

INVERSE PROBLEMS FOR INVARIANT ALGEBRAIC CURVES: EXPLICIT COMPUTATIONS

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ABSTRACT. Given an algebraic curve in the complex affine plane, we describe how to determine all planar polynomial vector fields which leave this curve invariant. If all (finite) singular points of the curve are nondegenerate, we give an explicit expression for these vector fields. In the general setting we provide an algorithmic approach, and as an alternative we discuss sigma processes.

1. INTRODUCTION

The question to determine the invariant algebraic curves of a given planar polynomial vector field, or to decide that no such curves exist, is part of a problem set forth by Poincaré, and is also essential in deciding whether the vector field admits a Darboux integrating factor. While there are many partial answers for this problem, a complete solution still seems beyond reach.

A solution of the inverse problem - to determine all polynomial vector fields that admit a prescribed collection of invariant algebraic curves (or just one possibly reducible curve) - seems to be essential in order to obtain a proper understanding of the situation. In this paper we present a solution to this inverse problem, using mainly results and tools from elementary commutative algebra. Our results indicate that it is sensible to consider the affine plane in its own right, as well as a subset of the projective plane. For the sake of completeness and to emphasize that our approach works very naturally, the paper also includes streamlined alternative proofs for some theorems that have been published previously.

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The vector fields admitting the given curve obviously form a linear space. We identify two subspaces that contain only vector fields of a known explicit form and we show that the quotient by either of these subspaces is finite dimensional. Assuming certain nondegeneracy conditions for the affine curve, we show that the whole linear space equals one of these subspaces. Generally we give dimension estimates, and we describe an algorithmic approach to determining the quotient space (indicating by examples that this is feasible). Finally we discuss blow-ups. In principle these allow to transform every affine curve to one that satisfies the nondegeneracy conditions. This opens another path towards the determination of the quotient space, and provides more structural insight.

2. BASICS AND KNOWN FACTS

We consider a complex polynomial vector field

$$(1) \quad X = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y},$$

sometimes also written as

$$X = \begin{pmatrix} P \\ Q \end{pmatrix},$$

and a non-constant polynomial f with prime factors f_1, \dots, f_r .

The complex zero set of f (which is generally a reducible curve in \mathbb{C}^2) is invariant for the vector field if and only if there is a polynomial L (called the cofactor of f) such that

$$(2) \quad Xf = L \cdot f, \quad \text{or} \quad P \cdot f_x + Q \cdot f_y = L \cdot f$$

equivalently there are polynomials L_1, \dots, L_r such that

$$Xf_i = L_i \cdot f_i, \quad 1 \leq i \leq r.$$

We will briefly say that in this case *the vector field X admits f* , or *admits the curve given by $f = 0$* . In the following we will assume $f = f_1 \cdots f_r$, which causes no loss of generality (therefore $L = L_1 + \dots + L_r$). The respective zero sets of f and the f_i in \mathbb{C}^2 will be denoted by C and C_i . As usual we call a point z with $f(z) = f_x(z) = f_y(z) = 0$ a *singular point of C* , and similarly for the C_i . The *Hamiltonian vector field of f* is defined by

$$X_f = -f_y \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial y}.$$

For a given f the vector fields admitting f form a linear space \mathcal{V}_f . Our goal is to determine this space precisely. First, we collect some facts and

properties that are known from previous work. (As for references, see [2], [3], [8], [10].) Clearly vector fields of the form

$$(3) \quad X = a \cdot X_f + f \cdot \tilde{X}$$

with an arbitrary polynomial a and an arbitrary polynomial vector field \tilde{X} admit the invariant curve C . These vector fields will be called the *trivial vector fields admitting f* . They form a subspace of \mathcal{V}_f which will be called \mathcal{V}_f^0 . For reducible f one can refine this to see that all vector fields of the form

$$(4) \quad X = \sum_i a_i \frac{f}{f_i} \cdot X_{f_i} + f \cdot \tilde{X}$$

admit f . These vector fields form a subspace of \mathcal{V}_f which will be denoted \mathcal{V}_f^1 . Note that $\mathcal{V}_f^0 \subseteq \mathcal{V}_f^1$. In “generic” geometric settings one has $\mathcal{V}_f = \mathcal{V}_f^1$; see [3] and an improvement in [2, Theorem 3.4] (in conjunction with [2, Theorem 3.6]). We will reprove the latter result using a different approach.

3. THE IDEAL OF COFACTORS

In this section we clarify the structure of the linear space \mathcal{V}_f of polynomial vector fields admitting f . In part the exposition is parallel to [10], Section 2, with some improvements and streamlining. We start with an auxiliary result.

Lemma 1. *Let g be a nonconstant polynomial with no multiple prime factors, and s be a greatest common divisor of the partial derivatives g_x and g_y . If s is not constant then g is constant and nonzero on every irreducible component of the vanishing set of s . In particular, g and s have no common zeros and there exist polynomials u and v such that $u \cdot s + v \cdot g = 1$.*

Proof. Let C^* be an irreducible component of the vanishing set of s , corresponding to a prime factor s^* of s , and $\gamma = (\gamma_1, \gamma_2)$ a local parametrization of C^* near a nonsingular point. Since the partial derivatives of g vanish on C^* , one has

$$\frac{d}{dt}(g(\gamma(t))) = g_x(\gamma(t))\dot{\gamma}_1(t) + g_y(\gamma(t))\dot{\gamma}_2(t) = 0$$

and thus g is locally constant on C^* , which implies constancy since C^* is irreducible.

Now assume that $g = 0$ on C^* . Then the vanishing set of s^* is contained in the vanishing set of g , and by irreducibility and Hilbert’s Nullstellensatz one finds that s^* divides g . Thus $g = s^* \cdot h$ with s^* and h relatively prime.

This implies

$$\begin{aligned} g_x &= s^* \cdot h_x + s_x^* \cdot h \\ g_y &= s^* \cdot h_y + s_y^* \cdot h. \end{aligned}$$

Therefore s^* divides both its partial derivatives, which is impossible from degree considerations; we have a contradiction.

The second assertion is clear with Hilbert's Nullstellensatz. \square

Now we turn to equation (2) and consider the cofactor L . For given f the cofactors of all vector fields admitting f clearly form an ideal. By definition of ideal quotients one obtains

$$L \in \langle f_x, f_y \rangle : \langle f \rangle$$

as a necessary and sufficient condition for the existence of P and Q such that (2) holds. We first clarify the role of the trivial vector fields admitting f .

Proposition 2. *A polynomial vector field X satisfies equation (2) with $L \in \langle f_x, f_y \rangle$ if and only if $X \in \mathcal{V}_f^0$.*

Proof. Evaluating (2) for $L = \tilde{P} \cdot f_x + \tilde{Q} \cdot f_y$ one finds

$$(P - f \cdot \tilde{P}) \cdot f_x + (Q - f \cdot \tilde{Q}) \cdot f_y = 0,$$

hence

$$\begin{aligned} P - f \cdot \tilde{P} &= \tilde{a} \cdot (-f_y/s) \\ Q - f \cdot \tilde{Q} &= \tilde{a} \cdot (f_x/s) \end{aligned}$$

with a g.c.d s of the partial derivatives of f , and some polynomial \tilde{a} . Thus we have a representation

$$X = \frac{\tilde{a}}{s} \cdot X_f + f \cdot \tilde{X},$$

and with $\tilde{a} = u \cdot s \cdot \tilde{a} + v \cdot f \cdot \tilde{a}$ from Lemma 1 we obtain

$$X = u\tilde{a} \cdot X_f + f \cdot \left(v\tilde{a} \cdot \frac{1}{s} X_f + \tilde{X} \right),$$

and the proof is finished. \square

By standard arguments we obtain the first Corollary.

Corollary 3. *The map sending a vector field admitting f to its cofactor induces an isomorphism*

$$\mathcal{V}_f / \mathcal{V}_f^0 \cong (\langle f_x, f_y \rangle : \langle f \rangle) / \langle f_x, f_y \rangle.$$

Corollary 4. For $r > 1$ let $p_i := X_{f_i}(f/f_i) = -\{f_i, f/f_i\}$, where $\{.,.\}$ denotes the standard Poisson bracket. Then $X \in \mathcal{V}_f^1$ if and only if

$$L \in \langle f_x, f_y, p_1, \dots, p_{r-1} \rangle$$

holds for the associated cofactor, and

$$\mathcal{V}_f/\mathcal{V}_f^1 \cong (\langle f_x, f_y \rangle : \langle f \rangle) / \langle f_x, f_y, p_1, \dots, p_{r-1} \rangle.$$

Proof. Let $Y_i := f/f_i \cdot X_{f_i}$, with associated cofactor $L_i = \frac{1}{f_i} X_{f_i}(f) = X_{f_i}(f/f_i) = p_i$. Thus $X = \sum a_i Y_i + f \tilde{X} \in \mathcal{V}_f^1$ implies $L \in \langle f_x, f_y, p_1, \dots, p_r \rangle$, and p_r can be discarded in view of $p_1 + \dots + p_r = 0$.

For the reverse inclusion assume that X has cofactor $L = \sum b_i p_i + u \cdot f_x + v \cdot f_y$, and set $\hat{X} = \sum b_i Y_i + f \cdot \begin{pmatrix} u \\ v \end{pmatrix}$. Then C is invariant for the vector field $X - \hat{X}$ with zero cofactor, thus $X - \hat{X} = \frac{\tilde{a}}{s} X_f$, as in the proof of Proposition 2. Then the arguments in this proof show $X - \hat{X} \in \mathcal{V}_f^0 \subset \mathcal{V}_f^1$. \square

In particular Proposition 2 shows that $\mathcal{V}_f = \mathcal{V}_f^0$ when $\langle f_x, f_y \rangle = \langle 1 \rangle$. In the following we will proceed as in [10], assuming that $\langle f_x, f_y \rangle \neq \langle 1 \rangle$ and considering a primary decomposition

$$(5) \quad \langle f_x, f_y \rangle = \mathbf{q}_1 \cap \dots \cap \mathbf{q}_s$$

with primary ideals \mathbf{q}_i and associated prime ideals \mathbf{p}_i . It is well known that either \mathbf{p}_i is generated by an irreducible polynomial (and thus corresponds to an irreducible curve), or that \mathbf{p}_i is maximal and corresponds to a point in the complex plane. By a standard result from commutative algebra

$$\langle f_x, f_y \rangle : \langle f \rangle = (\mathbf{q}_1 : \langle f \rangle) \cap \dots \cap (\mathbf{q}_s : \langle f \rangle)$$

see [10]. We quote one more result from [10].

Proposition 5. Let the notation be as in (5). Then $\mathbf{q}_i : \langle f \rangle \neq \mathbf{q}_i$ if and only if \mathbf{p}_i corresponds to a singular point of C . In particular $\mathcal{V}_f = \mathcal{V}_f^0$ if C has no singular point.

Now assume that C has singular points, and that precisely the ideals $\mathbf{q}_1, \dots, \mathbf{q}_m$, with some $m \geq 1$, correspond to these singular points. Let

$$(6) \quad \mathbf{a} := \mathbf{q}_1 \cap \dots \cap \mathbf{q}_m, \quad \mathbf{b} := \mathbf{q}_{m+1} \cap \dots \cap \mathbf{q}_s$$

with $\mathbf{b} : \langle f \rangle = \mathbf{b}$, and understanding $\mathbf{b} = \mathbb{C}[x, y]$ in case $s = m$. Lemma 1 shows that \mathbf{a} and \mathbf{b} have no common zero, thus $\mathbf{a} + \mathbf{b} = \mathbb{C}[x, y]$. The same argument shows

$$\mathbf{q}_i + \bigcap_{j \neq i} \mathbf{q}_j = \mathbb{C}[x, y], \quad 1 \leq i \leq m.$$

It should be noted that $\mathbb{C}[x, y]/\mathbf{a}$ is finite dimensional, see [10]. The following result is an improved version of Theorem 2.3 of [10] which clarifies the structure of \mathcal{V}_f and provides a basis for computations.

Theorem 6. *Let the hypotheses and notation be as above.*

(a) *The maps*

$$\begin{aligned} (\langle f_x, f_y \rangle : \langle f \rangle) / \langle f_x, f_y \rangle &\rightarrow (\mathbf{a} : \langle f \rangle) / \mathbf{a} \\ g + \langle f_x, f_y \rangle &\mapsto g + \mathbf{a} \\ \text{and} \\ (\langle f_x, f_y \rangle : \langle f \rangle) / \langle f_x, f_y \rangle &\rightarrow \bigoplus_{1 \leq i \leq m} (\mathbf{q}_i : \langle f \rangle) / \mathbf{q}_i \\ g + \langle f_x, f_y \rangle &\mapsto (g + \mathbf{q}_1, \dots, g + \mathbf{q}_m) \end{aligned}$$

are isomorphisms.

(b) *The kernel of the map*

$$\begin{aligned} \mathbb{C}[x, y]/\mathbf{a} &\rightarrow \mathbb{C}[x, y]/\mathbf{a} \\ g + \mathbf{a} &\mapsto f \cdot g + \mathbf{a} \end{aligned}$$

is equal to $(\mathbf{a} : \langle f \rangle) / \mathbf{a}$. This gives rise to an isomorphism

$$(\langle f_x, f_y \rangle : \langle f \rangle) / \langle f_x, f_y \rangle \cong \mathbb{C}[x, y] / \langle f, f_x, f_y \rangle.$$

(c) *Assume that $f_x = s \cdot h_1$, $f_y = s \cdot h_2$ with a g.c.d. s of the partial derivatives, and $\mathbf{c} := \langle h_1, h_2 \rangle$. Then $\mathbb{C}[x, y]/\mathbf{c}$ has finite dimension, and the invariance condition (2) holds if and only if*

$$L = s \cdot \widehat{L} \quad \text{with} \quad \widehat{L} \in \mathbf{c} : \langle f \rangle.$$

The map

$$\begin{aligned} \mathbb{C}[x, y]/\mathbf{c} &\rightarrow \mathbb{C}[x, y]/\mathbf{c} \\ g + \mathbf{c} &\mapsto f \cdot g + \mathbf{c} \end{aligned}$$

has kernel $(\mathbf{c} : \langle f \rangle) / \mathbf{c}$.

Proof. The Chinese Remainder Theorem implies that

$$\begin{aligned} \mathbb{C}[x, y] / (\mathbf{a} \cap \mathbf{b}) &\rightarrow \mathbb{C}[x, y] / \mathbf{a} \oplus \mathbb{C}[x, y] / \mathbf{b} \\ g + \mathbf{a} \cap \mathbf{b} &\mapsto (g + \mathbf{a}, g + \mathbf{b}) \end{aligned}$$

is an isomorphism. For an ideal $\mathbf{h} \supseteq \mathbf{a} \cap \mathbf{b}$ this induces the corresponding isomorphism

$$(7) \quad \mathbf{h} / (\mathbf{a} \cap \mathbf{b}) \cong (\mathbf{h} + \mathbf{a}) / \mathbf{a} \oplus (\mathbf{h} + \mathbf{b}) / \mathbf{b}.$$

Now $\langle f_x, f_y \rangle : \langle f \rangle = (\mathbf{a} : \langle f \rangle) \cap \mathbf{b}$, and therefore (recalling $\mathbf{a} + \mathbf{b} = \mathbb{C}[\mathbf{x}, \mathbf{y}]$)

$$\begin{aligned} (\langle f_x, f_y \rangle : \langle f \rangle) + \mathbf{a} &= \mathbf{a} : \langle f \rangle \\ (\langle f_x, f_y \rangle : \langle f \rangle) + \mathbf{b} &= \mathbf{b}. \end{aligned}$$

Using (7) with $\mathbf{h} = \langle f_x, f_y \rangle : \langle f \rangle$ shows the first assertion of (a). The second assertion of (a) follows readily with

$$\mathbb{C}[x, y] / (\mathbf{q}_1 \cap \dots \cap \mathbf{q}_m) \cong \bigoplus_{1 \leq i \leq m} \mathbb{C}[x, y] / \mathbf{q}_i$$

and analogous arguments.

The kernel of the map in part (b) equals $(\mathbf{a} + \langle f \rangle) / \mathbf{a}$ by definition. To prove the isomorphism property, note that this map between finite dimensional vector spaces has image $(\langle f \rangle + \mathbf{a}) / \mathbf{a}$. Consideration of the cokernel yields

$$\begin{aligned} (\mathbf{a} : \langle f \rangle) / \mathbf{a} &\cong (\mathbb{C}[x, y] / \mathbf{a}) / ((\langle f \rangle + \mathbf{a}) / \mathbf{a}) \\ &\cong \mathbb{C}[x, y] / (\langle f \rangle + \mathbf{a}). \end{aligned}$$

In view of (a) there remains to show that

$$\langle f, f_x, f_y \rangle = \langle f \rangle + \mathbf{a} = \langle f \rangle + \mathbf{a} \cap \mathbf{b}.$$

But by Lemma 1, f and \mathbf{b} have no common zeros, hence there are polynomials $g^* \in \mathbb{C}[x, y]$ and $b^* \in \mathbf{b}$ such that $g^* \cdot f + b^* = 1$. Now for $g \cdot f + a \in \langle f \rangle + \mathbf{a}$ one finds

$$\begin{aligned} g \cdot f + a &= (g \cdot f + a)(g^* \cdot f + b^*) \\ &= (g \cdot g^* \cdot f + a \cdot g^* + b^* \cdot g) \cdot f + a \cdot b^* \in \langle f \rangle + \mathbf{a} \cap \mathbf{b}. \end{aligned}$$

So statement (b) follows.

Finally for statement (c) Lemma 1 shows the assertion about the invariance condition. The rest is then clear. \square

Corollary 7. *For any polynomial f with simple prime factors, the dimension of $\mathcal{V}_f / \mathcal{V}_f^0$ is finite and equal to the dimension of any of the spaces in Theorem 6.*

Proof. This follows from Corollary 3 and Theorem 6 if $\langle f_x, f_y \rangle \neq \langle 1 \rangle$ and directly in the remaining case. \square

Some general consequences of Proposition 5 and Corollary 7 can be stated in explicit form. Recall that a singular point of C has multiplicity one if and only if its associated ideal in the primary decomposition (5) is maximal; other characterizations are given by invertibility of the Hessian of f , or by a local transversality condition for intersections, see [10].

Theorem 8. *The following statements hold.*

- (a) *The dimension of $\mathcal{V}_f/\mathcal{V}_f^0$ is greater than or equal to the number of singular points of C , with equality holding if every singular point of C has multiplicity one. In particular $\mathcal{V}_f = \mathcal{V}_f^0$ when C has no singular points.*
- (b) *If C has the singular points z_1, \dots, z_m , all of multiplicity one, then the map*

$$\begin{aligned} \mathcal{V}_f/\mathcal{V}_f^0 &\rightarrow \mathbb{C}^m \\ X + \mathcal{V}_f^0 &\mapsto (L(z_1), \dots, L(z_m)) \end{aligned}$$

(with L the cofactor of f for the vector field X) is a well-defined isomorphism of vector spaces.

- (c) *If all C_i are smooth, and if all intersections of distinct irreducible curves have multiplicity one, then $\mathcal{V}_f = \mathcal{V}_f^1$.*

Proof. Statement (a) is obvious.

For statement (b), first note that the cofactor of a trivial vector field vanishes at any singular point of C by Proposition 2, hence the map is well-defined. Now fix a singular point z_i . Since z_i is simple, the corresponding ideal \mathfrak{q}_i is maximal, and necessarily $\mathfrak{q}_i : \langle f \rangle = \mathbb{C}[x, y]$. Therefore there exists a vector field Y_i admitting f with cofactor K_i and $K_i(z_i) \neq 0$. Multiplying Y_i by suitable polynomials yields a vector field whose cofactor does not vanish at z_i but vanishes at every other z_j . This finishes the proof of statement (b).

For statement (c) there only remains to show for that every singular point z there exists an $X_z \in \mathcal{V}_f^1$ whose cofactor L_z does not vanish in z . Now z is a common zero of two distinct polynomials f_i and f_j , and $f_k(z) \neq 0$ for all other k . Define

$$X_z := \frac{f}{f_i} X_{f_i}, \quad \text{with} \quad L_z = \sum_{\ell \neq i} \frac{f}{f_i f_\ell} X_{f_i}(f_\ell),$$

then $L_z(z)$ is a nonzero multiple of $X_{f_i}(f_j)(z)$, and the latter is nonzero because of the transversality condition. \square

Statement (c) of this theorem shows that $\mathcal{V}_f = \mathcal{V}_f^1$ if certain geometric nondegeneracy conditions hold for the curves. This was also proved in [2], using a different approach.

For the dimension of $\mathcal{V}_f/\mathcal{V}_f^1$ we obtain an explicit expression.

Proposition 9. *If $\mathbb{C}[x, y]/\langle f_x, f_y \rangle$ is finite dimensional then*

$$\dim \mathcal{V}_f/\mathcal{V}_f^1 = \dim \mathbb{C}[x, y]/\langle f_x, f_y, p_1, \dots, p_{r-1} \rangle - \dim \langle f, f_x, f_y \rangle / \langle f_x, f_y \rangle.$$

Proof. From Corollary 4 we know that

$$\mathcal{V}_f/\mathcal{V}_f^1 \cong (\langle f_x, f_y \rangle : \langle f \rangle) / \langle f_x, f_y, p_1, \dots, p_{r-1} \rangle.$$

Now consider the linear map

$$\mathbb{C}[x, y] / \langle f_x, f_y, p_1, \dots, p_{r-1} \rangle \rightarrow \mathbb{C}[x, y] / \langle f_x, f_y \rangle$$

induced by $g \mapsto g \cdot f$. Note that this map is well defined because $p_i f \in \langle f_x, f_y \rangle$ for $i = 1, \dots, r-1$ (recall that p_i 's are cofactors). The kernel of this map is $\langle f_x, f_y \rangle / \langle f_x, f_y, p_1, \dots, p_{r-1} \rangle$ and the image is equal to $\langle f, f_x, f_y \rangle / \langle f_x, f_y \rangle$. Since the sum of the dimensions of the kernel and the image is equal to $\dim \mathbb{C}[x, y] / \langle f_x, f_y, p_1, \dots, p_{r-1} \rangle$, the assertion follows. \square

Example: Let $r \geq 3$ and let $f = f_1 \cdots f_r$ be homogeneous (thus f has degree r). Then,

$$\dim \mathcal{V}_f / \mathcal{V}_f^1 \geq \frac{(r-1)(r-2)}{2}.$$

To see this, note $f = x f_x + y f_y \in \langle f_x, f_y \rangle$; thus

$$\dim \mathcal{V}_f / \mathcal{V}_f^1 = \dim (\mathbb{C}[x, y] / \langle f_x, f_y, p_1, \dots, p_{r-1} \rangle).$$

Since all generators of $\langle f_x, f_y, p_1, \dots, p_{r-1} \rangle$ are homogeneous polynomials of degree $\geq r-2$, any set of polynomials of degree less than $r-2$ is linearly independent modulo $\langle f_x, f_y, p_1, \dots, p_{r-1} \rangle$ if and only if it is linearly independent. (For $r = 3$ one easily verifies that $\dim \mathcal{V}_f / \mathcal{V}_f^1 = 1$.)

4. COMPUTATIONAL ISSUES

In contrast with [2] and [10] we will discuss computational issues here, and we will illustrate that the computations are in fact feasible. For the problem of vector fields admitting the invariant curve C we obtain precise information about the dimension modulo trivial vector fields, and an algorithm to determine a basis of $\mathcal{V}_f / \mathcal{V}_f^0$. All the necessary operations are standard procedures (e.g. for Gröbner bases), and many of these are built-in features of various computer algebra systems. The last step may be critical for some standard systems, but it uses only built-in features of a specialized system like SINGULAR. [7] Let us describe the simple algorithmic nature of the approach in greater detail. The notation is as in Theorem 6.

The algorithmic approach. Assume that a polynomial f is given, with no multiple prime factors.

1. Let $f_x = s \cdot h_1$, $f_y = s \cdot h_2$ with a g.c.d. s of the partial derivatives. (This is algorithmically accessible, e.g. via resultants.) Then (2) holds if and only if

$$L = s \cdot \widehat{L} \quad \text{with} \quad \widehat{L} \in \mathbf{c} : \langle f \rangle.$$

2. Let G be a Gröbner basis of \mathbf{c} with respect to some fixed monomial ordering. Then there are only finitely many monomials m_1, \dots, m_d which

are not multiples of some leading monomial in G . According to Cox et al. [4], Ch. 5, §3, Prop. 1, the dimension of $\mathbb{C}[x, y]/\mathbf{c}$ is equal to the number of these monomials, and the classes $m_i + \mathbf{c}$ form a basis.

3. Now consider the map

$$\begin{aligned} M_f : \mathbb{C}[x, y]/\mathbf{c} &\rightarrow \mathbb{C}[x, y]/\mathbf{c} \\ g + \mathbf{c} &\mapsto f \cdot g + \mathbf{c} \end{aligned}$$

which, by definition, has the kernel $(\mathbf{c} : \langle f \rangle) / \mathbf{c}$. The matrix representation of M_f with respect to the basis $(m_i + \mathbf{c})$ can be obtained by fully reducing the polynomials $f \cdot m_j$ with respect to the Gröbner basis G , since this automatically yields a representation

$$f \cdot m_j \equiv \sum_i \alpha_{ij} \cdot m_i \pmod{\mathbf{c}}$$

and thus the desired matrix $A = (\alpha_{ij})$. Now standard linear algebra yields the kernel of M_f , and thus representatives of $(\mathbf{c} : \langle f \rangle) / \mathbf{c}$ as linear combinations of the m_i .

4. If some elements of $\mathbf{c} : \langle f \rangle$ are known, e.g. by way of (4), then one may replace \mathbf{c} by some larger ideal \mathbf{c}^* , and determine a basis of $(\mathbf{c} : \langle f \rangle) / \mathbf{c}^*$.

5. From cofactors to vector fields: Every element of the Gröbner basis G is a combination of h_1 and h_2 . Given some $\hat{L} = \sum \beta_j \cdot m_j$ from the kernel in 3. or 4. above, express $f \cdot \hat{L}$ as a combination of the elements of G , and thus as a combination of h_1 and h_2 . (Actually, the algorithm may run in a slightly different fashion; see Decker and Lossen [6], sections 2.2 and 3.6.) This yields the components of a vector field with cofactor $L = s \cdot \hat{L}$.

In particular the algorithm allows us to easily decide whether some subspace is already equal to \mathcal{V}_f . We look at two examples (with degenerate singular points), in order to illustrate that the computations are indeed unproblematic. The computations were carried out using MAPLE and SINGULAR.

Example 1. For the polynomial

$$f = y \cdot (y - x^2) \cdot (x - y^2)$$

we compute $\dim \mathcal{V}_f / \mathcal{V}_f^0$. One obtains the following Gröbner basis with respect to the grevlex order (using the terminology of [4]):

$$\langle f, f_x, f_y \rangle = \langle y^3 - 2xy + \underline{x^3}, -xy^2 + \underline{y^4}, y^2 - 3x^2y + 2\underline{xy^3}, -y^3 + \underline{x^2y^2} \rangle.$$

The leading monomials have been underlined. Now the monomials which are not multiples of some leading monomial above can readily be listed:

$$1, x, x^2, y, xy, xy^2, y^2, xy^2, y^3.$$

According to Corollary 7 the dimension of $\mathcal{V}_f/\mathcal{V}_f^0$ is equal to 9.

Example 2. Let $f = (y - x^2)(y - x^3)$. Then, with respect to the grevlex order, a Gröbner basis of $\langle f_x, f_y \rangle$ is given by

$$-9y^2 + 29xy - 40y + 20\underline{x^2}; 108\underline{y^3} + 85xy - 193y^2; 12\underline{xy^2} + 5xy - 17y^2$$

with leading monomials underlined. Therefore a basis of the quotient space is formed by the monomials

$$1; x; y; xy; y^2;$$

and reduction to normal form yields

$$\begin{aligned} f \cdot 1 &\equiv -\frac{1}{20}xy + \frac{1}{20}y^2, \\ f \cdot x &\equiv -\frac{1}{30}xy + \frac{1}{30}y^2, \\ f \cdot y &\equiv -\frac{1}{54}xy + \frac{1}{54}y^2, \\ f \cdot xy &\equiv -\frac{1}{81}xy + \frac{1}{81}y^2, \\ f \cdot y^2 &\equiv -\frac{5}{729}xy + \frac{5}{729}y^2. \end{aligned}$$

Thus we obtain the matrix of M_f :

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1/20 & -1/30 & -1/54 & -1/81 & -5/729 \\ 1/20 & 1/30 & 1/54 & 1/81 & 5/729 \end{pmatrix}$$

and solving for the kernel shows that the polynomials

$$2 - 3x; 5x - 9y; 2y - 3xy; 5xy - 9y^2$$

span the ideal of cofactors modulo $\langle f_x, f_y \rangle$.

We invoke additional knowledge according to 4.: The vector field $Y := f_2 \cdot X_{f_1}$ admits f with cofactor $L^* := 3x^2 - 2x$, and multiples of Y will yield the corresponding multiples of L^* . Reduction to normal form yields

$$\begin{aligned} L^* \cdot 1 &\equiv -2x + 6y - \frac{87}{20}xy + \frac{27}{20}y^2, \\ L^* \cdot x &\equiv -4y + \frac{29}{4}xy - \frac{9}{4}y^2, \\ L^* \cdot y &\equiv -\frac{5}{4}xy + \frac{9}{4}y^2, \end{aligned}$$

and $L^* \cdot xy \equiv L^* \cdot y^2 \equiv L^* \cdot y$. Thus \mathcal{V}_f^1 has codimension one in \mathcal{V}_f , and only one additional vector field $Z = A\partial/\partial x + B\partial/\partial y$ with cofactor $2 - 3x$ (for instance), needs to be found. Using SINGULAR and proceeding with the `lift` command, exactly as in Decker and Lossen [6], subsection 3.6.1, one obtains

$$\begin{aligned} A = & -9/40 x^5 + 261/800 x^4 y - 1218891241/57600000 x^3 y^2 - 27/40 x^4 \\ & - 580752454969/4976640000 x^3 y - 1218891241/57600000 x^2 y^2 \\ & - 180423092156761/429981696000 x^3 - 580136595769/4976640000 x^2 y \\ & - 261/400 xy^2 + 1218891241/28800000 y^3 \\ & - 180444591241561/429981696000 x^2 + 9/10 xy \\ & + 582376083769/2488320000 y^2 + 1/2x + 180229600393561/214990848000 y \end{aligned}$$

and

$$\begin{aligned} B = & -9/8 x^6 + 261/160 x^5 y - 1218891241/11520000 x^4 y^2 - 9/4 x^5 \\ & - 581704237369/995328000 x^4 y - 783/800 x^3 y^2 \\ & + 1218891241/19200000 x^2 y^3 - 180229600393561/85996339200 x^4 + 9/5 x^3 y \\ & + 581293664569/1658880000 x^2 y^2 + 1218891241/28800000 xy^3 + 1/2 x^3 \\ & + 180358594902361/143327232000 x^2 y + 582376083769/2488320000 xy^2 \\ & + 179907114121561/214990848000 xy + y. \end{aligned}$$

As this example illustrates, the algorithmic approach is feasible. As should be expected, the computer algebra system may produce quite large output. Modifying the vector field $X = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$ by elements of \mathcal{V}_f^1 of the form

$$a(y - x^2) \begin{pmatrix} 1 \\ 3x^2 \end{pmatrix} + b(y - x^3) \begin{pmatrix} 1 \\ 2x \end{pmatrix}$$

with suitable a and b will yield $\tilde{X} = \tilde{A}\frac{\partial}{\partial x} + \tilde{B}\frac{\partial}{\partial y}$ where

$$\begin{aligned} \tilde{A} &= \frac{1}{4}(2x - 3x^4 + y), \\ \tilde{B} &= \frac{1}{4}(2x^3 + 3x^5 + 4y - 9x^3 y); \end{aligned}$$

which is more palatable.

Thus, while things are settled from an algorithmic prospective, more structural insight would be welcome.

5. SIGMA PROCESSES

In order to obtain a better structural understanding of the situation, blow ups may be helpful. For this purpose we turn to sigma processes; see Shafarevich [9]. Using a suitable finite succession of such processes one will break up singular points of multiplicity > 1 , as well as self-intersections of irreducible curves; thus one can reduce the geometry to the nondegenerate setting from Theorem 8. This follows from the Bendixson–Seidenberg

theorem; see Anosov et al. [1]. The prototype (up to affine coordinate transformations) is described next.

Proposition 10. *Consider the map*

$$\Phi : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ xy \end{pmatrix}.$$

- (a) *Given irreducible, pairwise relatively prime polynomials f_1, \dots, f_r , with $f := f_1 \cdots f_r$, such that 0 is a singular point for C and a polynomial vector field $X = (P, Q)$ on \mathbb{C}^2 , the following are equivalent:*
- (i) *The zero set of f is invariant for X with cofactor K .*
 - (ii) *The vector field*

$$\hat{X} = \frac{1}{x} \cdot \begin{pmatrix} xP(x, xy) \\ -yP(x, xy) + Q(x, xy) \end{pmatrix}$$

is polynomial, and the zero set of $\hat{f} := f(x, xy)$ is invariant for \hat{X} with cofactor $\hat{K}(x, y) = K(x, xy)$.

- (b) *If x is not among the prime factors of f then $\hat{f} = f_0^{*s} \cdot f_1^* \cdots f_r^*$ with $f_0^* = x$, $f_i^*(x, y) = x^{-s_i} f_i(x, xy)$ irreducible for $1 \leq i \leq r$, and $s = s_1 + \cdots + s_r > 1$.*
- (c) *For any polynomial g one has*

$$\hat{X}_g = \frac{1}{x} X_{\hat{g}}$$

(possibly with rational vector fields).

- (d) *If $X \in \mathcal{V}_f^1$ then $\hat{X} \in \mathcal{V}_{\hat{f}}^1$, with $f^* := f_0^* \cdot f_1^* \cdots f_r^*$.*

Proof. Let $\hat{X} = \hat{P} \frac{\partial}{\partial x} + \hat{Q} \frac{\partial}{\partial y}$. Then statement (a) follows from the construction

$$\begin{pmatrix} \hat{P}(x, y) \\ \hat{Q}(x, y) \end{pmatrix} = D\Phi(x, y)^{-1} \begin{pmatrix} P(\Phi(x, y)) \\ Q(\Phi(x, y)) \end{pmatrix},$$

which implies

$$X(g) \circ \Phi = \hat{X}(g \circ \Phi)$$

for all g . \hat{X} is polynomial because 0 is a singular point of f , and therefore stationary for X .

For statement (b) note that $\hat{f}_i(x, y) = f_i(x, xy) = x^{s_i} f_i^*(x, y)$, with x^{s_i} the highest power of x that divides \hat{f}_i , and f_i^* is irreducible. One has $s_i > 0$ if and only if $f_i(0) = 0$, and $s_i > 1$ if and only if 0 is a singular point for f_i .

Statements (c) and (d) follow by elementary computations. (Note that, while $\frac{1}{x} X_{f_i^*}$ is rational whenever $f_i(0) \neq 0$, every $\frac{1}{x} \frac{f_i^*}{f_i^*} X_{f_i^*}$ is polynomial due to $s \geq 2$).

□

We record the identities

$$(8) \quad \begin{aligned} P(x, y) &= \hat{P}\left(x, \frac{y}{x}\right), \\ Q(x, y) &= \frac{y}{x} \hat{P}\left(x, \frac{y}{x}\right) + x \hat{Q}\left(x, \frac{y}{x}\right), \end{aligned}$$

for later use. In principle one can use sigma processes to transfer complicated geometries in the affine plane to simpler ones, until Theorem 8(c) is applicable. The problem is to revert the blow-ups to the original setting, because there exist many vector fields that are not blow ups (as implied by (8)). Thus one gains general structural insight, but some problems remain.

To illustrate the applicability of blow-ups, we consider one (relatively simple) class of examples.

Proposition 11. *Let $f = f_1$ be irreducible such that 0 is the only singular point of C , and assume that $f^* = x^s f_1^*$ satisfies the hypothesis of Theorem 8(c).*

(a) *Then every vector field X that admits f satisfies*

$$\hat{X} = A \cdot f_1^* \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + B \cdot x \cdot \begin{pmatrix} -f_{1y}^* \\ f_{1x}^* \end{pmatrix} + x \cdot f_1^* \begin{pmatrix} U \\ 0 \end{pmatrix}$$

with suitable polynomials A , B and U .

(b) *One has*

$$\begin{aligned} P(x, xy) &= x \cdot (-B f_{1y}^* + U f_1^*) \\ Q(x, xy) &= xy \cdot (-B f_{1y}^* + U f_1^*) + x \cdot (A f_1^* + B x f_{1x}^*) \end{aligned}$$

and the cofactor K of f for the vector field X satisfies

$$K(x, xy) = s(-B f_{1y}^* + U f_1^*) + A f_{1y}^* + x U f_{1x}^*.$$

Proof. Statement (a) follows directly from Theorem 8(c); just note that any term $x \cdot f_1^* \cdot \begin{pmatrix} 0 \\ V \end{pmatrix}$ can be incorporated into the first one.

Statement (b) follows by straightforward computation. \square

One can put these results to practical use as follows. Setting $x = 0$, one finds

$$\begin{aligned} K(0, 0) &= s(-B(0, y) f_{1y}^*(0, y) + U(0, y) f_1^*(0, y)) + A(0, y) f_{1y}^* \\ &= (-sB(0, y) + A(0, y)) f_{1y}^*(0, y) + sU(0, y) f_1^*(0, y). \end{aligned}$$

So the right hand side of this identity must be constant. Generally, there is a systematic test whether a polynomial $h(x, y)$ can be written as $g(x, xy)$, with some polynomial g : Setting $h = \sum_{\ell \geq 0} h_\ell(y) x^\ell$, the necessary and sufficient condition is that h_ℓ has degree $\leq \ell$ for all ℓ .

Thus one may proceed to evaluate the identities in Proposition 11(b) degree by degree in x . This is a heuristic approach, not an algorithm (since

termination is not guaranteed). But the heuristic turns to be useful in several instances.

Example 1. Consider $f(x, y) = x^2 + x^m y^n - y^2$, with $m \geq 2$ and $m + n \geq 3$. This polynomial is irreducible, and we have only one singular point at $(0, 0)$, with $\dim \mathcal{V}_f / \mathcal{V}_f^0 = 1$. Note the remarks preceding Theorem 8. Now $\hat{f}(x, y) = x^2(1 + x^{m+n-2}y^n - y^2)$, and if X admits f then

$$\begin{aligned} \hat{X} = & A \cdot (1 + x^{m+n-2}y^n - y^2) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & + B \cdot x \cdot \begin{pmatrix} 2y - nx^{m+n-2}y^{n-1} \\ (m+n-2)x^{m+n-3}y^n \end{pmatrix} + x(1 + x^{m+n-2}y^n - y^2) \cdot \begin{pmatrix} U \\ 0 \end{pmatrix}. \end{aligned}$$

Assuming $A = 0$ we get

$$\begin{aligned} K(x, xy) = & 2B \cdot (2y - nx^{m+n-2}y^{n-1}) + 2U \cdot (1 + x^{m+n-2}y^n - y^2) \\ & + x \cdot U \cdot (m+n-2)x^{m+n-3}y^n, \end{aligned}$$

thus

$$K(0, 0) = 2B(0, y) \cdot 2y + 2U(0, y) \cdot (1 - y^2).$$

The simplest choice is $B(0, y) = y, U(0, y) = 2$. Trying $B = y$ and $U = 2$ (i.e., B and U are independent of x) we find

$$K(x, xy) = 4 + (6 - 2n)x^{m+n-2}y^n$$

so

$$K(x, y) = 4 + (6 - 2n)x^{m-2}y^n$$

and moreover

$$X = \begin{pmatrix} 2x + (2 - n)x^{m-1}y^n \\ 2y + mx^{m-2}y^{n+1} \end{pmatrix}$$

by elementary computation. (Note that there must be a cofactor K with $K(0, 0) \neq 0$), according to Theorem 8. There is no general guarantee that the heuristic works with B and U functions of y alone; but this would be the first attempt in any sense. If necessary, incorporate higher x -powers.)

Example 2. We consider $f(x, y) = y(x + y)(x - y) + x^4$. This polynomial is irreducible, and by computations as in Section 4 Example 2, one finds that $\dim \mathcal{V}_f / \mathcal{V}_f^0 = 4$, and that the polynomials

$$1 + 6y; \quad x(1 + 6y); \quad y(1 + 6y); \quad y^2(1 + 6y)$$

span the ideal of cofactors modulo $\langle f_x, f_y \rangle$. Rather than taking the algorithmic approach, we use sigma processes here in order to find vector fields.

We have $\hat{f} = x^3(y - y^3 + x) = x^3 f_1^*$, thus f^* satisfies Theorem 8(c) and a vector field admitting f^* has the form

$$A \cdot (y - y^3 + x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + Bx \cdot \begin{pmatrix} 3y^2 - 1 \\ 1 \end{pmatrix} + x(y - y^3 + x) \cdot \begin{pmatrix} U \\ 0 \end{pmatrix}.$$

Assuming $A = 0$, we obtain

$$K(x, xy) = 3(B(0, y) \cdot (3y^2 - 1) + U \cdot (y - y^3 + x)) + xU.$$

Therefore $K(0, 0) = 3(B(0, y) \cdot (3y^2 - 1) + U(0, y) \cdot (y - y^3))$. The Euclidean algorithm suggests

$$B(0, y) = \frac{3}{2}y^2 - 1, \quad U(0, y) = \frac{9}{2}y.$$

With the working hypothesis $B_x = U_x = 0$, we have $B(x, y) = \frac{3}{2}y^2 - 1$, $U(x, y) = \frac{9}{2}y$, and $K(x, xy) = 3(1 + \frac{9}{2}y) = 3 + 18xy$, which yields

$$K(x, y) = 3(1 + 6y).$$

By elementary computations the vector field

$$\tilde{Y} = (\frac{3}{2}y^2 - 1)x \cdot \begin{pmatrix} 3y^2 - 1 \\ 1 \end{pmatrix} + x(y - y^3 + x) \cdot \begin{pmatrix} \frac{9}{2}y \\ 0 \end{pmatrix}$$

turns out to be the blow-up of

$$Y = \begin{pmatrix} x + \frac{9}{2}xy \\ y + 6y^2 - x^2 \end{pmatrix}.$$

Thus we have found $Y(f) = 3(1 + 6y) \cdot f$; and this solves our problem: The vector field Y provides the cofactor $1 + 6y$ (up to a scalar); Thus $x \cdot Y$, $y \cdot Y$ and $y^2 \cdot Y$ will give the remaining cofactors.

These examples are not sufficient to decide about the relative merit of sigma process versus the algorithmic approach. But they indicate that further work with sigma processes maybe promising.

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