NET SPACES AND BOUNDEDNESS OF INTEGRAL OPERATORS

ERLAN NURSULTANOV AND SERGEY TIKHONOV

Abstract. In this paper we introduce new functional spaces which we call the net spaces. Using their properties, the necessary and sufficient conditions for the integral operators to be of strong or weak-type are obtained. The estimates of the norm of the convolution operator in weighted Lebesgue spaces are presented.

1. Introduction

Let $(\Omega, \mathfrak{F}, \mu)$ be a measurable space, where μ is a σ-additive measure, $\mathfrak{F}$ is the algebra of the measurable sets with identity Ω. The space $L_p(\Omega, \mu)$ is the collection of all those measurable functions f satisfying

$$\|f\|_{L_p(\Omega,\mu)} = \left(\int_\Omega |f(x)|^p \, d\mu\right)^{\frac{1}{p}} < \infty.$$

The distribution of a measurable function f on Ω is defined by

$$m(\sigma, f) = \mu\{x \in \Omega : |f(x)| > \sigma\}.$$

Then $f^*(t) = \inf\{\sigma : m(\sigma, f) \leqslant t\}$ is the decreasing rearrangement of f.

Let $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$. The Lorentz space $L_{pq}(\Omega, \mu)$ is defined [BS, Ch. 4] by those measurable functions f such that

$$\|f\|_{L_{pq}} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty,$$

when $0 < q < \infty$, and

$$\|f\|_{L_{p\infty}} = \sup_t \; t^{1/p} f^*(t) < \infty,$$

when $q = \infty$. We also define

$$f^{**}(x) = \frac{1}{x} \int_0^x f^*(t) dt.$$

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In this paper we study the weak and strong (p,q)-boundedness of the integral operator

$$Tf(y)=\int_D K(x,y)f(x)d\nu_x \tag{1.1}$$

in weighted Lebesgue and Lorentz spaces. Here the kernel $K(x,y)$ is a $\nu\otimes\mu$-measurable locally integrable function on $D\times\Omega$.

The boundedness of the Hardy-type transforms, the potential operators, the Laplace and Fourier transforms, the Riemann-Liouville operators in the weighted spaces have been extensively studied; and in many cases the complete answers were presented as criteria in terms of weights (see the texts [CRS], [KP], [EKM] and the earlier papers [AH], [AS], [BH1], [BH2], [CF], [Mu]).

In the general case, from the results by Kantorovič-Vulih and Dunford-Pettis ([DP], [KV], [KA, Ch. 11,§1]), it follows that for $1\leqslant p<\infty$ and $1<q\leqslant\infty$

$$\|T\|_{L_p(D,\nu)\to L_\infty(\Omega,\mu)}=ess\sup_{y\in\Omega}\|K(\cdot,y)\|_{L_{p'}(D,\nu)}, \tag{1.2}$$

$$\|T\|_{L_1(D,\nu)\to L_q(\Omega,\mu)}=ess\sup_{x\in D}\|K(x,\cdot)\|_{L_q(\Omega,\mu)}. \tag{1.3}$$

Necessary and sufficient conditions for T to be bounded from $L_p(D,\nu)$ to $L_q(\Omega,\mu)$ can be obtained using the factorization technique ([Ch], [Ga], [GR], [H1], [H2], [Ja]). In particular, such results were given for $1<q<p<\infty$; $1\leqslant p\leqslant\infty$, $q=1$; and $p=\infty$, $1\leqslant q\leqslant\infty$.

We also recall the results on the boundedness of the convolution-type operator in the Lorentz spaces (Hölder-type inequalities): we refer to O'Neil [ON], Yap [Ya], and Blozinski [Bl]. In particular, the following Young-O'Neil inequality is known

$$\|Af\|_{L_{q,s}(\Omega,dx)}\leqslant C\|K\|_{L_{r,t_1}(\Omega-D,dx)}\|f\|_{L_{p,t_2}(D,dx)}, \tag{1.4}$$

where $1<p,q,r<\infty$, $1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$, $\frac{1}{t_1}+\frac{1}{t_2}\geqslant\frac{1}{s}$,

$$Af(y)=\int_D K(x-y)f(x)dx, \tag{1.5}$$

and $d\mu=d\nu=dx$ are the n-dimensional Lebesgue measures. In particular, we have

$$\|A\|_{L_p(D,dx)\to L_q(\Omega,dx)}\leqslant C\|K\|_{L_{r,\infty}(\Omega-D,dx)}. \tag{1.6}$$

However, in the case of non-homogeneous measures, operator (1.5) does not satisfy all requirements from [ON] and needs thorough investigation. For the power weights the Young-O'Neil inequality was generalized by Kerman [Ke].

By C, C_i, c we will denote positive constants that may be different on different occasions. Also, $F \asymp G$ means that $F \leqslant CG$ and $G \leqslant CF$.

This paper is organized as follows. In section 2 we define the new function spaces, which we call the net spaces N_{pq} and which are natural generalizations of the Lorentz spaces. It turns out that they have similar embedding and interpolation properties as the Lorentz spaces. Particularly, we have

$$\cdots \hookrightarrow N_{p1} \hookrightarrow \cdots \hookrightarrow N_{pp} \hookrightarrow \cdots \hookrightarrow N_{pq} \hookrightarrow \cdots \hookrightarrow N_{p\infty}, \qquad 1 \leqslant p \leqslant q.$$

Section 3 consists the general results on the boundedness of operator (1.1). The necessary and sufficient conditions for (1.1) to be of (p, q) quasi-weak and (p, q) weak-type are presented. We also give a sufficient condition for (1.1) to be bounded in the Lorentz spaces. We finish with Sections 4 and 5 which deals with the convolution operator in L_p and Lorentz spaces respectively. Two important cases: of regular kernel and regular measures and investigated. In particular, Theorems 4.1 and 5.1 imply Kerman's results.

2. The net space

First, let $(\Omega, \mathfrak{F}, \mu)$ be the measurable space and let M^* be the collection of all μ-measurable sets of positive measure, i.e.,

$$M^* := \{e \in \mathfrak{F} : 0 < \mu(e) < \infty\}.$$

Then we will call *the net* M a fix subset from M^*.

Let $f(x)$ be a measurable function on Ω such that $f(x)$ is integrable on any $e \in M$. We define *the average function* $\bar{f}(t, M)$ on $t \in (0, \infty)$ as follows

$$\bar{f}(t, M) := \bar{f}(t, M, \mu) = \sup \frac{1}{\mu(e)} \left| \int_e f(x) d\mu \right|,$$

where the supremum is taken over all sets $e \in M$ such that $\mu(e) > t$, $t \in (0, \infty)$. In the case of $\sup\limits_{e \in M} \mu(e) = \alpha < \infty$ we set $\bar{f}(t, M) = 0$ for $t > \alpha$.

We say that a μ-measurable function $f(x)$ belongs to the *Net space* $N_{p,q}(M, \mu)$, $0 < p, q \leqslant \infty$, if

$$\|f\|_{N_{pq}(M,\mu)} = \left(\int_0^\infty \left(t^{1/p} \bar{f}(t, M) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty$$

for $q < \infty$ and

$$\|f\|_{N_{p\infty}(M,\mu)} = \sup_{e \in M} \frac{1}{(\mu(e))^{1/p'}} \left| \int_e f(x) d\mu \right| < \infty$$

for $q = \infty$. Here and subsequently $p' = p/(p-1)$.

Properties.

(1) If $M_1 \subset M_2$, then $N_{pq}(M_2) \hookrightarrow N_{pq}(M_1)$.

(2) For $0 < q \leqslant q_1 \leqslant \infty$ we have $N_{pq}(M,\mu) \hookrightarrow N_{pq_1}(M,\mu)$.

(3) If the net M is such that $\sup\limits_{e\in M} \mu e = \alpha < \infty$, then for $0 < p < p_1 \leqslant \infty$ and $0 < q, q_1 \leqslant \infty$ we have $N_{p_1 q_1}(M,\mu) \hookrightarrow N_{pq}(M,\mu)$.

(4) The net space $N_{p,q}(M,\mu)$ is a quasi-normable space as the factor-space over the kernel $\{f : \int_e f(x)d\mu = 0,\, e \in M\}$. In the case of $q \geqslant 1$, $N_{p,q}(M,\mu)$ is a normable space.

Proof. Items (1)-(4) follow from the definition of the net space and a similar argument which we use to prove corresponding properties for the Lorentz spaces. $\square$

Example. **(1)** Let us consider two nets: M_0 and M_1 such that $M_0 = \{[a,b] \subset \mathbb{R} : a < b\} \subset M_1 = M^*$. Then the following inequalities are true (for the proof see Example 3.1 in the next section):

$$C_1\|f\|_{N_{pq}(M_0,dx)} \leqslant \|\hat{f}\|_{L_{p'q}(\mathbb{R},dx)} \leqslant C_2\|f\|_{N_{pq}(M_1,dx)}, \quad 1 < p \leqslant 2$$

where $\hat{f}$ is the Fourier transform of f. This example shows that choosing the appropriate nets, one can "catch" the quantity in which we are interested. We also note that $N_{p,q}(M,\mu)$ was introduced in [Nu] for the case when μ is n-dimensional Lebesgue measure.

(2) Let $M = \{[a,b] \subset \mathbb{R} : a < b\}$ and let $1 < p < \infty$, $0 < q \leqslant \infty$. The function $f(x) = (-1)^{[x]}$ ($[x]$ is an integer part of x) belongs to $N_{p,q}(M,\mu)$ but $|f|$ does not belong to $N_{p,q}(M,\mu)$.

Indeed, if $\{t\}$ is the fractional part of t, i.e., $\{t\} = t - [t]$, then

$$\bar{f}(t,M) = \begin{cases} \frac{\max(\{t\},1-\{t\})}{t} \leqslant \frac{1}{t}, & t \geqslant 1; \\ 1, & t \leqslant 1. \end{cases}$$

Therefore,

$$\|f\|_{N_{pq}(M,\mu)} \leqslant \left(\int_0^1 t^{\frac{q}{p}-1}dt + 2\int_1^\infty t^{(\frac{1}{p}-1)q-1}dt\right)^{1/q} < \infty.$$

Thus, generally speaking, the net space is not a lattice [BS, p. 6].

(3) Let $M = \{[a,b] : \mu([a,b]) = 1\}$. Then for any $0 < p, q \leqslant \infty$ we have

$$N_{pq}(M,\mu) \equiv \left\{ \mu\text{-measurable } f : \sup_k \left| \int\limits_k^{k+1} f(x)d\mu \right| < \infty \right\}.$$

(4) Let $1 < p < \infty, 0 < q \leqslant \infty$, $M = \{[0,a] : a > 0\}$, and the function $f_\Omega(x)$ is characteristic function of the set $\Omega = \bigcup_{k=1}^{\infty}[a_k, a_k + 1]$. If $a_k = (k \ln^{1/q} k)^{p'}$, then $f_\Omega \in N_{p,q+\varepsilon}(M)$ for any $\varepsilon > 0$ but $f_\Omega \notin N_{p,q}(M)$. Therefore, the space $N_{p,q}(M,\mu)$ is sensible to regard the distribution of the oscillation of functions.

(5) Let $1 < p < \infty, 0 < q \leqslant \infty$, and $M = \left\{[\sum_{k=1}^{n} k, \sum_{k=1}^{n+1} k] : n \in \mathbb{N}\right\}$. Then the net space $N_{pq}(M,\mu)$ is isomorphic to the space

$$\left\{a = \{a_n\}_{n=1}^{\infty} : \sum_{k=1}^{\infty}\left(k^{1/p}\sup_{n\geqslant k}|a_n|\right)^q \frac{1}{k} < \infty\right\}.$$

(6) Let $1 < p < \infty, 0 < q \leqslant \infty$, and $M = \{[0,n] : n \in \mathbb{N}\}$. Then the net space $N_{pq}(M,\mu)$ is isomorphic to the space

$$\left\{a = \{a_n\}_{n=1}^{\infty} : \sum_{k=1}^{\infty}\left(k^{1/p}\sup_{n\geqslant k}\frac{1}{n}\left|\sum_{i=1}^{n} a_i\right|\right)^q \frac{1}{k} < \infty\right\}.$$

(7) Suppose that $1 < p \leqslant \infty$ and $0 < q \leqslant \infty$; then $N_{p,q}(M^*,\mu) \equiv L_{p,q}(\Omega,\mu)$. This follows from the fact that

$$\|f\|_{L_{p,q}(\Omega)} \asymp \begin{cases} \left(\int\limits_0^\infty \left(t^{1/p} f^{**}(t)\right)^q \frac{dt}{t}\right)^{1/q}, & 1 < p < \infty,\ q < \infty; \\ \sup\limits_{t>0} t^{1/p} f^{**}(t), & 1 < p \leqslant \infty,\ q = \infty \end{cases}$$

and Lemma 2.1 below. For the case of $0 < p,q \leqslant \infty$, see [EO]. We also note that $N_{1q}(M,\mu) \neq L_{1q}(D,\mu)$ but $N_{1\infty}(M,\mu) = L_1(D,\mu)$.

We will need the following lemma.

Lemma 2.1. *Let f be a measurable and integrable on the elements of the net M^*. Then we have*

$$\bar{f}(t, M^*) \leqslant f^{**}(t) \leqslant 4\bar{f}(t/3, M^*), \qquad t > 0 \tag{2.1}$$

Proof. Let $t \in (0,\infty)$ and for any $e \in M^*$ such that $\mu(e) = t$ and for a function f we define the following sets $\omega_1 = \{x \in e : f(x) \geqslant 0\}$ and $\omega_2 = \{x \in e : f(x) < 0\}$. Then

$$\begin{aligned}\int_e |f(x)|d\mu &= \int_{\omega_1} f(x)d\mu - \int_{\omega_2} f(x)d\mu \\ &\leqslant 2\max\left\{\left|\int_{\omega_1} f(x)d\mu\right|, \left|\int_{\omega_2} f(x)d\mu\right|\right\}.\end{aligned}$$

We can assume that

$$\Big|\int_{\omega_1} f(x)d\mu\Big| \geqslant \Big|\int_{\omega_2} f(x)d\mu\Big|.$$

We consider two cases: 1). $\mu(\omega_1) \geqslant \frac{1}{2}\mu(\omega_2)$; 2). $\mu(\omega_1) < \frac{1}{2}\mu(\omega_2)$. In case 1),

$$\mu(\omega_1) \geqslant \frac{1}{2}\mu(\omega_2) \quad \text{implies} \quad \mu(\omega_1) \geqslant \frac{\mu(e)}{3} = \frac{t}{3}$$

and

$$\begin{aligned} \frac{1}{\mu(e)}\int_e |f(x)|d\mu &\leqslant \frac{2}{\mu(e)}\left|\int_{\omega_1} f(x)d\mu\right| \\ &\leqslant \frac{2}{\mu(\omega_1)}\left|\int_{\omega_1} f(x)d\mu\right| \leqslant 2\bar{f}(\frac{t}{3}, M^*). \end{aligned} \tag{2.2}$$

In case 2), we have $2\mu(\omega_1) < \mu(\omega_2)$, i.e., $\mu(\omega_2) > 2\mu(e)/3 = 2t/3$. Then there exist ω_2^1 and ω_2^2 from M^* such that $\mu(\omega_2^1 \bigcap \omega_2^2) = 0$, $\omega_2^1 \bigcup \omega_2^2 = \omega_2$, and $\mu(\omega_2^i) = \mu(\omega_2)/2 > t/3$.

Further, by definition of the set ω_2, we have

$$\begin{aligned} \Big|\int_{\omega_1} f(x)d\mu\Big| &\geqslant \Big|\int_{\omega_2} f(x)d\mu\Big| = \Big|\int_{\omega_2^1} f(x)d\mu\Big| + \Big|\int_{\omega_2^2} f(x)d\mu\Big| \\ &\geqslant 2\min\left(\Big|\int_{\omega_2^1} f(x)d\mu\Big|, \Big|\int_{\omega_2^2} f(x)d\mu\Big|\right) = 2\Big|\int_{\omega_2^{i_0}} f(x)d\mu\Big|. \end{aligned}$$

Here, $\omega_2^{i_0}$ are sets where the infimum is attained.

Let now $\omega = \omega_1 \bigcup \omega_2^{i_0}$, then $|\omega| > |e|/3$ and

$$\begin{aligned} \Big|\int_{\omega} f(x)d\mu\Big| &= \Big|\int_{\omega_1} f(x)d\mu + \int_{\omega_2^{i_0}} f(x)d\mu\Big| \\ &\geqslant \Big|\int_{\omega_1} f(x)d\mu\Big| - \Big|\int_{\omega_2^{i_0}} f(x)d\mu\Big| \geqslant \frac{1}{2}\Big|\int_{\omega_1} f(x)d\mu\Big|. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{\mu(e)}\int_e |f(x)|d\mu &\leqslant \frac{2}{\mu(e)}\Big|\int_{\omega_1} f(x)d\mu\Big| \\ &\leqslant \frac{4}{\mu(\omega)}\Big|\int_{\omega} f(x)d\mu\Big| \leqslant 4\bar{f}(t/3, M^*). \end{aligned}$$

Collecting this,

$$f^{**}(t) = \sup_{\mu(e)=t} \frac{1}{\mu(e)}\int_e |f(x)|d\mu,$$

and (2.2), we arrive at the right hand side estimate in (2.1). Let us prove the left hand side estimate.

$$\begin{aligned}
\bar{f}(t,M^*) &= \sup_{\mu(e)\geqslant t} \frac{1}{\mu(e)}\left|\int_e f(x)d\mu\right| \\
&\leqslant \sup_{\mu(e)\geqslant t} \frac{1}{\mu(e)}\int_e |f(x)|d\mu \\
&= \sup_{\mu(e)\geqslant t} \frac{1}{\mu(e)}\int_0^{\mu(e)} f^*(s)ds \\
&= \sup_{\mu(e)\geqslant t} \frac{1}{\mu(e)}\left(\int_0^t f^*(s)ds + \int_t^{\mu(e)} f^*(s)ds\right) \\
&\leqslant \sup_{\mu(e)\geqslant t} \frac{1}{\mu(e)}\left(\int_0^t f^*(s)ds + (\mu(e)-t)\frac{1}{t}\int_0^t f^*(s)ds\right) \\
&= \frac{1}{t}\int_0^t f^*(s)ds = f^{**}(t).
\end{aligned}$$

The proof is now complete. □

Interpolation. We next discuss the interpolation of the net spaces using the real method of Peetre [BS, Ch. 5]. If X_0 and X_1 are a pair of quasi-normed spaces which are continuously embedded in a linear Hausdorff space X, i.e., (X_0, X_1) is a compatible couple, their K-functional is defined for any $f \in X_0 + X_1$ by

$$K(f,t) := K(f,t;X_0;X_1) := \inf_{f=f_0+f_1} \|f_0\|_{X_0} + t\|f_1\|_{X_1}.$$

For each $0 < \theta < 1, 0 < q \leqslant \infty$, the space $X_{\theta,q} := (X_0;X_1)_{\theta,q}$ is the collection of all functions $f \in X_0 + X_1$ for which

$$\|f\|_{\theta,q} := \left(\int_0^\infty \left(t^{-\theta}K(f,t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{2.3}$$

is finite (with the usual adjustment on the right side of (2.3) when $q = \infty$). This is an interpolation space since it follows easily from the definition of the K-functional that each linear operator which is bounded on X_0 and X_1 is also bounded on $X_{\theta,q}$.

We are interested in interpolation for a pair of net spaces.

Theorem 2.1. *Let $0 < p_0 < p_1 \leqslant \infty$, $0 < q_0, q_1, q \leqslant \infty$, and let M be any net in $\mathfrak{F}$. We have*

$$(N_{p_0,q_0}(M,\mu), N_{p_1,q_1}(M,\mu))_{\theta,q} \hookrightarrow N_{p,q}(M,\mu),$$

where $0 < \theta < 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$.

Proof. Because of the embedding $N_{p,q}(M,\mu) \hookrightarrow N_{p,\infty}(M,\mu)$ it is sufficient to show

$$(N_{p_0,\infty}(M,\mu), N_{p_1,\infty}(M,\mu))_{\theta,q} \hookrightarrow N_{p,q}(M,\mu).$$

Let $f = f_0 + f_1$, where $f_0 \in N_{p_0,\infty}(M,\mu)$ and $f_1 \in N_{p_1,\infty}(M,\mu)$. Clearly

$$\bar{f}(t,M) \leqslant \bar{f}_0(t,M) + \bar{f}_1(t,M)$$

and then denoting $v(t) = t^{\frac{p_0 p_1}{p_1 - p_0}}$ we have

$$\sup_{v(t)\geqslant s>0} s^{1/p_0}\bar{f}(s,M) \leqslant \sup_{s>0} s^{1/p_0}\bar{f}_0(s,M) + \sup_{v(t)>s} s^{\frac{1}{p_0}-\frac{1}{p_1}+\frac{1}{p_1}}\bar{f}_1(s,M)$$

$$\leqslant \sup_{s>0} s^{1/p_0}\bar{f}_0(s,M) + t\sup_{s>0} s^{1/p_1}\bar{f}_1(s,M).$$

Further, taking the infimum over all f_0 and f_1, such that $f = f_0 + f_1$, we get

$$\sup_{v(t)\geqslant s>0} s^{1/p_0}\bar{f}(s) \leqslant K(t,f;N_{p_0\infty},N_{p_1\infty}).$$

Thus for $0 < q \leqslant \infty$

$$\left(\int_0^\infty \left(t^{-\theta}K(t,f)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \geqslant \left(\int_0^\infty \left(t^{-\theta}\sup_{v\geqslant s>0} s^{1/p_0}\bar{f}(s,M)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

$$= \left((1/p_0 - 1/p_1)\int_0^\infty \left(t^{-\theta(\frac{1}{p_0}-\frac{1}{p_1})}\sup_{t\geqslant s>0} s^{1/p_0}\bar{f}(s,M)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

$$\geqslant \left((1/p_0 - 1/p_1)\int_0^\infty \left(t^{\frac{1-\theta}{p_0}+\frac{\theta}{p_1}}\bar{f}(t,M)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} = c\|f\|_{N_{pq}(M,\mu)},$$

which finishes the proof. □

As a consequence we write the following interpolation result.

Corollary 2.1.1. *Let $0 < p_0 < p_1 \leqslant \infty$, $0 < q \leqslant \infty$, and $0 < \theta < 1$. Let (X_0, X_1) be a compatible couple and M be an arbitrary net in $\mathfrak{F}$. If a semiadditive operator T acts as follows:*

$$T\colon X_0 \to N_{p_0\infty}(M,\mu) \quad \textit{with the norm} \quad D_0,$$
$$T\colon X_1 \to N_{p_1\infty}(M,\mu) \quad \textit{with the norm} \quad D_1,$$

then we also have

$$T\colon (X_0,X_1)_{\theta q} \to N_{pq}(M,\mu) \quad \textit{with the norm} \quad \|T\| \leqslant cD_0^{1-\theta}D_1^{\theta},$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$.

3. The strong-type and weak-type boundedness of integral operators

Corollary 2.1.1 allows us to obtain the strong-type estimates using weak-type estimates. Next result gives the criterion for the integral operator to be a weak-type or quasi-weak type. First, we recall these definitions.

- An operator T is a (p,q) strong-type operator if $T\colon L_p(D,\nu) \to L_q(\Omega,\mu)$.
- An operator T is a (p,q) weak-type operator if $T\colon L_p(D,\nu) \to L_{q\infty}(\Omega,\mu)$.
- An operator T is a (p,q) quasi-weak-type operator if $T\colon L_{p\,1}(D,\nu) \to L_{q\infty}(\Omega,\mu)$.

Clearly, if T is a (p,q) strong-type operator, then it is a (p,q) weak-type operator, and if T is a (p,q) weak-type operator, then it is a (p,q) quasi-weak-type operator (for $1 \leqslant p$).

Theorem 3.1. *Let $1 < q \leqslant \infty$ and let M be any net from $\mathfrak{F}$. Suppose that $X(D,\nu)$ is a Banach space of measurable functions on D and the integral operator*

$$Tf(y) = \int_D K(x,y)f(x)d\nu_x \tag{3.1}$$

acts from $X(D,\nu)$ into $N_{q\infty}(M,\mu)$.

Then the necessary and sufficient condition for T to be bounded from $X(D,\nu)$ into $N_{q\infty}(M,\mu)$ is

$$\sup_{e\in M}\left\|\frac{1}{(\mu(e))^{1/q'}}\int_e K(\cdot,y)d\mu\right\|_{X^*} < \infty,$$

where X^* is the associate space of X, i.e.,

$$X^* = \left\{ g : \|g\|_{X^*} = \sup_{\|f\|_X \leqslant 1}\left|\int_D g(x)f(x)d\nu\right| < \infty\right\}.$$

Moreover,

$$\|T\|_{X\to N_{q\infty}} = \sup_{e\in M}\left\|\frac{1}{\mu(e)^{1/q'}}\int_e K(\cdot,y)d\mu\right\|_{X^*}.$$

Proof. Indeed, by the definition of the spaces $N_{q\infty}(M,\mu)$ and X^*, we have

$$\begin{aligned}
\|T\|_{X\to N_{q\infty}(M,\mu)} &= \sup_{\|f\|_X\leqslant 1} \|Tf\|_{N_{q\infty}(M,\mu)} \\
&= \sup_{\|f\|_X\leqslant 1} \sup_{e\in M} \frac{1}{(\mu(e))^{1/q'}} \left| \int_e \int_D K(x,y) f(x) d\nu d\mu \right| \\
&= \sup_{e\in M} \sup_{\|f\|_X\leqslant 1} \left| \int_D f(x) \frac{1}{(\mu(e))^{1/q'}} \int_e K(x,y) d\mu d\nu \right| \\
&= \sup_{e\in M} \left\| \frac{1}{\mu(e)^{1/q'}} \int_e K(\cdot,y) d\mu \right\|_{X^*}
\end{aligned}$$

and the result follows. □

Corollary 3.1.1. *Let $1 < p,q \leqslant \infty$, $M_1 = \{e \subset \Omega : 0 < \mu(e) < \infty\}$, and $M_2 = \{w \subset D : 0 < \nu(w) < \infty\}$. Then the necessary and sufficient condition for operator (3.1) to be (p,q) quasi-weak-type operator is*

$$\sup_{e\in M_1,\ \omega\in M_2} \frac{1}{(\mu(e))^{1/q'}(\nu(w))^{1/p}} \left| \int_e \int_\omega K(x,y) d\mu d\nu \right| < \infty.$$

Moreover, for $0 < r \leqslant 1$ we have

$$\begin{aligned}
\|T\|_{L_{pr}(D,\nu)\to L_{q\infty}(\Omega,\mu)} &\asymp \|T\|_{L_{p1}(D,\nu)\to L_{q\infty}(\Omega,\mu)} \\
&\asymp \sup_{e\in M_1,\ \omega\in M_2} \frac{1}{(\mu(e))^{1/q'}(\nu(w))^{1/p}} \left| \int_e \int_\omega K(x,y) d\mu d\nu \right|.
\end{aligned}$$

Proof. We use the properties of the net spaces and Theorem 3.1:

$$\begin{aligned}
\|T\|_{L_{pr}(D,\nu)\to L_{q\infty}(\Omega,\mu)} &\leqslant C\|T\|_{L_{p\,1}(D,\nu)\to L_{q\infty}(\Omega,\mu)} \\
&\asymp \|T\|_{L_{p\,1}(D,\nu)\to N_{q\infty}(M_1,\mu)} \\
&= \sup_{e\in M_1} \frac{1}{(\mu(e))^{1/q'}} \left\| \int_e K(\cdot,y) d\mu \right\|_{L_{p',\infty}} \\
&\asymp \sup_{e\in M_1} \frac{1}{(\mu(e))^{1/q'}} \left\| \int_e K(\cdot,y) d\mu \right\|_{N_{p',\infty}(M_2,\nu)} \\
&= \sup_{e\in M_1} \sup_{\omega\in M_2} \frac{1}{(\mu(e))^{1/q'}} \frac{1}{(\nu(\omega))^{1/p}} \left| \int_e \int_\omega K(x,y) d\mu d\nu \right|.
\end{aligned}$$

To prove the inverse estimate of $\|T\|_{L_{pr}\to L_{q\infty}}$, we take the test function

$$f_0(x) = \left(\frac{r}{p}\right)^{1/r} \frac{\chi_w(x)}{(\nu(w))^{1/p}}$$

and note that

$$\|f_0\|_{L_{pr}(D,\nu)} = 1.$$

This completes the proof. □

Corollary 3.1.2. *Let $1 \leqslant p \leqslant \infty$, $1 < q \leqslant \infty$, and $M_1 = \{e \subset \Omega : 0 < \mu(e) < \infty\}$. Then the necessary and sufficient condition for the operator T to be (p, q) weak-type operator is*

$$\sup_{e\in M_1} \frac{1}{(\mu(e))^{1/q'}} \Big(\int_D \Big|\int_e K(x,y)d\mu\Big|^{p'} d\nu\Big)^{\frac{1}{p'}} < \infty. \tag{3.2}$$

Proof. Similar to the proof of Corollary 3.1.1, we have

$$\begin{aligned}\|T\|_{L_p(D,\nu)\to L_{q\infty}(\Omega,\mu)} &\asymp \|T\|_{L_p(D,\nu)\to N_{q\infty}(M_1,\mu)} \\ &= \sup_{e\in M_1} \frac{1}{(\mu(e))^{1/q'}} \sup_{\|f\|_p=1} \Big|\int\limits_D f(x) \int_e K(x,y)d\mu d\nu\Big| \\ &= \sup_{e\in M_1} \frac{1}{(\mu(e))^{1/q'}} \Big(\int_D \Big|\int_e K(x,y)d\mu\Big|^{p'} d\nu\Big)^{\frac{1}{p'}}.\end{aligned}$$

This completes the proof. □

Remarks. We note that in the case of $\Omega = \mathbb{R}^n$ and when $\mu = \nu$ are n-dimensional Lebesgue measures, Corollaries 3.1.1 and 3.1.2 were obtained in [KN].

We also mention the following result from [CRS, 1.2]. Let $L \subset M(X)$ be a regular class, $T : L \to M(Y)$ be an order continuous sublinear operator (see definitions in [CRS]), and $0 < q_0 \leqslant 1, 0 < p_0 < \infty$. Then

a) if $q_0 \leqslant q_1 \leqslant p_1 < \infty$,

$$\|T\|_{L_{p_0q_0}\to L_{p_1q_1}} = \sup_{\chi_B} \frac{\|T\chi_B\|_{L_{p_1q_1}}}{\|\chi_B\|_{L_{p_0q_0}}}; \tag{3.3}$$

b) if $q_0 < p_1 < q_1 \leqslant \infty$,

$$\|T\|_{L_{p_0q_0}\to L_{p_1q_1}} \leqslant \left(\frac{p_1}{p_1 - q_0}\right)^{1/q_0} \sup_{\chi_B} \frac{\|T\chi_B\|_{L_{p_1q_1}}}{\|\chi_B\|_{L_{p_0q_0}}}. \tag{3.4}$$

By Bukhvalov's theorem (see [Bu]), the order continuous linear operator is an integral. Generally speaking, the reverse is not true. So in the case of $q_1 = \infty$, the results of Corollaries 3.1.1 and 3.1.2 supplement with estimates (1.2) and (3.3)-(3.4).

Corollary 3.1.3. *Let $1 < p \leqslant q < \infty$. Assume that $A \subset (0,1)^2$ is a convex set which is bounded by a closed polyline ∂A with vertices $a_1, a_2, ..., a_n$. Let also $A_0 := \bar{A} \setminus \{a_1, ..., a_n\}$. Then the necessary and sufficient condition for an integral operator T to be bounded from $L_p(D,\nu)$ into $L_q(\Omega,\mu)$*

for any $(\frac{1}{p}, \frac{1}{q}) \in A_0$ *is*

$$\sup_{e\in M_1,\ \omega\in M_2} \frac{1}{(\mu(e))^{1/q'}(\nu(w))^{1/p}} \Big| \int_e \int_\omega K(x,y) d\nu d\mu \Big| < \infty \tag{3.5}$$

for any $(\frac{1}{p}, \frac{1}{q}) \in \partial A \setminus \{a_1, ..., a_n\}$.

Proof. Note that for any point b of the set A_0 we can find points $b_0, b_1 \in \partial A \setminus \{a_1, ..., a_n\}$ such that $b = (1-\theta)b_0 + \theta b_1$ for $\theta \in (0,1)$. Then by Corollaries 3.1.1 and 3.1.2, condition (3.5) implies strong-type boundedness of the operator A at the point b. The necessity follows from the fact that the strong-type boundedness implies the weak-type boundedness. The proof is now complete. □

In a similar way one proves the following proposition.

Corollary 3.1.4. *Let* $1 < p \leqslant q < \infty$. *Let* A *be an open set from* $(0,1)^2$. *Then the necessary and sufficient condition for an integral operator* T *to be bounded from* $L_p(D,\nu)$ *to* $L_q(\Omega,\mu)$ *for any* $(\frac{1}{p}, \frac{1}{q}) \in A_0$ *is the accuracy of condition (3.5) for any* $(\frac{1}{p}, \frac{1}{q}) \in A$.

Now we present a simple proof of the Hausdorff-Young-type inequality in the Lorentz spaces (see, for example, [SW, Ch. V]).

Example 3.1. Let $1 < p < 2$ and $0 < q \leqslant \infty$. Then for the Fourier transform

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-iyx} dx$$

we have

$$C_1(p,q)\|Hf\|_{L_{pq}(\mathbb{R},dx)} \leqslant \|\hat{f}\|_{L_{p'q}(\mathbb{R},dx)} \leqslant C_2(p,q)\|f\|_{L_{pq}(\mathbb{R},dx)} \tag{3.6}$$

where $Hf(x)$ is the following Hardy-type operator

$$Hf(x) = \frac{1}{|x|} \int_{-|x|}^{|x|} f(t) dt$$

Proof. Let $1 < p < \infty$, $M_0 = \{[a,b] : a < b\}$, and $M_1 = \{e \subset \mathbb{R} : 0 < m(e) = |e| < \infty\}$, where m is the linear Lebesgue measure. Then

$$\begin{aligned}
\sup_{\substack{w\in M_0\\ e\in M_1}} \frac{1}{|w|^{\frac{1}{p}}|e|^{\frac{1}{p}}} \left| \int_w \int_e e^{-iyx} dx dy \right| &\leqslant \sup_{\substack{w\in M_0\\ e\in M_1}} \frac{1}{|w|^{\frac{1}{p}}|e|^{\frac{1}{p}}} \int_e \frac{\big|\sin|w|x\big|}{|x|} dx \\
&\leqslant \sup_{\substack{w\in M_0\\ e\in M_1}} \frac{1}{|w|^{\frac{1}{p}}|e|^{\frac{1}{p}}} \int_e \min\Big(|w|, 1/|x|\Big) dx \\
&\leqslant \sup_{e\in M_1} \frac{1}{|e|^{\frac{1}{p}}} \int_e \frac{1}{|x|^{\frac{1}{p'}}} dx = p.
\end{aligned}$$

Therefore, by Theorem 3.1, this implies for any $p \in (1,\infty)$

$$\|\hat{f}\|_{N_{p'\infty}(M_0,dx)} \leqslant C(p)\,\|f\|_{L_{p\,1}(\mathbb{R},dx)}.$$

Let $1<p<\infty$. The last inequality implies for $1<p_0<p<p_1<\infty$ and $Tf=\hat{f}$ that,

$$T\colon L_{p_0\,1}(\mathbb{R},dx) \to N_{p_0'\infty}(M,\mu) \quad \text{with the norm} \quad C(p_0),$$
$$T\colon L_{p_1\,1}(\mathbb{R},dx) \to N_{p_1'\infty}(M,\mu) \quad \text{with the norm} \quad C(p_1).$$

By Corollary 2.1.1,

$$\|\hat{f}\|_{N_{p'q}(M_0,dx)} \leqslant C(p,q)\,\|f\|_{L_{pq}(\mathbb{R},dx)}$$

and hence

$$\|f\|_{N_{p'q}(M_0,dx)} \leqslant C(p,q)\,\|\hat{f}\|_{L_{pq}(\mathbb{R},dx)}$$

and the left-hand side inequality in (3.6) follows.

To prove the right-hand side inequality, assume that $1<p<2$ and so $2<p'<\infty$. Then, because of the fact that $|\hat{\chi}_\omega(y)| \leqslant |\omega|$ and by Plancherel's theorem, we get

$$\begin{aligned}
\sup_{w\in M_1} \frac{1}{|w|^{\frac{1}{p}}} \left\| \int_w e^{-iyx}dx \right\|_{L_{p'}} &= \sup_{w\in M_1} \frac{1}{|w|^{\frac{1}{p}}} \left(\int_{\mathbb{R}} |\hat{\chi}_w|^2 |\hat{\chi}_w|^{p'-2} dx \right)^{\frac{1}{p'}} \\
&\leqslant \sup_{w\in M_1} \frac{|w|^{1-2/p'}}{|w|^{\frac{1}{p}}} \left(\int_{\mathbb{R}} |\hat{\chi}_w|^2 dx \right)^{\frac{1}{p'}} \\
&= \sup_{w\in M_1} \frac{|w|^{\frac{1}{p'}} |w|^{1-\frac{2}{p'}}}{|w|^{\frac{1}{p}}} = 1.
\end{aligned}$$

Hence for any $p\in(1,2)$ we write

$$\|\hat{f}\|_{L_{p'\infty}(\mathbb{R},dx)} \asymp \|\hat{f}\|_{N_{p'\infty}(M_1,dx)} \leqslant \|f\|_{L_p(\mathbb{R},dx)}.$$

Thus

$$\|\hat{f}\|_{L_{p'q}(\mathbb{R},dx)} \asymp \|\hat{f}\|_{N_{p'q}(M_1,dx)} \leqslant C\,\|f\|_{L_{pq}(\mathbb{R},dx)}.$$

The proof is now complete. □

Example 3.2. Let $1<p<2$ and $0<q\leqslant\infty$. Then if $|f(x)| \leqslant C|Hf(x)|$, then we have

$$\|\hat{f}\|_{L_{p'q}(\mathbb{R},dx)} \asymp \|f\|_{L_{pq}(\mathbb{R},dx)}. \tag{3.7}$$

Let $2<p<\infty$ and $0<q\leqslant\infty$. Then if $|\hat{f}(x)| \leqslant C|H\hat{f}(x)|$, then we also have (3.7).

If we assume a stronger condition like monotonicity or quasi-monotonicity[1] of $f(x)$ and $f(-x)$ (or of $\hat{f}(x)$ and $\hat{f}(-x)$) on $\mathbb{R}_+$, then (3.7) is true for $p \in (1, \infty)$. The proof easily follows from the proof in Example 3.1, and duality arguments (see also Sagher's papers [Sa] and [BS]).

Now we present the main result in this section which is the sufficient conditions for an integral operator to be bounded in the Lorentz spaces.

Let now $e \in M_1$ and $w \in M_2$. Define the following function

$$F(e,\omega) := F(e,\omega;K) := \frac{1}{\nu(\omega)^{\frac{1}{p}}} \frac{1}{\mu(e)^{\frac{1}{q'}}} \left| \int_e \int_\omega K(x,y) d\nu \, d\mu \right|.$$

Theorem 3.2. *Let $1 \leqslant r, \theta, h \leqslant \infty$, and $1/r + 1 = 1/h + 1/\theta$. Let also $M_1 = \{e \subset \Omega : 0 < \mu(e) < \infty\}$, $M_2 = \{w \subset D : 0 < \nu(w) < \infty\}$. If there exists $\gamma > 0$ such that*

$$B = \left(\int_0^\infty \Big(\sup_{\mu(e)/\nu^\gamma(w) = t} F(e,w) \Big)^h \frac{dt}{t} \right)^{\frac{1}{h}} < \infty \qquad \text{if} \quad h < \infty \tag{3.8}$$

and

$$B = \sup_{e \in M_1, \ \omega \in M_2} F(e,w) < \infty \qquad \text{if} \quad h = \infty,$$

then the integral operator

$$Tf(y) = \int\limits_D K(x,y) f(x) d\nu_x$$

is bounded from $L_{p\theta}(D,\nu)$ into $L_{qr}(\Omega,\mu)$, $1 < p, q < \infty$, and

$$\|T\|_{L_{p\theta} \to L_{qr}} \leqslant C(p,q) \, B. \tag{3.9}$$

Moreover, in the case of $r = \infty$ and $\theta = 1$, the condition $B < \infty$ is also necessary.

Proof. We define

$$\Phi(s,t) := \sup_{\substack{\mu(e)=t \\ e \in M_1}} \sup_{\substack{\nu(w)=s \\ w \in M_2}} F(e,w).$$

First, let us show the accuracy of

$$\sup_{\substack{\mu(e)=t \\ e \in M_1}} \frac{1}{\mu(e)} \left| \int\limits_e Tf(y) d\mu \right| \leqslant C t^{-1/q} \int\limits_0^\infty s^{1/p-1} f^*(s) \Phi(s,t) ds. \tag{3.10}$$

---

[1]A function $f(x)$ on $\mathbb{R}_+$ is quasi-monotone if there exists $\tau > 0$ such that $f(x)/x^\tau$ is monotone decreasing.

Indeed, using the Hardy inequality ($\int fg \leqslant \int f^* g^*$) and the monotonicity of K^*, we get

$$\sup_{\substack{\mu(e)=t \\ e\in M_1}} \frac{1}{\mu(e)} \left| \int_e Tf(y)d\mu \right| = t^{-1/q} \sup_{\substack{\mu(e)=t \\ e\in M_1}} \frac{1}{(\mu(e))^{1/q'}} \left| \int_D f(x) \int_e K(x,y) d\mu d\nu \right|$$

$$\leqslant t^{-1/q} \sup_{\substack{\mu(e)=t \\ e\in M_1}} \frac{1}{(\mu(e))^{1/q'}} \int_0^\infty f^*(s) \left(\int_e K(\cdot,y)d\mu \right)^* (s)\, ds$$

$$\leqslant t^{-1/q} \sup_{\substack{\mu(e)=t \\ e\in M_1}} \frac{1}{(\mu(e))^{1/q'}} \int_0^\infty f^*(s) \left(\int_e K(\cdot,y)d\mu \right)^{**} (s) ds.$$

Further, by Lemma 2.1, for $g(x) \equiv \int\limits_e K(x,y)d\mu$ we write

$$g^{**}(s) \leqslant 4 \sup_{\substack{\nu(w)=s/3 \\ w\in M_2}} \frac{1}{\nu(w)} \Big| \int\limits_w g(x) d\nu \Big|.$$

Then the left hand side of (3.10) is bounded by

$$12 t^{-1/q} \int_0^\infty f^*(3s) \sup_{\substack{\mu(e)=t \\ e\in M_1}} \sup_{\substack{\nu(w)=s \\ \omega\in M_2}} \frac{1}{(\mu(e))^{1/q'}} \frac{1}{(\nu(w))} \left| \int_w \int_e K(x,y) d\mu d\nu \right| ds$$

$$\leqslant 12 t^{-1/q} \int_0^\infty s^{1/p-1} f^*(s) \Phi(s,t) ds,$$

and (3.10) follows.

Then, using Lemma 2.1 and (3.10), we have

$$\|Tf\|_{L_{qr}} \asymp \left(\int_0^\infty \left(t^{1/q} (Tf)^{**}(t) \right)^r dt/t \right)^{1/r}$$

$$\leqslant C \left(\int_0^\infty \left(t^{1/q} \sup_{\substack{\mu(e)=t \\ e\in M_1}} \frac{1}{\mu(e)} \left| \int_e Tf(y) d\mu \right| \right)^r \frac{dt}{t} \right)^{1/r}$$

$$\leqslant C \left(\int_0^\infty \left(\int_0^\infty s^{1/p-1} f^*(s) \Phi(s,t) ds \right)^r \frac{dt}{t} \right)^{1/r}.$$

Further we use the following representation

$$s^{1/p} f^*(s) \Phi(s,t) = \left((s^{1/p} f^*)^{1-\theta/r} \right) \left((s^{1/p} f^*)^{\theta/r} \Phi(s,t) \right).$$

Then by the Hölder inequality with exponents h and $h'=\left(\frac{1}{\theta}-\frac{1}{r}\right)^{-1}$, we write[2]

$$\|Tf\|_{L_{qr}}\leqslant\left(\int_0^\infty\left[\left(\int_0^\infty(s^{1/p}f^*(s))^\theta\frac{ds}{s}\right)^{1/\theta-1/r}\right.\right.$$

$$\left.\left.\left(\int_0^\infty\left(s^{1/p}f^*(s)\right)^{\theta h/r}\Phi^h(s,t)\frac{ds}{s}\right)^{1/h}\right]^r\frac{dt}{t}\right)^{1/r}$$

$$=\|f\|_{L_{p\theta}}^{1-\theta/r}\left(\int_0^\infty\left(\int_0^\infty\left[s^{1/p}f^*(s)\right]^{\theta h/r}\Phi^h(s,t)\frac{ds}{s}\right)^{r/h}\frac{dt}{t}\right)^{1/r}.$$

Changing the variables $s\to st^\gamma$, using the Minkowski inequality and then changing the variables $st^\gamma\to t$ again, we get

$$\|Tf\|_{L_{qr}}\leqslant C\|f\|_{L_{p\theta}}^{1-\theta/r}\left(\int_0^\infty\left(\int_0^\infty\left[(st^\gamma)^{1/p}f^*(st^\gamma)\right]^{\theta h/r}\Phi^h(st^\gamma,t)\frac{ds}{s}\right)^{r/h}\frac{dt}{t}\right)^{1/r}$$

$$\leqslant C\|f\|_{L_{p\theta}}^{1-\theta/r}\left(\int_0^\infty\left(\int_0^\infty\left[(st^\gamma)^{1/p}f^*(st^\gamma)\right]^{\theta}\Phi^r(st^\gamma,t)\frac{dt}{t}\right)^{h/r}\frac{ds}{s}\right)^{1/h}$$

$$=C\|f\|_{L_{p\theta}}^{1-\theta/r}\left(\int_0^\infty\left(\int_0^\infty\left[t^{1/p}f^*(t)\right]^{\theta}\Phi^r(t,(t/s)^{\frac{1}{\gamma}})\frac{dt}{t}\right)^{h/r}\frac{ds}{s}\right)^{1/h}.$$

Noting

$$\Phi(t,(t/s)^{\frac{1}{\gamma}})\leqslant\sup_{\sigma/\tau^\gamma=s}\Phi(\sigma,\tau),$$

we write

$$\|Tf\|_{L_{q\,r}}\leqslant C\|f\|_{L_{p\theta}}\left(\int_0^\infty(\sup_{\sigma/\tau^\gamma=s}\Phi(\sigma,\tau))^h\frac{ds}{s}\right)^{1/h}$$

$$=C\|f\|_{L_{p\theta}}\left(\int_0^\infty(\sup_{\mu(e)/\nu^\gamma(w)=s}F(e,w))^h\frac{ds}{s}\right)^{1/h}.$$

If $\theta=1$ and $r=\infty$, the necessity of condition (3.9) follows from Corollary 3.1.1. The proof is now complete. $\square$

[2]In the case of $h=1$ or $h=\infty$ the proof is similar. We use the usual adjustment of $(\int|g|^q)^{1/q}$ when $q=\infty$.

Remark 3.1. From the proof of Theorem 3.2 one can observe that condition (3.8) can be changed to the following condition: there exist $c > 0$ and $\gamma > 0$ such that

$$B = \left(\int_0^\infty \left(\sup_{\mu(e)/c\nu^\gamma(w)=t} F(e,w) \right)^h \frac{dt}{t} \right)^{\frac{1}{h}} < \infty.$$

4. Convolution operator in Lebesgue spaces

First, we study the (p,q)-strong-type convolution operators. We will need the following lemma.

Lemma 4.1. *Let f, g, and K be measurable functions on $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(x)K(x-y)dxdy \leqslant \int_0^\infty f^*(s) \int_0^\infty g^*(t)K^{**}(|s-t|)dtds,$$

where $K^{**}(|s-t|) = \frac{1}{|s-t|}\int_0^{|s-t|} K^*(\xi)d\xi$.

Proof. We use the Hardy inequality on rearrangements:

$$\begin{aligned}
\left| \int_{-\infty}^\infty g(y)(f*K)(y)dy \right| &\leqslant \int_0^\infty g^*(s)(f*K)^{**}(s)ds \\
&= \int_0^\infty g^*(s) \sup_{|e|=s} \frac{1}{|e|} \int_e \left| \int_{-\infty}^\infty K(y-x)f(x)dx \right| dyds \\
&\leqslant \int_0^\infty g^*(s) \sup_{|e|=s} \int_0^\infty f^*(t) \sup_{|w|=t} \frac{1}{|e|}\frac{1}{|w|} \\
&\qquad \int_w \int_e |K(y-x)|dydxdtds \\
&\leqslant \int_0^\infty g^*(s) \int_0^\infty f^*(t) \sup_{|e|=s} \sup_{|w|=t} \frac{1}{|e|}\frac{1}{|w|} \\
&\qquad \int_w \int_e |K(y-x)|dydxdtds.
\end{aligned}$$

Then the inequality

$$\sup_{\substack{|w|=t \\ |e|=s}} \frac{1}{|e|}\frac{1}{|w|} \int_w \int_e |K(x-y)|dxdy \leqslant K^{**}(\max(s,t)) \leqslant K^{**}(|s-t|)$$

finishes the proof of Lemma 4.1. $\square$

Theorem 4.1. *Let $\alpha, \beta \in [0,1)$, $1 < p, q < \infty$, and $0 < \frac{1}{r} = 1 - \frac{1-\alpha}{p'} - \frac{1-\beta}{q}$. Suppose that measures μ and ν are defined as follows*

$$\mu(e) = \int_e \frac{dy}{|y|^\beta}, \qquad \nu(\omega) = \int_\omega \frac{dx}{|x|^\alpha} \tag{4.1}$$

Then the norm of the convolution operator

$$Af(y) = \int_{-\infty}^{+\infty} K(y-x) f(x) d\nu$$

satisfies

$$\|A\|_{L_p(\mathbb{R},\nu)\to L_q(\mathbb{R},\mu)} \leqslant c \sup_{e\in M^*} \frac{1}{|e|^{\frac{1}{r}}} \left| \int_e K(t)dt \right| =: cB,$$

where $|e|$ is the linear measure of e. Moreover, if the kernel $K(t)$ is non-negative, then

$$c \sup_{d>0} \frac{1}{d^{\frac{1}{r}}} \int_{-d}^{d} K(t)dt \leqslant \|A\|_{L_p(\mathbb{R},\nu)\to L_q(\mathbb{R},\mu)}. \tag{4.2}$$

Proof. Denote

$$\gamma = \frac{1-\beta}{1-\alpha} \quad \text{and} \quad \delta = \frac{(1-\alpha)^\gamma}{1-\beta}.$$

By Lemma 4.1,

$$\begin{aligned}
\sup_{\mu(e)=\delta(\nu(\omega))^\gamma t} F(e,\omega) &= \sup_{\mu(e)=\delta(\nu(\omega))^\gamma t} \frac{1}{\mu(e)^{\frac{1}{q'}} \nu(w)^{\frac{1}{p}}} \int_e \int_\omega \frac{K(x-y)dxdy}{|x|^\alpha |y|^\beta} \\
&\leqslant \sup_{\mu(e)=\delta(\nu(\omega))^\gamma t} \frac{1}{\mu(e)^{\frac{1}{q'}} \nu(w)^{\frac{1}{p}}} \int_0^{|e|} \int_0^{|\omega|} \frac{1}{|x|^\alpha |y|^\beta} \\
&\qquad \left(\frac{1}{|x-y|} \int_0^{|x-y|} K^*(t)dt \right) dxdy \\
&\leqslant B \sup_{\mu(e)=\delta(\nu(\omega))^\gamma t} \frac{1}{\mu(e)^{\frac{1}{q'}} \nu(w)^{\frac{1}{p}}} \\
&\qquad \int_0^{|e|} \int_0^{|\omega|} \frac{dxdy}{|x-y|^{1-\frac{1}{r}} |x|^\alpha |y|^\beta} \\
&= B \sup_{a>0} \frac{1}{\left(at^{\frac{1}{1-\beta}}\right)^{\frac{1-\beta}{q'}} \left(a^{(1-\alpha)}\right)^{\frac{1}{p}}} \\
&\qquad \int_0^{at^{\frac{1}{1-\beta}}} \int_0^a \frac{dxdy}{|x-y|^{1-\frac{1}{r}} |x|^\alpha |y|^\beta}.
\end{aligned}$$

In the last inequality we have used the fact that

$$\mu([0,at^{\frac{1}{1-\beta}}]) = \delta\big(\nu([0,a])\big)^{\gamma}\, t.$$

Furthermore, we remark that

$$\sup_{a>0} \frac{t^{-1/q'}}{a^{\frac{1-\alpha}{p}+\frac{1-\beta}{q'}}} \int_0^{at^{\frac{1}{1-\beta}}} \int_0^a \frac{dxdy}{|x-y|^{1-\frac{1}{r}}|x|^{\alpha}|y|^{\beta}} = t^{\frac{1}{q}} \int_0^1 \int_0^1 \frac{dzds}{|z-t^{\frac{1}{1-\beta}}s|^{\theta} z^{\alpha} s^{\beta}}.$$

Therefore,

$$\left(\int_0^\infty \left(\sup_{\mu(e)=\delta(\nu(\omega))^{\gamma}t} F(e,\omega)\right)^h \frac{dt}{t}\right)^{\frac{1}{h}}$$

$$\leqslant B\left(\int_0^\infty \left(t^{\frac{1}{q}} \int_0^1 \int_0^1 \frac{dzds}{|z-t^{\frac{1}{1-\beta}}s|^{1-\frac{1}{r}} z^{\alpha} s^{\beta}}\right)^h \frac{dt}{t}\right)^{\frac{1}{h}}$$

$$= CB\left(\int_0^\infty \left(y^{\frac{1-\beta}{q}} \int_0^1 \int_0^1 \frac{dzds}{|z-ys|^{1-\frac{1}{r}} z^{\alpha} s^{\beta}}\right)^h \frac{dy}{y}\right)^{\frac{1}{h}}$$

and by the Minkowski inequality,

$$\leqslant CB \int_0^1 \int_0^1 \left(\int_0^\infty \left(\frac{y^{\frac{1-\beta}{q}}}{|z-ys|^{1-\frac{1}{r}} z^{\alpha} s^{\beta}}\right)^h \frac{dy}{y}\right)^{\frac{1}{h}} dzds$$

$$= CB\left(\int_0^1 \int_0^1 \left(\int_0^\infty \left(\frac{xz}{s}\right)^{\frac{1-\beta}{q}} \frac{1}{z^{1-\frac{1}{r}}|1-x|^{1-\frac{1}{r}}}\right)^h \frac{dx}{x}\right)^{\frac{1}{h}} \frac{dz}{z^{\alpha}} \frac{ds}{s^{\beta}}$$

$$= CB\left(\int_0^1 z^{\frac{1-\alpha}{p}-1}dz\right)\left(\int_0^1 s^{\frac{1-\beta}{q'}-1}ds\right)\left(\int_0^\infty \left(\frac{x^{\frac{1-\beta}{q}}}{|1-x|^{\frac{1-\alpha}{p'}+\frac{1-\beta}{q}}}\right)^h \frac{dx}{x}\right)^{\frac{1}{h}}$$

$$= C_1 B.$$

Thus, remark 3.1 implies sufficient part.

Let us prove the estimate of $\|A\|$ from below. Suppose now that the operator A is bounded from $L_p(\mathbb{R},\nu)$ into $L_q(\mathbb{R},\mu)$, i.e.,

$$\|A\|_{L_p(\mathbb{R},\nu)\to L_q(\mathbb{R},\mu)} = \sup_{\substack{\|g\|_{L_{q'}(\mathbb{R},\mu)}=1 \\ \|f\|_{L_p(\mathbb{R},\nu)}=1}} \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} K(y-x) f(x)|x|^{-\alpha}|y|^{-\beta} dxdy < \infty.$$

For any $d>0$ we define the test functions:

$$g_0(y)=\chi_{[-d,d]}(y)/\|\chi_{[-d,d]}\|_{L_{q'}(\mathbb{R},\mu)}=C_2 d^{-\frac{1-\beta}{q'}}\chi_{[-d,d]}(y)$$

and

$$\begin{aligned} f_0(x)&=|x|^\alpha\chi_{[-2d,2d]}(x)/\||x|^\alpha\chi_{[-2d,2d]}\|_{L_p(\mathbb{R},\nu)}\\ &=|x|^\alpha\chi_{[-2d,2d]}(x)\left(\int_{-2d}^{2d}(|y|^\alpha)^p\,|y|^{-\alpha}dy\right)^{-\frac{1}{p}}\\ &=C_3 d^{\frac{1-\alpha}{p'}-1}|x|^\alpha\chi_{[-2d,2d]}(x). \end{aligned}$$

Therefore,

$$\|A\|_{L_p(\mathbb{R},\nu)\to L_q(\mathbb{R},\mu)}\geqslant C_4\frac{1}{d^{\frac{1-\beta}{q'}+1-\frac{1-\alpha}{p'}}}\int_{-d}^{d}|y|^{-\beta}\int_{-2d}^{2d}K(y-x)dxdy$$

Now using $K\geqslant 0$, we get for any $y\in[-d,d]$: $[-d,d]\subset[-2d,2d]+y$ and

$$\|A\|_{L_p(\mathbb{R},\nu)\to L_q(\mathbb{R},\mu)}\geqslant C_5\frac{1}{d^{1-\frac{1-\alpha}{p'}-\frac{1-\beta}{q}}}\int_{-d}^{d}K(z)dz$$

Taking into account arbitrary choice of $d>0$, we get (4.2). The proof is now complete. □

Now we introduce the following concept.

Definition. A measurable function $f(x)$ on $\mathbb{R}$ is said to be weak monotone if for any $x\in\mathbb{R}$ one has

$$|f(x)|\leqslant C\sup_{d\geqslant|x|}\frac{1}{d}\left|\int_{-d}^{d}f(y)dy\right|.$$

Note that if one of nonnegative functions $f(x)$ or $f(-x)$ is monotone decreasing or, more general, quasi-monotone on $\mathbb{R}_+$, then $f(x)$ is a weak monotone. On the other hand, the following functions are not quasi-monotone but weak monotone:

$$\frac{|\sin x|}{|x|^\alpha},\qquad\frac{\cos x}{|x|^\alpha},\qquad\alpha\in(0,1).$$

Corollary 4.1.1. *Let us assume that all conditions of Theorem 4.1 hold and K is a nonnegative weak monotone function on $\mathbb{R}$. Then the necessary and sufficient condition for the convolution operator Af to be bounded from $L_p(\mathbb{R},\nu)$ into $L_q(\mathbb{R},\mu)$ is*

$$\sup_{d>0}\frac{1}{d^{\frac{1}{r}}}\int_{-d}^{d}K(t)dt<\infty.$$

Proof. Since K is a weak monotone, we obtain

$$\sup_{e\in M^*}\frac{1}{|e|^{\frac{1}{r}}}\left|\int_e K(t)dt\right| \leqslant c\sup_{e\in M^*}\frac{1}{|e|^{\frac{1}{r}}}\left|\int_e \sup_{d\geqslant |t|}\frac{1}{d}\int_{-d}^{d}K(s)dsdt\right|$$

$$\leqslant c\sup_{d>0}\frac{1}{d^{\frac{1}{r}}}\int_{-d}^{d}K(s)ds\sup_{e\in M^*}\frac{1}{|e|^{\frac{1}{r}}}\left|\int_e\frac{1}{|t|^{1-1/r}}dt\right|$$

$$\leqslant c\sup_{d>0}\frac{1}{d^{\frac{1}{r}}}\int_{-d}^{d}K(s)ds,$$

and the required follows form Theorem 4.1. □

From the proof of Theorem 4.1 one can obtain its analogue for functions on an interval.

Theorem 4.2. *Let* $\alpha,\beta\in[0,1)$, $1<p,q<\infty$, $1+\frac{1}{q}=\frac{1}{p}+\frac{1}{h}$, *and* $0<\frac{1}{r}=1-\frac{1-\alpha}{p'}-\frac{1-\beta}{q}$. *Let* K *be a nonnegative weak monotone function on* $[-2a,2a]$. *Suppose that the measures* μ *and* ν *are defined as follows*

$$\mu(e)=\int_e\frac{dy}{|y|^\beta},\qquad \nu(\omega)=\int_\omega\frac{dx}{|x|^\alpha}$$

and

$$Af(y)=\int_{-a}^{a}K(x-y)f(x)\frac{dx}{|x|^\alpha}.$$

Then

$$\|A\|_{L_p([-a,a],\nu)\to L_q([-a,a],\mu)}\asymp\sup_{2a\geqslant d>0}\frac{1}{d^{\frac{1}{r}}}\int_{-d}^{d}K(t)dt<\infty.$$

As applications of this result, we obtain the sharp increase rate for the norm of the Cesàro operator from $L_p(|x|^\gamma)$ into $L_q(|x|^{-\beta})$ and a weighted version of the Nikol'skii inequality for trigonometrical polynomials (see also [MT]).

Corollary 4.2.1. *The Fejér kernel is defined as*

$$F_n(x)=\frac{1}{n}\sum_{k=0}^{n-1}D_k(x)=\frac{1}{n}\sum_{k=0}^{n-1}\sum_{\nu=-k}^{k}e^{2\pi i\nu x}=\frac{1}{n}\frac{\sin^2\pi(2n-1)x}{\sin^2\pi x}.$$

It is well known that

$$F_n(x)\geqslant 0\qquad \textit{and}\qquad \int_{-1/2}^{1/2}F_n(x)\,dx=1.$$

The Cesàro sum of 1*-periodic function* f *is written by*

$$\sigma_n(f,y)=\int_{-1/2}^{1/2} f(x)F_n\big(x-y\big)dx$$

and the n*-th partial sum of the Fourier series of* f *is*

$$S_n(f,y)=\int_{-1/2}^{1/2} f(x)D_n\big(x-y\big)dx.$$

Then for $1<p,q<\infty, 0\leqslant\beta<1, 0\leqslant\gamma<p-1$*, we have*

$$\sup_{\|f\|_{L_p([-1/2,1/2],|x|^{\gamma})}=1}\|\sigma_n(f)\|_{L_q([-1/2,1/2],|x|^{-\beta})}\asymp n^{\frac{1+\gamma}{p}-\frac{1-\beta}{q}}, \tag{4.3}$$

$$\sup_{\|f\|_{L_p([-1/2,1/2],|x|^{\gamma})}=1}\|S_n(f)\|_{L_q([-1/2,1/2],|x|^{-\beta})}\leqslant Cn^{\frac{1+\gamma}{p}-\frac{1-\beta}{q}}. \tag{4.4}$$

where $\|f\|_{L_q([-1/2,1/2],|x|^{-\beta})}=\Big(\int_{-1/2}^{1/2}|f(x)|^q|x|^{-\beta}dx\Big)^{\frac{1}{q}}$

Proof. One can note that

$$F_n(x)\leqslant\frac{c}{|x|}\int_{-|x|}^{|x|}F_n(t)dt,\qquad |x|\in(0,1/2]. \tag{4.5}$$

Indeed,

$$\frac{(\sin(2n-1)\pi x)^2}{n(\sin\pi x)^2}\leqslant\frac{\min((2n-1)^2,1/x^2)}{n}\leqslant 4\min(n,1/|x|),$$

If $x\geqslant 1/(2n-1)$,

$$\begin{aligned}\frac{1}{x}\int_0^x\frac{(\sin(2n-1)\pi t)^2}{n(\sin\pi t)^2}dt &\geqslant\frac{c}{xn}\sum_{k=1}^{m_0}\left(\frac{(2n-1)}{k}\right)^2\\ &\qquad\int_{(k-1)/(2n-1)}^{k/(2n-1)}\left(\sin(2n-1)\pi t\right)^2dt\\ &=\frac{c}{x}\sum_{k=1}^{m_0}\frac{1}{k^2}\int_{k-1}^{k}\left(\sin\pi y\right)^2dy\\ &\geqslant\frac{c}{x}\sum_{k=1}^{m_0}\frac{1}{k^2}\geqslant\frac{c}{x}\geqslant c\min(n,1/x).\end{aligned}$$

If $0<x<1/(2n-1)$,

$$\frac{1}{x}\int_0^x\frac{(\sin(2n-1)\pi t)^2}{n(\sin\pi t)^2}dt\geqslant\frac{n}{3}\geqslant\frac{\min(n,1/x)}{3}.$$

Therefore, (4.5) follows. Then Theorem 4.2 implies

$$\|A\|_{L_p([-a,a],\nu)\to L_q([-a,a],\mu)} \asymp \sup_{2a\geqslant d>0} \frac{1}{d^{\frac{1}{r}}} \int_{-d}^{d} K(t)dt < \infty.$$

So, for $\alpha = \gamma/(p-1)$ we get

$$\begin{aligned}
&\sup_{\|f\|_{L_p([-\pi,\pi],|x|^{\gamma})}=1} \|\sigma_n(f)\|_{L_q([-\pi,\pi],|x|^{-\beta})} \\
&\qquad = \sup_{\|g\|_{L_p([-\pi,\pi],|x|^{-\alpha})}=1} \left\|\sigma_n(g(\cdot)|\cdot|^{-\alpha})\right\|_{L_q([-\pi,\pi],|x|^{-\beta})} \\
&\qquad \asymp \sup_{2a\geqslant d>0} \frac{1}{d^{1-\frac{1-\alpha}{p'}-\frac{1-\beta}{q}}} \int_{-d}^{d} F_n(t)dt.
\end{aligned}$$

Let us now show that

$$\sup_{2a\geqslant d>0} \frac{1}{d^{\frac{1}{r}}} \int_{-d}^{d} F_n(t)dt \asymp n^{1/r}$$

From the proof above we have

$$\sup_{2a\geqslant d>0} \frac{1}{d^{1/r}} \int_{-d}^{d} F_n(t)dt \geqslant c \sup_{2a\geqslant d\geqslant 1/n} d^{1-1/r} \min(n, 1/d) = cn^{1/r}$$

On the other hand,

$$\sup_{2a\geqslant d>0} \frac{1}{d^{1/r}} \int_{-d}^{d} F_n(t)dt \leqslant c \sup_{2a\geqslant d>0} \frac{1}{d^{1/r}} \int_{0}^{d} \min(n, 1/t)dt.$$

Then we write $\sup_{1/n\geqslant d>0} d^{-1/r} dn = n^{1/r}$ and

$$\begin{aligned}
&\sup_{2a\geqslant d>1/n} \frac{1}{d^{1/r}} \left(\int_{0}^{1/n} + \int_{1/n}^{d}\right) \min(n, 1/t)dt \\
&\qquad \leqslant c \sup_{2a\geqslant d>1/n} d^{-1/r} + cn^{1/r} \sup_{2a\geqslant d>1/n} (nd)^{-1/r}(1+\ln(nd)) \leqslant cn^{1/r}.
\end{aligned}$$

Therefore, the following is true: if $1 < p, q < \infty$, $0 \leqslant \beta < 1$, and $0 \leqslant \gamma < p-1$, then

$$\sup_{\|f\|_{L_p([-\pi,\pi],|x|^{\gamma})}=1} \|\sigma_n(f)\|_{L_q([-\pi,\pi],|x|^{-\beta})} \asymp n^{1-\frac{1-\alpha}{p'}-\frac{1-\beta}{q}} = n^{\frac{1+\gamma}{p}-\frac{1-\beta}{q}}.$$

Let us prove now the estimate for $S_n(f,x)$. We consider the following operator

$$|S_n|(f,y) = \int_{-1/2}^{1/2} f(x)|D_n(x-y)|dx$$
$$= \int_{-1/2}^{1/2} f(x)\left|\frac{\sin(2n-1)\pi(x-y)}{\sin\pi(x-y)}\right| dx.$$

It is clear that $\|S_n(f)\|_{L_p\to L_q} \leqslant \||S_n|(f)\|_{L_p\to L_q}$. As in the proof of (4.5), one gets

$$\left|\frac{\sin(2n-1)\pi x}{\sin\pi x}\right| \leqslant 2\min(n, 1/|x|)$$

and

$$\sup_{1\geqslant x\geqslant d} \frac{1}{x}\int_0^x \left|\frac{\sin(2n-1)\pi t}{\sin\pi t}\right| dt \geqslant c\min(n, 1/d).$$

Hence, Theorem 4.2 implies

$$\sup_{\|f\|_{L_p([-\pi,\pi],|x|^\gamma)}=1} \left\||S_n|(f)\right\|_{L_q([-\pi,\pi],|x|^{-\beta})}$$
$$= \sup_{\|g\|_{L_p([-\pi,\pi],|x|^{-\alpha})}=1} \left\||S_n|(g(\cdot)|\cdot|^{-\alpha})\right\|_{L_q([-\pi,\pi],|x|^{-\beta})}$$
$$\asymp \sup_{2a\geqslant d>0} \frac{1}{d^{1-\frac{1-\alpha}{p'}-\frac{1-\beta}{q}}}\int_{-d}^{d}\left|\frac{\sin(2n-1)\pi t}{\sin\pi t}\right| dt$$
$$\asymp n^{1-\frac{1-\alpha}{p'}-\frac{1-\beta}{q}} = n^{\frac{1+\gamma}{p}-\frac{1-\beta}{q}}.$$

Thus, (4.3) and (4.4) are verified. □

5. Convolution operator in Lorentz spaces

The following theorem provides the sufficient conditions for the convolution operator to be bounded in a Lorentz space.

Theorem 5.1. *Let $1 < p,q < \infty$ and let the measures μ and ν be on measurable sets D and Ω from $\mathbb{R}^n$ respectively. Assume that a function $K(z)$ defined on $D-\Omega = \{z = x-y : x\in D, y\in\Omega\}$ satisfies the condition* *that there exists* $\gamma > 0$ *such that*

$$\sup_{e\in M_1} \frac{1}{(\mu(e))^{1/q'-1/\gamma p'}} \left|\int_e K(x-y)d\mu_y\right| \leqslant B \qquad \textit{for a.e.} \quad x \in D,$$

$$\sup_{w\in M_2} \frac{1}{(\nu(w))^{1/p-\gamma/q}} \left|\int_w K(x-y)d\nu_x\right| \leqslant B \qquad \textit{for a.e.} \quad y \in \Omega,$$

where $M_1 = \Big\{e \subset \Omega : 0 < \mu(e) < \infty\Big\}$ *and* $M_2 = \Big\{w \subset D : 0 < \nu(w) < \infty\Big\}$. *Then*

$$Af(y) = \int\limits_D K(x-y)f(x)d\nu_x$$

is bounded from $L_{p,h_1}(D,\nu)$ *to* $L_{q,h_2}(\Omega,\mu)$ *with* $1 \leqslant h_1 \leqslant h_2 \leqslant \infty$ *and, moreover,*

$$\|A\|_{L_{p,h_1}(D,\nu)\to L_{q,h_2}(\Omega,\mu)} \leqslant CB,$$

where $C = C(p,q,h_1,h_2)$.

Proof. By Theorem 3.2 it is sufficient to prove

$$I = \left(\int_0^\infty \Big(\sup_{\mu(e)/\nu^\gamma(w)=s} F(e,w;K)\Big)^r \frac{ds}{s}\right)^{\frac{1}{r}} \leqslant CB, \tag{5.1}$$

where

$$F(e,\omega;K) = \frac{1}{\nu(\omega)^{\frac{1}{p}}} \frac{1}{\mu(e)^{\frac{1}{q'}}} \left|\int_e\int_\omega K(x-y)d\nu\, d\mu\right|,$$

$\frac{1}{r} = \frac{1}{h_2} + 1 - \frac{1}{h_1} \leqslant 1$. To prove (5.1), we write $I^r = \int_0^1 + \int_1^\infty =: I_1^r + I_2^r$ and estimate each term separately.

$$I_1 \leqslant \left(\int_0^1 \left(\sup_{\mu(e)/\nu^\gamma(w)=s} \frac{1}{(\mu(e))^{1/q'}} \frac{1}{\nu(w)^{1/p}} \left|\int_e\int_w K(x-y)d\nu d\mu\right|\right)^r \frac{ds}{s}\right)^{1/r}$$

$$\leqslant ess\sup_{y\in\Omega} \sup_{\nu(w)>0} \frac{1}{\nu(w)^{1/p-\gamma/q}}$$

$$\left|\int_w K(x-y)d\nu\right| \left(\int_0^1 \Big(\sup_{\mu(e)/\nu^\gamma(w)=s} \frac{\mu(e)^{1/q}}{\nu(w)^{\gamma/q}}\Big)^r \frac{ds}{s}\right)^{1/r}$$

$$\leqslant B\left(\int_0^1 s^{r/q}\frac{ds}{s}\right)^{1/r} = (q/r)^{1/r}B.$$

Further,

$$I_2 \leqslant \left(\int\limits_1^\infty \Big(\sup_{\mu(e)/\nu^\gamma(w)=s} \frac{1}{(\mu(e))^{1/q'}} \frac{1}{\nu(w)^{1/p}} \Big| \int\limits_w \int\limits_e K(x-y) d\nu d\mu \Big| \Big)^r \frac{ds}{s} \right)^{\frac{1}{r}}$$

$$\leqslant ess \sup_{x\in D} \sup_{\mu(e)>0} \frac{1}{(\mu(e))^{1/q'-1/\gamma p'}}$$

$$\left| \int\limits_e K(x-y) d\mu \right| \left(\int\limits_1^\infty \left(\sup_{\mu(e)/\nu^\gamma(w)=s} \frac{\nu(w)^{1/p'}}{\mu(e)^{1/\gamma p'}} \right)^r \frac{ds}{s} \right)^{1/r}$$

$$\leqslant (\gamma p'/r)^{1/r} B.$$

Thus $I \leqslant CB$, which finishes the proof. □

Now let us investigate the necessary conditions for boundedness of the convolution operator in Lorentz spaces. Let positive locally integrable functions $\nu(x)$ and $\mu(x)$ on $\mathbb{R}$ define the following measures

$$\mu(e) = \int_e \mu(y) dy \quad \text{and} \quad \nu(w) = \int_w \nu(x) dx. \tag{5.2}$$

Let $e = [a, b]$, then we denote by $[e]_1, [e]_2$, and $[e]_3$ the following parts of e: $[e]_1 = [a, a + \frac{b-a}{3}]$, $[e]_2 = [a + \frac{b-a}{3}, b - \frac{b-a}{3}]$, and $[e]_3 = [b - \frac{b-a}{3}, b]$. For the function $\mu(x) > 0$ on $\mathbb{R}$ we define the net

$$M_\mu = \left\{ [a, b] : \quad \frac{\mu(y)}{2} \leqslant \mu(x) \leqslant 2\mu(y), \quad \forall x, y \in [a, b] \right\}.$$

Example 5.1. Suppose $\mu(x) = \frac{1}{|x|^\beta}$, $\beta > 0$; then

$$M_\mu = \Big\{ [a, b] : 0 < a < b \leqslant 2^{\frac{1}{\beta}} a < \infty \Big\} \bigcup \Big\{ [a, b] : -\infty < 2^{\frac{1}{\beta}} b \leqslant a < b < 0 \Big\}.$$

Let $|e|$ be a length of an interval e. Note that for any $e \in M_\mu$ and $u \in [-\frac{|e|}{3}, \frac{|e|}{3}]$ one has

$$\mu(e) \leqslant 6\mu([e]_2 + u), \tag{5.3}$$

where $[e]_2 + u = \{x + u : x \in [e]_2\}$. Indeed,

$$\mu(e) = \int_e \mu(y)dy = \int_{[e]_1} \mu(y)dy + \int_{[e]_2} \mu(y)dy + \int_{[e]_3} \mu(y)dy$$

$$= \int_{[e_2]+u} \left(\mu(y - \frac{|e|}{3} - u) + \mu(y - u) + \mu(y + \frac{|e|}{3} - u) \right) dy$$

$$\leqslant 6 \int_{[e]_2+u} \mu(y)dy = 6\mu([e]_2 + u).$$

Theorem 5.2. *Let $1 < p < q < \infty$, $1 \leqslant h_1 \leqslant h_2 \leqslant \infty$, and let μ, ν be defined by (5.2). Suppose ν and μ satisfy for some $\gamma > 0$*

$$\begin{aligned} &\mu([0, |w|/3]) \geqslant c_1 \nu^\gamma(w) \quad \forall\, w \in M_\nu, \\ &\nu^\gamma([0, |e|/3]) \geqslant c_2 \mu(e) \qquad \forall\, e \in M_\mu, \end{aligned} \tag{5.4}$$

and suppose that the convolution operator

$$Ag(y) = \int\limits_D K(y - x)g(x)d\nu_x, \qquad K(x) \geqslant 0,$$

is bounded from $L_{p,h_1}(\mathbb{R}, \nu)$ into $L_{q,h_2}(\mathbb{R}, \mu)$; then

$$\sup_{e \in M_\mu} \sup_{0<y<\frac{|e|}{3}} \frac{1}{(\mu([e]_2 + y))^{1/q' - 1/\gamma p'}}$$

$$\int_{[e]_2} K(z)\mu(z + y)dz \leqslant c\|A\|_{L_{p,h_1}(\mathbb{R},\nu) \to L_{q,h_2}(\mathbb{R},\mu)},$$

$$\sup_{w \in M_\nu} \sup_{0<x<\frac{|w|}{3}} \frac{1}{(\nu([w]_2 + x))^{1/p - \gamma/q}}$$

$$\int_{[w]_2} K(z)\nu(x + z)dz \leqslant c\|A\|_{L_{p,h_1}(\mathbb{R},\nu) \to L_{q,h_2}(\mathbb{R},\mu)},$$

Conversely, suppose $M_1 = \{e : 0 < \mu(e) < \infty\}$ and $M_2 = \{w : 0 < \nu(w) < \infty\}$ and one of the following conditions holds true:

$$\sup_{e \in M_1} \sup_{y \in \mathbb{R}} \frac{1}{(\mu(e + y))^{1/q' - 1/\gamma p'}} \int_e K(z)\mu(y + z)dz \leqslant B$$

or

$$\sup_{w \in M_2} \sup_{x \in \mathbb{R}} \frac{1}{(\nu(x - w))^{1/p - \gamma/q}} \int_w K(z)\nu(x - z)dz \leqslant B;$$

then the operator A is bounded from $L_{p,h_1}(\mathbb{R}, \nu)$ into $L_{q,h_2}(\mathbb{R}, \mu)$ and

$$\|A\|_{L_{p,h_1}(\mathbb{R},\nu) \to L_{q,h_2}(\mathbb{R},\mu)} \leqslant cB.$$

Remark 5.1. The functions $\mu(y)=\frac{1}{|y|^\beta}, \beta\in[0,1)$ and $\nu(x)=\frac{1}{|x|^\alpha}, \alpha\in[0,1)$ satisfy conditions (5.4) with $\gamma=\frac{1-\beta}{1-\alpha}$.

Proof of Theorem. 5.2 By Corollary 3.1.1, we have

$$\|A\|_{L_{ph_1}(\nu)\longrightarrow L_{qh_2}(\mu)}\geqslant \\ c\sup_{e\in M_1}\sup_{w\in M_2}\frac{1}{(\mu(e))^{1/q'}}\frac{1}{(\nu(w))^{1/p}}\int_e\int_w K(y-x)\mu(y)\nu(x)dxdy.$$

Let $e\in M_\mu$. Applying (5.3-5.4) and noting that for any $0<x\leqslant\frac{|e|}{3}$ one has $[e]_2\subset e-x$, we write

$$\begin{aligned}
&\|A\|_{L_{ph_1}(\nu)\longrightarrow L_{qh_2}(\mu)}\\
&\geqslant \frac{c_2^{1/p'}}{\nu([0,\frac{|e|}{3}])}\int_0^{\frac{|e|}{3}}\nu(x)\frac{1}{(\mu(e))^{\frac{1}{q'}-\frac{1}{\gamma p'}}}\int_e K(y-x)\mu(y)dydx\\
&\geqslant \frac{c_2^{1/p'}}{\mu([0,\frac{|e|}{3}])}\int_0^{\frac{|e|}{3}}\nu(x)\frac{1}{(\mu(e))^{\frac{1}{q'}-\frac{1}{\gamma p'}}}\int_{[e]_2} K(z)\mu(z+x)dzdx\\
&\geqslant \frac{c_2^{1/p'}}{2\nu([0,\frac{|e|}{3}])}\int_0^{\frac{|e|}{3}}\nu(x)\left(\frac{1}{(\mu e)^{\frac{1}{q'}-\frac{1}{\gamma p'}}}\int_{[e]_2} K(z)\mu(z)dz\right)dx\\
&= \frac{c_2^{1/p'}}{2(\mu e)^{\frac{1}{q'}-\frac{1}{\gamma p'}}}\int_{[e]_2} K(z)\mu(z)dz\\
&\geqslant \frac{c_2^{1/p'}}{4\cdot 6^{\frac{1}{q'}-\frac{1}{\gamma p'}}(\mu([e]_2+x))^{\frac{1}{q'}-\frac{1}{\gamma p'}}}\int_{[e]_2} K(z)\mu(z+x)dz,
\end{aligned}$$

where $x\in[0,|e|/3]$. Therefore, taking into account arbitrary choice of $e\in M_\mu$, we have

$$\|A\|_{L_{ph_1}(\nu)\longrightarrow L_{qh_2}(\mu)}\geqslant \\ c\sup_{e\in M_\mu}\sup_{x\in[0,\frac{|e|}{3}]}\frac{1}{(\mu([e]_2+x))^{\frac{1}{q'}-\frac{1}{\gamma p'}}}\int_{[w]_2}K(z)\mu(z+x)dz.$$

Similarly,

$$\|A\|_{L_{ph_1}(\nu)\longrightarrow L_{qh_2}(\mu)}\geqslant \\ c\sup_{w\in M_\nu}\sup_{y\in[0,\frac{|w|}{3}]}\frac{1}{(\nu(y-[w]_2))^{\frac{1}{p}-\frac{\gamma}{q}}}\int_{[w]_2}K(z)\nu(y-z)dz.$$

Let us now prove the second part of the theorem. By Theorem 3.2,

$$\|A\|_{L_{ph_1}(\nu)\longrightarrow L_{qh_2}(\mu)} \leqslant c \sup_{e\in M_1}\sup_{x\in\mathbb{R}} \frac{1}{(\mu(e))^{\frac{1}{q'}-\frac{1}{\gamma p'}}}\int_e K(y-x)\mu(y)dy$$

$$= c\sup_{x\in\mathbb{R}}\sup_{w\in M_1} \frac{1}{(\mu(e+x))^{\frac{1}{q'}-\frac{1}{\gamma p'}}}\int_e K(z)\mu(z+x)dz.$$

$$\|A\|_{L_{ph_1}(\nu)\longrightarrow L_{qh_2}(\mu)} \leqslant c \sup_{w\in M_2}\sup_{y\in\mathbb{R}} \frac{1}{(\nu(y-w))^{\frac{1}{p}-\frac{\gamma}{q}}}\int_w K(z)\nu(y-z)dz.$$

The proof is now complete. □

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E. Nursultanov
Kazakh Branch of Moscow State University
Munatpasova, 7
010010Astana Kazakhstan
E-mail address: er-nurs@yandex.ru

S. Tikhonov
Scuola Normale Superiore
Piazza dei Cavalieri, 7
56126 Pisa Italy
E-mail address: s.tikhonov@sns.it