TREE AUTOMORPHISMS AND QUASI-ISOMETRIES OF THOMPSON'S GROUP F

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Abstract. We prove that automorphisms of the infinite binary rooted tree $\mathcal{T}_2$ do not yield quasi-isometries of Thompson's group F, except for the map which reverses orientation on the unit interval, a natural outer automorphism of F. This map, together with the identity map, forms a subgroup of $Aut(\mathcal{T}_2)$ consisting of 2-adic automorphisms, following standard terminology used in the study of branch groups. However, for more general p, we show that the analgous groups of p-adic tree automorphisms do not give rise to quasi-isometries of $F(p)$.

1. Introduction

Given a finitely generated group G, the quasi-isometry group of G is:

$$QI(G) = \{f \colon G \to G \mid f \text{ is a quasi-isometry}\}/\sim$$

where $f \sim h$ if they differ by a bounded amount in the supremum norm. Quasi-isometry groups are often large and difficult to compute. Here, we investigate the quasi-isometry group of Thompson's group F. Recent work of Burillo, Cleary and Röver [7] describes a family of quasi-isometries of F derived from the abstract commensurator of the group, and from the outer automorphism group described earlier by Brin [3]. They prove that the commensurator group of F embeds into the quasi-isometry group. In particular, these quasi-isometries do not arise as automorphisms of the group.

Little else is known about self-quasi-isometries of Thompson's group F or of the classification of groups quasi-isometric to F. A group is always quasi-isometric to its finite index subgroups; descriptions of all finite index subgroups of F are given in Burillo, Cleary and Röver [7] and Bleak and Wassink [2]. Two well-known questions motivate the exploration of $QI(F)$. The first asks whether F and $F \times \mathbb{Z}$ are quasi-isometric. As a group, F

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contains numerous subgroups isomorphic to $F^n \times \mathbb{Z}^m$, for $m, n \in \mathbb{Z}^+ \cup \{0\}$, which are quasi-isometrically embedded [6, 10, 13, 14]. The second question asks whether $F(p)$ and $F(q)$ are quasi-isometric, where $F(p)$ and $F(q)$ are the analogues of $F = F(2)$ defined over trees of constant valence $p+1$ away from the root caret. For an introduction to F we refer the reader to Cannon, Floyd and Parry [8] and for $F(p)$ to [5].

Each element of Thompson's group F corresponds uniquely to a pair of reduced finite rooted binary trees. Understanding the self-maps of underlying geometric structures has been an effective technique for analyzing some quasi-isometry groups. From this viewpoint, it is natural to ask when a tree automorphism will induce a quasi-isometry of F. We follow [1] and [12] in our notation and terminology for tree automorphisms, and refer the reader to those articles for more details. We let $\mathcal{T}_k$ denote the infinite k-ary rooted tree. The automorphism group $Aut(\mathcal{T}_k)$ is the iterated wreath product $Aut(\mathcal{T}_k) = S_k \wr (S_k \wr (S_k \wr \cdots))$ where S_k denotes the symmetric group on k letters. It is clear that the action of $f \in Aut(\mathcal{T}_k)$ on $\mathcal{T}_k$ preserves the levels of $\mathcal{T}_k$, and permutes by $\sigma_u \in S_k$ the k downward directed edges at each vertex u. Following [1], we say that f is represented on $\mathcal{T}_k$ by *decorating* each vertex u with the permutation σ_u, and that the decorated tree is the *portrait* of F. Equivalently, the vertex permutation σ_u is referred to as the *vertex activity* of f at u. A tree automorphism has a unique portrait, and given a portrait, we can reconstruct the automorphism by performing the specified permutations.

In the case of Thompson's group, we take $\mathcal{T} = \mathcal{T}_2$ to have valence 3 at all vertices other than the root vertex, which has valence 2. Thus the portrait of $f \in Aut(\mathcal{T})$ consists of decorating the vertices of $\mathcal{T}$ with either the identity permutation or the transposition (01), which induces a flip on the descending edges and interchanges the left and right subtrees of the vertex. Here, we completely determine which tree automorphisms induce quasi-isometries of F, and find that only one nontrivial quasi-isometry arises in this way– the outer automorphism of F which reverses orientation on the unit interval. A quasi-isometry need not preserve any group structure, much less any structure relating to the pair of trees representing a given element. Applying work of Fordham [11] we can say that a quasi-isometry must coarsely preserve the number of carets in each tree in the pair of trees representing an element of F.

We end with an example placing our results in the more general context of automorphims of other rooted regular trees, and the generalized Thompson's groups $F(p)$. The tree automorphism which induces the quasi-isometry of $F(2)$ is an example of a 2-adic automorphism in $Aut(\mathcal{T}_2)$, following following standard terminology used in the study of branch groups. We provide an example to show that not every element of the corresponding subgroup of p-adic automorphisms in $Aut(T_p)$ yields a quasi-isometry of the generalized Thompson group $F(p)$.

2. Thompson's group F

Thompson's group F is a finitely generated group with a wide range of combinatorial interpretations. The standard infinite presentation of F is given by:

$$\langle x_0, x_1, \ldots, | x_i^{-1} x_j x_i = x_{j+1} \quad \text{if } i < j \rangle$$

which immediately yields the standard finite presentation for F:

$$\langle x_0, x_1 | [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle.$$

Elements of Thompson's group F are understood in many equivalent ways, as follows.

(1) As words in either the finite or infinite presentation for the group given above.
(2) As orientation-preserving piecewise-linear homeomorphisms from $[0, 1]$ to itself with only finitely many singularities of slope. These singularities are required to have coordinates in the set of dyadic rationals. Away from these singularities, all slopes must be powers of 2.
(3) As pairs of finite rooted binary trees, each having the same number of leaves, as described in [8]. We call this a *tree pair diagram* representing an element.

We refer the reader to [8] for an explicit description of the equivalence of these interpretations of elements of F.

The trees we consider are built and labeled as follows. Let T be a rooted binary tree. A vertex together with its two downward directed edges is called a *caret*, so we regard T as composed of a collection of *carets*. A *leaf* is a vertex of valence one, and each edge of a caret may have another caret attached to it; in this case we say that the caret has a *right or left child* depending on where this new caret is attached. We consider the leaves of T numbered from left to right, beginning with 0. Figure 1 presents an example of a pair of trees representing an element of F with numbered leaves.

A caret is called *right* (resp. *left*) if one of its edges lies on the right (resp. left) side of the tree. All other carets are *interior*. The level of the caret is the number of edges in the shortest path connecting its vertex to the root vertex.

Figure 1. Tree pair diagram for $x_0^2 x_1 x_2 x_4 x_5 x_7 x_8 x_9^{-1} x_7^{-1} x_3^{-1} x_2^{-1} x_0^{-2}$ with leaves numbered from left to right.

We may require the trees in a tree pair diagram representing $w \in F$ to be *reduced*. Each element of F can be represented by infinitely many tree pair diagrams, but there is a unique reduced one. Let T_- and T_+ denote finite rooted binary trees, and $w = (T_-, T_+)$. A tree pair diagram is *unreduced* if each of T_- and T_+ contain a caret with two exposed leaves numbered m and $m+1$, and it is *reduced* otherwise. A caret with both leaves exposed is called an *exposed caret*. To obtain the reduced tree pair diagram representing a group element from an unreduced diagram, we remove the pairs of exposed carets with identical leaf numbers from both trees, repeating this procedure if necessary. When we write (T_-, T_+) to represent an element of F, we are assuming, unless noted otherwise, that the tree pair diagram is reduced.

We can refer to a position in a rooted binary tree using a vertex address, which is defined inductively as follows. The root vertex has the empty label. Given a vertex with label s, the left child of the vertex is labeled $s0$ and right child of the vertex labeled $s1$. For example, node 3 in the left-hand tree of Figure 1 has address 011.

In order to multiply two tree pair diagrams (T_-, T_+) and (S_-, S_+), unreduced representatives (T'_-, T'_+) and (S'_-, S'_+) of the two elements are created in which $T_- = S_+$. We define the product $(T_-, T_+)(S_-, S_+)$ to be the possibly unreduced tree pair diagram (S'_-, T'_+).

Given a tree pair diagram (T_-, T_+) representing an element $w \in F$, one would like to know the word length of w with respect to the standard finite generating set $\{x_0, x_1\}$. Fordham [11] developed a remarkable method for measuring word length in F with respect to this generating set which relies solely on combinatorial information regarding the configuration of the carets in the reduced tree pair diagram representing the element. In [5] and [9], this word length is shown to be proportional to the number of carets in either tree of the reduced pair representing the group element. If $N(w)$ is this number of carets, we will refer to the inequality given in [9], namely

$$N(w) - 2 \leq |w|_F \leq 4N(w) - 8 \tag{1}$$

where $|w|_F$ denotes this word length. In general, we let $d_{\{x_0,x_1\}}(v, w)$ denote the word length $|v^{-1}w|_F$ of $v^{-1}w$ with respect to the generating set $\{x_0, x_1\}$.

3. Tree automorphisms and quasi-isometries of F

We begin by recalling the definition of a quasi-isometry.

Definition 3.1. *Let X and Y be metric spaces. A map $f\colon X \to Y$ is a (K, C)-quasi-isometry if the following conditions hold.*

(1) *For all $x, y \in X$, we have $\frac{1}{K} d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq K d_X(x, y) + C$.*
(2) *There is a constant C' so that the C' neighborhood of $f(X)$ contains Y.*

A map which satisfies the first condition in Definition 3.1 but not the second is called a *quasi-isometric embedding.*

We consider automorphisms of the infinite binary rooted tree $\mathcal{T} = \mathcal{T}_2$ as described in Section 1, via portraits given by decorated infinite binary trees. In this case, all vertex permutations are either identity or the transposition (01). Recall that this transposition interchanges the left and right subtrees of the vertex. Two tree automorphisms f_1 and f_1 in $Aut(\mathcal{T})$ with portraits $\mathcal{L}_1$ and $\mathcal{L}_2$ determine a self-map of Thompson's group F, as follows. Given $w = (T_-, T_+) \in F$, we place T_- at the root of $\mathcal{T}$, and rearrange T_- according to the vertex permutations in $\mathcal{L}_1$ to obtain a new tree T'_-. We obtain a map ϕ_1 in this way. We do the same with T_+ and $\mathcal{L}_2$ to obtain T'_+ and the map ϕ_2. Then $\phi = (\phi_1, \phi_2)$ is a map from F to itself given by $\phi(T_-, T_+) = (T'_-, T'_+)$. (We will leave out the specific reference to the portraits $\mathcal{L}_1$ and $\mathcal{L}_2$ when they are clear.) We are interested in when the map ϕ determines a quasi-isometry of F.

We first show that if $\mathcal{L}_1 \neq \mathcal{L}_2$ then the induced map ϕ cannot be a quasi-isometry. The proofs of Proposition 3.5 and Lemma 3.5 below use the following fact about the change in caret numbering in a single tree after a transposition at a given vertex is performed.

Fact 3.2. Let T be a finite binary rooted tree, and ϕ a map induced on the infinite binary rooted tree $\mathcal{T}$ by $f \in Aut(\mathcal{T})$ with portrait $\mathcal{L}$. Let L and R be the left and right subtrees, respectively, of a vertex m in T. The leaf numbers of any exposed carets in L are smaller than the leaf numbers of any exposed carets in R.

Now we consider the tree $\phi(T)$. The leaf numbers of any exposed leaves in $\phi(R)$ will be smaller than the leaf numbers of any exposed carets in $\phi(L)$ if and only if the vertex permutation at $\phi(m)$ is a transposition.

We let T be a finite rooted binary tree, and ϕ a map induced by $f \in Aut(\mathcal{T})$ with portrait $\mathcal{L}$, as above Let ϕ^{-1} denote the inverse of ϕ, that is the map on finite rooted binary trees induced by f^{-1}. We note two facts about T and $\phi(T)$:

Fact 3.3. The trees T and $\phi(T)$ have the same number of exposed carets.

Fact 3.4. *Let $\tau \in \mathcal{T}_2$. If the vertex permutation at v induced by ϕ is τ, then the vertex permutation at $\phi^{-1}(v)$ induced by ϕ^{-1} is also τ.*

We begin by considering two tree automorphisms, $f_1, f_2 \in Aut(T)$, with portraits $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and the induced map $\phi = \phi_{(\mathcal{L}_1,\mathcal{L}_2)} \colon F \to F$. Our first result is that when $f_1 \neq f_2$, the map ϕ is never a quasi-isometry.

Proposition 3.5. *Let f_1 and f_2 be tree automorphisms with portraits $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, and induced map $\phi = \phi_{(\mathcal{L}_1,\mathcal{L}_2)} \colon F \to F$. If $\mathcal{L}_1 \neq \mathcal{L}_2$ then ϕ is not a quasi-isometry of F.*

Proof. We choose s to be the vertex closest to the root caret whose vertex permutations in $\mathcal{L}_1$ and $\mathcal{L}_2$ differ. If there are several such vertices, we choose the one on its level which is closest to the left side of the tree when the vertices are considered from left to right. We assume that the vertex permutation $\sigma_s \in \mathcal{L}_1$ is the transposition, and in $\mathcal{L}_2$ it is the identity.

We consider the vertex $s \in \mathcal{T} = \mathcal{T}_2$, the infinite rooted binary tree, and the string of carets $\mathcal{S}$ with root s, each of which is the left child of the previous caret. We consider two cases, based on the vertex permutations in this string of carets under f_1 and f_2: either there is some caret t after which all vertex activity is the same for f_1 and f_2, or this is not the case.

Case 1: Suppose first that after vertex t in this string $\mathcal{S}$, all vertex permutations are identical. We construct the following pair of trees. Let M' be the minimal tree containing the caret with vertex t. Let B denote the subtree of the string $\mathcal{S}$ of carets with root vertex the left leaf of t and length m. In the argument below, we will vary the length m of this string. Notice that by Fact 3.4, the subtrees $\phi_1^{-1}(B)$ and $\phi_2^{-1}(B)$ are identical as subtrees, although they have different root vertices. Let $M = M'$ with the subtree B attached to the left leaf of vertex t.

Now consider $(\phi_1^{-1}(M'), \phi_2^{-1}(M'))$. If this is not a reduced tree pair diagram, then the single exposed caret in each tree has identical leaf numbers, say k and $k+1$. But when we consider $(\phi_1^{-1}(M), \phi_2^{-1}(M))$, because the vertex permutations at t are different, so are the vertex permutations in ϕ_1^{-1} and ϕ_2^{-1} at vertices $\phi_1^{-1}(t)$ and $\phi_2^{-1}(t)$, respectively. Since $\phi_1^{-1}(B) = \phi_2^{-1}(B)$, in one of the trees in $(\phi_1^{-1}(M), \phi_2^{-1}(M))$, the leaf numbering in $\phi_i^{-1}(B)$ begins at k, and in the other tree it begins at $k+1$. Thus the tree pair diagram $(\phi_1^{-1}(M), \phi_2^{-1}(M))$ is reduced. It is easy to see that when $\phi = (\phi_1, \phi_2)$ is applied to this tree pair diagram, we obtain the identity element of F. Thus by varying the length of B, we can contradict any fixed quasi-isometry constants.

Now suppose that $(\phi_1^{-1}(M'), \phi_2^{-1}(M'))$ is a reduced tree pair diagram. This means that there are at least two vertices along $\mathcal{S}$ (including s) where the vertex permutations differ; that is, we rule out the case $s = t$. If $s = t$, then M' has as its single exposed caret the caret with vertex s. Since both leaves of s are exposed, and the vertex permutations are identical at all other vertices of M', we see that $\phi_1^{-1}(M') = \phi_2^{-1}(M')$ and thus the tree pair diagram is not reduced.

If $(\phi_1^{-1}(M'), \phi_2^{-1}(M'))$ is a reduced tree pair diagram and we make an argument analogous to the one above, by attaching the subtree B to the left leaf of the exposed caret in M' to form a tree M, we may be in the situation where the exposed carets in $\phi_1^{-1}(M)$ and $\phi_2^{-1}(M)$ have identical leaf numbers, causing the entire subtree $\phi_i^{-1}(B)$ to be removed when the tree pair diagram $(\phi_1^{-1}(M), \phi_2^{-1}(M))$ is reduced. If this is not the case, we make an argument identical to that given above. If this is the case, we alter our trees as follows.

Suppose that in $\mathcal{L}_1$, the vertex permutation $\sigma_s = (01)$. To form M, attach two carets to the right leaf of s, one the right child of the other, as well as the subtree B to the left leaf of t. When we now compute $\phi_1^{-1}(M)$ and $\phi_2^{-1}(M)$, these right carets appear before the exposed caret in $\phi_1^{-1}(M)$ but not in $\phi_2^{-1}(M)$. This has the effect of increasing the numbering of the exposed leaves in $\phi_1^{-1}(B)$ by two, so the resulting tree pair diagram $(\phi_1^{-1}(M), \phi_2^{-1}(M))$ will be reduced. Note that this only works when the first tree pair diagram we considered above was unreduced. We again note that the word length of this element if proportional to the length of B, and its image under $\phi = (\phi_1\phi_2)$ is the identity. By varying the size of B we can contradict any fixed quasi-isometry constants.

Case 2: Now we suppose that there is never a vertex t along $\mathcal{S}$ so that the two portraits $\mathcal{L}_1$ and $\mathcal{L}_2$ are identical for all vertices in $\mathcal{S}$ descended from t. Let t be a vertex along $\mathcal{S}$ at which the two portraits differ. Suppose that in $\mathcal{L}_1$, $\sigma_t = (01)$ and in $\mathcal{L}_2$ σ_t is the identity. Let M' be as above the minimal tree containing the caret with vertex t.

Given quasi-isometry constants (K, C), if we can find t sufficiently far down the string of carets $\mathcal{S}$ so that $(\phi_1^{-1}(M'), \phi_2^{-1}(M'))$ is a reduced tree pair diagram, then we are done. Applying [10] we know that the word length of $(\phi_1^{-1}(M'), \phi_2^{-1}(M'))$ is roughly proportional to the number of carets in the substring of $\mathcal{S}$, and $\phi = (\phi_1, \phi_2)$ maps this group element to the identity. Since there is never a vertex along $\mathcal{S}$ after which the two portraits are identical, we have an infinite sequence of vertices t which may be used to make this argument.

If $(\phi_1^{-1}(M'), \phi_2^{-1}(M'))$ is not a reduced pair of trees, then let the exposed caret in each tree have leaf numbers a and $a+1$. Form the tree M by adding a single additional caret from $\mathcal{S}$ to M' at the left child of t. Applying the vertex permutations σ_t from the portraits of $\mathcal{L}_1^{-1}$ and $\mathcal{L}_2^{-1}$, we see that the trees in $(\phi_1^{-1}(M), \phi_2^{-1}(M))$ no longer form an unreduced pair, as the leaf numbers of the exposed caret in one tree are $a, a+1$ and in the other are $a+1, a+2$. We then make the same argument to finish this case as in the previous case. □

We next determine when a map $\phi = \phi(\mathcal{L}, \mathcal{L})\colon F \to F$ induced by $f \in Aut(\mathcal{T})$ with portrait $\mathcal{L}$ is a quasi-isometry of F. The following lemma is a special case of Theorem 3.9. We include it to illustrate the fundamental idea of the proofs below.

Lemma 3.6. *Let f be a tree automorphism with portrait $\mathcal{L}$ which performs a permutation at the root vertex, and is the identity at all other vertices. As above, let $\phi\colon F \to F$ be the map induced on F by $\mathcal{L}$. Then ϕ is not a quasi-isometry of F.*

FIGURE 2. The group element $w = (T_-, T_+)$, where the subtree A (which has n carets) will be removed when the element $\phi(w)$ is reduced. In this case, ϕ is the map induced by the tree automorphism which performs a transposition at the root vertex and is the identity at all other vertices.

FIGURE 3. The unreduced tree pair diagram representing $\phi(w)$, where $w \in F$ is the group element given in Figure 2.

Proof. Consider the example $w = (T_-, T_+)$ given in Figure 2, where we let the subtree A contain n carets. The smallest leaf number of an exposed leaf of T_- in subtree A is clearly 1, while in T_+ it is 2. Thus the pair of trees in the figure is reduced, even though the exact form of the subtree A is not given. It is not possible for a tree to have two exposed carets whose leaves are numbered a and $a+1$ simultaneously with $a+1$ and $a+2$.

The tree pair diagram representing $\phi(w)$ is given in Figure 3. It is clear by construction that the smallest leaf number in subtree A in either tree in Figure 3 is 5, and thus the entire subtree A is removed when this tree pair diagram is reduced.

Note that the identity element is fixed under the map ϕ. Thus, by making the subtree A sufficiently large, we can produce examples of elements which contradict any fixed quasi-isometry constants. □

We can prove the analogous lemma where $\mathcal{L}$ is the portrait of a tree automorphism whose activity at every vertex other than the root is the transposition using a similar example.

Lemma 3.7. *Let f be a tree automorphism with portrait $\mathcal{L}$ which is the identity at the root vertex, and performs the transposition at all other vertices. As above, let $\phi\colon F \to F$ be the map induced on F by $\mathcal{L}$. Then ϕ is not a quasi-isometry of F.*

Proof. We follow the proof of Lemma 3.6, using the pair of trees (T_-, T_+) constructed as follows. To form T_-, begin with the root caret, and attach a caret to each leaf. Number these leaves from 0 through 3 and attach the generic subtree A to leaf 2. To form T_+, begin with the root caret and attach the generic subtree A to its right leaf. To the left leaf of the root, add a single caret, and then a right child to that new caret. We claim that the tree pair diagram (T_-, T_+) is unreduced, as the leaf numbering of the exposed carets in A begins at 2 in T_- and at 1 in T_+. Thus by inequality 1, the word length of the group element represented by (T_-, T_+) is at least the number of carets in A. By construction, when we reduce the tree pair diagram we get by applying ϕ to this pair of trees, the numbering of the exposed leaves in the subtrees $\phi(A)$ is identical, and these subtrees are removed, leaving an element of fixed length. Again, since word length is roughly proportional tothe number of carets in reduced tree pair diagrams, we can contradictany possible choice of quasi-isometry constants. □

Proposition 3.8. *Let f be a tree automorphism with portrait $\mathcal{L}$ which performs a transposition at every vertex, and $\phi\colon F \to F$ the map induced by $\mathcal{L}$ on F. Then ϕ is a quasi-isometry of F.*

Proof. This corresponds to the outer automorphism of order 2 which reverses orientation of the intervals, described in Brin [3], and thus is obviously a quasi-isometry. This quasi-isometry is not a bounded distance from the identity, see [7]. □

We note that we can also prove Lemma 3.7 by considering the tree automorphism as the composition of the orientation-reversing quasi-isometry of Lemma 3.8 with the map given in Lemma 3.6 which is not a quasi-isometry.

The description of those tree automorphisms which induce quasi-isometries of F is completed by the following:

Theorem 3.9. *Let f be a tree automorphism with portrait $\mathcal{L}$ in which not all vertex permutations are identical. Then the induced map $\phi\colon F \to F$ is not a quasi-isometry of F.*

Proof. We choose s to be the vertex closest to the root caret whose vertex permutation is the identity. If there are several such vertices, we choose the one on its level which is closest to the left side of the tree when the vertices are considered from left to right. If s is not the root, then we let r be the parent vertex of s, whose vertex permutation is necessarily a transposition. We divide the proof into several cases; in each case we will construct elements $v,\ w \in F$ so that $d_{\{x_0,x_1\}}(v,w)$ is arbitrarily large, but the distance in the word metric of their images $d_{\{x_0,x_1\}}(\phi(v),\phi(w))$ is fixed and thus we can contradict any fixed quasi-isometry constants.

In the cases where s is not the root vertex, we let M be the minimal tree containing a caret with vertex r. In the cases below, the letter A will refer to an unspecified large subtree which is attached to M in a particular manner; we will increase the size of A to create arbitrarily large examples of group elements v and w.

Case 1: Includes all situations when s is not theroot, except when r is a right caret whose left child is s. We will construct $v = (T_-, T_+)$ and $w = (S_-, S_+)$ as follows. Take the tree $\phi^{-1}(M)$ and attach to the leaf of $\phi^{-1}(r)$ which does not terminate at $\phi^{-1}(s)$ the subtree (a) pictured in Figure 4. To any leaf preceding the subtree A in this tree, add a single caret c, and call the resulting tree T_-. There is such a leaf exactly because we rule out the situation where r is a right caret with vertex s the terminus of its left leaf. To form S_-, add to the same vertex of $\phi^{-1}(M)$ the subtree (b) pictured in Figure 4. Notice that the two trees T_- and S_- have the same number n of carets. As a result of the caret c in T_- which does not occur in S_-, the leaf numbering of the subtree A in T_- begins with some number k, and in S_- with $k-1$.

Figure 4. The subtrees which are attached to the leaf of r which is not the vertex s in the construction of group elements in Theorem 3.9.

We let R be the tree consisting of n right carets, each the right child of the previous one. Let $T_+ = S_+ = \phi^{-1}(R)$. We do not claim that either pair (T_-, T_+) or (S_-, S_+) is reduced; however, we can use these potentially unreduced tree pair diagrams to compute $d_{\{x_0,x_1\}}(v,w) = |v^{-1}w|_F$. To do this, we consider the trees in the order S_- S_+ T_+ T_- and notice that by construction, the middle trees are identical. Thus the word length of the pair (S_-, T_-) yields the distance between the original elements v and w. The fact that the leaf numbering of the preimage of A in S_- begins with $k-1$ and in T_- it begins with k means that when this tree pair diagram is reduced, no carets from the subtree $\phi(A)$ will be removed. Thus the number of carets in the reduced tree pair diagram representing $v^{-1}w$ is at least the number of carets in $\phi(A)$, and from inequality 1 we see that the word length of $v^{-1}w$ is also at least the number of carets in A. By varying A we can construct elements v and w with arbitrarily large distance between them in this word metric.

We now compute $d_{\{x_0,x_1\}}(\phi(v), \phi(w))$. Recall that s was chosen to be the first instance where the vertex permutation is the identity; thus the vertex permutations at all vertices of $\phi^{-1}(M)$ are transpositions. Since the caret c was added to a leaf preceding the subtree A in T_-, by Fact 3.2, in $\phi^{-1}(T_-)$ the exposed leaves of the caret $\phi(c)$ will have higher numbers than any exposed leaves in the subtree $\phi(A)$. To finish describing $\phi(T_-)$, note that when we perform the vertex permutation $\sigma_r = (01)$, we move the root of the added subtree (a) from Figure 4 to the vertex s, where it remains. Whatever the form of the subtree $\phi(A)$, its root remains the left child of s. The tree $\phi(S_-)$ is formed analogously, using the subtree (b) from Figure 4 and without the additional caret c. The important point is that now the leaves of caret c in $\phi(T_-)$ are numbered after the leaves in $\phi(A)$; this guarantees that the leaf numbering in $\phi(A)$ in both $\phi(T_-)$ and $\phi(S_-)$ is identical.

To compute $d_{\{x_0,x_1\}}(\phi(v), \phi(w))$, we consider the trees in the order $\phi(S_-)$ $\phi(S_+)$ $\phi(T_+)$ $\phi(T_-)$. Again, the middle trees are identical, and we consider the unreduced tree pair diagram $(\phi(S_-), \phi(T_-)$. To reduce this pair of trees, at least the entire subtree A must be removed, leaving a group element of fixed word length. By increasing the size of the subtree A, we can contradict any fixed quasi-isometry constants.

Case 2: Now we consider when s is not the root and r is a right caret whose left child is$s$. In this case the minimal tree M containing r consists entirely of right carets. To form T_- and S_-, we add the subtrees (c) and (d), respectively, which are pictured in Figure 4 to two copies of $\phi^{-1}(M)$ at leaf 0, creating T_- and S_-. Additionally, we add a single caret c to T_- attached to any leaf with greater leaf number than those in A. The trees T_- and S_- have the same number, n, of carets. We let R be the tree consisting entirely of n right carets, each the right child of the previous one. Let $\mathcal{T}_+ = S_+ = \phi^{-1}(R)$. Given the construction of T_- and S_-, it is easy to see that the leftmost leaf of the subtree A in T_- has leaf number 2, and the leftmost leaf of the subtree A in S_- has leaf number 3. Following the reasoning in Case 1, we conclude that $d_{\{x_0,x_1\}}(v, w)$ is at least the number of carets in A.

To compute $d_{\{x_0,x_1\}}(\phi(v), \phi(w))$, we consider the images of the different trees under ϕ. By construction, in $\phi(T_-)$ and $\phi(S_-)$, the images of the subtrees (c) and (d), respectively, from Figure 4 (which contain the subtree

A) are attached to the vertex s, which is the left child of r. Moreover, the exposed leaves of the caret $\phi(c)$ are now numbered before the exposed leaves in the subtree $\phi(A)$ in $\phi(T_-)$. As a result, the leaf numbering in $\phi(A)$ in both $\phi(T_-)$ and $\phi(S_-)$ is identical. As $\phi(T_+) = \phi(S_-)$, the word length of the pair $(\phi(S_-), \phi(T_-))$ will be the distance between the two group elements $\phi(v)$ and $\phi(w)$. When this tree pair diagram is reduced, the entire subtree $\phi(A)$ will be removed from both trees, leaving an element whose length is determined by the location of r. Again, by increasing the size of the subtree A, we can contradict any possible fixed quasi-isometry constants.

Case 3: s is the root caret. Suppose that ϕ were a quasi-isometry of F. Let ψ be the quasi-isometry of F induced by the tree automorphism g whose vertex activity at every vertex is the transposition (01), which was discussed in Lemma 3.8. Then $\psi \circ \phi$ would be a quasi-isometry of F, corresponding to the map induced by the tree automorphism $f \circ g$. Notice that the vertex activity at the root caret of $f \circ g$ is the transposition (01), and by the choice of f there is at least one vertex whose permutation is the identity. It follows from the previous two cases of this proof that $\psi \circ \phi$ is not a quasi-isometry, contradicting our assumption. Thus ϕ is not a quasi-isometry of F. □

4. Higher valence trees and $F(p)$

We end with examples placing the above results in the context of the generalized Thompson's groups $F(p)$, and the automorphism group $Aut(\mathcal{T}_p)$ of the infinite rooted tree with constant valence $p+1$ away from the root vertex, which has valence p. In Section 3 we showed that the only automorphism of T_2 which yields a quasi-isometry of $F = F(2)$ satisfies the property that each vertex permutation is the transposition $(01) \in S_2$. In the context of tree automorphisms, these are 2-adic automorphisms. When p is prime, we consider the cyclic permutation $\pi = (0123 \cdots p-1)$. A p-adic automorphism of T_p is one that performs a power of π at every vertex of T_p. (For more information on p-adic automorphisms of T_p we refer the reader to [15].) In this context we need not rely on p being prime, and we consider the analogous subgroup of $Aut(\mathcal{T}_n)$ for general $n > 2$.

The group $F(n)$, for any integer $n \geq 2$, is the group of piecewise-linear orientation preserving maps of the closed unit interval to itself, in which the coordinates of all discontinuities of slope lie in $\mathbb{Z}[\frac{1}{n}]$ and the slopes of all linear pieces are integer powers of n. Equivalently, group elements are described by equivalence classes of pairs of finite rooted trees, where each vertex except the root caret has valence $n+1$, and the root has valence n. A reduction condition analogous to the one for $F(2)$ holds, and thus each group element corresponds uniquely to a reduced tree pair diagram. We refer the reader to Burillo, Cleary and Stein [5] for background on $F(p)$, as well as for a proof that the word length of $w \in F(p)$ is proportional to the number of carets in a reduced tree pair diagram for w, analogous to inequality 1.

A natural question to ask is whether non-trivial p-adic automorphisms which have the same permutation at each vertex induce quasi-isometries of $F(p)$ in a manner analogous to the case of $F = F(2)$ and T_2 above. We show in an elementary example that such p-adic automorphisms of $Aut(T_p)$ do not always induce quasi-isometries of $F(p)$.

We consider $F(3)$, and the permutation $\pi = (012)$. If we take the group element represented by the pair of trees (T_-, T_+) in Figure 5, where A is an arbitrary large subtree, and perform the map ϕ induced by the tree automorphism whose vertex activity is π at each vertex, we obtain a tree pair diagram $\phi(T_-, T_+) = (S_-, S_+)$ which is unreduced, with $\phi(A)$ both identical in shape and leaf numbering in both trees, as the automorphisms are self-similar. When this tree pair diagram is reduced, we obtain a fixed, short element. By choosing the original subtree A as large as required, we can contradict any fixed quasi-isometry constants.

Figure 5. A reduced pair of trees representing an element of $F(3)$, where A represents an arbitrary subtree.

Thus, this 3-adic tree automorphism does not yield a quasi-isometry of $F(3)$. The corresponding construction works for all n, showing that there are n-adic automorphisms which are not quasi-isometries of $F(n)$ when $n > 2$. The automorphism $f \in Aut(T_k)$ whose vertex permutation at every vertex is the order reversing permutation of $\{1, \ldots, n\}$ again yields an outer automorphism of F, and thus a quasi-isometry, but is not p-adic for $p > 2$.

We end with the comment that to date, no one has provided a context in which to understand geometrically the groups of quasi-isometries of $F(p)$, including the exotic outer automorphisms described by Brin and Guzman [4].

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