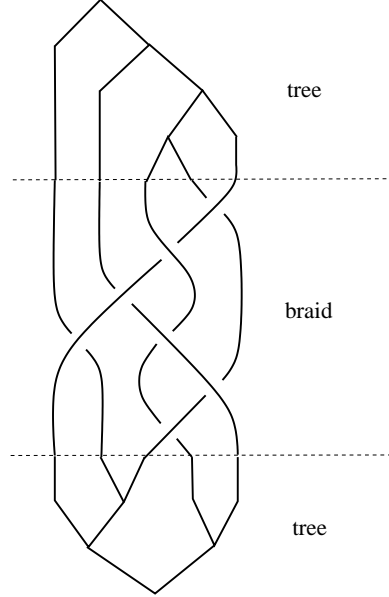
FIGURE 1. A diagram for an element in BV

which have immediate common parent nodes in both trees, we obtain elements as equivalence classes of braided tree pair diagrams. There is a unique minimal size representative in each equivalence class. We multiply elements by concatenation of two appropriately chosen tree-braid-tree diagram representatives. By successive splittings of strands, we make the bottom tree of the first representative coincide with the top tree of the second representative, and then we cancel those trees and concatenate the resulting braids. An example of multiplying elements of BV is shown in Figure 2. The interested reader can see many explicit details and examples in Brady, Burillo, Cleary and Stein [3], as well as the work of Brin [4] and Dehornoy [8].

We can consider, as done in [3], the group $\widehat{BV}$ as the subgroup of BV that has the last strand unbraided. Though our results below are all stated for BV, there are immediate natural analogous results for $\widehat{BV}$ which simply omit the generators τ_i and any relations involving τ_i. Presentations for both groups are given in [3]; here is a presentation for BV.

Theorem 1.1 (Brady, Burillo, Cleary, Stein [3]). *BV has generators x_i, σ_i, τ_i, and relators*

(A) $x_j x_i = x_i x_{j+1}$ *for $j > i$*
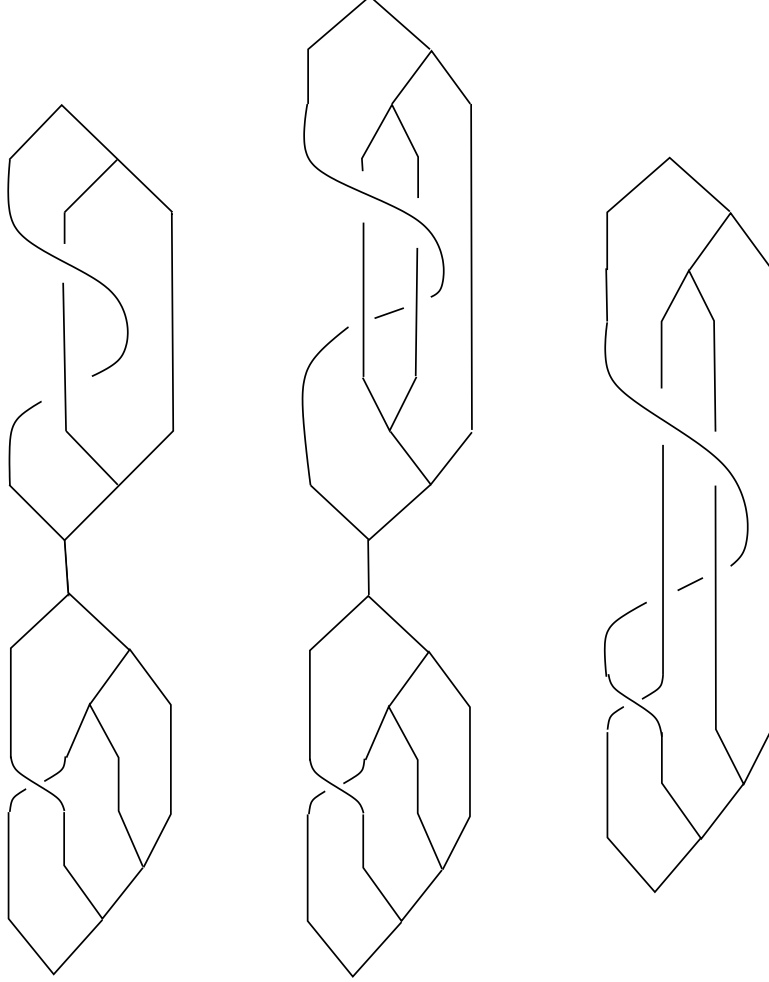
FIGURE 2. An example of multiplication in BV

(B1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ *for* $j - i \geq 2$
(B2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
(B3) $\sigma_i \tau_j = \tau_j \sigma_i$ *for* $j - i \geq 2$
(B4) $\sigma_i \tau_{i+1} \sigma_i = \tau_{i+1} \sigma_i \tau_{i+1}$.
(C1) $\sigma_i x_j = x_j \sigma_i$, *for* $i < j$
(C2) $\sigma_i x_i = x_{i-1} \sigma_{i+1} \sigma_i$
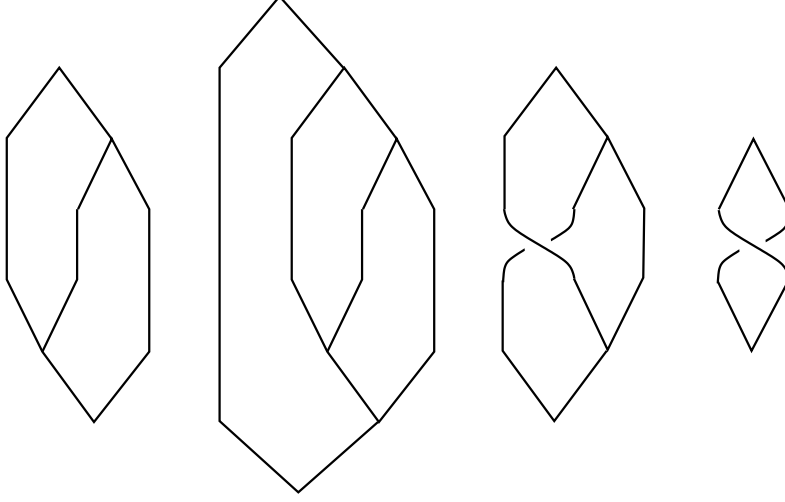(C3) $\sigma_i x_j = x_j \sigma_{i+1}$, *for* $i \geq j + 2$

FIGURE 3. The generators of BV, from left to right, x_0, x_1, σ_1, τ_1

(C4) $\sigma_{i+1}x_i = x_{i+1}\sigma_{i+1}\sigma_{i+2}$
(D1) $\tau_i x_j = x_j\tau_{i+1}, \ for \ i - j \geq 2$
(D2) $\tau_i x_{i-1} = \sigma_i\tau_{i+1}$
(D3) $\tau_i = x_{i-1}\tau_{i+1}\sigma_i$.

In this presentation, the generators x_i correspond to the generators of Thompson's group F (see [7]). The generators σ_i and τ_i are the braid generators, and all of them are given by tree-pair-tree diagrams where the braid is a one-crossing braids and the trees are all-right trees (trees which only have right child nodes.) The braided generators are:

- generators σ_i: crosses strand i over strand $i + 1$ between all-right trees with $i + 2$ leaves,
- generators τ_i: crosses strand i over strand $i + 1$ between all-right trees with $i + 1$ leaves.

The subgroup $\widehat{BV}$ is generated by all generators which have the last strand unbraided; that is, given by words x_i and σ_i.

These presentations are infinite, but both groups admit finite presentations. We use finite generating sets for understanding the metrics on these groups. The group BV is generated by x_0, x_1, σ_1 and τ_1, and its subgroup $\widehat{BV}$ is generated by x_0, x_1, and σ_1. Figure 3 shows the standard four generators.

A word in the generators of BV corresponds potentially to the concatenation of many tree-braid-tree diagrams, but it can always be rewritten, using the relators, as a word which corresponds to a single tree-braid-tree diagram, see [3]. Such a word would be of the type

$$(x_{i_1}^{r_1} x_{i_2}^{r_2} \ldots x_{i_n}^{r_n})(Br)(x_{j_m}^{-s_m} \ldots x_{j_2}^{-s_2} x_{j_1}^{-s_1})$$

where Br is a braid, expressed as product of generators σ_i and τ_i. The minimal (reduced) representative diagram as a word in the generators for an element, will be used to compute and to estimate the metric. Of course, the minimal length word for an element may likely not correspond to a diagram given in tree-braid-tree form.

For each n, the ordinary n-strand braid groups B_n, generated by the crossing of adjacent strands (usually denoted σ_i), naturally are all subgroups of BV merely by considering the n-strands as the leaves of all-right trees.

2. Motivating examples.

In Thompson's group F, whose elements are pairs of trees, word length is proportional to the number of nodes in the trees. In the ordinary n-strand braid groups, word length with respect to the standard $n-1$ generators is given by the number of crossings in a minimal braid diagram. Thus, it is reasonable to suspect in our setting, where the elements are combinations of trees and braids, the number of nodes and number of crossings have to be considered. The main difference is that here, if the relevant trees are not of the same shape, we may need to split strands to be able to multiply elements, and splitting strands will increase the total number of crossings. The following examples show that there can be interactions which rule out the possibility that word length is estimated simply as being proportional to the number of strands and the number of crossings.

Example 2.1. *The reduced tree-braid-tree diagram for the element $(x_1\tau_1)^{2n}$ has $2n+3$ nodes and $2n(n+2)^2$ crossings.*

This follows from a standard induction argument on n. For $n = 1$ we see 5 nodes and 18 crossings. It is not hard to check the values for a general n once we see the particular shape of these elements– the interleaved splitting and braiding quickly leads to diagrams with many crossings. See Figure 4 for the example of the element $(x_1\tau_1)^6$, which has word length 12, and a total of 150 crossings

Another family of examples of interesting elements is given by the families of Garside elements, which are half-twists of all strands. These elements and their powers can play a central role in various normal forms for ordinary braids, see Birman and Brendle [2]. The Garside element

block

repeated

six times

Total of six sets of five strands
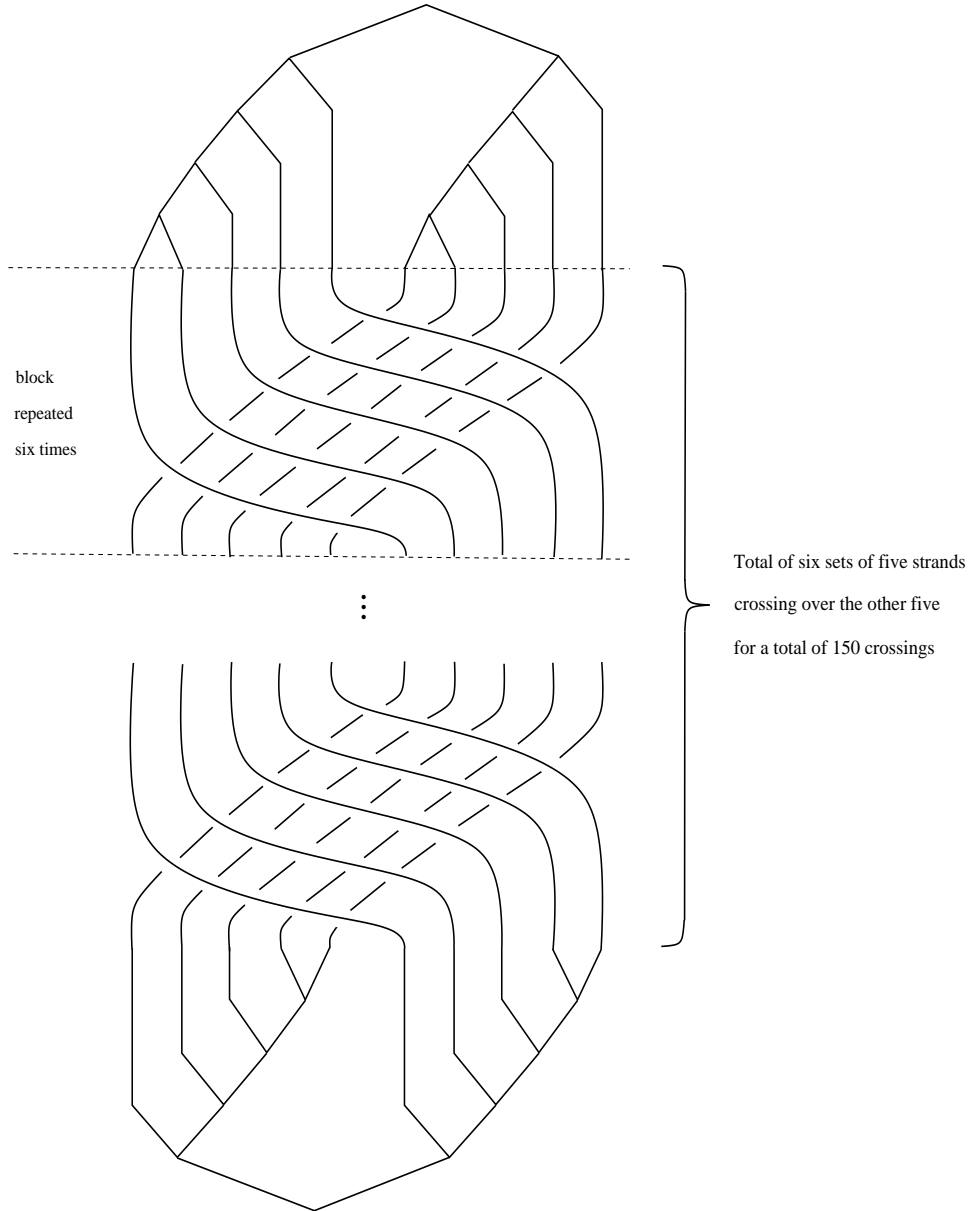
crossing over the other five

for a total of 150 crossings

FIGURE 4. The element $(x_1\tau_1)^6$, with word length 12, with 9 nodes and 6 blocks where the first 5 strands go over the last 5 strands, giving a total of 150 crossings.

$\Delta_n = \sigma_1\sigma_2\cdots\sigma_n\sigma_1\cdots\sigma_{n-1}\cdots\sigma_1\sigma_2\sigma_1$ is the longest positive permutation braid in B_n and the left-multiple by x_0^{-3} of the Garside element Δ_4 is shown as an element of BV in Figure 5. Each strand crosses every other strand exactly once, giving $\binom{n}{2}$ crossings and thus the word length in B_n is quadratic in the number of strands, even when the number of generators grows linearly with the number of strands. Direct substitution of those expressions into generators of BV to find word length in BV is far from optimal, though, as shown by this example.

Example 2.2. *The Garside element Δ_{n+1} in the standard B_{n+1} subgroup of BV has length at most $6n - 7$ in BV for $n \geq 2$.*

This is again seen easily. The Garside element Δ_2 in B_2 is exactly τ_1, and we see that products of the form $\tau_1\tau_2\cdots\tau_n = x_0^{n-1}\Delta_{n+1}$ giving that $\Delta_{n+1} = x_0^{1-n}\tau_1\cdots\tau_n$. To substitute in terms of the standard finite generating set, first we note that $\tau_n = x_0^{2-n}\tau_2 x_0^{n-2}$ for $n \geq 2$ and then that $\tau_2 = x_0^{-1}\tau_1\sigma_1^{-1}$. Substituting to express Δ_{n+1} in terms of the finite generating set, we have $\Delta_{n+1} = x_0^{1-n}\tau_1\tau_2(x_0^{-1}\tau_2 x_0)(x_0^{-2}\tau_2 x_0^2)\cdots(x_0^{-n+2}\tau_2 x_0^{n-2})$ which has length $6n - 7$ when τ_2 is expressed in terms of the 4 generators, for $n \geq 2$.

So in the $n - 1$ generator subgroup B_n of BV, the length of Δ_n grows quadratically with n, yet in BV itself, with only 4 generators, we can obtain the Garside element Δ_n as the product of generators whose length grows at most linearly with the number of generators.

3. Metric properties

The examples in the previous section indicate that overall, the number of crossings is not a good indicator of the length, since there are elements with large number of crossings and small length. But the number of crossings can still give an upper bound on the length.

Theorem 3.1. *Given an element w in BV or $\widehat{BV}$ which has a reduced tree-pair-tree diagram representative with n nodes and k crossings then there exists a constant C such that the length $|w|$ is at most $C(n + nk)$.*

Proof. We consider then the reduced tree-braid-tree diagram of the element w. It can be split into three diagrams using all-right trees, giving a split into a positive element of F, a braid on two all-right trees, and a negative element of F. The elements of F have length at most proportional to n (see [5]). For the middle braid in all-right trees, and with k crossings, we can split it into k elements, each one of them with the same all-right tree on top and bottom, and one single crossing. Each one of these elements is then a representative for a generator σ_i or τ_i, for any $i \geq 1$. We only need to go back to the expression of each σ_i and τ_i in terms of the finite
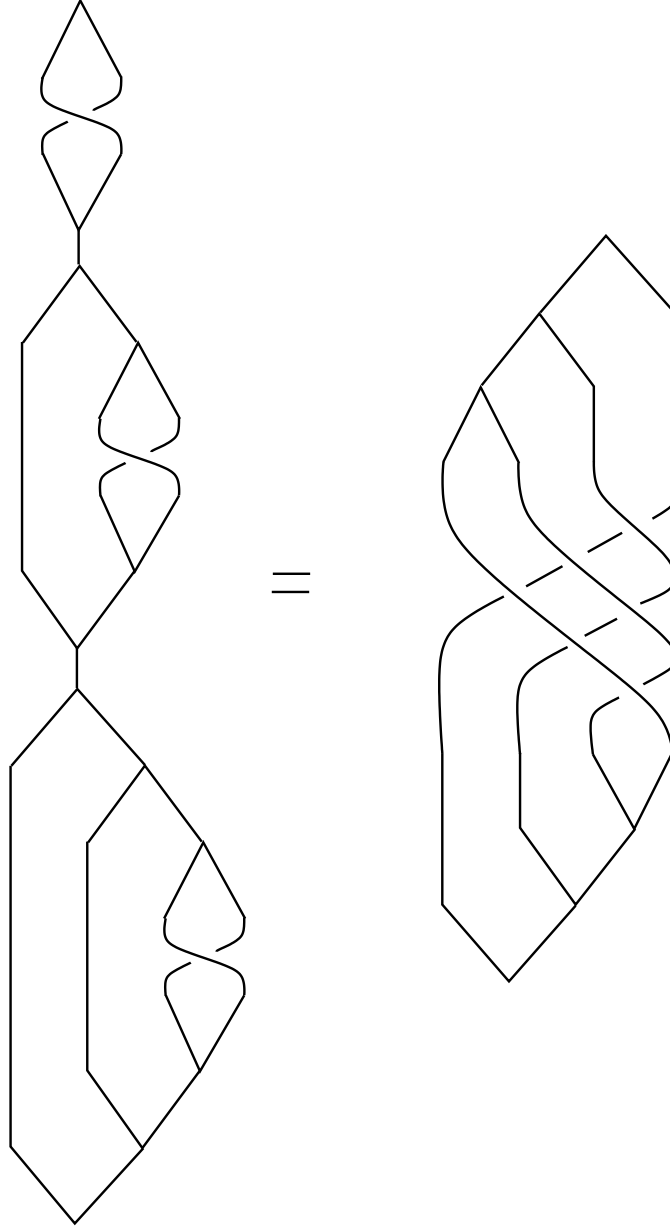
FIGURE 5. Constructing the Garside element Δ_4 as a product of τ_i. Here we show $\tau_1 \tau_2 \tau_3 = x_0^{-3} \Delta_4$.

set of generators $x_0, x_1, \sigma_1, \tau_1$, and verify that, if $i \leq n$, each σ_i or τ_i can be written with a number of generators which is at most linear in n. Thus substitution gives the stated upper bound on word length.   □

Examples 2.1 and 2.2 from the previous section show that the number of crossings may grow cubically or quadratically with word length. So to find lower bounds, we need to consider quantities in the diagram which are more closely related to the length than the number of crossings. These quantities are the number of nodes and the maximum number of crossings between any pair of strands.

We let w be an element given in a reduced tree-braid-tree diagram representative, and let n be the number of nodes. We let s_{ij}, for $i, j = 1, 2, \ldots, n+1$ be the number of times the i-th and j-th strands cross each other, and we set the maximum number of crossings between any pair of strands as

$$s = \max\{s_{ij} \,|\, i, j = 1, 2, \ldots, n+1\}.$$

First, we remark that the number of nodes in a reduced diagram gives a lower bound for the word length.

Proposition 3.2. *Given an element w in BV or $\widehat{BV}$ which has a reduced diagram representative with n nodes then there exists a constant C such that the length $|w|$ is at least Cn.*

This follows by standard arguments analogous to those used for other groups of the Thompson class. In the case of BV, multiplication by any of the generators can add at most two nodes to a reduced tree braid tree diagram, so an element with n nodes will require at least $\frac{n}{2}$ generators.

Furthermore, the maximal number of crossings for a pair of strands also gives a lower bound for the metric.

Proposition 3.3. *Given an element w in BV or $\widehat{BV}$ which has a reduced tree-braid-tree diagram representative where s is the maximal number of crossings of any pair of strands. Then the length $|w|$ is at least s.*

The proof becomes straightforward once we observe that representatives of the generators x_0 and x_1 never have any crossings, and representatives for the braid generators are restricted by the following lemma.

Lemma 3.4. *In any tree-braid-tree representative of τ_1 or σ_1 with n nodes, any pair of strands cross only once. That is, for any representative of τ_1 or σ_1, we have $s = 1$.*

Proof. The reduced representatives for σ_1 and τ_1 have one single crossing. We obtain other representatives from this one by splitting strands. But if

a pair of strands cross only once, after the process of splitting any strand, in the resulting braid, any given pair of strands still cross only once.   □

Hence, Proposition 3.3 follows since representatives for x_0 and x_1 have no crossings, and each time we multiply by τ_1 or σ_1, the corresponding generator we use will add at most one crossing to each pair of strands.

There is an elementary relation between the total number of crossings k and s, the maximal number of crossings of any pair of strands, which is given multiplying by the number of pairs, $\binom{n+1}{2}$. Hence, we have that

$$k \leq \frac{n(n+1)}{2} s.$$

This leads to a lower bound on word length in terms of the number of crossings as follows:

Proposition 3.5. *There exists a constant C such that if an element w of BV or $\widehat{BV}$ has length N, the number of crossings on its reduced tree-braid-tree diagram representative satisfies $k \leq CN^3$.*

Proof. We consider what happens when we multiply by a generator. The number of nodes n grows by at most 2, since a generator has at most 3 nodes and every element has at least the root node. And in view of Lemma 3.4, s can grow by only 1. Hence both n and s grow linearly with length, and hence, the inequality linking the total number of crossings k to them finishes the proof.   □

Example 2.1, where the total number of crossings grows cubically with word length, shows that the cubic bound in Proposition 3.5 is optimal.

From here we deduce these final results for the bounds, where we give lower and upper bounds for the metric based on the quantities k and s.

Theorem 3.6. *For an element w of BV or $\widehat{BV}$ in tree-braid-tree form with n nodes, k total crossings, and with the maximum number of crossings of a pair of strands is s, there are constants C_1, C_2, C_3, C_4 for which the metric satisfies the following inequalities:*

$$C_1 \max\{n, \sqrt[3]{k}\} \leq C_2 \max\{n, s\} \leq |w| \leq C_3(n + nk) \leq C_4(n + n^3 s)$$

References

[1] Jean-Camille Birget. The groups of Richard Thompson and complexity. *Internat. J. Algebra Comput.*, 14(5-6):569–626, 2004. International Conference on Semigroups and Groups in honor of the 65th birthday of Prof. John Rhodes.

[2] Joan S. Birman and Tara E. Brendle. Braids: a survey. In *Handbook of knot theory*, pages 19–103. Elsevier B. V., Amsterdam, 2005.

[3] Thomas Brady, José Burillo, Sean Cleary, and Melanie Stein. Pure braid subgroups of braided Thompson's groups. To appear in Publ. Mat.

[4] Matthew G. Brin. The algebra of strand splitting I. A braided version of Thompson's group V. To appear in J. of Group Theory.

[5] José Burillo, Sean Cleary, and Melanie Stein. Metrics and embeddings of generalizations of Thompson's group F. *Trans. Amer. Math. Soc.*, 353(4):1677–1689 (electronic), 2001.

[6] José Burillo, Sean Cleary, Melanie Stein, and Jennifer Taback. Combinatorial and metric properties for Thompson's group T. To appear in Trans. AMS.

[7] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.

[8] Patrick Dehornoy. The group of parenthesized braids. *Adv. Math.*, 205(2):354–409, 2006.

[9] S. Blake Fordham. Minimal length elements of Thompson's group F. *Geom. Dedicata*, 99:179–220, 2003.

DEPARTAMENT DE MATEMÀTICA APLICADA IV, UNIVERSITAT POLITÈCNICA DE CATALUNYA, ESCOLA POLITÈCNICA SUPERIOR DE CASTELLDEFELS, 08860 CASTELLDEFELS, BARCELONA, SPAIN
   E-mail address: `burillo@mat.upc.es`

DEPARTMENT OF MATHEMATICS, THE CITY COLLEGE OF NEW YORK & THE CUNY GRADUATE CENTER, NEW YORK, NY 10031
   E-mail address: `cleary@sci.ccny.cuny.edu`