

GENERALIZED MULTIREOLUTION ANALYSES WITH GIVEN MULTIPLICITY FUNCTIONS

LAWRENCE W. BAGGETT, NADIA S. LARSEN, KATHY D. MERRILL,
JUDITH A. PACKER, AND IAIN RAEBURN

ABSTRACT. Generalized multiresolution analyses are increasing sequences of subspaces of a Hilbert space \mathcal{H} that fail to be multiresolution analyses in the sense of wavelet theory because the core subspace does not have an orthonormal basis generated by a fixed scaling function. Previous authors have studied a multiplicity function m which, loosely speaking, measures the failure of the GMRA to be an MRA. When the Hilbert space \mathcal{H} is $L^2(\mathbb{R}^n)$, the possible multiplicity functions have been characterized by Baggett and Merrill. Here we start with a function m satisfying a consistency condition which is known to be necessary, and build a GMRA in an abstract Hilbert space with multiplicity function m .

INTRODUCTION

Classically, a multiresolution analysis (MRA) for $L^2(\mathbb{R})$ is an increasing sequence of closed subspaces V_n of $L^2(\mathbb{R})$ such that the complements $W_n := V_{n+1} \ominus V_n$ give a direct sum decomposition $L^2(\mathbb{R}) = \bigoplus_{n \in \mathbb{Z}} W_n$, such that W_{n+1} is the dilation of W_n , and such that there is a scaling function ϕ whose integer translates form an orthonormal basis for V_0 . MRAs have been extensively used in signal processing, where the spaces W_n are used inductively to add detail to an approximation of a signal $f \in L^2(\mathbb{R})$, and in other parts of mathematics, including analysis on fractals and probability (as discussed in the recent book [13], for example). From our point of view,

2000 *Mathematics Subject Classification.* Primary 42C40, 47A20.

Key words and phrases. Generalized multiresolution analysis, multiresolution analysis, filter, wavelet, isometry, direct limit.

Nadia Larsen was supported by the Research Council of Norway. Judith Packer was supported by the National Science Foundation. Iain Raeburn was supported by the Australian Research Council, through the ARC Centre for Complex Dynamic Systems and Control, and by the Centre de Recerca Matemàtica at the Universitat Autònoma de Barcelona.

the importance of MRAs lies in their fundamental role in the construction of wavelets (see [16]). The idea is that if there is also a function ψ whose translates form an orthonormal basis for W_0 , then the dilates of this basis form an orthonormal basis for $L^2(\mathbb{R})$ — in other words, ψ is a wavelet. However, not all MRAs give rise to wavelets, and not all wavelets are associated to MRAs; a famous example of a wavelet which does not arise from an MRA is the Journé wavelet.

Generalized multiresolution analyses were introduced by Baggett, Medina and Merrill in [4], where they were used to construct many examples of wavelet sets, including the one whose characteristic function is the Fourier transform of the Journé wavelet. A generalized multiresolution analysis (GMRA) for a Hilbert space \mathcal{H} consists of an increasing sequence of closed subspaces V_n such that the complements $W_n := V_{n+1} \ominus V_n$ give a direct-sum decomposition $\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} W_n$ in which the W_n for $n \geq 0$ are invariant under a representation π of an abelian group Γ on \mathcal{H} , and in which W_{n+1} is the dilation of W_n . The representation theory of abelian groups associates to the representations $\pi|_{V_0}$ and $\pi|_{W_0}$ integer-valued multiplicity functions m and \tilde{m} on the dual group $\hat{\Gamma}$. In this paper, we consider the question of which functions m and \tilde{m} can arise as multiplicity functions of GMRA's.

Previous work on this question has focused on the case $\mathcal{H} = L^2(\mathbb{R}^n)$, with the group \mathbb{Z}^n acting by translation and the dilation implemented by an integer matrix A whose eigenvalues λ satisfy $|\lambda| > 1$. In this case, Baggett and Merrill showed that m is associated to a GMRA if and only if m satisfies a consistency condition, described in detail below, and a technical condition on the translates of the support of m [5], which was discovered independently by Bownik, Rzeszutnik and Speegle [6] in their characterization of the dimension function of a wavelet. When a GMRA in $L^2(\mathbb{R}^n)$ has an associated (multi-) wavelet, one or more functions ψ_k such that the translates $\pi_n(\psi_k)$ form an orthonormal basis for W_0 (so that \tilde{m} is constant), the characterizations in [5] and [6] coincide.

Here we show that the second technical condition from [5] and [6] is particular to $L^2(\mathbb{R}^n)$: provided one is willing to consider GMRA's in abstract Hilbert spaces, there are surprisingly few restrictions on m and \tilde{m} apart from the consistency condition of [5]. Our new results include a general construction of filters for multiplicity functions (Proposition 7), and a criterion for the purity of an associated isometry which improves a key result in [3] (Theorem 8).

We begin in §1 by discussing GMRA's and multiplicity functions, and stating our main theorem. We work in an abstract Hilbert space, with a countable abelian group Γ of translations and a dilation operator which is

compatible with an endomorphism α of Γ . In §2, we revisit the direct-limit construction from [14] to see what extra input we need to ensure that the direct limit carries the necessary translation group and dilation operator (Theorem 5). Then in §3 we prove our main theorem. We first show that our multiplicity function m admits a low-pass filter, which is a matrix H of functions on $\widehat{\Gamma}$ satisfying relations, introduced in [3], which generalize those of quadrature mirror filters. From H we build an isometry S_H on a Hilbert space \mathcal{K} , following an idea which goes back at least to [7], and Theorem 8 says that when the filter is low-pass, S_H is a pure isometry. Then, when we apply the construction of Theorem 5 to this isometry, we obtain a direct-limit Hilbert space which has the required GMRA.

Since we think Theorem 8 and its proof are likely to be of independent interest, we have made them the focus of a separate section. Our proof follows the general strategy suggested in [3, Lemma 3.3], but here we have been able to replace some of the grittier estimates with exact calculations, and those which remain are much sharper. The crux of the argument is the almost everywhere pointwise convergence of a sequence of averages, which we achieve by applying the reverse martingale convergence theorem. (We thank Dan Stroock for suggesting to us that we were dealing with a reverse martingale; in retrospect, we could also have learned this from [11].) In the final section we discuss some examples which show that our results have broader scope than those of [5] and [6].

Notation and standing assumptions. Throughout this paper, Γ is a countable abelian group with compact dual $\widehat{\Gamma}$, and λ denotes normalised Haar measure on $\widehat{\Gamma}$. We fix an injective endomorphism α of Γ such that $\alpha(\Gamma)$ has finite index N in Γ , and we write α^* for the endomorphism of $\widehat{\Gamma}$ onto itself defined by $\alpha^*(\omega) = \omega \circ \alpha$, and note that $|\ker \alpha^*| = N$. We assume that $\bigcup_{n \geq 1} \ker \alpha^{*n}$ is dense in $\widehat{\Gamma}$ (or equivalently, that $\bigcap_{n \geq 1} \alpha^n(\Gamma) = \{0\}$). If σ is a subset of $\widehat{\Gamma}$, then χ_σ denotes the characteristic function $\chi_\sigma: \widehat{\Gamma} \rightarrow \mathbb{C}$ which is 1 on σ and 0 elsewhere.

All Hilbert spaces in the paper are separable.

1. MULTIPLICITY FUNCTIONS AND THE MAIN THEOREM

Let $\pi: \Gamma \rightarrow U(\mathcal{H})$ be a unitary representation, and let δ be a unitary operator on \mathcal{H} such that

$$\delta^{-1} \pi_\gamma \delta = \pi_{\alpha(\gamma)} \quad \text{for all } \gamma \in \Gamma.$$

A *generalized multiresolution analysis* (or GMRA) relative to π and δ is a sequence $\{V_n : n \in \mathbb{Z}\}$ of closed subspaces of \mathcal{H} with the following properties:

- (a) $V_n \subset V_{n+1}$ for all n ,
- (b) $V_{n+1} = \delta(V_n)$ for all n ,
- (c) $\bigcup_{n=0}^{\infty} V_n$ is dense in \mathcal{H} and $\bigcap_{n=-\infty}^0 V_n = \{0\}$, and
- (d) V_0 is invariant under π .

GMRA's were introduced in [4], and have since been studied in [5] and [3]; other authors had previously observed that much of the theory still works when we use Γ in place of the classical translation group \mathbb{Z}^n (see [1] and [9], for example).

Property (d) of a GMRA implies that $\rho := \pi|_{V_0}$ is a unitary representation of Γ , and Stone's theorem on unitary representations of abelian groups together with the multiplicity theory for projection valued measures [15] gives a Borel measure μ on $\widehat{\Gamma}$, unique Borel subsets $\sigma_1 \supseteq \sigma_2 \supseteq \dots$ of $\widehat{\Gamma}$, and a unitary operator $J: V_0 \rightarrow \bigoplus_i L^2(\sigma_i, \mu)$ satisfying

$$[J(\rho_\gamma(v))]_i(\omega) = \omega(\gamma)[J(v)]_i(\omega)$$

for $v \in V_0$, $\gamma \in \Gamma$ and μ -almost all $\omega \in \widehat{\Gamma}$ (see [4, Proposition 1]).

When $\mathcal{H} = L^2(\mathbb{R}^d)$, Γ is the lattice \mathbb{Z}^d , $\alpha(k) = Ak$ and π is the representation determined by translation, the measure μ is necessarily absolutely continuous with respect to the Haar measure on the torus $\mathbb{T}^d \equiv \widehat{\mathbb{Z}^d}$ (see [4, Propositions 2 and 3]). This absolute continuity does not necessarily hold in general, but here we are interested in the converse, and we assume that our measures μ are absolutely continuous with respect to the Haar measure λ on $\widehat{\Gamma}$.

With the above conventions, the function $m = \sum \chi_{\sigma_i}$ is called the *multiplicity function* of the GMRA. Properties (a), (b) and (d) in the definition of a GMRA imply that the subspace $W_0 = V_1 \ominus V_0$ also is invariant under π , and hence determines a unitary representation $\tilde{\rho}$ of Γ on W_0 . As above, Stone's theorem gives a measure $\tilde{\mu}$ on $\widehat{\Gamma}$, subsets $\tilde{\sigma}_1 \supseteq \tilde{\sigma}_2 \supseteq \dots$ of $\widehat{\Gamma}$, and a unitary map $\tilde{J}: W_0 \rightarrow \bigoplus_k L^2(\tilde{\sigma}_k, \tilde{\mu})$ such that

$$[\tilde{J}(\tilde{\rho}_\gamma(v))]_k(\omega) = \omega(\gamma)[\tilde{J}(v)]_k(\omega)$$

for $v \in W_0$, $\gamma \in \Gamma$ and $\tilde{\mu}$ -almost all $\omega \in \widehat{\Gamma}$. We write \tilde{m} for the corresponding *complementary multiplicity function* given by

$$\tilde{m}(\omega) = \sum_k \chi_{\tilde{\sigma}_k}(\omega).$$

We now prove that the multiplicity functions m and \tilde{m} of the GMRA $\{V_n : n \in \mathbb{Z}\}$ satisfy the following consistency equation for $(\mu + \tilde{\mu})$ -almost

all $\omega \in \widehat{\Gamma}$:

$$(1) \quad m(\omega) + \tilde{m}(\omega) = \sum_{\alpha^*(\zeta)=\omega} m(\zeta).$$

This will follow immediately from Lemma 1 below, which gives the consistency equation under slightly more general assumptions reflecting the dependency between (1) and the properties (a), (b) and (d) alone in the definition of a GMRA.

Lemma 1. *Let ρ and $\tilde{\rho}$ be representations of Γ on closed subspaces V and W of a Hilbert space \mathcal{H} such that there is a unitary δ on \mathcal{H} satisfying the following conditions:*

(i) $\delta(V) = V \oplus W$, and

(ii) $\delta^{-1}(\rho \oplus \tilde{\rho})_\gamma \delta|_V = \rho_{\alpha(\gamma)}$ for all $\gamma \in \Gamma$.

Let m , μ and \tilde{m} , $\tilde{\mu}$ be the multiplicity functions and the associated Borel measures given by Stone's theorem for ρ and respectively $\tilde{\rho}$. Then $m(\omega) + \tilde{m}(\omega) = \sum_{\alpha^*(\zeta)=\omega} m(\zeta)$ for $(\mu + \tilde{\mu})$ -almost all $\omega \in \widehat{\Gamma}$.

Proof. We begin by recalling an additional consequence of Stone's theorem. Suppose that π is a representation of the abelian group Γ acting in a Hilbert space \mathcal{V} , and let a Borel measure ν and Borel subsets $\{\tau_i\}$ be as in the statement of Stone's theorem. Suppose $\{\tau'_l\}$ is another collection of (not necessarily nested) Borel subsets of $\widehat{\Gamma}$, and suppose J^1 is a unitary operator from \mathcal{V} onto $\bigoplus_l L^2(\tau'_l)$ satisfying

$$[J^1(\pi_\gamma(f))](\omega) = \omega(\gamma)[J^1(f)](\omega)$$

for all $f \in \mathcal{V}$, all $\gamma \in \Gamma$, and ν -almost all $\omega \in \widehat{\Gamma}$. Then

$$\sum \chi_{\tau_i}(\omega) = \sum_l \chi_{\tau'_l}(\omega)$$

for ν -almost all $\omega \in \widehat{\Gamma}$. (This is really part of the proof of Stone's theorem. In fact, Stone's theorem is essentially the same as the canonical decomposition theorem for projection-valued measures; see [15].)

Let σ_i , J and $\tilde{\sigma}_k$, \tilde{J} be the nested Borel subsets and unitaries given by Stone's theorem applied to ρ and respectively $\tilde{\rho}$. Let m' be the multiplicity function associated to $\rho \oplus \tilde{\rho}$ on $V \oplus W$. Define a unitary J^1 from $V \oplus W$ to $\bigoplus_i L^2(\sigma_i) \oplus \bigoplus_k L^2(\tilde{\sigma}_k)$ by

$$[J^1(f \oplus g)](\omega) = [J(f)](\omega) \oplus [\tilde{J}(g)](\omega).$$

The additional consequence of Stone's theorem described above implies that for almost all ω , we have

$$m'(\omega) = \sum_i \chi_{\sigma_i}(\omega) + \sum_k \chi_{\tilde{\sigma}_k}(\omega) = m(\omega) + \tilde{m}(\omega).$$

Thus to verify the equation claimed in the lemma, it suffices to prove that

$$(2) \quad m'(\omega) = \sum_{\alpha^*(\zeta)=\omega} m(\zeta).$$

Let s be a Borel cross-section for the quotient map of $\hat{\Gamma}$ onto $\hat{\Gamma}/\ker(\alpha^*)$. For each i and each η in the kernel of α^* , define

$$\tau_{i,\eta} = \{\omega \in \hat{\Gamma} : s(\omega)\eta \in \sigma_i\},$$

and using (i) define $J' : V \oplus W \rightarrow \bigoplus_{i,\eta} L^2(\tau_{i,\eta})$ by

$$[J'(f \oplus g)]_{i,\eta}(\omega) = [J(\delta^{-1}(f \oplus g))]_i(s(\omega)\eta).$$

For $f \in V$ and $g \in W$ we then have

$$\begin{aligned} [J'((\rho \oplus \tilde{\rho})_\gamma(f \oplus g))]_{i,\eta}(\omega) &= [J(\delta^{-1}((\rho \oplus \tilde{\rho})_\gamma(f \oplus g)))]_i(s(\omega)\eta) \\ &= [J(\rho_{\alpha(\gamma)}(\delta^{-1}(f \oplus g)))]_i(s(\omega)\eta) \text{ by (ii)} \\ &= [s(\omega)\eta](\alpha(\gamma))[J(\delta^{-1}(f \oplus g))]_i(s(\omega)\eta) \\ &= [\alpha^*(s(\omega)\eta)](\gamma)[J'(f \oplus g)]_{i,\eta}(\omega) \\ &= \omega(\gamma)[J'(f \oplus g)]_{i,\eta}(\omega). \end{aligned}$$

Therefore, again by the additional consequence of Stone's theorem described above, the multiplicity function m' is given by

$$\begin{aligned} m'(\omega) &= \sum_{i,\eta} \chi_{\tau_{i,\eta}}(\omega) = \sum_{i,\eta} \chi_{\sigma_i}(s(\omega)\eta) \\ &= \sum_{\eta} \sum_i \chi_{\sigma_i}(s(\omega)\eta) = \sum_{\eta} m(s(\omega)\eta) \\ &= \sum_{\alpha^*(\zeta)=\omega} m(\zeta), \end{aligned}$$

as was sought in (2). \square

Remark 2. We look at the above definitions in the familiar setting of $L^2(\mathbb{R})$. Suppose that π is the representation of $\Gamma = \mathbb{Z}$ by translations on $L^2(\mathbb{R})$ and δ is a dilation operator. When W_0 is generated by a wavelet ψ , so that the translates $\{\pi_n\psi\}$ form an orthonormal basis for W_0 , the representation $\pi|_{W_0}$ is equivalent to the representation by multiplication operators on $L^2(\mathbb{T})$ and

the complementary multiplicity function \tilde{m} is identically 1. When there is a scaling function ϕ such that $\{\pi_n \phi\}$ is an orthonormal basis for V_0 , so that the GMRA is an MRA, we also have m identically equal to 1. However, the Journé wavelet provides an example of a wavelet such that the corresponding GMRA is not an MRA, and the multiplicity function m is not constant. The function m for the Journé wavelet is explicitly worked out in [8] (and see also Example 13 below).

In this paper, we ask what functions can arise as multiplicity functions, and our main result is the following theorem.

Theorem 3. *Suppose $c \in \mathbb{N}$ and $m: \hat{\Gamma} \rightarrow \{0, 1, \dots, c\}$ is a Borel function which satisfies the consistency inequality*

$$(3) \quad m(\omega) \leq \sum_{\alpha^*(\zeta)=\omega} m(\zeta),$$

and define $\tilde{m}: \hat{\Gamma} \rightarrow \{0, 1, \dots, c\}$ by

$$(4) \quad \tilde{m}(\omega) = \sum_{\alpha^*(\zeta)=\omega} m(\zeta) - m(\omega).$$

Suppose that there is a positive integer a satisfying $m(\omega) - \tilde{m}(\omega) \leq a \leq m(\omega)$ for all ω near 1. Then there is a GMRA which has m and \tilde{m} as the associated multiplicity and complementary multiplicity functions.

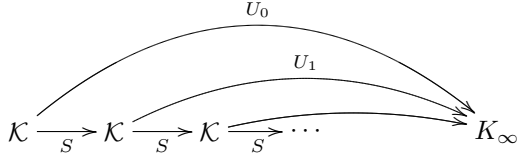
Remark 4. In many examples (such as Examples 13 and 14 below), the multiplicity function m attains its maximum value c throughout a neighborhood of 1, and then $a = c$ satisfies the hypothesis of the theorem.

2. CONSTRUCTION OF A GMRA FROM A PURE ISOMETRY

Our construction is based on the direct-limit construction of multiresolution analyses in [14]. Similar constructions have been used elsewhere, often described as dilations (in [12], for example); the key point of our approach is the emphasis on universal properties of the construction. In this paper our emphasis is on the construction of GMRA's, and in a sequel we plan to discuss further applications of this circle of ideas to the construction of frames and wavelets.

We begin by extending the construction of [14] to accommodate the action of the abstract translation group Γ .

Theorem 5. *Suppose S is an isometry on a Hilbert space \mathcal{K} , and let $(\mathcal{K}_\infty, U_n)$ be the direct limit of the direct system (H_n, T_n) in which each Hilbert space $H_n = \mathcal{K}$ and each $T_n = S$:*



(a) There is a unitary operator S_∞ on \mathcal{K}_∞ such that

$$S_\infty(U_n h) = U_n(S h) = U_{n-1} h \quad \text{for every } h \in H_n := \mathcal{K}.$$

(b) The subspaces $V_n := U_n \mathcal{K}$ of \mathcal{K}_∞ satisfy

(i) $V_n \subset V_{n+1}$;

(ii) $\bigcup_{n=0}^\infty V_n$ is dense in \mathcal{K}_∞ ;

(iii) S_∞ is a unitary isomorphism of V_{n+1} onto V_n .

(c) For $n < 0$, define $V_n := S_\infty^{|n|}(V_0)$. Then $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$ if and only if S is a pure isometry (in the sense that $\bigcap_{n=1}^\infty S^n \mathcal{K} = \{0\}$).

(d) If ρ is a unitary representation of Γ on \mathcal{K} such that

$$(5) \quad S \rho_\gamma = \rho_{\alpha(\gamma)} S \quad \text{for } \gamma \in \Gamma,$$

then there exists a unitary representation π of Γ on \mathcal{K}_∞ such that all the subspaces V_n , for $n \geq 0$, are invariant under π and

$$(6) \quad S_\infty \pi_\gamma = \pi_{\alpha(\gamma)} S_\infty \quad \text{for } \gamma \in \Gamma.$$

Proof. The construction of S_∞ is described on page 37 of [14], and it is proved there that S_∞ is unitary. Since $U_n \mathcal{K} = U_{n+1} S \mathcal{K}$, we have $V_n \subset V_{n+1}$; that the union of the subspaces $V_n = U_n \mathcal{K}$ is dense is a standard property of the direct limit. The equation

$$S_\infty(U_{n+1} h) = U_{n+1}(S h) = U_n h$$

shows that S_∞ is an isomorphism of V_{n+1} onto V_n .

For (c), we notice that for $n < 0$,

$$V_n = S_\infty^{|n|} V_0 = S_\infty^{|n|} U_0 \mathcal{K} = U_0 S^{|n|} \mathcal{K},$$

so

$$\begin{aligned} \bigcap_{n \in \mathbb{Z}} V_n = \{0\} &\iff \bigcap_{k=1}^\infty V_{-k} = \{0\} \\ &\iff \bigcap_{k=1}^\infty U_0 S^k \mathcal{K} = \{0\} \\ &\iff \bigcap_{k=1}^\infty S^k \mathcal{K} = \{0\}. \end{aligned}$$

Since $\bigcap_{k=1}^\infty S^k \mathcal{K}$ is the largest subspace of \mathcal{K} on which S is unitary, this proves (c).

The intertwining relation (5) implies that

$$\begin{array}{ccccccc}
 \mathcal{K} & \xrightarrow{S} & \mathcal{K} & \xrightarrow{S} & \mathcal{K} & \xrightarrow{S} & \cdots & \cdots & \longrightarrow & \mathcal{K}_\infty \\
 \downarrow \rho_\gamma & & \downarrow \rho_{\alpha(\gamma)} & & \downarrow \rho_{\alpha^2(\gamma)} & & & & & \downarrow \pi_\gamma \\
 \mathcal{K} & \xrightarrow{S} & \mathcal{K} & \xrightarrow{S} & \mathcal{K} & \xrightarrow{S} & \cdots & \cdots & \longrightarrow & \mathcal{K}_\infty,
 \end{array}$$

is a commutative diagram of isometries, and hence the universal property of the direct limit gives the existence of a unique isometry π_γ such that $\pi_\gamma \circ U_n = U_n \circ \rho_{\alpha^n(\gamma)}$, which implies immediately that $V_n = U_n \mathcal{K}$ is invariant under π_γ . Uniqueness implies that $\pi_{-\gamma}$ is an inverse for π_γ and that $\pi_{\gamma\tau} = \pi_\gamma \pi_\tau$, so π is a unitary representation of Γ . Finally, we have

$$\begin{aligned}
 S_\infty \pi_\gamma U_n &= S_\infty U_n \rho_{\alpha^n(\gamma)} = U_n S \rho_{\alpha^n(\gamma)} \\
 &= U_n \rho_{\alpha^{n+1}(\gamma)} S = \pi_{\alpha(\gamma)} U_n S \\
 &= \pi_{\alpha(\gamma)} S_\infty U_n,
 \end{aligned}$$

which establishes (6). \square

Corollary 6. *If S is a pure isometry on \mathcal{K} , then the subspaces V_n of \mathcal{K}_∞ form a generalized multiresolution analysis with respect to $\pi: \Gamma \rightarrow U(\mathcal{K}_\infty)$ and $\delta := S_\infty^{-1}$.*

3. PROOF OF THE MAIN THEOREM

Let $m: \widehat{\Gamma} \rightarrow \mathbb{Z}$ be a Borel function such that $0 \leq m(\omega) \leq c$ for all ω , and for $0 \leq i \leq c$ write $\sigma_i := \{\omega \in \widehat{\Gamma} : m(\omega) \geq i\}$. A *filter relative to m and α^** is a matrix-valued Borel function $H = [h_{i,j}]: \widehat{\Gamma} \rightarrow M_c(\mathbb{C})$ such that $h_{i,j}$ vanishes outside σ_j and

$$(7) \quad \sum_{\alpha^*(\zeta)=\omega} H(\zeta) H^*(\zeta) = N \Sigma(\omega) \quad \text{for almost all } \omega \in \widehat{\Gamma},$$

where $\Sigma(\omega)$ is the diagonal matrix with entries $\chi_{\sigma_i}(\omega)$. Such a filter is *low-pass of rank a* if H is continuous near 1 and $H(1)$ has block form

$$H(1) = \begin{pmatrix} N^{1/2} 1_a & 0 \\ 0 & 0 \end{pmatrix},$$

where 1_a denotes the 1×1 identity matrix.

Crucial for our argument is that, when m satisfies the hypotheses of Theorem 3, there are always compatible low-pass filters.

Proposition 7. *Suppose that the positive integer a satisfies*

$$(8) \quad m(\omega) - \tilde{m}(\omega) \leq a \leq m(\omega) \quad \text{for all } \omega \text{ near } 1 \text{ in } \widehat{\Gamma}.$$

Then there is a filter H relative to m and α^ which is low-pass of rank a .*

Proof. We begin by writing the filter equations (7) in the form

$$(9) \quad \sum_j \sum_{\zeta \in \ker \alpha^*} h_{i,j}(\omega\zeta) \overline{h_{i',j}(\omega\zeta)} = N \delta_{i,i'} \chi_{\sigma_i}(\alpha^*(\omega)).$$

We choose a Borel cross-section s for α^* , and write $C = s(\widehat{\Gamma})$; then every element in $\widehat{\Gamma}$ can be written in a unique way as $\omega\zeta$ for some $\omega \in C$ and $\zeta \in \ker \alpha^*$, and to build a filter it suffices to construct c functions

$$h_i(\omega) = \{h_{i,j}(\omega\zeta) : 1 \leq j \leq m(\omega), \zeta \in \ker \alpha^*\}$$

from C to \mathbb{C}^{cN} such that the $h_{i,j}(\omega\zeta)$ vanish unless $\omega\zeta \in \sigma_j$ and (9) holds for every $\omega \in C$. Equation (9) is equivalent to asking that the vectors $(h_i(\omega))$ in \mathbb{C}^{cN} are orthogonal of norm $N^{1/2} \chi_{\sigma_i}(\alpha^*(\omega))$.

Let U be a neighborhood of 1 such that (8) holds for $\omega \in U$, and shrink U to ensure that the sets $\{U\zeta : \zeta \in \ker \alpha^*\}$ are pairwise disjoint.

From the continuity of α^* , there exist neighborhoods V and W of the identity, both contained in U , such that α^* maps W onto V , and since $U \cap U\zeta = \emptyset$ for $\zeta \neq 1$, α^* is a homeomorphism of W onto V . We may suppose without loss of generality that $W \subset C$.

For $\omega \in W$ and $i \leq a$, we define $h_i(\omega)$ by

$$h_{i,j}(\omega\zeta) = \begin{cases} \sqrt{N} & \text{if } i = j \leq a \text{ and } \zeta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This will ensure that our filter is continuous at 1 and is low-pass of rank a . For $i > m(\alpha^*(\omega))$, we must set $h_i(\omega) = 0$ for all ω . For $a < i \leq m(\alpha^*(\omega))$, the entries $h_{i,j}(\omega)$ must be 0 for $\omega \in W$. We also need to take $h_{i,j}(\omega\zeta) = 0$ unless $\omega\zeta \in \sigma_j$, which is equivalent to $m(\omega\zeta) \geq j$. Thus for each ζ , there are $m(\omega\zeta)$ j s for which $h_{i,j}(\omega\zeta)$ can be non-zero, and hence $\sum_{\zeta \neq 1} m(\omega\zeta)$ potentially non-zero elements.

Since $\omega \in W$ implies $\alpha^*(\omega) \in V$, and since V is contained in U , we have

$$m(\alpha^*(\omega)) - a \leq \tilde{m}(\alpha^*(\omega)) = \sum_{\zeta \in \ker \alpha^*} m(\omega\zeta) - m(\omega) = \sum_{\zeta \in \ker \alpha^*, \zeta \neq 1} m(\omega\zeta).$$

Thus the number of components in $h_i(\omega)$ which can be non-zero is greater than or equal to the required number $m(\alpha^*(\omega)) - a$ of orthogonal vectors $h_i(\omega)$, and it is possible to find such vectors. Since there are only finitely many possible sets of values of $m(\alpha^*(\omega))$ and $m(\omega\zeta)$, and we can use the

same vectors for $h_i(\omega)$ when these values are all the same, we can find simple functions h_i with the required properties.

Defining the vectors $h_i(\omega)$ for $\omega \in C \setminus W$ is easier, since now we just need to define $h_i(\omega)$ for $i \leq m(\alpha^*(\omega))$, and we have $\sum_{\zeta \in \ker \alpha^*} m(\omega\zeta) \geq m(\alpha^*(\omega))$ non-zero entries to play with. \square

Our main technical result shows that low-pass filters give rise to pure isometries. In an attempt to clarify our overall strategy, we will postpone the proof of this result till the next section.

Theorem 8. *Suppose that $m: \widehat{\Gamma} \rightarrow \{0, 1, \dots, c\}$ is Borel, and that H is a filter relative to m and α^* . Let \mathcal{K} be the Hilbert space defined by*

$$(10) \quad \mathcal{K} = \bigoplus_i L^2(\sigma_i),$$

and define an operator S_H on \mathcal{K} by

$$(11) \quad (S_H f)(\omega) = H^t(\omega) f(\alpha^*(\omega)).$$

Then S_H is an isometry on \mathcal{K} . If the filter H is low-pass of some rank a between 1 and c , then S_H is a pure isometry.

We now have all the ingredients to prove our main theorem.

Proof of Theorem 3. Proposition 7 gives us a low-pass filter H of rank a . Let S_H be the pure isometry on $\mathcal{K} = \bigoplus_i L^2(\sigma_i)$ discussed in Theorem 8. Define a representation ρ of Γ on \mathcal{K} by

$$\rho_\gamma(f)(\omega) = \omega(\gamma) f(\omega).$$

Then

$$\begin{aligned} S_H(\rho_\gamma(f))(\omega) &= H^t(\omega) \rho_\gamma(f)(\alpha^*(\omega)) \\ &= H^t(\omega) \alpha^*(\omega)(\gamma) f(\alpha^*(\omega)) \\ &= H^t(\omega) \omega(\alpha(\gamma)) f(\alpha^*(\omega)) \\ &= \rho_{\alpha(\gamma)}(S_H f)(\omega). \end{aligned}$$

Now applying Theorem 5 gives us a direct limit Hilbert space $(\mathcal{K}_\infty, U_n)$, a representation $\pi: \Gamma \rightarrow U(\mathcal{K}_\infty)$, and a dilation operator $\delta = S_\infty^{-1}$, such that $\{V_n\} := \{U_n \mathcal{K}\}$ is a GMRA relative to π and δ .

The canonical embedding U_0 is an isomorphism of $\mathcal{K} = \bigoplus_i L^2(\sigma_i)$ onto V_0 which intertwines ρ and $\pi|_{V_0}$, so this GMRA has multiplicity function m . It follows from equation (1) that \tilde{m} must be the complementary multiplicity function. \square

This completes the proof of Theorem 3, modulo our obligation to provide a proof of Theorem 8.

4. LOW-PASS FILTERS AND PURE ISOMETRIES

In this section, we pay our debts by proving Theorem 8.

Since $H^t(\omega)_{ij} = h_{j,i}(\omega)$ has support in σ_i , $S_H f$ belongs to \mathcal{K} . A computation shows that the adjoint of S_H is given by

$$(S_H^* f)(\omega) = \frac{1}{N} \sum_{\alpha^*(\zeta) = \omega} \overline{H(\zeta)} f(\zeta),$$

and it is then easy to check that $S_H^* S_H f = f$ when f belongs to \mathcal{K} . So S_H is an isometry.

From here we assume that H is low-pass, and aim to prove that S_H is a pure isometry, or in other words that $\bigcap_{n=0}^{\infty} S_H^n \mathcal{K} = \{0\}$. We assume that this is not true, and look for a contradiction. Since every non-zero Hilbert space contains a unit vector, we can find a unit vector f in $\bigcap_{n=0}^{\infty} S_H^n \mathcal{K}$. To arrive at our contradiction, we consider the sequence $f_n := S_H^{*n} f$; since S_H is unitary on $\bigcap_{n=0}^{\infty} S_H^n \mathcal{K}$ with inverse S_H^* , $\{f_n\}$ is a sequence of unit vectors in $\bigcap_{n=0}^{\infty} S_H^n \mathcal{K}$.

We will need to deal with the powers of S_H and S_H^* , and simple induction arguments yield the following explicit formulas:

$$\begin{aligned} (S_H^n f)(\omega) &= \left(\prod_{k=0}^{n-1} H^t(\alpha^{*k}(\omega)) \right) f(\alpha^{*n}(\omega)), \text{ and} \\ (S_H^{*n} f)(\omega) &= \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta) = \omega} \left(\prod_{k=n-1}^0 \overline{H(\alpha^{*k}(\zeta))} \right) f(\zeta). \end{aligned}$$

We view elements $g \in \mathcal{K}$ as functions from $\widehat{\Gamma}$ to \mathbb{C}^c whose i th coordinate g_i has support in σ_i , and write $\|g(\zeta)\|$ for the norm of the vector $g(\zeta) \in \mathbb{C}^c$. Then for each $g \in \mathcal{K}$, the function $\zeta \mapsto \|g(\zeta)\|^2$ is integrable on $\widehat{\Gamma}$. We want to identify the integrable functions associated to our sequence $f_n = S_H^{*n} f$. A crucial step in the calculation is the following extension of the filter identity (7). In the following formula (12) it is crucial that the products are interpreted in the correct order: the middle terms, for example, are the ones for which $k = 0$ and $l = 0$.

Lemma 9. *For every $n \geq 1$, we have*

$$\begin{aligned} (12) \quad \sum_{\alpha^{*n}(\zeta) = \omega} \left(\prod_{k=n-1}^0 H(\alpha^{*k}(\zeta)) \right) \left(\prod_{l=0}^{n-1} H^*(\alpha^{*l}(\zeta)) \right) \\ = N^n \Sigma(\omega) \text{ for almost all } \omega. \end{aligned}$$

Proof. For $n = 1$ we recover the usual filter identity (7). Suppose (12) is true for $n \geq 1$. Then

$$\begin{aligned}
& \sum_{\alpha^{*(n+1)}(\zeta)=\omega} \left(\prod_{k=n}^0 H(\alpha^{*k}(\zeta)) \right) \left(\prod_{l=0}^n H^*(\alpha^{*l}(\zeta)) \right) \\
&= \sum_{\alpha^{*n}(\eta)=\omega} \sum_{\alpha^*(\zeta)=\eta} \left(\prod_{k=n}^1 H(\alpha^{*k}(\zeta)) \right) \\
&\quad H(\zeta) H^*(\zeta) \left(\prod_{l=1}^n H^*(\alpha^{*l}(\zeta)) \right) \\
&= \sum_{\alpha^{*n}(\eta)=\omega} \left(\prod_{k=n-1}^0 H(\alpha^{*k}(\eta)) \right) \\
&\quad \left(\sum_{\alpha^*(\zeta)=\eta} H(\zeta) H^*(\zeta) \right) \left(\prod_{l=0}^{n-1} H^*(\alpha^{*l}(\eta)) \right) \\
&= \sum_{\alpha^{*n}(\eta)=\omega} \left(\prod_{k=n-1}^0 H(\alpha^{*k}(\eta)) \right) N\Sigma(\eta) \left(\prod_{l=0}^{n-1} H^*(\alpha^{*l}(\eta)) \right).
\end{aligned}$$

Now notice that for each i , both the (i, j) entry $h_{i,j}(\eta)$ in $H(\eta)$ and the (j, i) entry in $H^*(\eta)$ vanish unless $\eta \in \sigma_j$, in which case the (j, j) entry in $\Sigma(\eta)$ is 1. So the $\Sigma(\eta)$ in the middle has no effect, and we deduce from the inductive hypothesis that the last expression reduces to $N(N^n\Sigma(\omega)) = N^{n+1}\Sigma(\omega)$. \square

Lemma 10. *For almost all ω we have*

$$(13) \quad \|f_n(\omega)\|^2 = \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta)=\omega} \|f(\zeta)\|^2.$$

Proof. We write $(v|w)$ for the usual inner product on \mathbb{C}^c . Then since $f \in S_H^n \mathcal{K} = S_H^n S_H^{*n} \mathcal{K}$, we have

$$\begin{aligned}
(14) \quad \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta)=\omega} \|f(\zeta)\|^2 &= \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta)=\omega} (f(\zeta) | f(\zeta)) \\
&= \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta)=\omega} (S_H^n S_H^{*n} f(\zeta) | S_H^n S_H^{*n} f(\zeta))
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta)=\omega} (S_H^n f_n(\zeta) | S_H^n f_n(\zeta)) \\
&= \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta)=\omega} \left(\prod_{l=0}^{n-1} H^t(\alpha^{*l}(\zeta)) f_n(\alpha^{*n}\zeta) \right) \\
&\quad \prod_{k=0}^{n-1} H^t(\alpha^{*k}(\zeta)) f_n(\alpha^{*n}\zeta) \\
&= \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta)=\omega} \left(\left(\prod_{k=n-1}^0 \overline{H(\alpha^{*k}(\zeta))} \right) \right. \\
&\quad \left. \left(\prod_{l=0}^{n-1} H^t(\alpha^{*l}(\zeta)) \right) f_n(\omega) \right) f_n(\omega) \\
&= (\Sigma(\omega) f_n(\omega) | f_n(\omega)),
\end{aligned}$$

where at the last step we used the conjugate of (12). Now we deduce from the original filter equation (7) that

$$h_{i,j}(\zeta) \neq 0 \implies \alpha^*(\zeta) \in \sigma_i.$$

Then the i th entry $[f_n(\omega)]_i$ satisfies

$$[f_n(\omega)]_i = \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta)=\omega} \sum_{j=i}^c \overline{h_{i,j}(\alpha^{*(n-1)}(\zeta))} \left[\left(\prod_{k=n-2}^0 \overline{H(\alpha^{*k}(\zeta))} \right) f(\zeta) \right]_j,$$

and hence vanishes unless $\alpha^*(\alpha^{*(n-1)}(\zeta)) = \omega$ is in σ_i . Thus $\Sigma(\omega) f_n(\omega) = f_n(\omega)$, and the calculation (14) gives the result. \square

We can rewrite the formula (13) as

$$(15) \quad \|f_n(\alpha^{*n}(\omega))\|^2 = \frac{1}{N^n} \sum_{\eta \in \ker \alpha^{*n}} \|f(\omega\eta)\|^2,$$

and we claim that the right-hand side $X_n(\omega)$ of (15) is the expectation $E(\|f\|^2 | \mathcal{B}_n)$ of f with respect to the subalgebra $\mathcal{B}_n := (\alpha^*)^{-n}(\mathcal{B})$ of the Borel σ -algebra \mathcal{B} . To see this, we note that \mathcal{B}_n is the σ -algebra of Borel sets which are invariant under the action of $\ker \alpha^{*n}$, so that X_n is certainly

\mathcal{B}_n -measurable, and for $B \in \mathcal{B}_n$, we have

$$\begin{aligned} \int_B X_n(\omega) d\omega &= \frac{1}{N^n} \sum_{\zeta \in \ker \alpha^{*n}} \int_{B\zeta^{-1}} \|f(\omega)\|^2 d\omega \\ &= \frac{1}{N^n} \sum_{\zeta \in \ker \alpha^{*n}} \int_B \|f(\omega)\|^2 d\omega \\ &= \int_B \|f(\omega)\|^2 d\omega; \end{aligned}$$

in other words, X_n has the properties which characterise $E(\|f\|^2 | \mathcal{B}_n)$ (see the observation at the top of page 18 of [17]), and hence $X_n = E(\|f\|^2 | \mathcal{B}_n)$. Further, if $B \in \mathcal{B}_{n+1} \subset \mathcal{B}_n$, then we have

$$\int_B X_{n+1}(\omega) d\omega = \int_B \|f(\omega)\|^2 d\omega = \int_B X_n(\omega) d\omega,$$

and hence $X_{n+1} = E(X_n | \mathcal{B}_{n+1})$. Thus the family $\{X_n\}$ satisfies the hypotheses of the reverse martingale convergence theorem (as in [10, Theorem 10.6.1], for example), and we can deduce from that theorem that X_n converges almost everywhere to the expectation $E(\|f\|^2 | \mathcal{B}_\infty)$ associated to $\mathcal{B}_\infty := \bigcap_{n \geq 1} \mathcal{B}_n$.

To identify $E(\|f\|^2 | \mathcal{B}_\infty)$, we need the following standard lemma.

Lemma 11. *If $B \in \mathcal{B}_\infty$, then $\lambda(B)$ is either 0 or 1.*

Proof. Notice that B is invariant under multiplication by elements of $\ker \alpha^{*n}$ for every $n \geq 1$. Suppose $\gamma \in \Gamma \setminus \{0\}$. Since $\bigcup_{n \geq 1} \ker \alpha^{*n}$ is dense in $\widehat{\Gamma}$, two characters of $\widehat{\Gamma}$ which agree on $\bigcup_{n \geq 1} \ker \alpha^{*n}$ must agree on all of $\widehat{\Gamma}$. Thus there exist n and $\zeta \in \ker \alpha^{*n}$ such that $\zeta(\gamma) \neq 1$. Then the Fourier coefficients of the characteristic function χ_B satisfy

$$\begin{aligned} \widehat{\chi}_B(\gamma) &= \int_{\widehat{\Gamma}} \chi_B(\omega) \omega(\gamma) d\omega = \int_{\widehat{\Gamma}} \chi_B(\zeta\omega) (\zeta\omega)(\gamma) d\omega \\ &= \zeta(\gamma) \int_{\widehat{\Gamma}} \chi_{\zeta^{-1}B}(\omega) \omega(\gamma) d\omega = \zeta(\gamma) \int_{\widehat{\Gamma}} \chi_B(\omega) \omega(\gamma) d\omega \\ &= \zeta(\gamma) \widehat{\chi}_B(\gamma), \end{aligned}$$

and hence $\widehat{\chi}_B(\gamma) = 0$. Thus $\widehat{\chi}_B(\gamma) = 0$ for every non-zero γ , and χ_B is either 0 or 1 in $L^1(\widehat{\Gamma})$, which implies the result. \square

So $\mathcal{B}_\infty = \{B \in \mathcal{B} : \lambda(B) = 0 \text{ or } 1\}$, and the expectation $E(\|f\|^2 | \mathcal{B}_\infty)$ is the constant function $\int \|f(\omega)\|^2 d\omega$. Since our f is a unit vector, we have now proved the following Proposition.

Proposition 12. *For almost all $\omega \in \widehat{\Gamma}$, we have*

$$\|f_n(\alpha^{*n}(\omega))\|^2 = \frac{1}{N^n} \sum_{\eta \in \ker \alpha^{*n}} \|f(\omega\eta)\|^2 \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

We are now ready to get the contradiction which will prove that S_H is a pure isometry. We fix $\delta > 0$. We view H as a block matrix $H = (H_{i,j})$ for the decomposition $\mathbb{C}^c = \mathbb{C}^a \oplus \mathbb{C}^{c-a}$, and choose a neighborhood V of the identity such that for each $\omega \in V$ we have

$$\|H_{1,1}(\omega) - \sqrt{N}1_a\| < \delta \quad \text{and} \quad \|H_{i,j}(\omega)\| < \delta \quad \text{for } (i,j) \neq (1,1).$$

Next, we choose a neighborhood W of the identity such that W , $\alpha^*(W)$ and $\alpha^{*2}(W)$ are all contained in V .

By Egorov's theorem, there exists a set E whose complement has measure less than $\lambda(W)/4$ and an integer M such that, for all $n \geq M$ and all $\omega \in E$,

$$1 - \delta < \|f_n(\alpha^{*n}(\omega))\| < 1 + \delta.$$

Since α^* is measure-preserving in the sense that $\lambda((\alpha^*)^{-1}(E)) = \lambda(E)$, the set

$$A := W \cap (\alpha^*)^{-M}(E) \cap (\alpha^*)^{-(M+1)}(E) \cap (\alpha^*)^{-(M+2)}(E)$$

has positive measure. It then follows that $\alpha^{*M}(\omega)$, $\alpha^{*(M+1)}(\omega)$, and $\alpha^{*(M+2)}(\omega)$ all belong to $V \cap E$ for every $\omega \in A$.

We now fix $\omega \in A$, write $v = ([v]_1, [v]_2)$ for the block decomposition of $v \in \mathbb{C}^c$, and make lower and upper estimates for $\|[f_{M+1}(\alpha^{*(M+1)}(\omega))]_1\|$. For the lower estimate, we observe that

$$\begin{aligned} \|[f_{M+1}(\alpha^{*(M+1)}(\omega))]_2\| &\leq \sum_{j=1}^2 \|H_{2,j}^t(\alpha^{*(M+1)}(\omega)) [f_{M+2}(\alpha^{*(M+2)}(\omega))]_j\| \\ &\leq 2\delta(1 + \delta), \end{aligned}$$

and deduce that

$$\begin{aligned} \|[f_{M+1}(\alpha^{*(M+1)}(\omega))]_1\| &\geq \|f_{M+1}(\alpha^{*(M+1)}(\omega))\| - \|[f_{M+1}(\alpha^{*(M+1)}(\omega))]_2\| \\ &\geq 1 - \delta - 2\delta(1 + \delta). \end{aligned}$$

For the upper estimate, we write

$$(16) \quad [f_M(\alpha^{*M}(\omega))]_1 = \sum_{j=1}^2 H_{1,j}(\alpha^{*M}(\omega)) [f_{M+1}(\alpha^{*(M+1)}(\omega))]_j,$$

rewrite the first summand on the right as

$$N^{1/2} [f_{M+1}(\alpha^{*(M+1)}(\omega))]_1 + (H_{1,1}(\alpha^{*M}(\omega)) - N^{1/2}1_a) [f_{M+1}(\alpha^{*(M+1)}(\omega))]_1,$$

and turn (16) round to get the estimate

$$\begin{aligned} N^{1/2} \| [f_{M+1}(\alpha^{*(M+1)}(\omega))]_1 \| &\leq 1 + \delta + 2\delta(1 + \delta) \\ &= (1 + \delta)(1 + 2\delta). \end{aligned}$$

Combining the upper and lower estimates shows that for every $\delta > 0$ we must have

$$1 - \delta - 2\delta(1 + \delta) \leq N^{-1/2}(1 + \delta)(1 + 2\delta),$$

which we can see is impossible by letting $\delta \rightarrow 0$.

This completes the proof of Theorem 8.

5. EXAMPLES

We give several examples showing how low-pass filters can occur for different values of a in Proposition 7 and Theorem 8 even for the same multiplicity function m . For simplicity of notation, we identify the multiplicative group \mathbb{T} with the additive set \mathbb{R}/\mathbb{Z} , where we choose the coset representatives of \mathbb{R}/\mathbb{Z} in \mathbb{R} to be $[-1/2, 1/2)$. This agrees with the notation in [3].

Example 13. Consider the multiplicity function for dilation by 2 in \mathbb{R} corresponding to the Journé wavelet, previously studied in [8], [2] and [3]. The multiplicity function for this minimally supported frequency wavelet is given by

$$m(x) = \begin{cases} 2 & \text{if } x \in [-\frac{1}{7}, \frac{1}{7}) \\ 1 & \text{if } x \in \pm[\frac{1}{7}, \frac{2}{7}) \cup \pm[\frac{3}{7}, \frac{1}{2}) \\ 0 & \text{otherwise.} \end{cases}$$

Filters which give rise to the Journé wavelet and satisfy the low-pass condition of rank $a = 1$ were constructed in [8]. However, for this m the number $a = 2$ also satisfies the conditions of Proposition 7, and we can also find filters which satisfy the low-pass condition of rank $a = 2$. Indeed,

$$h_{1,1} = \sqrt{2}\chi_{[-\frac{2}{7}, -\frac{1}{4}) \cup [-\frac{1}{7}, \frac{1}{7}) \cup [\frac{1}{4}, \frac{2}{7})}, \quad h_{1,2} = h_{2,1} = 0, \quad \text{and} \quad h_{2,2} = \sqrt{2}\chi_{[-\frac{1}{14}, \frac{1}{14})}$$

have the required properties. If we consider the 2×2 matrix $H = [h_{i,j}]$ as in Theorem 8 and view the infinite product $\prod_{j=1}^{\infty} [2^{-1/2}H(2^{-j}x)]$ as a 2×2 matrix with entries in $L^{\infty}(\mathbb{R})$, we find that it is a diagonal matrix with diagonal entries

$$\hat{\phi}_1 := \chi_{[-\frac{4}{7}, -\frac{1}{2}) \cup [-\frac{2}{7}, \frac{2}{7}) \cup [\frac{1}{2}, \frac{4}{7})} \quad \text{and} \quad \hat{\phi}_2 := \chi_{[-\frac{1}{7}, \frac{1}{7})}.$$

Then the shift invariant subspace of $L^2(\mathbb{R})$ generated by the inverse Fourier transforms ϕ_1 and ϕ_2 has multiplicity function

$$m' = \chi_{[-\frac{1}{2}, -\frac{3}{7}) \cup [-\frac{2}{7}, \frac{2}{7}) \cup [\frac{3}{7}, \frac{1}{2})},$$

which is a degenerate version of the original multiplicity function m . Thus one does not have an analogue of Theorem 3.4 of [2] in this situation, and one must rely on the direct limit Hilbert space instead of $L^2(\mathbb{R})$ to construct our GMRA.

Example 14. The Journé wavelet set corresponds to the case $n = 1$ in [4, §4, Example 1]. If we consider the case $n = 2$ in this family of examples, we obtain the wavelet set $W = \pm[\frac{4}{15}, \frac{1}{2}) \cup \pm[4, \frac{64}{15})$ and the multiplicity function given by

$$m(x) = \begin{cases} 3 & \text{if } x \in [-\frac{1}{15}, \frac{1}{15}) \\ 2 & \text{if } x \in \pm[\frac{1}{15}, \frac{2}{15}) \\ 1 & \text{if } x \in \pm[\frac{2}{15}, \frac{4}{15}) \cup \pm[\frac{7}{15}, \frac{1}{2}) \\ 0 & \text{otherwise.} \end{cases}$$

Thus $m(x) - \tilde{m}(x) = 2$ in a neighborhood of $x = 0$, and $a = 1$ does not satisfy the hypothesis of Proposition 7 or Theorem 8. However, the values $a = 2$ and $a = 3$ satisfy the hypothesis of Proposition 7, and we can construct filters corresponding to these two values.

For $a = 2$, we can take

$$H = [h_{i,j}] = \sqrt{2} \begin{pmatrix} \chi_{[-\frac{2}{15}, \frac{2}{15}) \cup \pm[\frac{1}{4}, \frac{4}{15})} & 0 & 0 \\ 0 & \chi_{[-\frac{1}{15}, \frac{1}{15})} & 0 \\ \chi_{\pm(\frac{7}{15}, \frac{1}{2}]} & 0 & 0 \end{pmatrix},$$

which is low-pass of rank $a = 2$ for dilation by 2. A different low-pass filter of rank $a = 2$ for this m is

$$H = [h_{i,j}] = \sqrt{2} \begin{pmatrix} \chi_{[-\frac{2}{15}, \frac{2}{15}) \cup \pm[\frac{1}{4}, \frac{4}{15})} & 0 & 0 \\ 0 & \chi_{[-\frac{1}{30}, \frac{1}{30})} & \chi_{\pm[\frac{1}{30}, \frac{1}{15})} \\ \chi_{\pm(\frac{13}{30}, \frac{1}{2}]} & 0 & 0 \end{pmatrix}.$$

The following family satisfies the low-pass condition of rank $a = 3$ for dilation by 2, with the same choice of m :

$$H = [h_{i,j}] = \sqrt{2} \begin{pmatrix} \chi_{[-\frac{2}{15}, \frac{2}{15}) \cup \pm[\frac{1}{4}, \frac{4}{15})} & 0 & 0 \\ 0 & \chi_{[-\frac{1}{15}, \frac{1}{15})} & 0 \\ 0 & 0 & \chi_{[-\frac{1}{30}, \frac{1}{30})} \end{pmatrix}.$$

In all of these cases, the infinite product $\prod_{j=1}^{\infty} [2^{-1/2} H(2^{-j}x)]$ gives functions whose inverse Fourier transforms in $L^2(\mathbb{R})$ generate a shift invariant subspace with multiplicity function a degenerate form of the original m .

REFERENCES

- [1] L. Baggett, A. Carey, W. Moran and P. Oehring, *General existence theorems for orthonormal wavelets, an abstract approach*, Publ. Res. Inst. Math. Sci. Kyoto Univ., **31** (1995), 95–111.
- [2] L. W. Baggett, J. E. Courter and K. D. Merrill, *The construction of wavelets from generalized conjugate mirror filters in $L^2(\mathbb{R}^n)$* , Appl. Comput. Harmon. Anal., **13** (2002), 201–223.
- [3] L. W. Baggett, P. E. T. Jorgensen, K. D. Merrill and J. A. Packer, *Construction of Parseval wavelets from redundant filter systems*, J. Math. Phys., **46** (2005), #083502, 1–28.
- [4] L. W. Baggett, H. A. Medina and K. D. Merrill, *Generalized multiresolution analyses and a construction procedure for all wavelet sets in \mathbb{R}^n* , J. Fourier Anal. Appl., **5** (1999), 563–573.
- [5] L. W. Baggett and K. D. Merrill, *Abstract harmonic analysis and wavelets in \mathbb{R}^n* , in The Functional and Harmonic Analysis of Wavelets and Frames, Contemp. Math., vol. 24, Amer. Math. Soc., Providence, 1999, pages 17–27.
- [6] M. Bownik, Z. Rzeszutnik, and D. Speegle, *A characterization of dimension functions of wavelets*, Appl. Comput. Harmon. Anal., **10** (2001), 71–92.
- [7] O. Bratteli and P. E. T. Jorgensen, *Isometries, shifts, Cuntz algebras and multiresolution analyses of scale N* , Integral Equations & Operator Theory, **28** (1997), 382–443.
- [8] J. Courter, *Construction of dilation- d wavelets*, in The Functional and Harmonic Analysis of Wavelets and Frames, Contemp. Math., vol. 247, Amer. Math. Soc., Providence, 1999, pages 183–205.
- [9] S. Dahlke, *Multiresolution analysis and wavelets on locally compact abelian groups*, Wavelets, Images, and Surface Fittings, A.K. Peters, Wellesley, 1994, pages 141–156.
- [10] R. M. Dudley, *Real Analysis and Probability*, Cambridge Univ. Press, 2002.
- [11] D. E. Dutkay and P. E. T. Jorgensen, *Hilbert spaces built on a similarity and on dynamical renormalization*, J. Math. Phys., **47** (2006), #053504, 1–20.
- [12] D. E. Dutkay and P. E. T. Jorgensen, *Martingales, endomorphisms and covariant systems of operators on Hilbert space*, J. Operator Theory, **58** (2007), 269–310.
- [13] P. E. T. Jorgensen, *Analysis and probability: wavelets, signals, fractals*, Graduate Texts in Math., vol. 234, Springer, New York, 2006.
- [14] N. S. Larsen and I. Raeburn, *From filters to wavelets via direct limits*, in Operator Theory, Operator Algebras and Applications, Contemp. Math., vol. 414, Amer. Math. Soc., Providence, 2006, pages 35–40.
- [15] G. W. Mackey, *Induced representations of locally compact groups. II. The Frobenius reciprocity theorem*, Ann. of Math., **58** (1953), 193–221.
- [16] S. G. Mallat, *Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$* , Trans. Amer. Math. Soc., **315** (1989), 69–87.
- [17] K. Petersen, *Ergodic Theory*, Cambridge Univ Press, 1983.

LAWRENCE BAGGETT
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COLORADO, BOULDER
COLORADO 80309, USA
E-mail address: `baggett@euclid.colorado.edu`

NADIA S. LARSEN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSLO, BLINDERN
NO-0316 OSLO, NORWAY
E-mail address: `nadiasl@math.uio.no`

KATHY MERRILL
DEPARTMENT OF MATHEMATICS
COLORADO COLLEGE, COLORADO SPRINGS
COLORADO, 80903, USA
E-mail address: `kmerrill@coloradocollege.edu`

JUDITH PACKER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COLORADO, BOULDER
COLORADO 80309, USA
E-mail address: `packer@euclid.colorado.edu`

IAIN RAE BURN
SCHOOL OF MATHEMATICS AND APPLIED STATISTICS
UNIVERSITY OF WOLLONGONG
NSW 2522, AUSTRALIA
E-mail address: `raeburn@uow.edu.au`