NOTE ON HOMOLOGY OF EXPANDING FOLIATIONS

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ABSTRACT. This note contains some remarks about the homologies that can be associated to a foliation which is invariant and uniformly expanded by a diffeomorphism. We construct a family of 'dynamical' closed currents supported on the foliation which help us relate the geometric volume growth of the leaves under the diffeomorphism with the map induced on homology in the case when these currents have nonzero homology.

1. INTRODUCTION

Let M be a n-dimensional compact connected oriented Riemannian manifold and W a k-dimensional oriented foliation on M with C^r leaves, $r \geq 1$. One is interested in the homologies that can be associated to this foliation (the 'direction' of the leaves inside the manifold). One can consider the closed currents supported on the foliation, or equivalently the transverse measures, and consider their homologies.

There are foliations which don't support any closed currents, like the center unstable foliation of an Anosov flow. However if the leaves of the foliation have sub exponential growth, there are always closed currents supported on them (see [Pl]). This is the case when the foliation is uniformly expanded by a C^r diffeomorphism f.

We will restrict our attention to this case, the foliation is preserved by the diffeomorphism and there are constants $C_1 > 0$, $\lambda_1 > 1$ such that

$$m(Df^n|_{TW}) \geq C_1^{-1}\lambda_1^n,$$

where $m(T) = \|T^{-1}\|^{-1}$ is the minimal norm of T.

The dynamical (geometrical) volume growth of the foliation W under the diffeomorphism f is defined as follows:

$$\chi(W,f) = \sup_D \limsup_{n\to\infty} \frac{\log(\mathrm{vol}(f^n(D)))}{n},$$

where the supremum is taken over disks D inside the leaves of W. This is used in the study of the entropy of diffeomorphisms (see [Ne], [Yo], [HSX]).

One would like to extract information on the volume growth of f on W from the homologies of the closed currents supported on the foliation. A necessary condition for that is the existence of such closed currents with nonzero homology. This is not always the case, for example all the closed currents supported on the unstable foliation of an Anosov flow have zero homology. A sufficient condition for the existence of currents with nonzero homology is the existence of a closed k-form non degenerate on the foliation.

We will further restrict our attention to a subset of closed currents supported on W, defined 'dynamically'. They are obtained by starting with a 'simple' non-closed current supported on W, pushing it forward under the map induced by f, and renormalizing with a convenient constant. Under some regularity conditions the limit currents we obtain are closed. Thus we get an alternative way of showing that an uniformly expanding oriented foliation supports closed currents (or transversal measures) without using Plante's result, and we give an algorithm to construct them. We will describe this construction in Section 2.

In Section 3 we consider the case when these dynamical closed currents have nonzero homology and show how to relate the geometrical volume growth of W with the action of f_* on the homology. The basic idea used here is the following: the map induced by f on homology is linear and easy to describe, and as the pushed forward currents become almost closed, they should 'follow' the dynamics of this projectivized linear map. This result complements the one by Yano on the entropy conjecture relative to a foliation (see [Ya]).

In the last section we present two examples. In the first one we show how the dynamical closed currents can indeed form a nontrivial (also homological) subset of the set of all closed currents supported on W. In the second example we show that some regularity conditions on the starting simple non-closed currents are indeed required. Taking just the current of integration on a disk inside W may not be good enough, because when we iterate it the ratio between the volume of the boundary and the volume of the disk may not converge to zero.

Other studies of the homology of foliations (and relation with entropy for Anosov maps) are done for example in [Pl], [RS], [Sc], [SW], [Su]. The remarks in this paper complement the results from [HSX] and [SX], which contain applications of these type of results to the study of (non) absolute continuity of invariant foliations and entropy of partially hyperbolic diffeomorphisms.

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2. Dynamical closed currents on expanding foliations

In this section we present the construction of the dynamical closed currents supported on the foliation and what we call the topological growth of the foliation.

Suppose that the positive real constants $\lambda_i, \Lambda_i, C_i$ for $1 \leq i \leq k$ satisfy:

$$\frac{1}{C_i}\lambda_i^n \leq m(\Lambda^i Df^n|_{TW}) \leq \|\Lambda^i Df^n|_{TW}\| \leq C_i\Lambda_i^n \ , \ \forall n \in \mathbb{N}.$$

In other words, λ_i and Λ_i are lower and upper bounds for the i-dimensional rate of expansion on W. We will assume that W is uniformly expanded by f, i. e. $\lambda_1 > 1$.

If the foliation is one-dimensional, or if $\Lambda_{k-1} < \lambda_k$, we can consider any k-dimensional disk inside a leaf of a foliation, iterate it under f, and take the current of integration over it divided by the volume of the iterate of the disk. Using Stokes theorem it is easy to prove that every limit current we obtain must be closed.

However this may not be always the case, so in order to make sure that the limit currents are closed in general we have to modify it, to make sure that there is no mass accumulating on the boundary.

Let $0 < \alpha < 1$ such that

$$\lambda_1^{\frac{1}{\alpha}-1} > \frac{\Lambda_{k-1}}{\lambda_k}.$$

Let B be a closed C^1 disk inside a leaf $W(x)$ of the foliation. Let d be the Riemannian distance on M and $\rho : B \to \mathbb{R}_+$ be a C^1 function such that:

(1) ρ^α is Lipschitz, i. e. there exist $L > 0$ such that $|\rho^\alpha(x) - \rho^\alpha(y)| \leq Ld(x,y)$, $\forall x, y \in B$;
(2) ρ vanishes on the boundary of B, $\rho|_{\partial B} = 0$;
(3) ρ is the density of a probability measure on B with respect to the Lebesgue measure on B, i.e. $\int_B \rho d\mu = 1$.

For example ρ can be a bump function on B which approaches 0 at the boundary of B like $x^{\frac{1}{\alpha}}$.

Define the currents:

$$C_1(\omega) = \int_B \rho\omega \ , \ \forall \omega \in \Omega_k(M);$$

$$C_n(\omega) = \frac{1}{D_n}\int_B \rho f^{*n}\omega \ , \ \forall \omega \in \Omega_k(M)$$

where $D_n = \int_B \rho |df^n| d\mu = \int_{f^n(B)} \rho \circ f^{-n} d\mu$ is a normalizing constant. Here $|df^n|$ is the Jacobian of f^n restricted to TW and $\Omega_k(M)$ is the set of k-forms on M.

One can easily check that the sequence C_n is uniformly bounded so it has subsequences with weak limits. Also C_n is obtained actually by pushing forward C_1 by f^n and dividing by D_n:

$$C_n(\omega) = \frac{1}{D_n}(f^n_* C_1)(\omega) = \frac{1}{D_n} C_1(f^{*n}\omega) = \frac{1}{D_n}\int_{f^n(B)} (\rho\circ f^{-n})\omega.$$

We now claim that every such limit current is closed.

Theorem 2.1. *If C is a weak limit of C_n then C is a closed current.*

Proof. We have to prove that C applied to any exact k-form is 0. It is enough to show that

$$\lim_{n\to\infty} C_n(d\omega) = 0\ ,\ \forall\omega\in\Omega_{k-1}(M).$$

We will use the following simple lemma:

Lemma 2.2. *If $g:\mathbb{R}^k\to\mathbb{R}_+$ is differentiable with g^α Lipschitz with Lipschitz constant L, $0<\alpha<1$ then the inequality $\frac{\|dg\|}{g}\geq t$ implies $\|dg\|\leq\left(\frac{L}{\alpha}\right)^{\frac{1}{\alpha}} t^{1-\frac{1}{\alpha}}$.*

Proof. If g^α is Lipschitz with constant L then

$$\|dg\|g^{\alpha-1}\leq\frac{L}{\alpha}$$

so

$$\left(\frac{\|dg\|}{g}\right)^{1-\alpha}\|dg\|^\alpha\leq\frac{L}{\alpha}$$

and then

$$\|dg\|\leq\left(\frac{L}{\alpha}\right)^{\frac{1}{\alpha}} t^{1-\frac{1}{\alpha}}.$$

□

Now we continue with the proof of the theorem. We have the following:

$$\begin{aligned} C_n(d\omega) &= \frac{1}{D_n}\int_B \rho\, df^{*n}\omega = \frac{1}{D_n}\int_B (d(\rho f^{*n}\omega) - d\rho\wedge f^{*n}\omega) \\ &= \frac{1}{D_n}\left(\int_{\partial B}\rho f^{*n}\omega - \int_B d\rho\wedge f^{*n}\omega\right) = -\frac{1}{D_n}\int_B d\rho\wedge f^{*n}\omega \end{aligned}$$

and

$$|C_n(d\omega)|\leq\frac{1}{D_n}\int_B\|d\rho\|\|f^{*n}\omega\|d\mu.$$

There is a constant $C>0$ such that

$$\|f^{*n}\omega\|\leq C\Lambda_{k-1}^n\ ,\ \|f^{*n}\omega\|\leq C\frac{|df^n|}{\lambda_1^n}.$$

Let $\lambda < \lambda_1$ and $\beta < 1$ such that $\beta\lambda^{\frac{1}{\alpha}-1} > \frac{\Lambda_{k-1}}{\lambda_k}$ and using the lemma with $g = \rho$, $t = \lambda^n$ and $s = \left(\frac{L}{\alpha}\right)^{\frac{1}{\alpha}}$ we get:

$$\begin{aligned} |C_n(d\omega)| &\leq \frac{1}{D_n}\int_{\left[\frac{\|d\rho\|}{\rho}<\lambda^n\right]} C\|d\rho\|\frac{|df^n|}{\lambda_1^n}d\mu + \frac{1}{D_n}\int_{\left[\|d\rho\|\leq s\lambda^{n(1-\frac{1}{\alpha})}\right]} C\|d\rho\|\Lambda_{k-1}^n d\mu \\ &\leq \frac{1}{D_n}\int_B C\rho\lambda^n\frac{|df^n|}{\lambda_1^n}d\mu + \frac{1}{D_n}\int_B Cs\left(\lambda^{1-\frac{1}{\alpha}}\Lambda_{k-1}\right)^n d\mu \leq \\ &\leq C\left(\frac{\lambda}{\lambda_1}\right)^n + C'\frac{\left(\lambda^{1-\frac{1}{\alpha}}\Lambda_{k-1}\right)^n}{D_n} \leq C\left(\frac{\lambda}{\lambda_1}\right)^n + C'\frac{\lambda_k^n}{D_n}\beta^n \\ &\leq C\left(\frac{\lambda}{\lambda_1}\right)^n + C'\beta^n \end{aligned}$$

and this converges to 0 as n tends to infinity, q.e.d.

□

Remark 1. *This result can be generalized to more complicated currents which locally can be written as*

$$C(\omega) = \int_T \left(\int_{D(y)} \rho_y(x)\omega\right) d\mu_T$$

where T is a transversal to the foliation, $D(y)$ is the local leaf passing through $y \in T$, ρ_y satisfies the conditions above uniformly in y and μ_T is a Borel measure on T.

Remark 2. *Better regularity conditions would imply a faster rate of convergence, up to the order of $\frac{1}{\lambda_1^n}$.*

Remark 3. *The existence of the foliation is not important in this argument, it is enough to have a submanifold which is uniformly expanded by f. The limit currents will be closed, however they may be all zero, so one needs some extra assumptions replacing the orientability of the foliation to get nontrivial limits.*

Question 1. *What is the optimal regularity for ρ such that the limit currents are closed?*

Define $H(W)$ to be the span of the set of homologies corresponding to all the limit currents for all the disks D inside the foliation and densities ρ satisfying the required conditions. Obviously $H(W)$ is invariant under f_*. We define the *topological growth of W under f* to be the logarithm of the spectral radius of f_* restricted to $H(W)$

$$\tau(W, f) = \log \operatorname{sp}(f_*|_{H(W)}),$$

or equivalently the logarithm of the largest absolute value of the eigenvalues.

3. Limit currents with non-trivial homology

We now consider the homologies associated to the closed limit currents described in the previous section. We are interested in the case when this homologies are nontrivial.

Fix ω_i, $1 \leq i \leq d = \dim H^k(M, \mathbb{R})$ closed forms such that $\{[\omega_i] : 1 \leq i \leq d\}$ is a basis for $H^k(M, \mathbb{R})$. Then $f^*(\omega_i)$ is also a closed form so there are real numbers a_i^j such that

$$[f^*(\omega_i)] = \sum_{j=1}^{d} a_i^j [\omega_j].$$

Consequently $f^*(\omega_i) - \sum_{j=1}^{d} a_i^j \omega_j$ is an exact form so we can choose $\widetilde{\omega}_i \in \Omega_{k-1}(M)$ such that

$$f^*(\omega_i) - \sum_{j=1}^{d} a_i^j \omega_j = d\widetilde{\omega}_i.$$

Now for any k-current C we can associate a unique homology class $h(C) \in H_k(M, \mathbb{R})$ such that

$$C(\omega_i) = \langle h(C), [\omega_i] \rangle \quad \forall i \in \{1, 2, \ldots, d\}.$$

The map h is linear, continuous, depends of course on the choice of ω_i and restricted to the closed currents is the natural projection to the homology. One can then check that

$$\begin{aligned}
\langle h(f_*(C)), [\omega_i] \rangle &= f_*(C)(\omega_i) = C(f^*(\omega_i)) = C(\sum_{j=1}^{d} a_i^j \omega_j + d\widetilde{\omega}_i) \\
&= \sum_{j=1}^{d} a_i^j C(\omega_j) + C(d\widetilde{\omega}_i) = \left\langle h(C), \sum_{j=1}^{d} a_i^j [\omega_j] \right\rangle + C(d\widetilde{\omega}_i) \\
&= \langle h(C), [f^*(\omega_i)] \rangle + C(d\widetilde{\omega}_i) = \langle f_*(h(C)), [\omega_i] \rangle + C(d\widetilde{\omega}_i)
\end{aligned}$$

or

$$h(f_*(C)) = f_*(h(C)) + h_\epsilon(C)$$

where $h_\epsilon(C)$ is such that $\langle h_\epsilon(C), [\omega_i] \rangle = C(d\widetilde{\omega}_i)$. This means that if C is almost closed, i. e. $C(d\widetilde{\omega}_i)$ are close to zero, then $h_\epsilon(C)$ is small and f_* almost commutes with h.

This is the case for the currents C_n from the previous section. Because $f_*(C_n) = \frac{D_{n+1}}{D_n} C_{n+1}$ we get

$$h(C_{n+1}) = \frac{D_n}{D_{n+1}} h(f_*(C_n)) = \frac{D_n}{D_{n+1}} f_*(h(C_n)) + \frac{D_n}{D_{n+1}} h_\epsilon(C_n).$$

In this case $h_\epsilon(C_n) \to 0$ and $\frac{D_n}{D_{n+1}}$ is uniformly bounded from above and below,

$$C_k^{-2} \Lambda_k^{-1} \leq \frac{D_n}{D_{n+1}} \leq C_k^2 \lambda_k^{-1}.$$

More generally one can prove that for any natural number $l > 1$ we have

$$h(C_{n+l}) = \frac{D_n}{D_{n+l}} f_*^l(h(C_n)) + h_{\epsilon,l}(C_n)$$

where $h_{\epsilon,l}(C_n) \to 0$ and

$$C_k^{-2} \Lambda_k^{-l} \leq \frac{D_n}{D_{n+l}} \leq C_k^2 \lambda_k^{-l}.$$

Suppose that all the limit currents (corresponding to some density ρ) have nontrivial homology. This is the case when there is a closed k-form non-degenerate on the foliation. Or there is a family of closed k-forms such that for any leaf of the foliation at least one of these forms is non-degenerate on the leaf. Then for n sufficiently large $h(C_n)$ is bounded away from zero and if we let $\|h(C_n)\| = d_n$, because f_* and h almost commute, we get that $\frac{h(C_n)}{d_n}$ can be identified with a pseudo-orbit of the projectivized linear map induced by f on the homology. Furthermore the jumps become arbitrarily small. Consider the Lyapunov decomposition of $H_k(M, \mathbb{R})$, or the finest dominated splitting with respect to f_*:

$$H_k(M, \mathbb{R}) = E_{\mu_1} \oplus E_{\mu_2} \oplus \cdots \oplus E_{\mu_z}$$

where $0 < \mu_1 < \mu_2 < \cdots < \mu_z$ and E_{μ_i} contains the generalized eigenspaces of the eigenvalues with absolute value μ_i. Also, for every unit vector $v \in E_{\mu_i}$ we have:

$$\lim_{l \to \infty} \frac{\log(\|f_*^l(v)\|)}{l} = \log \mu_i,$$

and the limit is uniform with respect to v.

Now, using invariant cones and the fact that the jumps of the pseudo-orbit are arbitrarily small, we get that all the limits of the sequence $\frac{h(C_n)}{d_n}$ must be in one of these invariant subspaces, say E_μ.

We know that:

$$\frac{h(C_{n+l})}{d_{n+l}} = \frac{D_n d_n}{D_{n+l} d_{n+l}} f_*^l \left(\frac{h(C_n)}{d_n} \right) + \frac{h_{\epsilon,l}(C_n)}{d_{n+l}}.$$

Using the fact that $\|\frac{h(C_{n+l})}{d_{n+l}}\| = 1$ and $\frac{h_{\epsilon,l}(C_n)}{d_{n+l}} \to 0$ we get that

$$\lim_{n\to\infty} \frac{D_n d_n}{D_{n+l} d_{n+l}} \|f_*^l\left(\frac{h(C_n)}{d_n}\right)\| = 1.$$

Taking the logarithm and dividing by l we get:

$$\lim_{n\to\infty}\left(\frac{1}{l}\log\frac{D_n d_n}{D_{n+l}d_{n+l}} + \frac{\log\|f_*^l(\frac{h(C_n)}{d_n})\|}{l}\right) = 0.$$

For any $\epsilon > 0$ there exist a natural number l such that for any unit vector $v \in E_\mu$ we have

$$\left|\frac{\log\|f_*^l(v)\|}{l} - \log\mu\right| < \epsilon.$$

Because $h(C_n)$ comes arbitrarily close to E_μ there exist $n_0 > 0$ such that

$$\left|\frac{\log\|f_*^l(\frac{h(C_n)}{d_n})\|}{l} - \log\mu\right| < 2\epsilon, \quad \forall n > n_0.$$

Then there exist $n_1 > n_0$ such that

$$\left|\frac{1}{l}\log\frac{D_{n+l}d_{n+l}}{D_n d_n} - \log\mu\right| < 3\epsilon, \quad \forall n > n_1.$$

applying this formula for $n, n+l, n+2l, \dots, n+(i-1)l$, adding and dividing by i we get:

$$\left|\frac{1}{il}\log(D_{n+il}d_{n+il}) - \frac{1}{il}\log(D_n d_n) - \log\mu\right| < 3\epsilon, \quad \forall n > n_1.$$

But $\lim_{i\to\infty}\frac{n+il}{il} = 1$ and $\lim_{i\to\infty}\frac{1}{il}\log(D_n d_n) = 0$ so there exist $n_2 > n_1$ such that

$$\left|\frac{\log(D_n d_n)}{n} - \log\mu\right| < 4\epsilon, \quad \forall n > n_2.$$

This is true for any $\epsilon > 0$, so $\lim_{n\to\infty}\frac{\log(D_n d_n)}{n} = \log\mu$ and because d_n is bounded away from zero and infinity one gets

$$\lim_{n\to\infty}\frac{\log(D_n)}{n} = \log\mu.$$

We proved the following theorem:

Theorem 3.1. *If all the limit currents corresponding to some density ρ are non-trivial, then the homologies of the limit currents are combinations of (generalized) eigenvectors corresponding to eigenvalues with the same absolute value μ and*

$$\lim_{n\to\infty}\frac{\log(D_n)}{n} = \log\mu.$$

As a consequence we also get the following theorem:

Theorem 3.2. *If all the limit currents for all the densities ρ have nontrivial homologies, then the topological growth of the foliation is equal to the dynamical volume growth of the foliation:*

$$\tau(W,f) = \chi(W,f) = \sup_\rho \lim_{n\to\infty} \frac{\log D_n}{n}$$

where the supremum is taken over densities ρ satisfying the conditions in the previous section.

Proof. From the previous theorem we easily get that

$$\tau(W,f) = \sup_\rho \lim_{n\to\infty} \frac{\log D_n}{n},$$

so all we have to prove is that

$$\chi(W,f) = \sup_\rho \lim_{n\to\infty} \frac{\log D_n}{n}.$$

But for any ρ supported on the disk D we have

$$D_n \leq \sup_{x\in D} \rho(x) \mathrm{vol}(f^n(D))$$

so

$$\lim_{n\to\infty} \frac{\log D_n}{n} \leq \lim_{n\to\infty} \frac{\log(\mathrm{vol}(f^n(D)))}{n}$$

and consequently

$$\sup_\rho \lim_{n\to\infty} \frac{\log D_n}{n} \leq \chi(W,f).$$

Also for every disk D' on the foliation there exist a density ρ' satisfying the required conditions, supported on a larger disk $D'' \supset D'$ and such that $\rho'|_{D'} = c$ for some fixed $c > 0$, so

$$\mathrm{vol}(f^n(D')) \leq \frac{D'_n}{c},$$

and then

$$\lim_{n\to\infty} \frac{\log(\mathrm{vol}(f^n(D')))}{n} \leq \lim_{n\to\infty} \frac{\log D'_n}{n}$$

so

$$\chi(W,f) \leq \sup_\rho \lim_{n\to\infty} \frac{\log D_n}{n}.$$

The required equality follows. □

Question 2. *Is it possible to get limit currents with both zero and nonzero homologies?*

Remark 4. *If we replace the volume growth of W by the absolute volume growth*

$$\bar{\chi}(W,f) = \limsup_{n\to\infty} \frac{1}{n} \log(\sup_{x\in M} vol(f^n(B_y))),$$

where B_y is the ball of radius 1 centered at y on the leaf of W passing through y, then one always gets the inequality:

$$\tau(W,f) \le \bar{\chi}(W,f).$$

Also one has:

$$\chi(W,f) \le \bar{\chi}(W,f) \le h(f),$$

which gives the entropy conjecture relative to the foliation W (see [HSX],[Ya]*).*

4. Examples

4.1. **First example.** The first examples show that there may be closed currents supported on the foliation which are not dynamical closed currents. This is because the dynamical closed currents can't be 'wandering'.

Consider a hyperbolic linear automorphism A of the two-torus $\mathbb{T}^2$ and $B = A \times A^2$ on $\mathbb{T}^4$. This map has an unstable foliation W^u and weak unstable and strong unstable sub-foliations W^{wu} and W^{su}. We also denote by A and B the linear maps on $\mathbb{R}^2$ and $\mathbb{R}^4$ and E^u, E^{wu}, E^{su} the corresponding subspaces in $\mathbb{R}^4$. Let $\beta : \mathbb{T}^1 \to \mathbb{T}^1$ be the map defined as follows: consider the map B acting on oriented lines passing through the origin in the unstable plane E^u and identify it with a map on the circle (β is actually the double cover of the projectivization of $B|_{E^u}$). This map will have two repellers corresponding to W^{wu} and two attractors corresponding to W^{su}. Now let $C = B \times \beta$ be a diffeomorphism of $\mathbb{T}^5$. Define a foliation W on $\mathbb{T}^5$ by lines in the following way: each $t \in \mathbb{T}^1$ corresponds to an oriented line in E^u, and let $\mathbb{T}^4 \times \{t\}$ be foliated by lines parallel to this line. It is easy to see that this is an invariant foliation which is uniformly expanded by C (it sub-foliates W^u). Then the dynamical closed currents have homologies only in the directions of E^{wu} and E^{su} but there are closed currents supported on W with homologies in all the directions in E^u.

One can also replace B by B', where $B'(u,v) = (Au + v, Av)$. There is again the same W^u and E^u, but no strong and weak unstable foliations. However W^{su} and E^{su} are still invariant. Define an invariant sub-foliation W' of W^u in the same way as before, using B'. Then $H(W')$ corresponds to E^{su}, and again there are closed currents supported on W' with homologies in all the directions in E^u.

Question 3. *Is $H(W)$ the limit set of f_* (projectively) restricted to the set of homologies of general closed currents supported on W?*

4.2. **Second example.** We will present here an example where we show that if we start with a current of integration on a disk (ρ is the indicator function for the disk) and we apply the algorithm (apply f_* and then divide by the volume) we may not end up with closed limit currents.

Let A be a linear Anosov automorphism of the torus $\mathbb{T}^2$ with the expanding eigenvalue $b > 1$. Consider B the product of A with a north pole (N)-south pole (S) map of the circle $\mathbb{T}^1$ - α:

$$B(x,y) = (A(x), \alpha(y)) \quad , \ \forall x \in \mathbb{T}^2, y \in \mathbb{T}^1.$$

There exist a fixed point x_0 for A and consequently (x_0, N) is a fixed point for B. Let b be the eigenvalue of $D\alpha$ at N. We will consider actually that we have a chart on $\mathbb{T}^1$ with $N = 0$, $S = 1 = -1$ and

$$D\alpha = b \qquad \text{on} \qquad \left[-\frac{1}{2b}, \frac{1}{2b}\right]; \tag{1}$$

$$0 < D\alpha < b \qquad \text{on} \qquad \left(\frac{1}{2b}, 1\right) \cup \left(-1, -\frac{1}{2b}\right). \tag{2}$$

We also consider charts (x^1, x^2) around x_0 such that $x_0 = (0,0)$ and $\frac{\partial}{\partial x^1}$ and $\frac{\partial}{\partial x^2}$ are the unstable and the stable bundles of A on $\mathbb{T}^2$. The product will be a chart (x^1, x^2, y) on a neighborhood of (x_0, N). We remark that the curve $(t, 0, t), t \in \left[0, \frac{1}{2b}\right]$ is uniformly expanded by DB at a rate of approximately b^n. It has one endpoint fixed at (x_0, N) and the rest accumulates to the unstable manifold of (x_0, S) in $\mathbb{T}^2 \times \{S\}$. Now we consider M' to be the unit tangent bundle of a surface with constant negative curvature and ϕ_t the geodesic flow on M'. Suppose that the time t map of the geodesic flow ϕ_t expands the unstable manifolds by a factor of e^{kt}. Now take the skew product f on $M = \mathbb{T}^3 \times M'$ given by:

$$f(x,y,z) = (B(x,y), \phi_{T(y)}(z)) \quad , \quad \forall x \in \mathbb{T}^2, y \in \mathbb{T}^1, z \in M',$$

where $T : \mathbb{T}^1 \to \mathbb{R}_+$ is such that:

$$e^{kT(0)} = d; \tag{3}$$

$$e^{kT(y)} = de^{-my} \ \text{ for } \ y \in \left[0, \frac{1}{2b}\right]; \tag{4}$$

$$1 < e^{kT(y)} < c \ \text{ for } \ y \in \left(\frac{1}{2b}, \frac{1}{2}\right), \tag{5}$$

where $1 < c < cb < d$ and $de^{-\frac{m}{2b}} = c$ or $m = 2b \ln \frac{d}{c}$. Extend T for $-1 \leq y \leq 0$ such that T is smooth and strictly positive.

Now we consider an rectangle D in M formed by the product of a piece of an unstable manifold of length 1 of the geodesic flow on M' and the curve we considered before $(t,0,t), t \in \left[0, \frac{1}{2b}\right]$ on $\mathbb{T}^3$. This rectangle is uniformly expanded by f. We will prove that the ratio

$$\frac{\mathrm{Vol}(\partial f^n(D))}{\mathrm{Vol}(f^n(D))}$$

is bounded away from zero. Here the volume is taken with respect to the Riemannian metric on M' crossed with the Riemannian metric on $\mathbb{T}^3$ given by the chart described before.

By considering only the piece of the unstable manifold of the geodesic flow which is in $\{N\} \times M'$ and is expanded at a rate of d^n we get that

$$\mathrm{Vol}(\partial f^n(D)) > d^n.$$

Now we will estimate the volume of $f^n(d)$. For this we divide the rectangle D into n smaller rectangles corresponding to t (or y) in the intervals

$$\left[\frac{1}{2b^2}, \frac{1}{2b}\right], \left[\frac{1}{2b^3}, \frac{1}{2b^2}\right], \dots, \left[\frac{1}{2b^n}, \frac{1}{2b^{n-1}}\right], \left[0, \frac{1}{2b^n}\right].$$

For each $t \in \left[\frac{1}{2b^{k+1}}, \frac{1}{2b^k}\right]$, $1 \le k \le n-1$, the Jacobian of f^n restricted to the tangent plane to the disk D at the point corresponding to (t,s) where s is some parametrization of the unstable piece in M' is smaller than

$$b^n \prod_{i=0}^{n-1} e^{kT(\alpha^i(t))}.$$

Now

$$\prod_{i=0}^{n-1} e^{kT(\alpha^i(t))} \le \prod_{i=0}^{k-1} e^{kT(b^i t)} c^{n-k} = d^k c^{n-k} e^{-mt\frac{b^k-1}{b-1}} \le d^k c^{n-k} e^{-mtb^{k-1}}.$$

Consequently

$$|Df^n(t,s)|_{TD}| \le b^n d^k c^{n-k} e^{-mtb^{k-1}} \text{ if } \frac{1}{2b^{k+1}} \le t \le \frac{1}{2b^k}, \quad 1 \le k \le n-1.$$

Also for $0 \le t \le \frac{1}{2b^n}$ one can check that the similar inequality holds for $k = n$. The volume of $f^n(D)$ is bounded by the integral of the Jacobian:

$$\begin{aligned} \mathrm{Vol}(f^n(D)) &\le \sum_{k=1}^{n-1} b^n d^k c^{n-k} \int_{\frac{1}{2b^{k+1}}}^{\frac{1}{2b^k}} e^{-mtb^{k-1}} dt + b^n d^n \int_0^{\frac{1}{2b^n}} e^{-mtb^{n-1}} dt \\ &\le C \sum_{k=1}^{n} b^{n-k} d^k c^{n-k} = C b^n c^n \sum_{k=1}^{n} \left(\frac{d}{bc}\right)^k \le C' d^n \end{aligned}$$

where C, C' are some fixed strictly positive constants. For the last inequality the condition $d > bc$ is required. Combining the inequalities for the volume of the disk and of the boundary we get:

$$\frac{\mathrm{Vol}(f^n(\partial D))}{\mathrm{Vol}(f^n(D))} \geq \frac{1}{C'} > 0.$$

In order to prove that the limit currents are not closed one can choose an 1-form which can be written $\omega = y\omega'$ for $0 \leq y \leq 1$, where ω' is an 1-form non-degenerate on the unstable foliation of the geodesic flow. Then integrating $d\omega = dy \wedge \omega' + y d\omega'$ on $f^n(D)$ we get something greater (if we choose the right orientation) then integrating just on the set corresponding to $0 \leq y \leq \frac{1}{2b^n}$. But on this set $dy \wedge \omega'$ is equivalent to the volume form on $f^n(D)$ and $yd\omega'$ is degenerate, so the integral is greater than $C''d^n$ for some constant $C'' > 0$ so

$$|C_n(\omega)| \geq \frac{C''}{C'} > 0.$$

We remark that the disk D in our example can be made part of an invariant foliation in the following way: extend the segment $(t, 0, t)$ for $t \in \left(-\frac{1}{2b}, \frac{1}{2b}\right)$, translate this segment in the $\mathbb{T}^2$ direction to every point in $\mathbb{T}^2 \times \{N\}$, and iterate by B. If we add the unstable foliation of $\mathbb{T}^2 \times \{S\}$ we get a foliation of $\mathbb{T}^3$ invariant under B. Take the product of this foliation with the unstable foliation of the geodesic flow.

Question 4. *Is there a similar example for the unstable foliation of a partially hyperbolic diffeomorphism?*

Remark 5. *In this example the volume growth of D is equal to the volume growth of ∂D. One can actually find examples where the boundary grows strictly faster than the disk, by using a slight modification of the example of volume growth greater than the entropy:*

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ uniformly expanding such that $f(x, y) = (\mu x, \lambda y)$ on a small compact neighborhood of 0 and $f(x, y) = (\mu x, \nu y)$ outside a larger compact neighborhood of 0, for $\lambda > \mu^k > \mu^{k-1} > \nu > 1$and $k > 2$. Now consider the curve $y = x^k \sin\frac{1}{x}$ for $x \in [0, 1]$ and complete it to some simple closed curve ∂D which is the boundary of the disk D. Now the volume growth of D is $\log\mu + \log\nu$ and the volume growth of ∂D is $\frac{\log\lambda}{k} > \log\mu + \log\nu$.

This example can be easily done on the unstable manifold of some fixed point of a diffeomorphism on the torus. However, this does not imply that the limit currents obtained by pushing forward D and rescaling are not closed.

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