CLUSTER ALGEBRAS, QUIVER REPRESENTATIONS AND TRIANGULATED CATEGORIES

BERNHARD KELLER

Abstract. This is an introduction to some aspects of Fomin-Zelevinsky's cluster algebras and their links with the representation theory of quivers and with Calabi-Yau triangulated categories. It is based on lectures given by the author at summer schools held in 2006 (Bavaria) and 2008 (Jerusalem). In addition to by now classical material, we present the outline of a proof of the periodicity conjecture for pairs of Dynkin diagrams (details will appear elsewhere) and recent results on the interpretation of mutations as derived equivalences.

Contents

1. Introduction

1.1. **Context.** Cluster algebras were invented by S. Fomin and A. Zelevinsky [51] in the spring of the year 2000 in a project whose aim it was to develop a combinatorial approach to the results obtained by G. Lusztig concerning total positivity in algebraic groups [104] on the one hand and canonical bases in quantum groups [103] on the other hand (let us stress

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that canonical bases were discovered independently and simultaneously by M. Kashiwara [85]). Despite great progress during the last few years [53] [16] [56], we are still relatively far from these initial aims. Presently, the best results on the link between cluster algebras and canonical bases are probably those of C. Geiss, B. Leclerc and J. Schröer [65] [66] [63] [62] [64] but even they cannot construct canonical bases from cluster variables for the moment. Despite these difficulties, the theory of cluster algebras has witnessed spectacular growth thanks notably to the many links that have been discovered with a wide range of subjects including

- Poisson geometry [70] [71] ...,
- integrable systems [55] ...,
- higher Teichmüller spaces [44] [45] [46] [47] ...,
- combinatorics and the study of combinatorial polyhedra like the Stasheff associahedra [35] [34] [100] [49] [109] [50] ...,
- commutative and non commutative algebraic geometry, in particular the study of stability conditions in the sense of Bridgeland [21] [23] [20] [24], Calabi-Yau algebras [72] [36], Donaldson-Thomas invariants [121] [96] [97] [99] ...,
- and last not least the representation theory of quivers and finite-dimensional algebras, cf. for example the surveys [9] [112] [114].

We refer to the introductory papers [54] [127] [129] [130] [131] and to the cluster algebras portal [48] for more information on cluster algebras and their links with other parts of mathematics.

The link between cluster algebras and quiver representations follows the spirit of categorification: One tries to interpret cluster algebras as combinatorial (perhaps K-theoretic) invariants associated with categories of representations. Thanks to the rich structure of these categories, one can then hope to prove results on cluster algebras which seem beyond the scope of the purely combinatorial methods. It turns out that the link becomes especially beautiful if we use a *triangulated categories* constructed from categories of quiver representations.

1.2. **Contents.** We start with an informal presentation of Fomin-Zelevinsky's classification theorem and of the cluster algebras (without coefficients) associated with Dynkin diagrams. Then we successively introduce quiver mutations, the cluster algebra associated with a quiver, and the cluster algebra with coefficients associated with an 'ice quiver' (a quiver some of whose vertices are frozen). We illustrate cluster algebras with coefficients on a number of examples appearing as coordinate algebras of homogeneous varieties.

Sections 5, 6 and 7 are devoted to the (additive) categorification of cluster algebras. We start by recalling basic notions from the representation theory of quivers. Then we present a fundamental link between indecomposable representations and cluster variables: the Caldero-Chapoton formula. After a brief reminder on derived categories in general, we give the canonical presentation in terms of generators and relations of the derived category of a Dynkin quiver. This yields in particular a presentation for the module category, which we use to sketch Caldero-Chapoton's proof of their formula. Then we introduce the cluster category and survey its many links to the cluster algebra in the finite case. Most of these links are still valid, mutatis mutandis, in the acyclic case, as we see in section 6. Surprisingly enough, one can go even further and categorify interesting classes of cluster algebras using generalizations of the cluster category, which are still triangulated categories and Calabi-Yau of dimension 2. We present this relatively recent theory in section 7. In section 8, we apply it to sketch a proof of the periodicity conjecture for pairs of Dynkin diagrams. In the final section 9, we give an interpretation of quiver mutation in terms of derived equivalences. We use this framework to establish links between various ways of lifting the mutation operation from combinatorics to linear or homological algebra: mutation of cluster-tilting objects, spherical collections and decorated representations.

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2. An informal introduction to cluster-finite cluster algebras

2.1. **The classification theorem.** Let us start with a remark on terminology: a *cluster* is a group of similar things or people positioned or occurring closely together [119], as in the combination 'star cluster'. In French, "star cluster' is translated as 'amas d'étoiles', whence the term 'algèbre amassée" for cluster algebra.

We postpone the precise definition of a cluster algebra to section 3. For the moment, the following description will suffice: A cluster algebra is a commutative $\mathbb{Q}$-algebra endowed with a family of distinguished generators (the *cluster variables*) grouped into overlapping subsets (the *clusters*) of fixed finite cardinality, which are constructed recursively using *mutations*.

The set of cluster variables in a cluster algebra may be finite or infinite. The first important result of the theory is the classification of those cluster algebras where it is finite: the *cluster-finite* cluster algebras. This is the

Classification Theorem 2.1 (Fomin-Zelevinsky [53])**.** *The cluster-finite cluster algebras are parametrized by the finite root systems (like semisimple complex Lie algebras).*

It follows that for each Dynkin diagram Δ, there is a canonical cluster algebra $\mathcal{A}_\Delta$. It turns out that $\mathcal{A}_\Delta$ occurs naturally as a subalgebra of the field of rational functions $\mathbb{Q}(x_1, \ldots, x_n)$, where n is the number of vertices of Δ. Since $\mathcal{A}_\Delta$ is generated by its cluster variables (like any cluster algebra), it suffices to produce the (finite) list of these variables in order to describe $\mathcal{A}_\Delta$. Now for the algebras $\mathcal{A}_\Delta$, the recursive construction via mutations mentioned above simplifies considerably. In fact, it turns out that one can *directly construct the cluster variables* without first constructing the clusters. This is made possible by

2.2. **The knitting algorithm.** The general algorithm will become clear from the following three examples. We start with the simplest non trivial Dynkin diagram:

$$\Delta = A_2 : \ \circ \!-\!\!\!-\!\!\!-\!\!\!-\! \circ \ .$$

We first choose a numbering of its vertices and an orientation of its edges:

$$\vec{\Delta} = \vec{A}_2 : \ 1 \longrightarrow 2 \ .$$

Now we draw the so-called *repetition* (or *Bratteli diagram*) $\mathbb{Z}\vec{\Delta}$ associated with Δ: We first draw the product $\mathbb{Z} \times \vec{\Delta}$ made up of a countable number of copies of Δ (drawn slanted upwards); then for each arrow $\alpha : i \to j$ of Δ, we add a new family of arrows $(n, \alpha^*) : (n, j) \to (n+1, i)$, $n \in \mathbb{Z}$ (drawn slanted downwards). We refer to section 5.5 for the formal definition. Here is the result for $\vec{\Delta} = \vec{A}_2$:

$$\begin{tikzcd}
\cdots & & \circ \arrow[dr] & & \circ \arrow[dr] & & \circ \arrow[dr] & & \circ \arrow[dr] & & \circ & \cdots \\
& \circ \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & &
\end{tikzcd}$$

We will now assign a cluster variable to each vertex of the repetition. We start by assigning x_1 and x_2 to the vertices of the zeroth copy of $\vec{\Delta}$. Next, we construct new variables $x_1', x_2', x_1'', \ldots$ by 'knitting' from the left to the right (an analogous procedure can be used to go from the right to the left).

$$\begin{tikzcd}[column sep=small]
\cdots & & x_2 \arrow[dr] & & x_2' \arrow[dr] & & x_2'' \arrow[dr] & & \circ \arrow[dr] & & \circ & \cdots \\
& x_1 \arrow[ur] & & x_1' \arrow[ur] & & x_1'' \arrow[ur] & & x_1''' \arrow[ur] & & \circ \arrow[ur] & &
\end{tikzcd}$$

To compute x_1', we consider its immediate predecessor x_2, add 1 to it and divide the result by the left translate of x_1', to wit the variable x_1. This yields

$$x_1' = \frac{1+x_2}{x_1}.$$

Similarly, we compute x_2' by adding 1 to its predecessor x_1' and dividing the result by the left translate x_2:

$$x_2' = \frac{1+x_1'}{x_2} = \frac{x_1+1+x_2}{x_1x_2}.$$

Using the same rule for x_1'' we obtain

$$x_1'' = \frac{1+x_2'}{x_1'} = \left(\frac{x_1x_2+x_1+1+x_2}{x_1x_2}\right) / \left(\frac{1+x_2}{x_1}\right) = \frac{1+x_1}{x_2}.$$

Here something remarkable has happened: The numerator $x_1x_2+x_1+1+x_2$ is actually divisible by $1+x_2$ so that the denominator remains a monomial (contrary to what one might expect). We continue with

$$x_2'' = \frac{1+x_1''}{x_2'} = \left(\frac{x_2+1+x_1}{x_2}\right) / \left(\frac{x_1+1+x_2}{x_1x_2}\right) = x_1 \,,$$

a result which is perhaps even more surprising. Finally, we get

$$x_1''' = \frac{1+x_2''}{x_1''} = (1+x_1)/\left(\frac{1+x_1}{x_2}\right) = x_2.$$

Clearly, from here on, the pattern will repeat. We could have computed 'towards the left' and would have found the same repeating pattern. In conclusion, there are the 5 cluster variables x_1, x_2, x_1', x_2' and x_1'' and the cluster algebra $\mathcal{A}_{A_2}$ is the $\mathbb{Q}$-subalgebra (not the subfield!) of $\mathbb{Q}(x_1, x_2)$ generated by these 5 variables.

Before going on to a more complicated example, let us record the remarkable phenomena we have observed:

(1) All denominators of all cluster variables are monomials. In other words, the cluster variables are Laurent polynomials. This *Laurent phenomenon* holds for all cluster variables in all cluster algebras, as shown by Fomin and Zelevinsky [52].

(2) The computation is *periodic* and thus only yields finitely many cluster variables. Of course, this was to be expected by the classification theorem above. In fact, the procedure generalizes easily from Dynkin diagrams to arbitrary trees, and then periodicity characterizes Dynkin diagrams among trees.

(3) *Numerology*: We have obtained 5 cluster variables. Now we have $5 = 2 + 3$ and this decomposition does correspond to a natural partition of the set of cluster variables into the two *initial* cluster variables x_1 and x_2 and the three non initial ones x_1', x_2' and x_1''. The latter are in natural bijection with the positive roots α_1, $\alpha_1+\alpha_2$ and α_2 of the root system of type A_2 with simple roots α_1 and α_2. To see this, it suffices to look at the denominators of the three variables: The denominator $x_1^{d_1}x_2^{d_2}$ corresponds to the root $d_1\alpha_1+d_2\alpha_2$. It was proved by Fomin-Zelevinsky [53] that this generalizes to arbitrary Dynkin diagrams. In particular, the number of cluster variables in the cluster algebra $\mathcal{A}_\Delta$ always equals the sum of the rank and the number of positive roots of Δ.

Let us now consider the example A_3: We choose the following linear orientation:

$$1 \longrightarrow 2 \longrightarrow 3\,.$$

The associated repetition looks as follows:

$$\begin{tikzcd}
\circ \arrow[dr] & & x_3 \arrow[dr] & & x_3' \arrow[dr] & & x_1 \arrow[dr] & & x_3' \arrow[dr] & \\
& x_2 \arrow[ur] \arrow[dr] & & x_2' \arrow[ur] \arrow[dr] & & x_2'' \arrow[ur] \arrow[dr] & & x_2 \arrow[ur] \arrow[dr] & & x_2' \\
x_1 \arrow[ur] & & x_1' \arrow[ur] & & x_1'' \arrow[ur] & & x_1''' \arrow[ur] & & x_3 \arrow[ur] &
\end{tikzcd}$$

The computation of x_1' is as before:

$$x_1' = \frac{1 + x_2}{x_1}.$$

However, to compute x_2', we have to modify the rule, since x_2' has *two* immediate predecessors with associated variables x_1' and x_3. In the formula, we simply take the *product* over all immediate predecessors:

$$x_2' = \frac{1 + x_1' x_3}{x_2} = \frac{x_1 + x_3 + x_2 x_3}{x_2 x_3}.$$

Similarly, for the following variables x_3', x_1'', ... We obtain the periodic pattern shown in the diagram above. In total, we find $9 = 3 + 6$ cluster variables, namely

$$x_1\ ,\ x_2\ ,\ x_3, \frac{1+x_2}{x_1}\ ,\ \frac{x_1+x_3+x_2x_3}{x_1x_2}\ ,\ \frac{x_1+x_1x_2+x_3+x_2x_3}{x_1x_2x_3}\ ,$$
$$\frac{x_1+x_3}{x_2}\ ,\ \frac{x_1+x_1x_2+x_3}{x_2x_3}\ ,\ \frac{1+x_2}{x_3}\ .$$

The cluster algebra $\mathcal{A}_{A_3}$ is the subalgebra of the field $\mathbb{Q}(x_1, x_2, x_3)$ generated by these variables. Again we observe that all denominators are monomials. Notice also that $9 = 3+6$ and that 3 is the rank of the root system associated with A_3 and 6 its number of positive roots. Moreover, if we look at the denominators of the non initial cluster variables (those other than x_1, x_2, x_3), we see a natural bijection with the positive roots

$$\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \alpha_2, \alpha_2+\alpha_3, \alpha_3$$

of the root system of A_3, where α_1, α_2, α_3 denote the three simple roots.

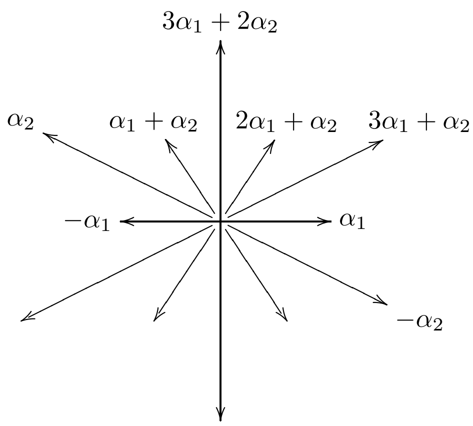
Finally, let us consider the non simply laced Dynkin diagram $\Delta = G_2$:

$$\circ \overset{(3,1)}{\text{---}} \circ\ .$$

The associated Cartan matrix is

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

and the associated root system of rank 2 looks as follows:



We choose an orientation of the valued edge of G_2 to obtain the following valued oriented graph:

$$\vec{\Delta}:\ 1 \xrightarrow{(3,1)} 2\ .$$

Now the repetition also becomes a valued oriented graph

$$\begin{tikzcd}[column sep=small]
& x_2 \arrow[dr, "{1,3}"] & & x_2' \arrow[dr, "{1,3}"] & & x_2'' \arrow[dr, "{1,3}"] & & x_2''' \arrow[dr, "{1,3}"] & & x_2 \\
x_1 \arrow[ur, "{3,1}"] & & x_1' \arrow[ur, "{3,1}"] & & x_1'' \arrow[ur, "{3,1}"] & & x_1''' \arrow[ur, "{3,1}"] & & x_1 \arrow[ur, "{3,1}"] &
\end{tikzcd}$$

The mutation rule is a variation on the one we are already familiar with: In the recursion formula, each predecessor p of a cluster variable x has to be raised to the power indicated by the valuation 'closest' to p. Thus, we have for example

$$x_1' = \frac{1+x_2}{x_1}\ ,\ x_2' = \frac{1+(x_1')^3}{x_2} = \frac{1+x_1^3+3x_2+3x_2^2+x_2^3}{x_1^3x_2}\ ,$$

$$x_1' = \frac{1+x_2'}{x_1'} = \frac{\dots}{x_1^2x_2}\ ,$$

where we can read off the denominators from the decompositions of the positive roots as linear combinations of simple roots given above. We find $8 = 2+6$ cluster variables, which together generate the cluster algebra $\mathcal{A}_{G_2}$ as a subalgebra of $\mathbb{Q}(x_1, x_2)$.

3. Symmetric cluster algebras without coefficients

In this section, we will construct the cluster algebra associated with an antisymmetric matrix with integer coefficients. Instead of using matrices, we will use quivers (without loops or 2-cycles), since they are easy to visualize and well-suited to our later purposes.

3.1. **Quivers.** Let us recall that a *quiver* Q is an oriented graph. Thus, it is a quadruple given by a set Q_0 (the set of vertices), a set Q_1 (the set of arrows) and two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ which take an arrow to its source respectively its target. Our quivers are 'abstract graphs' but in practice we draw them as in this example:

$$\begin{tikzcd}
Q: & & 3 \arrow[dl, "\lambda"'] & & 5 \arrow[loop left, "\alpha"] \arrow[rr, shift left=2] \arrow[rr] \arrow[rr, shift right=2] & & 6 \\
& 1 \arrow[rr, "\nu"'] & & 2 \arrow[ul, "\mu"'] \arrow[r, shift left, "\beta"] & 4. \arrow[l, shift left, "\gamma"] & &
\end{tikzcd}$$

A *loop* in a quiver Q is an arrow α whose source coincides with its target; a 2-*cycle* is a pair of distinct arrows $\beta \neq \gamma$ such that the source of β equals the target of γ and vice versa. It is clear how to define 3-*cycles*, *connected* *components* .... A quiver is *finite* if both, its set of vertices and its set of arrows, are finite.

3.2. **Seeds and mutations.** Fix an integer $n \geq 1$. A *seed* is a pair (R, u), where

a) R is a finite quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$;
b) u is a free generating set $\{u_1, \ldots, u_n\}$ of the field $\mathbb{Q}(x_1, \ldots, x_n)$ of fractions of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ in n indeterminates.

Notice that in the quiver R of a seed, all arrows between any two given vertices point in the same direction (since R does not have 2-cycles). Let (R, u) be a seed and k a vertex of R. The *mutation* $\mu_k(R, u)$ of (R, u) at k is the seed (R', u'), where

a) R' is obtained from R as follows:
   1) reverse all arrows incident with k;
   2) for all vertices $i \neq j$ distinct from k, modify the number of arrows between i and j as follows:

R	R'
$\begin{tikzcd} i \arrow[rr, "r"] \arrow[dr, "s"'] & & j \\ & k \arrow[ur, "t"'] & \end{tikzcd}$	$\begin{tikzcd} i \arrow[rr, "r+st"] & & j \arrow[dl, "t"] \\ & k \arrow[ul, "s"] & \end{tikzcd}$
$\begin{tikzcd} i \arrow[rr, "r"] & & j \arrow[dl, "t"] \\ & k \arrow[ul, "s"] & \end{tikzcd}$	$\begin{tikzcd} i \arrow[rr, "r-st"] \arrow[dr, "s"'] & & j \\ & k \arrow[ur, "t"'] & \end{tikzcd}$

   where r, s, t are non negative integers, an arrow $i \xrightarrow{l} j$ with $l \geq 0$ means that l arrows go from i to j and an arrow $i \xrightarrow{l} j$ with $l \leq 0$ means that $-l$ arrows go from j to i.

b) u' is obtained from u by replacing the element u_k with

$$u'_k = \frac{1}{u_k} \left(\prod_{\text{arrows } i \to k} u_i + \prod_{\text{arrows } k \to j} u_j \right). \tag{3.2.1}$$

In the *exchange relation* (3.2.1), if there are no arrows from i with target k, the product is taken over the empty set and equals 1. It is not hard to see that $\mu_k(R, u)$ is indeed a seed and that μ_k is an involution: we have $\mu_k(\mu_k(R, u)) = (R, u)$. Notice that the expression given in (3.2.1) for u'_k is *subtraction-free.*

To a quiver R without loops or 2-cycles with vertex set $\{1, \ldots, n\}$ there corresponds the $n \times n$ antisymmetric integer matrix B whose entry b_{ij} is the number of arrows $i \to j$ minus the number of arrows $j \to i$ in R (notice that at least one of these numbers is zero since R does not have 2-cycles). Clearly, this correspondence yields a bijection. Under this bijection, the matrix B' corresponding to the mutation $\mu_k(R)$ has the entries

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{else,} \end{cases}$$

where $[x]_+ = \max(x, 0)$. This is matrix mutation as it was defined by Fomin-Zelevinsky in their seminal paper [51], *cf.* also [56].

3.3. **Examples of seed and quiver mutations.** Let R be the cyclic quiver

(3.3.1)
$$\begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$$

and $u = \{x_1, x_2, x_3\}$. If we mutate at $k = 1$, we obtain the quiver

$$\begin{tikzcd} & 1 \arrow[dl] & \\ 2 & & 3 \arrow[ul] \end{tikzcd}$$

and the set of fractions given by $u'_1 = (x_2 + x_3)/x_1$, $u'_2 = u_2 = x_2$ and $u'_3 = u_3 = x_3$. Now, if we mutate again at 1, we obtain the original seed. This is a general fact: Mutation at k is an involution. If, on the other hand, we mutate (R', u') at 2, we obtain the quiver

$$\begin{tikzcd} & 1 & \\ 2 \arrow[ur] & & 3 \arrow[ul] \end{tikzcd}$$

and the set u'' given by $u''_1 = u'_1 = (x_2 + x_3)/x_1$, $u'_2 = \frac{x_1+x_2+x_3}{x_1x_2}$ and $u''_3 = u'_3 = x_3$.

An important special case of quiver mutation is the mutation at a source (a vertex without incoming arrows) or a sink (a vertex without outgoing arrows). In this case, the mutation only reverses the arrows incident with the mutating vertex. It is easy to see that all orientations of a tree are mutation equivalent and that only sink and source mutations are needed to pass from one orientation to any other.

Let us consider the following, more complicated quiver glued together from four 3-cycles:

(3.3.2)
$$\begin{tikzcd}
& & 1 \arrow[dr] & & \\
& 2 \arrow[ur] \arrow[dr] & & 3 \arrow[ll] \arrow[ddr] & \\
& & 5 \arrow[ur] \arrow[dll] & & \\
4 \arrow[uur] & & & & 6. \arrow[ull]
\end{tikzcd}$$

If we successively perform mutations at the vertices 5, 3, 1 and 6, we obtain the sequence of quivers (we use [88])

$$\begin{tikzcd}
& & 1 \arrow[dr] & & \\
& 2 \arrow[ur] & & 3 \arrow[dl] & \\
& & 5 \arrow[ul] \arrow[drr] & & \\
4 \arrow[urr] & & & & 6 \arrow[llll]
\end{tikzcd}
\quad
\begin{tikzcd}
& & 1 \arrow[dd] & & \\
& 2 \arrow[ur] & & 3 \arrow[ul] & \\
& & 5 \arrow[ul] \arrow[ur] \arrow[drr] & & \\
4 \arrow[urr] & & & & 6 \arrow[llll]
\end{tikzcd}
\quad
\begin{tikzcd}
& & 1 \arrow[dl] \arrow[dr] & & \\
& 2 & & 3 & \\
& & 5 \arrow[uu] \arrow[drr] & & \\
4 \arrow[urr] & & & & 6 \arrow[llll]
\end{tikzcd}
\quad
\begin{tikzcd}
& & 1 \arrow[dl] \arrow[dr] & & \\
& 2 & & 3 & \\
& & 5 \arrow[uu] & & \\
4 \arrow[rrrr] & & & & 6. \arrow[ull]
\end{tikzcd}$$

Notice that the last quiver no longer has any oriented cycles and is in fact an orientation of the Dynkin diagram of type D_6. The sequence of new fractions appearing in these steps is

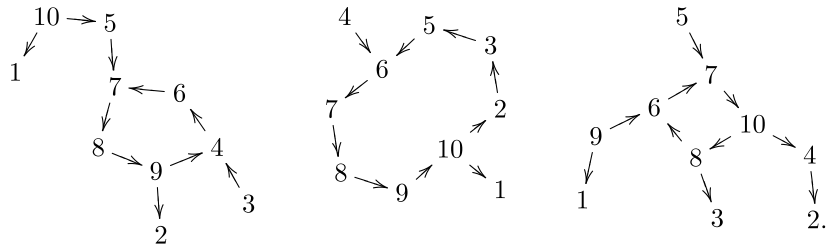
$$\begin{aligned}
u_5' &= \frac{x_3x_4 + x_2x_6}{x_5}, \quad u_3' = \frac{x_3x_4 + x_1x_5 + x_2x_6}{x_3x_5}, \\
u_1' &= \frac{x_2x_3x_4 + x_3^2x_4 + x_1x_2x_5 + x_2^2x_6 + x_2x_3x_6}{x_1x_3x_5}, \\
u_6' &= \frac{x_3x_4 + x_4x_5 + x_2x_6}{x_5x_6}.
\end{aligned}$$

It is remarkable that all the denominators appearing here are monomials and that all the coefficients in the numerators are positive.

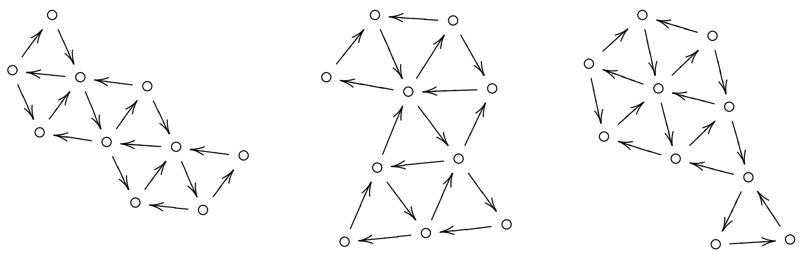
Finally, let us consider the quiver

(3.3.3)
$$\begin{tikzcd}
& & & 1 \arrow[dr] & & & \\
& & 2 \arrow[ur] \arrow[dr] & & 3 \arrow[ll] \arrow[dr] & & \\
& 4 \arrow[ur] \arrow[dr] & & 5 \arrow[ll] \arrow[ur] \arrow[dr] & & 6 \arrow[ll] \arrow[dr] & \\
7 \arrow[ur] & & 8 \arrow[ll] \arrow[ur] & & 9 \arrow[ll] \arrow[ur] & & 10. \arrow[ll]
\end{tikzcd}$$

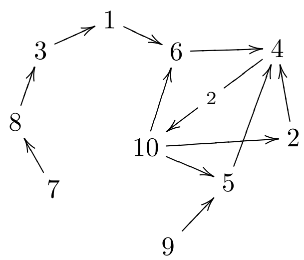
One can show [92] that it is impossible to transform it into a quiver without oriented cycles by a finite sequence of mutations. However, its mutation class (the set of all quivers obtained from it by iterated mutations) contains many quivers with just one oriented cycle, for example



In fact, in this example, the mutation class is finite and it can be completely computed using, for example, [88]: It consists of 5739 quivers up to isomorphism. The above quivers are members of the mutation class containing relatively few arrows. The initial quiver is the unique member of its mutation class with the largest number of arrows. Here are some other quivers in the mutation class with a relatively large number of arrows:



Only 84 among the 5739 quivers in the mutation class contain double arrows (and none contain arrows of multiplicity ≥ 3). Here is a typical example



The classification of the quivers with a finite mutation class is still open. Many examples are given in [50] and [39].

The quivers (3.3.1), (3.3.2) and (3.3.3) are part of a family which appears in the study of the cluster algebra structure on the coordinate algebra of the subgroup of upper unitriangular matrices in $SL(n, \mathbb{C})$, *cf.* section 4.6. The study of coordinate algebras on varieties associated with reductive algebraic groups (in particular, double Bruhat cells) has provided a major impetus for the development of cluster algebras, *cf.* [16].

3.4. **Definition of cluster algebras.** Let Q be a finite quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$. Consider the *initial seed* (Q, x) consisting of Q and the set x formed by the variables $x_1, \ldots, x_n$. Following [51] we define

- the *clusters with respect to* Q to be the sets u appearing in seeds (R, u) obtained from (Q, x) by iterated mutation,
- the *cluster variables* for Q to be the elements of all clusters,
- the *cluster algebra* $\mathcal{A}_Q$ to be the $\mathbb{Q}$-subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all the cluster variables.

Thus, the cluster algebra consists of all $\mathbb{Q}$-linear combinations of monomials in the cluster variables. It is useful to define yet another combinatorial object associated with this recursive construction: The *exchange graph* associated with Q is the graph whose vertices are the seeds modulo simultaneous renumbering of the vertices and the associated cluster variables and whose edges correspond to mutations.

A remarkable theorem due to Gekhtman-Shapiro-Vainshtein states that each cluster u occurs in a unique cluster (R, u), *cf.* [69].

Notice that the knitting algorithm only produced the cluster variables whereas this definition yields additional structure: the clusters.

3.5. **The example** A_2**.** Here the computation of the exchange graph is essentially equivalent to performing the knitting algorithm. If we denote the cluster variables by x_1, x_2, x_1', x_2' and x_1'' as in section 2.2, then the exchange graph is the pentagon

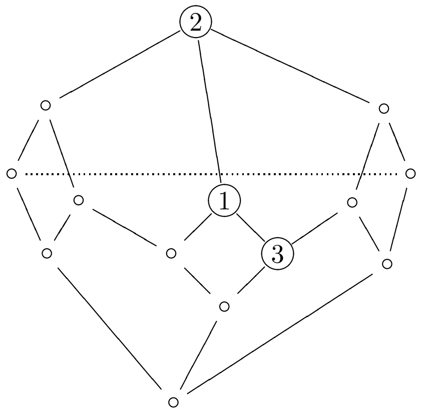
$$
\begin{tikzcd}
 & & (x_1' \leftarrow x_2) \arrow[dll, dash] \arrow[drr, dash] & & \\
(x_1 \to x_2) \arrow[dr, dash] & & & & (x_1' \to x_2') \arrow[dl, dash] \\
 & (x_1'' \to x_1) \arrow[rr, dash] & & (x_1'' \leftarrow x_1') &
\end{tikzcd}
$$

where we have written $x_1 \to x_2$ for the seed $(1 \to 2, \{x_1, x_2\})$. Notice that it is not possible to find a consistent labeling of the edges by 1's and 2's. The reason for this is that the vertices of the exchange graph are not the seeds but the seeds up to renumbering of vertices and variables. Here the clusters are precisely the pairs of consecutive variables in the cyclic ordering of $x_1, \ldots, x_1''$.

3.6. **The example A_3.** Let us consider the quiver

$$Q:\ 1 \longrightarrow 2 \longrightarrow 3$$

obtained by endowing the Dynkin diagram A_3 with a linear orientation. By applying the recursive construction to the initial seed (Q, x) one finds exactly fourteen seeds (modulo simultaneous renumbering of vertices and cluster variables). These are the vertices of the exchange graph, which is isomorphic to the third Stasheff associahedron [120] [35]:



The vertex labeled 1 corresponds to (Q, x), the vertex 2 to $\mu_2(Q, x)$, which is given by

$$\begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}\ ,\ \left\{ x_1, \frac{x_1 + x_3}{x_2}, x_3 \right\}\ ,$$

and the vertex 3 to $\mu_1(Q, x)$, which is given by

$$1 \longleftarrow 2 \longrightarrow 3\ ,\ \left\{ \frac{1 + x_3}{x_1}, x_2, x_3 \right\}.$$

As expected (section 2.2), we find a total of $3 + 6 = 9$ cluster variables, which correspond bijectively to the faces of the exchange graph. The clusters x_1, x_2, x_3 and x_1', x_2, x_3 also appear naturally as slices of the repetition, where by a *slice*, we mean a full connected subquiver containing a representative of each orbit under the horizontal translation (a subquiver is *full* if, with any two vertices, it contains all the arrows between them).

$$\begin{tikzcd}[column sep=small]
\circ \arrow[dr] & & x_3 \arrow[dr] & & x_3' \arrow[dr] & & x_1 \arrow[dr] & & x_3' \arrow[dr] & \\
& x_2 \arrow[ur] \arrow[dr] & & x_2' \arrow[ur] \arrow[dr] & & x_2'' \arrow[ur] \arrow[dr] & & x_2 \arrow[ur] \arrow[dr] & & x_2' \\
x_1 \arrow[ur] & & x_1' \arrow[ur] & & x_1'' \arrow[ur] & & x_1''' \arrow[ur] & & x_3 \arrow[ur] &
\end{tikzcd}$$

In fact, as it is easy to check, each slice yields a cluster. However, some clusters do not come from slices, for example the cluster x_1, x_3, x_1'' associated with the seed $\mu_2(Q, x)$.

3.7. **Cluster algebras with finitely many cluster variables.** The phenomena observed in the above examples are explained by the following key theorem:

Theorem 3.1 (Fomin-Zelevinsky [53]). *Let Q be a finite connected quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$. Let $\mathcal{A}_Q$ be the associated cluster algebra.*

a) *All cluster variables are Laurent polynomials, i.e. their denominators are monomials.*
b) *The number of cluster variables is finite iff Q is mutation equivalent to an orientation of a simply laced Dynkin diagram Δ. In this case, Δ is unique and the non initial cluster variables are in bijection with the positive roots of Δ; namely, if we denote the simple roots by $\alpha_1, \ldots, \alpha_n$, then for each positive root $\sum d_i \alpha_i$, there is a unique non initial cluster variable whose denominator is $\prod x_i^{d_i}$.*
c) *The knitting algorithm yields all cluster variables iff the quiver Q has two vertices or is an orientation of a simply laced Dynkin diagram Δ.*

The theorem can be extended to the non simply laced case if we work with valued quivers as in the example of G_2 in section 2.2.

It is not hard to check that the knitting algorithm yields exactly the cluster variables obtained by iterated mutations at sinks and sources. Remarkably, in the Dynkin case, all cluster variables can be obtained in this way.

The construction of the cluster algebra shows that if the quiver Q is mutation-equivalent to Q', then we have an isomorphism

$$\mathcal{A}_{Q'} \xrightarrow{\sim} \mathcal{A}_Q$$

preserving clusters and cluster variables. Thus, to prove that the condition in b) is sufficient, it suffices to show that $\mathcal{A}_Q$ is cluster-finite if the underlying graph of Q is a Dynkin diagram.

No normal form for mutation-equivalence is known in general and it is unkown how to decide whether two given quivers are mutation-equivalent. However, for certain restricted classes, the answer to this problem is known: Trivially, two quivers with two vertices are mutation-equivalent iff they are isomorphic. But it is already a non-trivial problem to decide when a quiver

$$\begin{tikzcd} 1 \arrow[rr, "r"] & & 2 \arrow[dl, "t"] \\ & 3 \arrow[ul, "s"] & \end{tikzcd} ,$$

where r, s and t are non negative integers, is mutation-equivalent to a quiver without a 3-cycle: As shown in [14], this is the case iff the 'Markoff inequality'

$$r^2 + s^2 + t^2 - rst > 4$$

holds or one among r, s and t is < 2.

For a general quiver Q, a criterion for $\mathcal{A}_Q$ to be cluster-finite in terms of quadratic forms was given in [13]. In practice, the quickest way to decide whether a concretely given quiver is cluster-finite and to determine its cluster-type is to compute its mutation-class using [88]. For example, the reader can easily check that for $3 \leq n \leq 8$, the following quiver glued together from $n-2$ triangles

$$\begin{tikzcd}
1 \arrow[d] & 3 \arrow[l] \arrow[d] & 5 \arrow[l] \arrow[d] & \cdots \arrow[l] & n-1 \arrow[l] \arrow[d] \\
2 \arrow[ur] & 4 \arrow[l] \arrow[ur] & 6 \arrow[l] \arrow[ur] & \cdots \arrow[l] \arrow[ur] & n \arrow[l]
\end{tikzcd}$$

is cluster-finite of respective cluster-type A_3, D_4, D_5, E_6, E_7 and E_8 and that it is not cluster-finite if $n > 8$.

4. Cluster algebras with coefficients

In their combinatorial properties, cluster algebras with coefficients are very similar to those without coefficients which we have considered up to now. The great virtue of cluster algebras with coefficients is that they proliferate in nature as algebras of coordinates on homogeneous varieties. We will define cluster algebras with coefficients and illustrate their ubiquity on several examples.

4.1. **Definition.** Let $1 \leq n \leq m$ be integers. An *ice quiver of type* (n, m) is a quiver $\widetilde{Q}$ with vertex set

$$\{1, \dots, m\} = \{1, \dots, n\} \cup \{n+1, \dots, m\}$$

such that there are no arrows between any vertices i, j which are strictly greater than n. The *principal part* of $\widetilde{Q}$ is the full subquiver Q of $\widetilde{Q}$ whose vertex set is $\{1, \dots, n\}$ (a subquiver is *full* if, with any two vertices, it contains all the arrows between them). The vertices $n+1$, ..., m are often called *frozen vertices*. The cluster algebra

$$\mathcal{A}_{\widetilde{Q}} \subset \mathbb{Q}(x_1, \dots, x_m)$$

is defined as before but

- only mutations with respect to vertices in the principal part are allowed and no arrows are drawn between the vertices $> n$,

- in a cluster

$$u = \{u_1, \ldots, u_n, x_{n+1}, \ldots, x_m\}$$

  only u_1, …, u_n are called cluster variables; the elements x_{n+1}, …, x_m are called *coefficients*; to make things clear, the set u is often called an *extended cluster*;
- the *cluster type of* $\widetilde{Q}$ is that of Q if it is defined.

Often, one also considers localizations of $\mathcal{A}_{\widetilde{Q}}$ obtained by inverting certain coefficients. Notice that the datum of $\widetilde{Q}$ corresponds to that of the integer $m \times n$-matrix $\widetilde{B}$ whose top $n \times n$-submatrix B is antisymmetric and whose entry b_{ij} equals the number of arrows $i \to j$ or the opposite of the number of arrows $j \to i$. The matrix B is called the *principal part* of $\widetilde{B}$. One can also consider valued ice quivers, which will correspond to $m \times n$-matrices whose principal part is antisymmetrizable.

4.2. **Example:** $SL(2, \mathbb{C})$**.** Let us consider the algebra of regular functions on the algebraic group $SL(2, \mathbb{C})$, *i.e.* the algebra

$$\mathbb{C}[a, b, c, d]/(ad - bc - 1).$$

We claim that this algebra *has a cluster algebra structure*, namely that it is isomorphic to the complexification of the cluster algebra with coefficients associated with the following ice quiver

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ \boxed{2} & & \boxed{3} \end{tikzcd}$$

where we have framed the frozen vertices. Indeed, here the principal part Q only consists of the vertex 1 and we can only perform one mutation, whose associated exchange relation reads

$$x_1 x_1' = 1 + x_2 x_3 \text{ or } x_1 x_1' - x_2 x_3 = 1.$$

We obtain an isomorphism as required by sending x_1 to a, x_1' to d, x_2 to b and x_3 to c. We describe this situation by saying that the quiver

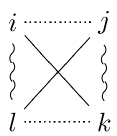
$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ \boxed{b} & & \boxed{c} \end{tikzcd}$$

whose vertices are labeled by the images of the corresponding variables, is an initial seed for a cluster structure on the algebra A. Notice that this cluster structure is not unique.

4.3. **Example: Planes in affine space.** As a second example, let us consider the algebra A of polynomial functions on the cone over the Grassmannian of planes in $\mathbb{C}^{n+3}$. This algebra is the $\mathbb{C}$-algebra generated by the Plücker coordinates x_{ij}, $1 \leq i < j \leq n+3$, subject to the Plücker relations: for each quadruple of integers $i < j < k < l$, we have

$$x_{ik}x_{jl} = x_{ij}x_{kl} + x_{jk}x_{il}.$$

Each plane P in $\mathbb{C}^{n+3}$ gives rise to a straight line in this cone, namely the one generated by the 2×2-minors x_{ij} of any $(n+3) \times 2$-matrix whose columns generate P. Notice that the monomials in the Plücker relation are naturally associated with the sides and the diagonals of the square



The relation expresses the product of the variables associated with the diagonals as the sum of the monomials associated with the two pairs of opposite sides.

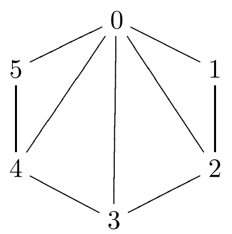
Now the idea is that the Plücker relations are exactly the exchange relations for a suitable structure of cluster algebra with coefficients on the coordinate ring. To formulate this more precisely, let us consider a regular $(n+3)$-gon in the plane with vertices numbered $1, \ldots, n+2$, and consider the variable x_{ij} as associated with the segment $[ij]$ joining the vertices i and j.

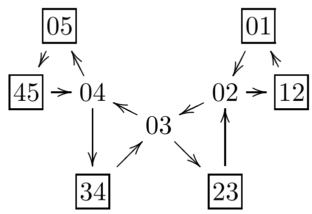
Proposition 4.1 ([53, Example 12.6])**.** *The algebra A has a structure of cluster algebra with coefficients such that*

- *the coefficients are the variables x_{ij} associated with the sides of the $(n+3)$-gon;*
- *the cluster variables are the variables x_{ij} associated with the diagonals of the $(n+3)$-gon;*
- *the clusters are the n-tuples of variables whose associated diagonals form a triangulation of the $(n+3)$-gon.*

Moreover, the exchange relations are exactly the Plücker relations and the cluster type is A_n.

Thus, a triangulation of the $(n+3)$-gon determines an initial seed for the cluster algebra and hence an ice quiver $\widetilde{Q}$ whose frozen vertices correspond to the sides of the $(n+3)$-gon and whose non frozen variables to the diagonals in the triangulation. The arrows of the quiver are determined by the exchange relations which appear when we wish to replace one diagonal $[ik]$ of the triangulation by its *flip*, i.e. the unique diagonal $[jl]$ different from $[ik]$ which does not cross any other diagonal of the triangulation. It is not hard to see that this means that the underlying graph of $\widetilde{Q}$ is the graph dual to the triangulation and that the orientation of the edges of this graph is induced by the choice of an orientation of the plane. Here is an example of a triangulation and the associated ice quiver:





4.4. **Example: The big cell of the Grassmannian.** We consider the cone over the big cell in the Grassmannian of k-dimensional subspaces of the space of rows $\mathbb{C}^n$, where $1 \leq k \leq n$ are fixed integers such that $l = n - k$ is greater or equal to 2. In more detail, let G be the group $SL(n, \mathbb{C})$ and P the subgroup of G formed by the block lower triangular matrices with diagonal blocks of sizes $k \times k$ and $l \times l$. The quotient $P \backslash G$ identifies with our Grassmannian. The big cell is the image under $\pi \colon G \to P \backslash G$ of the space of block upper triangular matrices whose diagonal is the identity matrix and whose upper right block is an arbitrary $k \times l$-matrix Y. The projection π induces an isomorphism between the space $M_{k \times l}(\mathbb{C})$ of these matrices and the big cell. In particular, the algebra A of regular functions on the big cell is the algebra of polynomials in the coefficients y_{ij}, $1 \leq i \leq k$, $1 \leq j \leq l$, of Y. Now for $1 \leq i \leq k$ and $1 \leq j \leq l$, let F_{ij} be the largest square submatrix of Y whose lower left corner is (i, j) and let $s(i, j)$ be its size. Put

$$f_{ij} = (-1)^{(k-i)(s(i,j)-1)} \det(F_{ij}).$$

Theorem 4.2 ([71])**.** *The algebra A has the structure of a cluster algebra with coefficients whose initial seed is given by*

$$\begin{tikzcd}
\boxed{f_{11}} \arrow[r] & f_{12} \arrow[r] \arrow[dl] & f_{13} \arrow[dl] & \cdots \arrow[r] & f_{1l} \arrow[dl] \\
\boxed{f_{21}} \arrow[u, dotted] \arrow[r] & f_{22} \arrow[u] \arrow[r] \arrow[dl] & f_{23} \arrow[u] \arrow[dl] & \cdots \arrow[r] & f_{2l} \arrow[u] \arrow[dl] \\
\vdots \arrow[u, dotted] & \vdots \arrow[u] \arrow[dl] & \vdots & & \vdots \arrow[u] \arrow[dl] \\
\boxed{f_{k1}} \arrow[u, dotted] \arrow[r, dotted] & \boxed{f_{k2}} \arrow[u] \arrow[r, dotted] & \boxed{f_{k3}} & \arrow[r, dotted] & \boxed{f_{kl}} \arrow[u]
\end{tikzcd}$$

The following table indicates when these algebras are cluster-finite and what their cluster-type is:

$k \setminus n$	2	3	4	5	6	7
2	A_1	A_2	A_3	A_4	A_5	A_6
3	A_2	D_4	E_6	E_8		
4	A_3	E_6				
5	A_4					

The homogeneous coordinate ring of the Grassmannian $Gr(k, n)$ itself also has a cluster algebra structure [118] and so have partial flag varieties, double Bruhat cells, Schubert varieties ..., *cf.* [63] [16].

4.5. **Compatible Poisson structures.** Recall that the group $SL(n, \mathbb{C})$ has a canonical Poisson structure given by the *Sklyanin bracket*, which is defined by

$$\omega(x_{ij}, x_{\alpha\beta}) = (\mathrm{sign}(\alpha - i) - \mathrm{sign}(\beta - j))x_{i\beta}x_{\alpha j}$$

where the x_{ij} are the coordinate functions on $SL(n, \mathbb{C})$. This bracket makes $G = SL(n, \mathbb{C})$ into a Poisson-Lie group and $P \setminus G$ into a Poisson G-variety for each subgroup P of G containing the subgroup B of lower triangular matrices. In particular, the big cell of the Grassmannian considered above inherits a Poisson bracket.

Theorem 4.3 ([70]). *This bracket is compatible with the cluster algebra structure in the sense that each extended cluster is a* log-canonical *coordinate system, i.e. we have*

$$\omega(u_i, u_j) = \omega_{ij}^{(u)} u_i u_j$$

for certain (integer) constants $\omega_{ij}^{(u)}$ *depending on the extended cluster* u. *Moreover, the coefficients are central for* ω.

This theorem admits the following generalization: Let $\widetilde{Q}$ be an ice quiver. Define the *cluster variety* $\mathcal{X}(\widetilde{Q})$ to be obtained by glueing the complex tori indexed by the clusters u

$$T^{(u)} = (\mathbb{C}^*)^n = \operatorname{Spec}(\mathbb{C}[u_1, u_1^{-1}, \ldots, u_m, u_m^{-1}])$$

using the exchange relations as glueing maps, where m is the number of vertices of $\widetilde{Q}$.

Theorem 4.4 ([70]). *Suppose that the principal part* Q *of* $\widetilde{Q}$ *is connected and that the matrix* $\widetilde{B}$ *associated with* $\widetilde{Q}$ *is of maximal rank. Then the vector space of Poisson structures on* $\mathcal{X}(\widetilde{Q})$ *compatible with the cluster algebra structure is of dimension*

$$1 + \binom{m-n}{2}.$$

Notice that in general, the cluster variety $\mathcal{X}(\widetilde{Q})$ is an open subset of the spectrum of the (complexified) cluster algebra. For example, for the cluster algebra associated with $SL(2, \mathbb{C})$ which we have considered above, the cluster variety is the union of the elements

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $SL(2, \mathbb{C})$ such that we have $abc \neq 0$ or $bcd \neq 0$. The cluster variety is always regular, but the spectrum of the cluster algebra may be singular. For example, the spectrum of the cluster algebra associated with the ice quiver

$$\begin{tikzcd} & x \arrow[dl, "2"'] \arrow[dr, "2"] & \\ \boxed{u} & & \boxed{v} \end{tikzcd}$$

is the hypersurface in $\mathbb{C}^4$ defined by the equation $xx' = u^2 + v^2$, which is singular at the origin. The corresponding cluster variety is obtained by removing the points with $x = x' = u^2 + v^2 = 0$ and is regular.

In the above theorem, the assumption that $\widetilde{B}$ be of full rank is essential. Otherwise, there may not exist any Poisson bracket compatible with the cluster algebra structure. However, as shown in [71], for any cluster algebra with coefficients, there are 'dual Poisson structures', namely certain 2-forms, which are compatible with the cluster algebra structure.

4.6. **Example: The maximal unipotent subgroup of $SL(n+1,\mathbb{C})$.** Let n be a non negative integer and N the subgroup of $SL(n+1,\mathbb{C})$ formed by the upper triangular matrices with all diagonal coefficients equal to 1. For $1 \leq i,j \leq n+1$ and $g \in N$, let $F_{ij}(g)$ be the maximal square submatrix of g whose lower left corner is (i,j). Let $f_{ij}(g)$ be the determinant of $F_{ij}(g)$. We consider the functions f_{ij} for $1 \leq i \leq n$ and $i+j \leq n+1$.

Theorem 4.5 ([16]). *The coordinate algebra $\mathbb{C}[N]$ has an upper cluster algebra structure whose initial seed is given by*

$$\begin{tikzcd}
f_{12} \arrow[r] & f_{13} \arrow[r] \arrow[dl] & f_{14} \arrow[r] \arrow[dl] & \cdots \arrow[r] & \boxed{f_{1n}} \arrow[dl, dotted] \\
f_{22} \arrow[r] \arrow[u] & f_{23} \arrow[r] \arrow[u] & \cdots & \boxed{f_{2,n-1}} \arrow[u] & \\
 & & & & \\
\vdots \arrow[r] & \cdots \arrow[dl, dotted] & & & \\
\boxed{f_{n-1,2}} \arrow[u] & & & &
\end{tikzcd}$$

We refer to [16] for the notion of 'upper' cluster algebra structure. It is not hard to check that this structure is of cluster type A_3 for $n=3$, D_6 for $n=4$ and cluster-infinite for $n \geq 5$. For $n=5$, this cluster algebra is related to the elliptic root system $E_8^{(1,1)}$ in the notations of Saito [116], *cf.* [65].

A theorem of Fekete [43] generalized in [15] claims that a square matrix of order $n+1$ is *totally positive* (*i.e.* all its minors are > 0) if and only if the following $(n+1)^2$ minors of g are positive: all minors occupying several initial rows and several consecutive columns, and all minors occupying several initial columns and several consecutive rows. It follows that an element g of N is totally positive if $f_{ij}(g) > 0$ for the f_{ij} belonging to the initial seed above. The same holds for the $u_1, \ldots, u_m$ in place of these f_{ij} for any cluster u of this cluster algebra because each exchange relation expresses the new variable *subtraction-free* in the old variables.

Geiss-Leclerc-Schröer have shown [66] that each monomial in the variables of an arbitrary cluster belongs to Lusztig's dual semicanonical basis of $\mathbb{C}[N]$ [105]. They also show that the dual semicanonical basis of $\mathbb{C}[N]$ is different from the dual canonical basis of Lusztig and Kashiwara except in types A_2, A_3 and A_4 [65].

5. Categorification via cluster categories: the finite case

5.1. Quiver representations and Gabriel's theorem. We refer to the books [115] [60] [5] and [4] for a wealth of information on the representation theory of quivers and finite-dimensional algebras. Here, we will only need very basic notions.

Let Q be a finite quiver without oriented cycles. For example, Q can be an orientation of a simply laced Dynkin diagram or the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & 3. \end{tikzcd}$$

Let k be an algebraically closed field. A *representation of* Q is a diagram of finite-dimensional vector spaces of the shape given by Q. More formally, a representation of Q is the datum V of

- a finite-dimensional vector space V_i for each vertex i or Q,
- a linear map $V_\alpha : V_i \to V_j$ for each arrow $\alpha : i \to j$ of Q.

Thus, in the above example, a representation of Q is a (not necessarily commutative) diagram

$$\begin{tikzcd} & V_2 \arrow[dr, "V_\beta"] & \\ V_1 \arrow[ur, "V_\alpha"] \arrow[rr, "V_\gamma"'] & & V_3 \end{tikzcd}$$

formed by three finite-dimensional vector spaces and three linear maps. A *morphism of representations* is a morphism of diagrams. More formally, a morphism of representations $f: V \to W$ is the datum of a linear map $f_i: V_i \to W_i$ for each vertex i of Q such that the square

$$\begin{tikzcd} V_i \arrow[r, "V_\alpha"] \arrow[d, "f_i"'] & V_j \arrow[d, "f_j"] \\ W_i \arrow[r, "W_\alpha"'] & W_j \end{tikzcd}$$

commutes for all arrows $\alpha: i \to j$ of Q. The *composition* of morphisms is defined in the natural way. We thus obtain the *category of representations* $\mathsf{rep}(Q)$. A morphism $f : V \to W$ of this category is an isomorphism iff its components f_i are invertible for all vertices i of Q_0.

For example, let Q be the quiver

$$1 \longrightarrow 2 \ ,$$

and

$$V\colon \ V_1 \xrightarrow{V_\alpha} V_2$$

a representation of Q. By choosing basis in the spaces V_1 and V_2 we find an isomorphism of representations

$$\begin{tikzcd} V_1 \arrow[r, "V_\alpha"] & V_2 \\ k^n \arrow[u] \arrow[r, "A"'] & k^p \arrow[u] \end{tikzcd} ,$$

where, by abuse of notation, we denote by A the multiplication by a $p \times n$-matrix A. We know that we have

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

for invertible matrices P and Q, where r is the rank of A. Let us denote the right hand side by $I_r \oplus 0$. Then we have an isomorphism of representations

$$\begin{tikzcd} k^n \arrow[r, "A"'] & k^p \\ k^n \arrow[u, "Q"] \arrow[r, "I_r \oplus 0"'] & k^p \arrow[u, "P^{-1}"'] \end{tikzcd}$$

We thus obtain a normal form for the representations of this quiver.

Now the category $\mathsf{rep}_k(Q)$ is in fact an *abelian category*: Its direct sums, kernels and cokernels are computed componentwise. Thus, if V and W are two representations, then the *direct sum* $V \oplus W$ is the representation given by

$$(V \oplus W)_i = V_i \oplus W_i \text{ and } (V \oplus W)_\alpha = V_\alpha \oplus W_\alpha \ ,$$

for all vertices i and all arrows α of Q. For example, the above representation in normal form is isomorphic to the direct sum

$$(\ k \xrightarrow{1} k\ )^r \oplus (\ k \longrightarrow 0\ )^{n-r} \oplus (\ 0 \longrightarrow k\ )^{p-r}.$$

The *kernel* of a morphism of representations $f\colon V \to W$ is given by

$$\mathsf{ker}(f)_i = \mathsf{ker}(f_i\colon V_i \to W_i)$$

endowed with the maps induced by the V_α and similarly for the cokernel. A *subrepresentation* V' of a representation V is given by a family of subspaces

$V_i' \subset V_i$, $i \in Q_0$, such that the image of V_i' under V_α is contained in V_j' for each arrow $\alpha \colon i \to j$ of Q. A sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

of representations is a *short exact sequence* if the sequence

$$0 \longrightarrow U_i \longrightarrow V_i \longrightarrow W_i \longrightarrow 0$$

is exact for each vertex i of Q.

A representation V is *simple* if it is non zero and if for each subrepresentation V' of V we have $V' = 0$ or $V/V' = 0$. Equivalently, a representation is simple if it has exactly two subrepresentations. A representation V is *indecomposable* if it is non zero and in each decomposition $V = V' \oplus V''$, we have $V' = 0$ or $V'' = 0$. Equivalently, a representation is indecomposable if it has exactly two direct factors.

In the above example, the representations

$$k \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow k$$

are simple. The representation

$$V = (\ k \xrightarrow{1} k\ )$$

is not simple: It has the non trivial subrepresentation $0 \longrightarrow k$. However, it is indecomposable. Indeed, each endomorphism $f \colon V \to V$ is given by two equal components $f_1 = f_2$ so that the endomorphism algebra of V is one-dimensional. If V was a direct sum $V' \oplus V''$ for two non-zero subspaces, the endomorphism algebra of V would contain the product of the endomorphism algebras of V' and V'' and thus would have to be at least of dimension 2. Since V is indecomposable, the exact sequence

$$0 \to (\ 0 \longrightarrow k\ ) \to (\ k \xrightarrow{1} k\ ) \to (\ k \longrightarrow 0\ ) \to 0$$

is not a split exact sequence.

If Q is an arbitrary quiver, for each vertex i, we define the representation S_i by

$$(S_i)_j = \begin{cases} k & i = j \\ 0 & \text{else.} \end{cases}$$

Then clearly the representations S_i are simple and pairwise non isomorphic. As an exercise, the reader may show that if Q does not have oriented cycles, then each representation admits a finite filtration whose subquotients are among the S_i. Thus, in this case, each simple representation is isomorphic to one of the representations S_i.

Recall that a (possibly non commutative) ring is *local* if its non invertible elements form an ideal.

Decomposition Theorem 5.1 (Azumaya-Fitting-Krull-Remak-Schmidt)**.**

a) *A representation is indecomposable iff its endomorphism algebra is local.*
b) *Each representation decomposes into a finite sum of indecomposable representations, unique up to isomorphism and permutation.*

As we have seen above, for quivers without oriented cycles, the classification of the simple representations is trivial. On the other hand, the problem of classifying the indecomposable representations is non trivial. Let us examine this problem in a few examples: For the quiver $1 \to 2$, we have checked the existence in part b) directly. The uniqueness in b) then implies that each indecomposable representation is isomorphic to exactly one of the representations S_1, S_2 and

$$k \xrightarrow{1} k \; .$$

Similarly, using elementary linear algebra it is not hard to check that each indecomposable representation of the quiver

$$\vec{A}_n : \; 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

is isomorphic to a representation $I[p,q]$, $1 \leq p < q \leq n$, which takes the vertices i in the interval $[p,q]$ to k, the arrows linking them to the identity and all other vertices to zero. In particular, the number of isomorphism classes of indecomposable representations of $\vec{A}_n$ is $n(n+1)/2$.

The representations of the quiver

$$\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$$

are the pairs (V_1, V_α) consisting of a finite-dimensional vector space and an endomorphism and the morphisms of representations are the 'intertwining operators'. It follows from the existence and uniqueness of the Jordan normal form that a system of representatives of the isomorphism classes of indecomposable representations is formed by the representations $(k^n, J_{n,\lambda})$, where $n \geq 1$ is an integer, λ a scalar and $J_{n,\lambda}$ the Jordan block of size n with eigenvalue λ.

The *Kronecker quiver*

$$\begin{tikzcd} 1 \arrow[r, shift left] \arrow[r, shift right] & 2 \end{tikzcd}$$

admits the following infinite family of pairwise non isomorphic representations:

$$\begin{tikzcd} k \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\mu"'] & k \end{tikzcd} \; ,$$

where $(\lambda : \mu)$ runs through the projective line.

Question 5.2. *For which quivers are there only finitely many isomorphism classes of indecomposable representations?*

To answer this question, we define the *dimension vector* of a representation V to be the sequence $\underline{\dim}\, V$ of the dimensions $\dim V_i$, $i \in Q_0$. For example, the dimension vectors of the indecomposable representations of $\vec{A}_2$ are the pairs

$$\underline{\dim}\, S_1 = [10]\,,\ \underline{\dim}\, S_2 = [01]\,,\ \underline{\dim}\, (k \to k) = [11].$$

We define the *Tits form*

$$q_Q : \mathbb{Z}^{Q_0} \to \mathbb{Z}$$

by

$$q_Q(v) = \sum_{i \in Q_0} v_i^2 - \sum_{\alpha \in Q_1} v_{s(\alpha)} v_{t(\alpha)}.$$

Notice that the Tits form does not depend on the orientation of the arrows of Q but only on its underlying graph. We say that the quiver Q is *representation-finite* if, up to isomorphism, it has only finitely many indecomposable representations. We say that a vector $v \in \mathbb{Z}^{Q_0}$ is a *root* of q_Q if $q_Q(v) = 1$ and that it is *positive* if its components are ≥ 0.

Theorem 5.3 (Gabriel [59])**.** *Let Q be a connected quiver and assume that k is algebraically closed. The following are equivalent:*

i) *Q is representation-finite;*
ii) *q_Q is positive definite;*
iii) *The underlying graph of Q is a simply laced Dynkin diagram Δ.*

Moreover, in this case, the map taking a representation to its dimension vector yields a bijection from the set of isomorphism classes of indecomposable representations to the set of positive roots of the Tits form q_Q.

It is not hard to check that if the conditions hold, the positive roots of q_Q are in turn in bijection with the positive roots of the root system Φ associated with Δ, via the map taking a positive root v of q_Q to the element

$$\sum_{i \in Q_0} v_i \alpha_i$$

of the root lattice of Φ.

Let us consider the example of the quiver $Q = \vec{A}_2$. In this case, the Tits form is given by

$$q_Q(v) = v_1^2 + v_2^2 - v_1 v_2.$$

It is positive definite and its positive roots are indeed precisely the dimension vectors

$$[01]\,,\ [10]\,,\ [11]$$

of the indecomposable representations.

Gabriel's theorem has been generalized to non algebraically closed ground fields by Dlab and Ringel [41]. Let us illustrate the main idea on one simple example: Consider the category of diagrams

$$V : \; V_1 \xrightarrow{f} V_2$$

where V_1 is a finite-dimensional real vector space, V_2 a finite-dimensional complex vector space and f an $\mathbb{R}$-linear map. Morphisms are given in the natural way. Then we have the following complete list of representatives of the isomorphism classes of indecomposables:

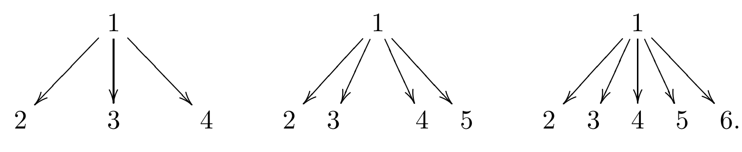
$$\mathbb{R} \to 0\,, \;\; \mathbb{R}^2 \to \mathbb{C}\,, \;\; \mathbb{R} \to \mathbb{C}\,, \;\; 0 \to \mathbb{C}.$$

The corresponding dimension vectors are

$$[10]\,, \;\; [21]\,, \;\; [11]\,, \;\; [01].$$

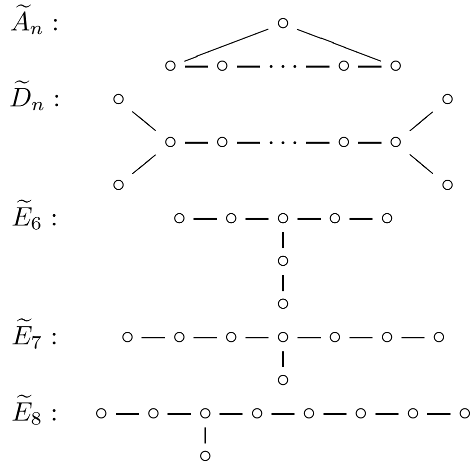
They correspond bijectively to the positive roots of the root system B_2.

5.2. **Tame and wild quivers.** The quivers with infinitely many isomorphism classes of indecomposables can be further subdivided into two important classes: A quiver is *tame* if it has infinitely many isomorphism classes of indecomposables but these occur in 'families of at most one parameter' (we refer to [115] [4] for the precise definition). The Kronecker quiver is a typical example. A quiver is *wild* if there are 'families of indecomposables of ≥ 2 parameters'. One can show that in this case, there are families of an arbitrary number of parameters and that the classification of the indecomposables over any fixed wild algebra would entail the classification of the the indecomposables over all finite-dimensional algebras. The following three quivers are representation-finite, tame and wild respectively:



Theorem 5.4 (Donovan-Freislich [42], Nazarova [110]). *Let Q be a connected quiver and assume that k is algebraically closed. Then Q is tame iff the underlying graph of Q is a simply laced extended Dynkin diagram.*

Let us recall the list of simply laced extended Dynkin quivers. In each case, the number of vertices of the diagram $\widetilde{D}_n$ equals $n+1$.



The following theorem is a first illustration of the close connection between cluster algebras and the representation theory of quivers. Let Q be a finite quiver without oriented cycles and let $\nu(Q)$ be the supremum of the multiplicities of the arrows occurring in all quivers mutation-equivalent to Q.

Theorem 5.5. a) *Q is representation-finite iff $\nu(Q)$ equals* 1.
b) *Q is tame iff $\nu(Q)$ equals* 2.
c) *Q is wild iff $\nu(Q) \geq 3$ iff $\nu(Q) = \infty$.*
d) *The mutation class of Q is finite iff Q has two vertices, is representation-finite or tame.*

Here, part a) follows from Gabriel's theorem and part (iii) of Theorem 1.8 in [53]. Part b) follows from parts a) and c) by exclusion of the third. For part c), let us first assume that Q is wild. Then it is proved at the end of the proof of theorem 3.1 in [12] that $\nu(Q) = \infty$. Conversely, let us assume that $\nu(Q) \geq 3$. Then using Theorem 5 of [30] we obtain that Q is wild. Part d) is proved in [12].

5.3. **The Caldero-Chapoton formula.** Let Δ be a simply laced Dynkin diagram and Q a quiver with underlying graph Δ. Suppose that the set of vertices of Δ and Q is the set of the natural numbers 1, 2, ..., n. We already know from part b) of theorem 3.1 that for each positive root

$$\alpha = \sum_{i=1}^{n} d_i \alpha_i$$

of the corresponding root system, there is a unique non initial cluster variable X_α with denominator
$$x_1^{d_1} \dots x_n^{d_n}.$$
By combining this with Gabriel's theorem, we get the

Corollary 5.6. *The map taking an indecomposable representation V with dimension vector (d_i) of Q to the unique non initial cluster variable X_V whose denominator is $x_1^{d_1} \dots x_n^{d_n}$ induces a bijection from the set of ismorphism classes of indecomposable representations to the set of non initial cluster variables.*

Let us consider this bijection for $Q = \vec{A}_2$:

$$S_2 = (0 \to k) \qquad P_1 = (k \to k) \qquad S_1 = (k \to 0)$$

$$X_{S_2} = \frac{1 + x_1}{x_2} \qquad X_{P_1} = \frac{x_1 + 1 + x_2}{x_1 x_2} \qquad S_{S_1} = \frac{1 + x_2}{x_1}$$

We observe that for the two simple representations, the numerator contains exactly two terms: the number of subrepresentations of the simple representation! Moreover, the representation P_1 has exactly three subrepresentations and the numerator of X_{P_1} contains three terms. In fact, it turns out that this phenomenon is general in type A. But now let us consider the following quiver with underlying graph D_4

$$\begin{tikzcd}[row sep=small] 3 \arrow[dr] & \\ 2 \arrow[r] & 4 \\ 1 \arrow[ur] & \end{tikzcd}$$

and the dimension vector d with $d_1 = d_2 = d_3 = 1$ and $d_4 = 2$. The unique (up to isomorphism) indecomposable representation V with dimension vector d consists of a plane V_4 together with three lines in general position $V_i \subset V_4$, $i = 1, 2, 3$. The corresponding cluster variable is

$$X_4 = \frac{1}{x_1 x_2 x_3 x_4^2} \left(1 + 3x_4 + 3x_4^2 + x_4^3 + 2x_1 x_2 x_3 + 3x_1 x_2 x_3 x_4 + x_1^2 x_2^2 x_3^2\right).$$

Its numerator contains a total of 14 monomials. On the other hand, it is easy to see that V_4 has only 13 types of submodules: twelve submodules are determined by their dimension vectors but for the dimension vector $e = (0, 0, 0, 1)$, we have a family of submodules: Each submodule of this dimension vector corresponds to the choice of a line in V_4. Thus for this dimension vector e, the family of submodules is parametrized by a projective line. Notice that the Euler characteristic of the projective line is 2 (since it is a sphere: the Riemann sphere). So if we attribute weight 1 to the submodules determined by their dimension vector and weight 2 to this

$\mathbb{P}^1$-family, we find a 'total submodule weight' equal to the number of monomials in the numerator. These considerations led Caldero-Chapoton [28] to the following definition, whose ingredients we describe below: Let Q be a finite quiver with vertices $1, \ldots, n$, and V a finite-dimensional representation of Q. Let d be the dimension vector of V. Define

$$CC(V) = \frac{1}{x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n}} \Big(\sum_{0 \leq e \leq d} \chi(\mathrm{Gr}_e(V)) \prod_{i=1}^{n} x_i^{\sum_{j \to i} e_j + \sum_{i \to j} (d_j - e_j)} \Big).$$

Here the sum is taken over all vectors $e \in \mathbb{N}^n$ such that $0 \leq e_i \leq d_i$ for all i. For each such vector e, the *quiver Grassmannian* $\mathrm{Gr}_e(V)$ is the variety of n-tuples of subspaces $U_i \subset V_i$ such that $\dim U_i = e_i$ and the U_i form a subrepresentation of V. By taking such a subrepresentation to the family of the U_i, we obtain a map

$$\mathrm{Gr}_e(V) \to \prod_{i=1}^{n} \mathrm{Gr}_{e_i}(V_i) \, ,$$

where $\mathrm{Gr}_{e_i}(V_i)$ denotes the ordinary Grassmannian of e_i-dimensional subspaces of V_i. Recall that the Grassmannian carries a canonical structure of projective variety. It is not hard to see that for a family of subspaces (U_i) the condition of being a subrepresentation is a closed condition so that the quiver Grassmannian identifies with a projective subvariety of the product of ordinary Grassmannians. If k is the field of complex numbers, the Euler characteristic χ is taken with respect to singular cohomology with coefficients in $\mathbb{Q}$ (or any other field). If k is an arbitrary algebraically closed field, we use étale cohomology to define χ. The most important properties of χ are (*cf. e.g.* section 7.4 in [68])

(1) χ is additive with respect to disjoint unions;
(2) if $p\colon E \to X$ is a morphism of algebraic varieties such that the Euler characteristic of the fiber over a point $x \in X$ does not depend on x, then $\chi(E)$ is the product of $\chi(X)$ by the Euler characteristic of the fiber over any point $x \in X$.

Theorem 5.7 (Caldero-Chapoton [28])**.** *Let Q be a Dynkin quiver and V an indecomposable representation. Then we have $CC(V) = X_V$, the cluster variable obtained from V by composing Fomin-Zelevinsky's bijection with Gabriel's.*

Caldero-Chapoton's proof of the theorem was by induction. One of the aims of the following sections is to explain 'on what' they did the induction.

5.4. **The derived category.** Let k be an algebraically closed field and Q a (possibly infinite) quiver without oriented cycles (we will impose more restrictive conditions on Q later). For example, Q could be the quiver

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "\gamma"'] & 3 \arrow[d, "\alpha"] \\ 2 \arrow[ur, "\beta"'] \arrow[r] & 4 \end{tikzcd}$$

A *path* of Q is a formal composition of ≥ 0 arrows. For example, the sequence $(4|\alpha|\beta|\gamma|1)$ is a path of length 3 in the above example (notice that we include the source and target vertices of the path in the notation). For each vertex i of Q, we have the *lazy path* $e_i = (i|i)$, the unique path of length 0 which starts at i and stops at i and does nothing in between. The *path category* has set of objects Q_0 (the set of vertices of Q) and, for any vertices i, j, the morphism space from i to j is the vector space whose basis consists of all paths from i to j. Composition is induced by composition of paths and the unit morphisms are the lazy paths. If Q is finite, we define the *path algebra* to be the matrix algebra

$$kQ = \bigoplus_{i,j \in Q_0} \mathsf{Hom}(i,j)$$

where multiplication is matrix multiplication. Equivalently, the path algebra has as a basis all paths and its product is given by concatenating composable paths and equating the product of non composable paths to zero. The path algebra has the sum of the lazy paths as its unit element

$$1 = \sum_{i \in Q_0} e_i .$$

The idempotent e_i yields the *projective right module*

$$P_i = e_i kQ.$$

The modules P_i generate the *category of k-finite-dimensional right modules* $\mathsf{mod}\, kQ$. Each arrow α from i to j yields a map $P_i \to P_j$ given by left multiplication by α. (If we were to consider – heaven forbid – left modules, the analogous map would be given by right multiplication by α and it would go in the direction opposite to that of α. Whence our preference for right modules).

Notice that we have an equivalence of categories

$$\mathsf{rep}_k(Q^{op}) \to \mathsf{mod}\, kQ$$

sending a representation V of the opposite quiver Q^{op} to the sum

$$\bigoplus_{i \in Q_0} V_i$$

endowed with the natural right action of the path algebra. Conversely, a kQ-module M gives rise to the representation V with $V_i = Me_i$ for each vertex i of Q and V_α given by right multiplication by α for each arrow α of Q. The category $\mathsf{mod}\, kQ$ is *abelian*, *i.e.* it is additive, has kernels and cokernels and for each morphism f the cokernel of its kernel is canonically isomorphic to the kernel of its cokernel.

The category $\mathsf{mod}\, kQ$ is *hereditary*. Recall from [33] that this means that submodules of projective modules are projective; equivalently, that all extension groups in degrees $i \geq 2$ vanish:

$$\mathsf{Ext}^i_{kQ}(L, M) = 0;$$

equivalently, that kQ is of global dimension ≤ 1; ... Thus, in the spirit of noncommutative algebraic geometry approached via abelian categories, we should think of $\mathsf{mod}\, kQ$ as a 'non commutative curve'.

We define $\mathcal{D}_Q$ to be the *bounded derived category* of the category $\mathsf{mod}\, kQ$. Thus, the objects of $\mathcal{D}_Q$ are the bounded complexes of (right) kQ-modules

$$\ldots \to 0 \to \ldots \to M^p \xrightarrow{d^p} M^{p+1} \to \ldots \to 0 \to \ldots .$$

Its morphisms are obtained from morphisms of complexes by formally inverting all quasi-isomorphisms. We refer to [122] [86] ... for in depth treatments of the fundamentals of this construction. Below, we will give a complete and elementary description of the category $\mathcal{D}_Q$ if Q is a Dynkin quiver. We have the following general facts: The functor

$$\mathsf{mod}\, kQ \to \mathcal{D}_Q$$

taking a module M to the complex concentrated in degree 0

$$\ldots \to 0 \to M \to 0 \to \ldots$$

is a fully faithful embedding. From now on, we will identify modules with complexes concentrated in degree 0. If L and M are two modules, then we have a canonical isomorphism

$$\mathsf{Ext}^i_{kQ}(L, M) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}_Q}(L, M[i])$$

for all $i \in \mathbb{Z}$, where $M[i]$ denotes the complex M shifted by i degrees to the left: $M[i]^p = M^{p+i}$, $p \in \mathbb{Z}$, and endowed with the differential $d_{M[i]} = (-1)^i d_M$. The category $\mathcal{D}_Q$ has all finite direct sums (and they are given by direct sums of complexes) and the decomposition theorem 5.1 holds. Moreover, each object is isomorphic to a direct sum of shifted copies of modules (this holds more generally in the derived category of any hereditary abelian category, for example the derived category of coherent sheaves on an algebraic curve). The category $\mathcal{D}_Q$ is abelian if and only if the quiver Q does not have any arrows. However, it is always *triangulated*. This means that it is k-linear (it is additive, and the morphism sets are endowed with k-vector space structures so that the composition is bilinear) and endowed with the following extra structure:

a) a *suspension (or shift) functor* $\Sigma\colon \mathcal{D}_Q \to \mathcal{D}_Q$, namely the functor taking a complex M to $M[1]$;
b) a class of *triangles* (sometimes called 'distinguished triangles'), namely the sequences
$$L \to M \to N \to \Sigma L$$
which are 'induced' by short exact sequences of complexes.

The class of triangles satisfies certain axioms, *cf. e.g.* [122]. The most important consequence of these axioms is that the triangles induce long exact sequences in the functors $\mathsf{Hom}(X, ?)$ and $\mathsf{Hom}(?, X)$, *i.e.* for each object X of $\mathcal{D}_Q$, the sequences
$$\dots (X, \Sigma^{-1}N) \to (X, L) \to (X, M) \to (X, N) \to (X, \Sigma L) \to \dots$$
and
$$\dots (\Sigma^{-1}N, X) \leftarrow (L, X) \leftarrow (M, X) \leftarrow (N, X) \leftarrow (\Sigma L, X) \leftarrow \dots$$
are exact.

5.5. **Presentation of the derived category of a Dynkin quiver.** From now on, we assume that Q is a Dynkin quiver. Let $\mathbb{Z}Q$ be its repetition (*cf.* section 2.2). So the vertices of $\mathbb{Z}Q$ are the pairs (p, i), where p is an integer and i a vertex of Q and the arrows of $\mathbb{Z}Q$ are obtained as follows: each arrow $\alpha\colon i \to j$ of Q yields the arrows
$$(p, \alpha)\colon (p, i) \to (p, j)\ ,\ p \in \mathbb{Z}\ ,$$
and the arrows
$$\sigma(p, \alpha)\colon (p-1, j) \to (p, i)\ ,\ p \in \mathbb{Z}.$$
We extend σ to a map defined on all arrows of $\mathbb{Z}Q$ by defining
$$\sigma(\sigma(p, \alpha)) = (p-1, \alpha).$$
We endow $\mathbb{Z}Q$ with the map σ and with the automorphism $\tau : \mathbb{Z}Q \to \mathbb{Z}Q$ taking (p, i) to $(p-1, i)$ and (p, α) to $(p-1, \alpha)$ for all vertices i of Q, all arrows α of Q and all integers p.

For a vertex v of $\mathbb{Z}Q$ the *mesh ending at* v is the full subquiver

(5.5.1)
$$\begin{tikzcd}
 & u_1 \arrow[ddr, "\alpha"] & \\
 & u_2 \arrow[dr] & \\
\tau v \arrow[uur, "\sigma(\alpha)"] \arrow[ur] \arrow[dr] & \vdots & v \\
 & u_s \arrow[ur] &
\end{tikzcd}$$

formed by v, $\tau(v)$ and all sources u of arrows $\alpha\colon u \to v$ of $\mathbb{Z}Q$ ending in v. We define the *mesh ideal* to be the (two-sided) ideal of the path category of $\mathbb{Z}Q$ which is generated by all *mesh relators*

$$r_v = \sum_{\text{arrows } \alpha\colon u \to v} \alpha\sigma(\alpha) ,$$

where v runs through the vertices of $\mathbb{Z}Q$. The *mesh category* is the quotient of the path category of $\mathbb{Z}Q$ by the mesh ideal.

Theorem 5.8 (Happel [75])**.** a) *There is a canonical bijection $v \mapsto M_v$ from the set of vertices of $\mathbb{Z}Q$ to the set of isomorphism classes of indecomposables of $\mathcal{D}_Q$ which takes the vertex $(1,i)$ to the indecomposable projective P_i.*

b) *Let $\mathsf{ind}\,\mathcal{D}_Q$ be the full subcategory of indecomposables of $\mathcal{D}Q$. The bijection of a) lifts to an equivalence of categories from the mesh category of $\mathbb{Z}Q$ to the category $\mathsf{ind}\,\mathcal{D}_Q$.*

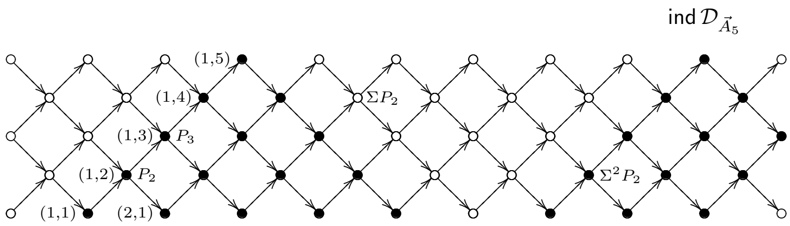


FIGURE 1. The repetition of type A_n

In figure 5.5, we see the repetition for $Q = \vec{A}_5$ and the map taking its vertices to the indecomposable objects of the derived category. The vertices marked $\bullet$ belonging to the left triangle are mapped to indecomposable modules. The vertex $(1,i)$ corresponds to the indecomposable projective P_i. The arrow $(1,i) \to (1,i+1)$, $1 \leq i \leq 5$, is mapped to the left multiplication by the arrow $i \to i+1$. The functor takes a mesh (5.5.1) to a triangle

(5.5.2) $$M_{\tau v} \longrightarrow \bigoplus_{i=1}^{s} M_{u_i} \longrightarrow M_v \longrightarrow \Sigma M_{\tau v}$$

called an *Auslander-Reiten triangle* or *almost split triangle*, *cf.* [76]. If M_v and $M_{\tau v}$ are modules, then so is the middle term and the triangle comes from an exact sequence of modules

$$0 \longrightarrow M_{\tau v} \longrightarrow \bigoplus_{i=1}^{s} M_{u_i} \longrightarrow M_v \longrightarrow 0$$

called an *Auslander-Reiten sequence* or *almost split sequence*, *cf.* [5]. These almost split triangles respectively sequences can be characterized intrinsically in $\mathcal{D}_Q$ respectively $\mathsf{mod}\,kQ$.

Recall that the Grothendieck group $K_0(\mathcal{T})$ of a triangulated category is the quotient of the free abelian group on the isomorphism classes $[X]$ of objects X of $\mathcal{T}$ by the subgroup generated by all elements

$$[X] - [Y] + [Z]$$

arising from triangles (X, Y, Z) of $\mathcal{T}$. In the case of $\mathcal{D}_Q$, the natural map

$$K_0(\operatorname{\mathsf{mod}} kQ) \to K_0(\mathcal{D}_Q)$$

is an isomorphism (its inverse sends a complex to the alternating sum of the classes of its homologies). Since $K_0(\operatorname{\mathsf{mod}} kQ)$ is free on the classes $[S_i]$ associated with the simple modules, the same holds for $K_0(\mathcal{D}_Q)$ so that its elements are given by n-tuples of integers. We write $\underline{\dim}\, M$ for the image in $K_0(\mathcal{D}_Q)$ of an object M of $K_0(\mathcal{D}_Q)$ and call $\underline{\dim}\, M$ the *dimension vector of* M. Then each triangle (5.5.2) yields an equality

$$\underline{\dim}\, M_v = \sum_{i=1}^{s} \underline{\dim}\, M_{u_i} - \underline{\dim}\, M_{\tau v}.$$

Using these equalities, we can easily determine $\underline{\dim}\, M$ for each indecomposable M starting from the known dimension vectors $\underline{\dim}\, P_i$, $1 \le i \le n$. In the above example, we find the dimension vectors listed in figure (2).

$$\begin{tikzcd}[column sep=tiny]
 & & & & 11111 \arrow[dr] & & -10000 \arrow[dr] & & -01000 \arrow[dr] & & -00100 \arrow[dr] & & -00010 \arrow[dr] & & -00001 \\
 & & & 11110 \arrow[ur] \arrow[dr] & & 01111 \arrow[ur] \arrow[dr] & & -11000 \arrow[ur] \arrow[dr] & & -01100 \arrow[ur] \arrow[dr] & & -00110 \arrow[ur] \arrow[dr] & & -00011 \arrow[ur] & \\
 & & 11100 \arrow[ur] \arrow[dr] & & 01110 \arrow[ur] \arrow[dr] & & 00111 \arrow[ur] \arrow[dr] & & -11100 \arrow[ur] \arrow[dr] & & -01110 \arrow[ur] \arrow[dr] & & -00111 \arrow[ur] & & \\
 & 11000 \arrow[ur] \arrow[dr] & & 01100 \arrow[ur] \arrow[dr] & & 00110 \arrow[ur] \arrow[dr] & & 00011 \arrow[ur] \arrow[dr] & & -11110 \arrow[ur] \arrow[dr] & & -01111 \arrow[ur] & & & \\
10000 \arrow[ur] & & 01000 \arrow[ur] & & 00100 \arrow[ur] & & 00010 \arrow[ur] & & 00001 \arrow[ur] & & -11111 \arrow[ur] & & & &
\end{tikzcd}$$

FIGURE 2. Some dimension vectors of indecomposables in $\mathcal{D}_{\vec{A}_5}$

Thanks to the theorem, the automorphism τ of the repetition yields a k-linear automorphism, still denoted by τ, of the derived category $\mathcal{D}_Q$. This automorphism has several intrinsic descriptions:

1) As shown in [61], it is the right derived functor of the left exact Coxeter functor $\operatorname{\mathsf{rep}}(Q^{op}) \to \operatorname{\mathsf{rep}}(Q^{op})$ introduced by Bernstein-Gelfand-Ponomarev [18] in their proof of Gabriel's theorem. If we identify $K_0(\mathcal{D}_Q)$ with the root lattice via Gabriel's theorem, then the automorphism induced by τ^{-1} equals the the Coxeter transformation c. As shown by Gabriel [61], the

identity $c^h = \mathbf{1}$, where h is the Coxeter number, lifts to an isomorphism of functors

$$\tau^{-h} \xrightarrow{\sim} \Sigma^2. \tag{5.5.3}$$

2) It can be expressed in terms of the Serre functor of $\mathcal{D}_Q$: Recall that for a k-linear triangulated category $\mathcal{T}$ with finite-dimensional morphism spaces, a Serre functor is an autoequivalence $S\colon \mathcal{T} \to \mathcal{T}$ such that the Serre duality formula holds: We have bifunctorial isomorphisms

$$D\,\mathsf{Hom}(X,Y) \xrightarrow{\sim} \mathsf{Hom}(X,SY)\ ,\ \ X,Y \in \mathcal{T}\ ,$$

where D is the duality $\mathsf{Hom}_k(?,k)$ over the ground field. Notice that this determines the functor S uniquely up to isomorphism. In the case of $\mathcal{D}_Q = \mathcal{D}^b(\mathsf{mod}\,kQ)$, it is not hard to prove that a Serre functor exists (it is given by the left derived functor of the tensor product by the bimodule $D(kQ)$). Now the autoequivalence τ, the suspension functor Σ and the Serre functor S are linked by the fundamental isomorphism

$$\tau\Sigma \xrightarrow{\sim} S. \tag{5.5.4}$$

5.6. **Caldero-Chapoton's proof.** The above description of the derived category yields in particular a description of the module category, which is a full subcategory of the derived category. This description was used by Caldero-Chapoton [28] to prove their formula. Let us sketch the main steps in their proof: Recall that we have defined a surjective map $v \mapsto X_v$ from the set of vertices of the repetition to the set of cluster variables such that

a) we have $X_{(0,i)} = x_i$ for $1 \leq i \leq n$ and
b) we have
$$X_{\tau v} X_v = 1 + \prod_{\text{arrows } w \to v} X_w$$
for all vertices v of the repetition.

We wish to show that we have

$$X_v = CC(M_v)$$

for all vertices v such that M_v is an indecomposable module. This is done by induction on the distance of v from the vertices $(1,i)$ in the quiver $\mathbb{Z}Q$. More precisely, one shows the following

a) We have $CC(P_i) = X_{(1,i)}$ for each indecomposable projective P_i. Here we use the fact that submodules of projectives are projective in order to explicitly compute $CC(P_i)$.
b1) For each split exact sequence
$$0 \to L \to E \to M \to 0\ ,$$

we have

$$CC(L)CC(M) = CC(E).$$

Thus, if $E = E_1 \oplus \dots E_s$ is a decomposition into indecomposables, then

$$CC(E) = \prod_{i=1}^{s} CC(E_i).$$

b2) If

$$0 \to L \to E \to M \to 0$$

is an almost split exact sequence, then we have

$$CC(E) + 1 = CC(L)CC(M).$$

It is now clear how to prove the equality $X_v = CC(M_v)$ by induction by proceeding from the projective indecomposables to the right.

5.7. **The cluster category.** The *cluster category*

$$\mathcal{C}_Q = \mathcal{D}_Q/(\tau^{-1}\Sigma)^{\mathbb{Z}} = \mathcal{D}_Q/(S^{-1}\Sigma^2)^{\mathbb{Z}}$$

is the orbit category of the derived category under the action of the cyclic group generated by the autoequivalence $\tau^{-1}\Sigma = S^{-1}\Sigma^2$. This means that the objects of $\mathcal{C}_Q$ are the same as those of the derived category $\mathcal{D}_Q$ and that for two objects X and Y, the morphism space from X to Y in $\mathcal{C}_Q$ is

$$\mathcal{C}_Q(X, Y) = \bigoplus_{p \in \mathbb{Z}} \mathcal{D}_Q(X, (S^{-1}\Sigma^2)^p Y).$$

Morphisms are composed in the natural way. This definition is due to Buan-Marsh-Reineke-Reiten-Todorov [10], who were trying to obtain a better understanding of the 'decorated quiver representations' introduced by Reineke-Marsh-Zelevinsky [107]. For quivers of type A, an equivalent category was defined independently by Caldero-Chapoton-Schiffler [29] using an entirely different description. Clearly the category $\mathcal{C}_Q$ is k-linear. It is not hard to check that its morphism spaces are finite-dimensional.

One can show [91] that $\mathcal{C}_Q$ admits a canonical structure of triangulated category such that the projection functor $\pi\colon \mathcal{D}_Q \to \mathcal{C}_Q$ becomes a triangle functor (in general, orbit categories of triangulated categories are no longer triangulated). The Serre functor S of $\mathcal{D}_Q$ clearly induces a Serre functor in $\mathcal{C}_Q$, which we still denote by S. Now, by the definition of $\mathcal{C}_Q$ (and its triangulated structure), we have an isomorphism of triangle functors

$$S \xrightarrow{\sim} \Sigma^2.$$

This means that $\mathcal{C}_Q$ is 2-*Calabi-Yau*. Indeed, for an integer $d \in \mathbb{Z}$, a triangulated category $\mathcal{T}$ with finite-dimensional morphism spaces is d-Calabi-Yau if it admits a Serre functor isomorphic as a triangle functor to the dth power of its suspension functor.

5.8. **From cluster categories to cluster algebras.** We keep the notations and hypotheses of the previous section. The suspension functor Σ and the Serre functor S induce automorphisms of the repetition $\mathbb{Z}Q$ which we still denote by Σ and S respectively. The orbit quiver $\mathbb{Z}Q/(\tau^{-1}\Sigma)^{\mathbb{Z}}$ inherits the automorphism τ and the map σ (defined on arrows only) and thus has a well-defined mesh category. Recall that we write $\mathsf{Ext}^1(X, Y)$ for $\mathsf{Hom}(X, \Sigma Y)$ in any triangulated category.

Theorem 5.9 ([10] [11])**.** a)
The decomposition theorem holds for the cluster category and the mesh category of $\mathbb{Z}Q/(\tau^{-1}\Sigma)^{\mathbb{Z}}$ is canonically equivalent to the full subcategory $\mathsf{ind}\,\mathcal{C}_Q$ of the indecomposables of $\mathcal{C}_Q$. Thus, we have an induced bijection $L \mapsto X_L$ from the set of isomorphism classes of indecomposables of $\mathcal{C}_Q$ to the set of all cluster variables of $\mathcal{A}_Q$ which takes the shifted projective ΣP_i to the initial variable x_i, $1 \le i \le n$.
Under this bijection, the clusters correspond to the *cluster-tilting sets*, *i.e.* the sets of pairwise non isomorphic indecomposables $T_1, \ldots, T_n$ such that we have

$$Ext^1(T_i, T_j) = 0$$

for all i, j.
If $T_1, \ldots, T_n$ is cluster-tilting, then the quiver (*cf.* below) of the endomorphism algebra of the sum $T = \bigoplus_{i=1}^n T_i$ does not have loops nor 2-cycles and the associated antisymmetric matrix is the exchange matrix of the unique seed containing the cluster $X_{T_1}, \ldots, X_{T_n}$.

In part b), the condition implies in particular that $\mathsf{Ext}^1(T_i, T_i)$ vanishes. However, for a Dynkin quiver Q, we have $\mathsf{Ext}^1(L, L) = 0$ for each indecomposable L of $\mathcal{C}_Q$. A *cluster-tilting object of* $\mathcal{C}_Q$ is the direct sum of the objects $T_1, \ldots, T_n$ of a cluster-tilting set. Since these are pairwise non-isomorphic indecomposables, the datum of T is equivalent to that of the T_i. A *cluster-tilted algebra of type* Q is the endomorphism algebra of a cluster-tilting object of $\mathcal{C}_Q$. In part c), the most subtle point is that the quiver does not have loops or 2-cycles [11]. Let us recall what one means by the *quiver of a finite-dimensional algebra over an algebraically closed field*:

Proposition-Definition 5.10 (Gabriel)**.** *Let B be a finite-dimensional algebra over the algebraically closed ground field k.*

a) *There exists a quiver Q_B , unique up to isomorphism, such that B is Morita equivalent to the algebra kQ_B/I, where I is an ideal of kQ_B contained in the square of the ideal generated by the arrows of Q_B.*

b) *The ideal I is not unique in general but we have $I = 0$ iff B is hereditary.*
c) *There is a bijection $i \mapsto S_i$ between the vertices of Q_B and the isomorphism classes of simple B-modules. The number of arrows from a vertex i to a vertex j equals the dimension of $\mathsf{Ext}^1_B(S_j, S_i)$.*

In our case, the algebra B is the endomorphism algebra of the sum T of the cluster-tilting set $T_1, \ldots, T_n$ in $\mathcal{C}_Q$. In this case, the Morita equivalence of a) even becomes an isomorphism (because the T_i are pairwise non isomorphic). For a suitable choice of this isomorphism, the idempotent e_i associated with the vertex i is sent to the identity of T_i and the images of the arrows from i to j yield a basis of the space of *irreducible morphisms*

$$\mathsf{irr}_T(T_i, T_j) = \mathsf{rad}_T(T_i, T_j)/\, \mathsf{rad}^2_T(T_i, T_j)\ ,$$

where $\mathsf{rad}_T(T_i, T_j)$ denotes the vector space of non isomorphisms from T_i to T_j (thanks to the locality of the endomorphism rings, this set is indeed closed under addition) and rad^2_T the subspace of non isomorphisms admitting a non trivial factorization:

$$\mathsf{rad}^2_T(T_i, T_j) = \sum_{r=1}^{n} \mathsf{rad}_T(T_r, T_j)\, \mathsf{rad}_T(T_i, T_r).$$

As an illustration of theorem 5.9, we consider the cluster-tilting set $T_1, \ldots, T_5$ in $\mathcal{C}_{\vec{A}_5}$ depicted in figure 3. Here the vertices labeled 0, 1, ..., 4 have to be identified with the vertices labeled 20, 21, ..., 24 (in this order) to obtain the orbit quiver $\mathbb{Z}Q/(\tau^{-1}\Sigma)^{\mathbb{Z}}$. In the orbit category, we have $\tau \xrightarrow{\sim} \Sigma$ so that ΣT_1 is the indecomposable associated to vertex 0, for example. Using this and the description of the morphisms in the mesh category, it is easy to check that we do have

$$\mathsf{Ext}^1(T_i, T_j) = 0$$

for all i, j. It is also easy to determine the spaces of morphisms

$$\mathsf{Hom}_{\mathcal{C}_Q}(T_i, T_j)$$

and the compositions of morphisms. Determining these is equivalent to determining the endomorphism algebra

$$\mathsf{End}(T) = \mathsf{Hom}(T, T) = \bigoplus_{i,j} \mathsf{Hom}(T_i, T_j).$$

This algebra is easily seen to be *isomorphic* to the algebra given by the following quiver Q'

$$\begin{tikzcd}
5 \arrow[dr, "\beta"'] & & 3 \arrow[ll, "\gamma"'] \\
& 2 \arrow[ur, "\alpha"'] \arrow[dr, "\beta"] & \\
1 \arrow[ur, "\alpha"] & & 4 \arrow[ll, "\gamma"]
\end{tikzcd}$$

with the relations

$$\alpha\beta = 0 \, , \ \beta\gamma = 0 \, , \ \gamma\alpha = 0.$$

Thus the quiver of $\mathsf{End}(T)$ is Q'. It encodes the exchange matrix of the associated cluster

$$\begin{aligned}
X_{T_1} &= \frac{1+x_2}{x_1} \\
X_{T_2} &= \frac{x_1x_2 + x_1x_4 + x_3x_4 + x_2x_3x_4}{x_1x_2x_3} \\
X_{T_3} &= \frac{x_1x_2x_3 + x_1x_2x_3x_4 + x_1x_2x_5 + x_1x_4x_5 + x_3x_4x_5 + x_2x_3x_4x_5}{x_1x_2x_3x_4x_5} \\
X_{T_4} &= \frac{x_2+x_4}{x_3} \\
X_{T_5} &= \frac{1+x_4}{x_5}.
\end{aligned}$$

5.9. **A K-theoretic interpretation of the exchange matrix.** Keep the notations and hypotheses of the preceding section. Let $T_1, \ldots, T_n$ be a cluster-tilting set, T the sum of the T_i and B its endomorphism algebra. For two finite-dimensional right B-modules L and M put

$$\begin{aligned}
\langle L, M\rangle_a = \dim \mathsf{Hom}(L,M) - \dim \mathsf{Ext}^1(L,M)& \\
- \dim \mathsf{Hom}(M,L) + \dim \mathsf{Ext}^1(M,L)&.
\end{aligned}$$

This is the antisymmetrization of a truncated Euler form. A priori it is defined on the split Grothendieck group of the category $\mathsf{mod}\, B$ (*i.e.* the quotient of the free abelian group on the isomorphism classes divided by the subgroup generated by all relations obtained from direct sums in $\mathsf{mod}\, B$).

Proposition 5.11 (Palu). *The form $\langle , \rangle_a$ descends to an antisymmetric form on $K_0(\mathsf{mod}\, B)$. Its matrix in the basis of the simples is the exchange matrix associated with the cluster corresponding to $T_1, \ldots, T_n$.*

5.10. **Mutation of cluster-tilting sets.** Let us recall two axioms of triangulated categories:

TR1 For each morphism $u\colon X \to Y$, there exists a triangle
$$X \xrightarrow{u} Y \to Z \to \Sigma X.$$

TR2 A sequence
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$
is a triangle if and only if the sequence
$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-u} \Sigma Y$$
is a triangle.

One can show that in TR1, the triangle is unique up to (non unique) isomorphism. In particular, up to isomorphism, the object Z is uniquely determined by u. Notice the sign in TR2. It follows from TR1 and TR2 that a given morphism also occurs as the second (respectively third) morphism in a triangle.

Now, with the notations and hypotheses of the preceding section, suppose that $T_1, \ldots, T_n$ is a cluster-tilting set and Q' the quiver of the endomorphism algebra B of the sum of the T_i. As explained after proposition-definition 5.10, we have a surjective algebra morphism
$$kQ' \to \bigoplus_{i,j} \mathsf{Hom}(T_i, T_j)$$
which takes the idempotent e_i to the identity of T_i and the arrows $i \to j$ to irreducible morphisms $T_i \to T_j$, for all vertices i, j of Q' (*cf.* the above example computation of B and $Q' = Q_B$).

Now let k be a vertex of Q' (the mutating vertex). We choose triangles
$$T_k \xrightarrow{u} \bigoplus_{\substack{\text{arrows}\\ k\to i}} T_i \to T_k^* \to \Sigma T_k$$
and
$${}^*T_k \to \bigoplus_{\substack{\text{arrows}\\ j\to k}} T_j \xrightarrow{v} T_k \to \Sigma\, {}^*T_k\ ,$$
where the component of u (respectively v) corresponding to an arrow $\alpha\colon k \to i$ (respectively $j \to k$) is the corresponding morphism $T_k \to T_i$ (respectively $T_j \to T_k$). These triangles are unique up to isomorphism and called the *exchange triangles* associated with k and $T_1, \ldots, T_n$.

Theorem 5.12 ([10])**.** a) *The objects T_k^* and ${}^*T_k$ are isomorphic.*

b) *The set obtained from $T_1, \ldots, T_n$ by replacing T_k with T_k^* is cluster-tilting and its associated cluster is the mutation at k of the cluster associated with $T_1, \ldots, T_n$.*
c) *Two indecomposables L and M appear as the the pair (T_k, T_k^*) associated with an exchange if and only if the space $\mathsf{Ext}^1(L, M)$ is one-dimensional. In this case, the exchange triangles are the unique (up to isomorphism) non split triangles*

$$L \to E \to M \to \Sigma L \text{ and } M \to E' \to L \to \Sigma M.$$

Let us extend the map $L \mapsto X_L$ from indecomposable to decomposable objects of $\mathcal{C}_Q$ by requiring that we have

$$X_N = X_{N_1} X_{N_2}$$

whenever $N = N_1 \oplus N_2$ (this is compatible with the muliplicativity of the Caldero-Chapoton map). We know that if $u_1, \ldots, u_n$ is a cluster and $B = (b_{ij})$ the associated exchange matrix, then the mutation at k yields the variable u_k' such that

$$u_k u_k' = \prod_{\substack{\text{arrows}\\ k\to i}} u_i + \prod_{\substack{\text{arrows}\\ j\to k}} u_j.$$

By combining this with the exchange triangles, we see that in the situation of c), we have

$$X_L X_M = X_E + X_{E'}.$$

We would like to generalize this identity to the case where the space $\mathsf{Ext}^1(L, M)$ is of higher dimension. For three objects L, M and N of $\mathcal{C}_Q$, let $Ext^1(L, M)_N$ be the subset of $\mathsf{Ext}^1(L, M)$ formed by those morphisms $\varepsilon \colon L \to \Sigma M$ such that in the triangle

$$M \to E \to L \xrightarrow{\varepsilon} \Sigma M\ ,$$

the object E is isomorphic to N (we do not fix an isomorphism). Notice that this subset is a cone (*i.e.* stable under multiplication by non zero scalars) in the vector space $\mathsf{Ext}^1(L, M)$.

Proposition 5.13 ([31])**.** *The subset $\mathsf{Ext}^1(L, M)_N$ is constructible in $\mathsf{Ext}^1(L, M)$. In particular, it is an algebraic variety (possibly reducible). It is empty for all but finitely isomorphism classes of objects N.*

If k is the field of complex numbers, we denote by χ the Euler characteristic with respect to singular cohomology with coefficients in a field. If k is an arbitrary algebraically closed field, we denote by χ the Euler characteristic with respect to étale cohomology with proper support.

Theorem 5.14 ([31]). *Suppose that L and M are objects of $\mathcal{C}_Q$ such that $\mathsf{Ext}^1(L,M) \neq 0$. Then we have*

$$X_L X_M = \sum_N \frac{\chi(\mathbb{P}\,\mathsf{Ext}^1(L,M)_N) + \chi(\mathbb{P}\,\mathsf{Ext}^1(M,L)_N)}{\chi(\mathbb{P}\,\mathsf{Ext}^1(L,M))}\, X_N\,,$$

where the sum is taken over all isomorphism classes of objects N of $\mathcal{C}_Q$.

Notice that in the theorem, the objects L and M may be decomposable so that X_L and X_M will not be cluster variables in general and the X_N do not form a linearly independent set in the cluster algebra. Thus, the formula should be considered as a relation rather than as an alternative definition for the multiplication of the cluster algebra. Notice that it nevertheless bears a close resemblance to the product formula in a dual Hall algebra: For two objects L and M in a finitary abelian category of finite global dimension, we have

$$[L] * [M] = \sum_{[N]} \frac{|\,\mathsf{Ext}^1(L,M)_N|}{|\,\mathsf{Ext}^1(L,M)|}\, [N]\,,$$

where the brackets denote isomorphism classes and the vertical bars the cardinalities of the underlying sets, *cf.* Proposition 1.5 of [117].

6. Categorification via cluster categories: the aycyclic case

6.1. **Categorification.** Let Q be a connected finite quiver without oriented cycles with vertex set $\{1,\ldots,n\}$. Let k be an algebraically closed field. We have seen in section 5.4 how to define the bounded derived category $\mathcal{D}_Q$. We still have a fully faithful functor from the mesh category of $\mathbb{Z}Q$ to the category of indecomposables of $\mathcal{D}_Q$ but this functor is very far from being essentially surjective. In fact, its image does not even contain the injective indecomposable kQ-modules. The methods of the preceding section therefore do not generalize but most of the results continue to hold. The derived category $\mathcal{D}_Q$ still has a Serre functor (the total left derived functor of the tensor product functor $? \otimes_B D(kQ)$). We can form the cluster category

$$\mathcal{C}_Q = \mathcal{D}_Q/(S^{-1}\Sigma^2)^{\mathbb{Z}}$$

as before and it is still a triangulated category in a canonical way such that the projection $\pi\colon \mathcal{D}_Q \to \mathcal{C}_Q$ becomes a triangle functor [91]. Moreover, the decomposition theorem 5.1 holds for $\mathcal{C}_Q$ and each object L of $\mathcal{C}_Q$ decomposes into a direct sum

$$L = \pi(M) \oplus \bigoplus_{i=1}^{n} \pi(\Sigma P_i)^{m_i}$$

for some module M and certain multiplicities m_i, $1 \leq i \leq n$, *cf.* [10]. We put

$$X_L = CC(M) \prod_{i=1}^{n} x_i^{m_i} ,$$

where $CC(M)$ is defined as in section 5.3 Notice that in general, X_L can only be expected to be an element of the fraction field $\mathbb{Q}(x_1, \ldots, x_n)$, not of the cluster algebra $\mathcal{A}_Q$ inside this field. (The exponents in the formula for X_L are perhaps more transparent in equation 7.5.1 below).

Theorem 6.1. *Let Q be a finite quiver without oriented cycles with vertex set $\{1, \ldots, n\}$.*

a) *The map $L \mapsto X_L$ induces a bijection from the set of isomorphism classes of rigid indecomposables of the cluster category $\mathcal{C}_Q$ onto the set of cluster variables of the cluster algebra $\mathcal{A}_Q$.*
b) *Under this bijection, the clusters correspond exactly to the cluster-tilting sets, i.e. the sets $T_1, \ldots, T_n$ of rigid indecomposables such that*

$$Ext^1(T_i, T_j) = 0$$

for all i, j.
c) *For a cluster-tilting set $T_1, \ldots, T_n$, the quiver of the endomorphism algebra of the sum of the T_i does not have loops nor 2-cycles and encodes the exchange matrix of the* [69] *seed containing the corresponding cluster.*
d) *If L and M are rigid indecomposables such that the space $\mathsf{Ext}^1(L, M)$ is one-dimensional, then we have the generalized exchange relation*

$$X_L X_M = X_B + X_{B'} \tag{6.1.1}$$

where B and B' are the middle terms of 'the' non split triangles

$$L \longrightarrow B \longrightarrow M \longrightarrow \Sigma L \quad \text{and} \quad M \longrightarrow B' \longrightarrow L \longrightarrow \Sigma M .$$

Parts a), b) and d) of the theorem are proved in [30] and part c) in [11]. The proofs build on work by many authors notably Buan-Marsh-Reiten-Todorov [25] Buan-Marsh-Reiten [11], Buan-Marsh-Reineke-Reiten-Todorov [10], Marsh-Reineke-Zelevinsky [107], ... and especially on Caldero-Chapoton's explicit formula for X_L proved in [28] for orientations of simply laced Dynkin diagrams. Another crucial ingredient of the proof is the Calabi-Yau property of the cluster category. An alternative proof of part c) was given by A. Hubery [81] for quivers whose underlying graph is an extended simply laced Dynkin diagram.

We describe the main steps of the proof of a). The mutation of cluster-tilting sets is defined using the construction of section 5.10.

1) If T is a cluster-tilting object, then the quiver Q_T of its endomorphism algebra does not have loops or 2-cycles. If T' is obtained from T by mutation at the summand T_1, then the quiver $Q_{T'}$ of the endomorphism algebra of T' is the mutation at the vertex 1 of the quiver Q_T, *cf.* [11].
2) Each rigid indecomposable is contained in a cluster-tilting set. Any two cluster-tilting sets are linked by a finite sequence of mutations. This is deduced in [10] from the work of Happel-Unger [78].
3) If (T_1, T_1^*) is an exchange pair and
$$T_1^* \to E \to T_1 \to \Sigma T_1^* \text{ and } T_1 \to E' \to T_1^* \to \Sigma T_k$$
are the exchange triangles, then we have
$$X_{T_1} X_{T_1^*} = X_E + X_{E'}.$$
This is shown in [30].

It follows from 1)-3) that the map $L \to X_L$ does take rigid indecomposables to cluster variables and that each cluster variable is obtained in this way. It remains to be shown that a rigid indecomposable L is determined up to isomorphism by X_L. This follows from

4) If M is a rigid indecomposable module, the denominator of X_M is $x_1^{d_1} \dots x_n^{d_n}$, *cf.* [30].

Indeed, a rigid indecomposable module M is determined, up to isomorphism, by its dimension vector.

We sum up the relations between the cluster algebra and the cluster category in the following table

cluster algebra	cluster category
multiplication	direct sum
addition	?
cluster variables	rigid indecomposables
clusters	cluster-tilting sets
mutation	mutation
exchange relation	exchange triangles
$xx^* = m + m'$	$T_k \to M \to T_k^* \to \Sigma T_k$
	$T_k^* \to M' \to T_k \to \Sigma T_k^*$

6.2. **Two applications.** Theorem 6.1 does shed new light on cluster algebras. In particular, we have the following

Corollary 6.2 (Caldero-Reineke [32])**.** *Suppose that Q does not have oriented cycles. Then all cluster variables of $\mathcal{A}_Q$ belong to $\mathbb{N}[x_1^{\pm}, \dots, x_n^{\pm}]$.*

This settles a conjecture of Fomin-Zelevinsky [51] in the case of cluster algebras associated with acyclic quivers, for cluster expansions in the initial cluster. The proof is based on Lusztig's [106] and in this sense it does not quite live up to the hopes that cluster theory ought to explain Lusztig's results. However, it does show that the conjecture is true for this important class of cluster algebras.

Here are two applications to the exchange graph of the cluster algebra associated with an acyclic quiver Q:

Corollary 6.3 ([30]). a) *For any cluster variable x, the set of seeds whose clusters contain x form a connected subgraph of the exchange graph.*
b) *The set of seeds whose quiver does not have oriented cycles form a connected subgraph (possibly empty) of the exchange graph.*

For acyclic cluster algebras, parts a) and b) confirm conjecture 4.14 parts (3) and (4) by Fomin-Zelevinsky in [54]. By b), the cluster algebra associated with a quiver without oriented cycles has a well-defined cluster-type.

6.3. **Cluster categories and singularities.** The construction of cluster categories may seem a bit artificial. Nevertheless, cluster categories do occur 'in nature'. In particular, certain triangulated categories associated with singularities are equivalent to cluster categories. We illustrate this on the following example: Let the cyclic group G of order 3 act on a three-dimensional complex vector space V by scalar multiplication with a primitive third root of unity. Let S be the completion at the origin of the coordinate algebra of V and let $R = S^G$ the fixed point algebra, corresponding to the completion of the singularity at the origin of the quotient $V//G$. The algebra R is a Gorenstein ring, *cf. e.g.* [124], and an isolated singularity of dimension 3, *cf. e.g.* Corollary 8.2 of [83]. The category $\mathrm{CM}(R)$ of maximal Cohen-Macaulay modules is an exact Frobenius category and its stable category $\underline{\mathrm{CM}}(R)$ is a triangulated category. By Auslander's results [6], *cf.* Lemma 3.10 of [125], it is 2-Calabi Yau. One can show that it is equivalent to the cluster category $\mathcal{C}_Q$ for the quiver

$$Q: \ 1 \ \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \ 2$$

by an equivalence which takes the cluster-tilting object $T = kQ$ to S considered as an R-module. This example can be found in [92], where it is deduced from an abstract characterization of cluster categories. A number of similar examples can be found in [26] and [94].

7. Categorification via 2-Calabi-Yau categories

The extension of the results of the preceding sections to quivers containing oriented cycles is the subject of ongoing research, *cf.* for example [40] [62]

[8]. Here we present an approach based on the fact that many arguments developed for cluster categories apply more generally to suitable triangulated categories, whose most important property it is to be Calabi-Yau of dimension 2. As an application, we will sketch a proof of the periodicity conjecture in section 8.

7.1. **Definition and main examples.** Let k be an algebraically closed field and let $\mathcal{C}$ a triangulated category with suspension functor Σ where all idempotents split (*i.e.* each idempotent endomorphism e of an object M is the projection onto M_1 along M_2 in a decomposition $M = M_1 \oplus M_2$). We assume that

1) $\mathcal{C}$ is Hom-finite (*i.e.* we have $\dim \mathcal{C}(L, M) < \infty$ for all L, M in $\mathcal{C}$) and the decomposition theorem 5.1 holds for $\mathcal{C}$;
2) $\mathcal{C}$ is 2-Calabi-Yau, *i.e.* we are given bifunctorial isomorphisms
$$D\mathcal{C}(L, M) \xrightarrow{\sim} \mathcal{C}(M, \Sigma^2 L)\ ,\ L, M \in \mathcal{C};$$
3) $\mathcal{C}$ admits a cluster-tilting object T, *i.e.*
   a) T is the sum of pairwise non-isomorphic indecomposables,
   b) T is rigid and
   c) for each object L of $\mathcal{C}$, if $\mathsf{Ext}^1(T, L)$ vanishes, then L belongs to the category $\mathsf{add}(T)$ of direct factors of finite direct sums of copies of T.

If all these assumptions hold, we say that $(\mathcal{C}, T)$ *is a* 2*-Calabi-Yau category with cluster tilting object.* If Q is a finite quiver, we say that a 2-Calabi-Yau category $\mathcal{C}$ with cluster-tilting object T is a 2*-Calabi-Yau realization of* Q if Q is the quiver of the endomorphism algebra of T.

For example, if $\mathcal{C}$ is the cluster category of a finite quiver Q without oriented cycles, conditions 1) and 2) hold and an object T is cluster-tilting in the above sense iff it is the direct sum of a cluster-tilting set $T_1, \ldots, T_n$, where n is the number of vertices of Q, *cf.* [10]. The 'initial' cluster-tilting object in this case is $T = kQ$ (the image in $\mathcal{C}_Q$ of the free module of rank one) and $(\mathcal{C}, kQ)$ is the *canonical* 2*-Calabi-Yau realization* of the quiver Q (without oriented cycles).

The second main class of examples comes from the work of Geiss-Leclerc-Schröer: Let Δ be a simply laced Dynkin diagram, $\vec{\Delta}$ a quiver with underlying graph Δ and $\overline{\Delta}$ the *doubled quiver* obtained from $\vec{\Delta}$ by adjoining an arrow $\alpha^* \colon j \to i$ for each arrow $\alpha \colon i \to j$. The *preprojective algebra* $\Lambda = \Lambda(\Delta)$ is the quotient of the path algebra of $\overline{\Delta}$ by the ideal generated by the relator
$$\sum_{\alpha \in Q_1} \alpha\alpha^* - \alpha^*\alpha.$$

For example, if $\vec{\Delta}$ is the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \ ,$$

then $\overline{\Delta}$ is the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\alpha^*"] \arrow[r, shift left, "\beta"] & 3 \arrow[l, shift left, "\beta^*"] \end{tikzcd}$$

and the ideal generated by the above sum of commutators is also generated by the elements

$$\alpha^*\alpha, \alpha\alpha^* - \beta^*\beta, \beta\beta^*.$$

It is classical, *cf. e.g.* [113], that the algebra $\Lambda = \Lambda(\Delta)$ is a finite-dimensional (!) selfinjective algebra (*i.e.* Λ is injective as a right Λ-module over itself). Let $\mathsf{mod}\,\Lambda$ denote the category of k-finite-dimensional right Λ-modules. The *stable module category* $\underline{\mathsf{mod}}\Lambda$ is the quotient of $\mathsf{mod}\,\Lambda$ by the ideal of all morphisms factoring through a projective module. This category carries a canonical triangulated structure (like any stable module category of a self-injective algebra): The suspension is constructed by choosing exact sequences of modules

$$0 \to L \to IL \to \Sigma L \to 0$$

where IL is injective (but not necessarily functorial in L; the object ΣL becomes functorial in L when we pass to the stable category). The triangles are by definition isomorphic to standard triangles obtained from exact sequences of modules as follows: Let

$$0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$$

be a short exact sequence of $\mathsf{mod}\,\Lambda$. Choose a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "\mathbf{1}"'] & M \arrow[r, "p"] \arrow[d, dotted] & N \arrow[r] \arrow[d, dotted, "e"] & 0 \\ 0 \arrow[r] & L \arrow[r] & IL \arrow[r] & \Sigma L \arrow[r] & 0 \end{tikzcd}$$

Then the image of (i, p, e) is a standard triangle in the stable module category. As shown in [37], the stable module category $\mathcal{C} = \underline{\mathsf{mod}}\Lambda$ is 2-Calabi-Yau and it is easy to check that assumption 1) holds.

Theorem 7.1 (Geiss-Leclerc-Schröer)**.** *The category $\mathcal{C} = \underline{\mathsf{mod}}\Lambda$ admits a cluster-tilting object T such that the quiver of $\mathsf{End}(T)$ is obtained from that of the category of indecomposable $k\Delta$-modules by deleting the injective vertices and adding an arrow $v \to \tau v$ for each non projective vertex v.*

Here, by the quiver of the category of indecomposable $k\Delta$-modules, we mean the full subquiver of the repetition $\mathbb{Z}Q$ which is formed by the vertices corresponding to modules (complexes concentrated in degree 0). Thus for $\Delta = \vec{A}_5$, this quiver is as follows:

$$\begin{tikzcd}
 & & & & P_5 = I_1 \arrow[dr] & & & & \\
 & & & P_4 \arrow[ur] \arrow[dr] & & I_2 \arrow[dr] & & & \\
 & & P_3 \arrow[ur] \arrow[dr] & & 7 \arrow[ur] \arrow[dr] & & I_3 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & 6 \arrow[ur] \arrow[dr] & & 10 \arrow[ur] \arrow[dr] & & I_4 \arrow[dr] & \\
P_1 \arrow[ur] & & 5 \arrow[ur] & & 9 \arrow[ur] & & 12 \arrow[ur] & & I_5
\end{tikzcd}$$

where we have marked the indecomposable projectives P_i and the indecomposable injectives I_j. If we remove the vertices corresponding to the indecomposable injectives (and all the arrows incident with them) and add an arrow $v \to \tau v$ for each vertex not corresponding to an indecomposable projective, we obtain the following quiver

$$\begin{tikzcd}
 & & & 0 \arrow[dr] & & & \\
 & & 1 \arrow[ur] \arrow[dr] & & 2 \arrow[ll] \arrow[dr] & & \\
 & 3 \arrow[ur] \arrow[dr] & & 4 \arrow[ll] \arrow[ur] \arrow[dr] & & 5 \arrow[ll] \arrow[dr] & \\
6 \arrow[ur] & & 7 \arrow[ll] \arrow[ur] & & 8 \arrow[ll] \arrow[ur] & & 9 \arrow[ll]
\end{tikzcd}$$

In a series of papers [65] [67] [66] [63] [62] [68] [64], Geiss-Leclerc-Schröer have obtained remarkable results for a class of quivers which are important in the study of (dual semi-)canonical bases. They use an analogue [68] of the Caldero-Chapoton map due ultimately to Lusztig [105]. The class they consider has been further enlarged by Buan-Iyama-Reiten-Scott [7]. Thanks to their results, an analogue of Caldero-Chapoton's formula and a weakened version of theorem 6.1 was proved in [58] for an even larger class.

7.2. **Calabi-Yau reduction.** Suppose that $(\mathcal{C}, T)$ is a 2-Calabi-Yau realization of a quiver Q so that for $1 \leq i \leq n$, the vertex i of Q corresponds to the indecomposable summand T_i of T. Let J be a subset of the set of vertices of Q and let Q' be the quiver obtained from Q by deleting all vertices in J and all arrows incident with one of these vertices. Let $\mathcal{U}$ be the full subcategory of $\mathcal{C}$ formed by the objects U such that $\mathsf{Ext}^1(T_j, U) = 0$ for all

$j \in J$. Note that all T_i, $1 \leq i \leq n$, belong to $\mathcal{U}$. Let $\langle T_j | j \in J \rangle$ denote the ideal of $\mathcal{U}$ generated by the identities of the objects T_j, $j \in J$. By imitating the construction of the triangulated structure on a stable category, one can endow the quotient

$$\mathcal{C}' = \mathcal{U}/\langle T_j | j \in J \rangle$$

with a canonical structure of triangulated category, *cf.* [83].

Theorem 7.2 (Iyama-Yoshino [83]). *The pair $(\mathcal{C}', T)$ is a 2-Calabi-Yau realization of the quiver Q'. Moreover, the projection $\mathcal{U} \to \mathcal{C}'$ induces a bijection between the cluster-tilting sets of $\mathcal{C}$ containing the T_j, $j \in J$, and the cluster-tilting sets of $\mathcal{C}'$.*

7.3. **Mutation.** Let $(\mathcal{C}, T)$ be a 2-Calabi-Yau category with cluster-tilting object. Let T_1 be an indecomposable direct factor of T.

Theorem 7.3 (Iyama-Yoshino [83]). *Up to isomorphism, there is a unique indecomposable object T_1^* not isomorphic to T_1 such that the object $\mu_1(T)$ obtained from T by replacing the indecomposable summand T_1 with T_1^* is cluster-tilting.*

We call $\mu_1(T)$ the *mutation* of T at T_1. If $\mathcal{C}$ is the cluster category of a finite quiver without oriented cycles, this operation specializes of course to the one defined in section 5.10. However, in general, the quiver Q of the endomorphism algebra of T may contain loops and 2-cycles and then the quiver of the endomorphism algebra of $\mu_1(T)$ is not determined by Q. Let us illustrate this phenomenon on the following example (taken from Proposition 2.6 of [26]): Let $\mathcal{C}$ be the orbit category of the bounded derived category $\mathcal{D}_{\vec{D}_6}$ under the action of the autoequivalence τ^2. Then $\mathcal{C}$ satisfies the assumptions 1) and 2) of section 7.1. Its category of indecomposables is equivalent to the mesh category of the quiver

$$\begin{tikzcd}
C \arrow[dr] & & C' \arrow[dr] & & C \arrow[dr] & \\
B \arrow[r] & 5 \arrow[r] \arrow[ur] \arrow[dr] & B' \arrow[r] & 11 \arrow[r] \arrow[ur] \arrow[dr] & B \arrow[r] & 17 \\
1 \arrow[ur] \arrow[dr] & & 7 \arrow[ur] \arrow[dr] & & 13 \arrow[ur] \arrow[dr] & \\
& 4 \arrow[ur] \arrow[dr] & & 10 \arrow[ur] \arrow[dr] & & 16 \\
A \arrow[ur] & & A' \arrow[ur] & & A \arrow[ur] &
\end{tikzcd}$$

where the vertices labeled A, 1, B, C, 4, 5 on the left have to be identified with the vertices labeled A, 13, B, C, 16, 17 on the right. In this case, there are exactly 6 indecomposable rigid objects, namely A, B, C, A', B' and C'.

There are exactly 6 cluster-tilting sets. The following is the exchange graph: Its vertices are cluster-tilting sets (we write AC instead of $\{A, C\}$) and its edges represent mutations.

$$\begin{tikzcd}
& AC \arrow[r, dash] \arrow[dl, dash] & AB \arrow[dr, dash] & \\
B'C \arrow[dr, dash] & & & C'B \arrow[dl, dash] \\
& B'A' \arrow[r, dash] & C'A' &
\end{tikzcd}$$

The quivers of the endomorphism algebras are as follows:

$$\begin{array}{rl}
AC, AB: & \begin{tikzcd} \circ \arrow[r, shift left] & \circ \arrow[l, shift left] \arrow[loop right] \end{tikzcd} \\
B'C, C'B: & \begin{tikzcd} \circ \arrow[r, shift left] & \circ \arrow[l, shift left] \end{tikzcd} \\
B'A', C'A': & \begin{tikzcd} \circ \arrow[loop left] \arrow[r, shift left] & \circ \arrow[l, shift left] \end{tikzcd}
\end{array}$$

In the setting of the above theorem, there are still exchange triangles as in theorem 5.12 but their description is different: Let $\mathcal{T}'$ be the full subcategory of $\mathcal{C}$ formed by the direct sums indecomposables T_i, where i is different from k. A *left $\mathcal{T}'$-approximation of T_k* is a morphism $f: T_k \to T'$ with T' int $\mathcal{T}'$ such that any morphism from T_k to an object of $\mathcal{T}'$ factors through f. A left $\mathcal{T}'$-approximation f is *minimal* if for each endomorphism g of T', the equality $gf = f$ implies that g is invertible. Dually, one defines (minimal) right $\mathcal{T}'$-approximations. It is not hard to show that they always exist.

Theorem 7.4 (Iyama-Yoshino [83]). *If (T_1, T_1^*) is an exchange pair, there are non split triangles, unique up to isomorphism,*

$$T_1 \xrightarrow{f} E' \to T_1^* \to \Sigma T_1 \quad \textit{and} \quad T_1^* \to E \xrightarrow{g} T_1 \to \Sigma T_1^*$$

such that f is a minimal left $\mathcal{T}'$-approximation and g a minimal right $\mathcal{T}'$-approximation.

7.4. **Simple mutations, reachable cluster-tilting objects.** Let $(\mathcal{C}, T)$ be a 2-CY category with cluster-tilting object and T_1 an indecomposable direct summand of T. Let (T_1, T_1^*) be the corresponding exchange pair and

$$T_1^* \to E \to T_1 \to \Sigma T_1^*$$

the exchange triangle. The long exact sequence induced in $\mathcal{C}(T_1, ?)$ by this triangle yields a short exact sequence

$$\mathcal{C}'_{\mathcal{T}}(T_1, T_1) \to \mathcal{C}(T_1, T_1) \to \mathsf{Ext}^1(T_1, T_1^*) \to 0\,,$$

where the leftmost term is the space of those endomorphisms of T_1 which factor through a sum of copies of T/T_1. Now the algebra $\mathcal{C}(T_1, T_1)$ is local and its residue field is k (since k is algebraically closed). We deduce the following lemma.

Lemma 7.5. *The quiver of the endomorphism algebra of T does not have a loop at the vertex corresponding to T_1 iff we have* $\dim \mathsf{Ext}^1(T_1, T_1^*) = 1$ *iff* $\mathsf{Ext}^1(T_1, T_*)$ *is a simple module over* $\mathcal{C}(T_1, T_1)$*. In this case, in the exchange triangles*

$$T_1^* \to E \to T_1 \to \Sigma T_1 \text{ and } T_1 \to E' \to T_1^* \to \Sigma T_1 \, ,$$

we have

$$E = \bigoplus_{\substack{arrows \\ i \to 1}} T_i \text{ and } E' = \bigoplus_{\substack{arrows \\ 1 \to j}} T_j .$$

We say that the mutation at T_1 is *simple* if the conditions of the lemma hold. If $\mathcal{C}$ is the cluster category of a finite quiver without oriented cycles, all mutations in $\mathcal{C}$ are simple, by part c) of theorem 6.1. By a theorem of Geiss-Leclerc-Schröer, if $\mathcal{C}$ is the stable module category of a preprojective algebra of Dynkin type, the quiver of any cluster-tilting object in $\mathcal{C}$ does not have loops nor 2-cycles. So again, all mutations are simple in this case.

Theorem 7.6 (Buan-Iyama-Reiten-Scott [7])**.** *Suppose that the quivers Q and Q' of the endomorphism algebras of T and $T' = \mu_1(T)$ do not have loops nor 2-cycles. Then Q' is the mutation of Q at the vertex 1.*

We define a cluster-tilting object T' to be *reachable from T* if there is a sequence of mutations

$$T = T^{(0)} \rightsquigarrow T^{(1)} \rightsquigarrow \ldots \rightsquigarrow T^{(N)} = T$$

such that the quiver of $\mathsf{End}(T^{(i)})$ does not have loops nor 2-cycles for all $1 \leq i \leq N$. We define a rigid indecomposable of $\mathcal{C}$ to be *reachable from T* if it is a direct summand of a reachable cluster-tilting object.

Corollary 7.7. *If a cluster-tilting object T' is reachable from T, then the quiver of the endomorphism algebra of T' is mutation-equivalent to the quiver of the endomorphism algebra of T. If $\mathcal{C}$ is a cluster-category or the stable module category of the preprojective algebra of a Dynkin diagram, then all quivers mutation-equivalent to Q are obtained in this way.*

7.5. **Combinatorial invariants.** Let $(\mathcal{C}, T)$ be a 2-Calabi-Yau category with cluster-tilting object. Let $\mathcal{T}$ be the full subcategory whose objects are all direct factors of finite direct sums of copies of T. Notice that $\mathcal{T}$ is equivalent to the category of finitely generated projective modules over the endomorphism algebra of T. Let $K_0(\mathcal{T})$ be the Grothendieck group of the additive category $\mathcal{T}$. Thus, the group $K_0(\mathcal{T})$ is free abelian on the isomorphism classes of the indecomposable summands of T.

Lemma 7.8 (Keller-Reiten [93]). *For each object L of $\mathcal{C}$, there is a triangle*

$$T_1 \to T_0 \to L \to \Sigma T_1$$

such that T_0 and T_1 belong to $\mathcal{T}$. The difference

$$[T_0] - [T_1]$$

considered as an element of $K_0(\mathcal{T})$ does not depend on the choice of this triangle.

In the situation of the lemma, we define the *index* $\mathsf{ind}(L)$ *of* L as the element $[T_0] - [T_1]$ of $K_0(\mathcal{T})$.

Theorem 7.9 (Dehy-Keller [38]). a) *Two rigid objects are isomorphic iff their indices are equal.*

b) *The indecomposable summands of a cluster-tilting object form a basis of $K_0(\mathcal{T})$. In particular, all cluster-tilting objects have the same number of pairwise non isomorphic indecomposable summands.*

Let B be the endomorphism algebra of T. For two finite-dimensional right B-modules L and M put

$$\langle L, M\rangle_a = \dim \mathsf{Hom}(L, M) - \dim \mathsf{Ext}^1(L, M) - \dim \mathsf{Hom}(M, L) + \dim \mathsf{Ext}^1(M, L).$$

This is the antisymmetrization of a truncated Euler form. A priori it is defined on the split Grothendieck group of the category $\mathsf{mod}\, B$ (*i.e.* the quotient of the free abelian group on the isomorphism classes divided by the subgroup generated by all relations obtained from direct sums in $\mathsf{mod}\, B$).

Proposition 7.10 (Palu [111]). *The form $\langle , \rangle_a$ descends to an antisymmetric form on $K_0(\mathsf{mod}\, B)$. Its matrix in the basis of the simples is the antisymmetric matrix associated with the quiver of B (loops and 2-cycles do not contribute to this matrix).*

Let $T_1, \ldots, T_n$ be the pairwise non isomorphic indecomposable direct summands of T. For $L \in \mathcal{C}$, we define the integer $g_i(L)$ to be the multiplicity of $[T_i]$ in the index $\mathsf{ind}(L)$, $1 \leq i \leq n$, and we define the element X'_L of the field $\mathbb{Q}(x_1, \ldots, x_n)$ by

$$X'_L = \prod_{i=1}^{n} x_i^{g_i(L)} \sum_e \chi(\mathsf{Gr}_e(\mathsf{Ext}^1(T, L))) \prod_{i=1}^{n} x_i^{\langle S_i, e\rangle_a}, \tag{7.5.1}$$

where S_i is the simple quotient of the indecomposable projective B-module $P_i = \mathsf{Hom}(T, T_i)$. Notice that we have $X'_{T_i} = x_i$, $1 \leq i \leq n$. If $\mathcal{C}$ is the cluster-category of a finite quiver Q without oriented cycles and $T = kQ$, then we have $X'_L = X_{\Sigma L}$ in the notations of section 6 and the formula for X'_L is essentially another expression for the Caldero-Chapoton formula.

Now let Q be the quiver of the endomorphism algebra of T in $\mathcal{C}$ and let $\mathcal{A}_Q$ be the associated cluster algebra.

Theorem 7.11 (Palu [111])**.** *If L and M are objects of $\mathcal{C}$ such that $\mathsf{Ext}^1(L, M)$ is one-dimensional, then we have*

$$X'_L X'_M = X'_E + X'_{E'} ,$$

where

$$L \to E \to M \to \Sigma L \text{ and } M \to E' \to L \to \Sigma M$$

are 'the' two non split triangles. Thus, if L is a rigid indecomposable reachable from T, then X'_L is a cluster variable of $\mathcal{A}_Q$.

Corollary 7.12. *Suppose that $\mathcal{C}$ is the cluster category $\mathcal{C}_Q$ of a finite quiver Q without oriented cycles. Let $\mathcal{A}_Q \subset \mathbb{Q}(x_1, \ldots, x_n)$ be the associated cluster algebra and $x \in \mathcal{A}_Q$ a cluster variable. Let $L \in \mathcal{C}_Q$ be the unique (up to isomorphism) indecomposable rigid object such that $x = X_{\Sigma L}$. Let $u_1, \ldots, u_n$ be an arbitrary cluster of $\mathcal{A}_Q$ and $T_1, \ldots, T_n$ the cluster-tilting set such that $u_i = X_{T_i}$, $1 \leq i \leq n$. Then the expression of x as a Laurent polynomial in $u_1, \ldots, u_n$ is given by*

$$x = X'_L(u_1, \ldots, u_n).$$

The expression for X'_L makes it natural to define the polynomial $F'_L \in \mathbb{Z}[y_1, \ldots y_n]$ by

$$F'_L = \sum_e \chi(\mathsf{Gr}_e(\mathsf{Ext}^1(T, L))) \prod_{j=1}^n y_j^{e_j}. \tag{7.5.2}$$

We then have

$$X'_L = \prod_{i=1}^n x_i^{g_i} F'_L(\prod_{i=1}^n x_i^{b_{i1}}, \ldots, \prod_{i=1}^n x_i^{b_{in}}).$$

The polynomial F'_L is related to Fomin-Zelevinsky's F-polynomials [55] [56], as we will see below.

7.6. **More mutants categorified.** Let Q be a finite quiver without loops nor 2-cycles with vertex set $\{1, \ldots, n\}$. Let $\mathbb{T}_n$ be the regular n-ary tree: Its edges are labeled by the integers $1, \ldots, n$ such that the n edges emanating from each vertex carry different labels. Let t_0 be a vertex of $\mathbf{T}_n$. To each vertex t of $\mathbb{T}_n$ we associate a seed $(Q_t, \mathbf{x}_t)$ (*cf.* section 3.2) such that at $t = t_0$, we have $Q_t = Q$ and $\mathbf{x}_t = \{x_1, \ldots, x_n\}$ and whenever t is linked to t' by an edge labeled i, we have $(Q_{t'}, \mathbf{x}_{t'}) = \mu_i(Q_t, \mathbf{x}_t)$.

Now assume that Q admits a 2-Calabi-Yau realization $(\mathcal{C}, T)$. According to the mutation theorem 7.6, we can associate a cluster-tilting object T_t to each vertex t of $\mathbb{T}_n$ such that at $t = t_0$, we have $T_t = T$ and that whenever t is linked to t' by an edge labeled i, we have $T_{t'} = \mu_i(T_t)$.

Now let

$$t_0 \xrightarrow{\ i_1\ } t_1 \xrightarrow{\ i_2\ } \cdots \xrightarrow{\ i_N\ } t_N$$

be a path in $\mathbb{T}_n$ and suppose that for each $1 \leq i \leq N$, the quiver of the endomorphism algebra of T_{t_i} does not have loops nor 2-cycles. Then it follows by induction from theorem 7.6 that the quiver of the endomorphism algebra of T_{t_i} is Q_{t_i}, $1 \leq i \leq N$, and from theorem 7.11 that the cluster $\mathbf{x}_{t_i}$ equals the image under $L \mapsto X'_L$ of the set of indecomposable direct factors of T_{t_i}.

Following [56], let us consider three other pieces of data associated with each vertex t of $\mathbb{T}_n$:

- the tropical Y-variables $y_{1,t}, \ldots, y_{n,t}$,
- the F-polynomials $F_{1,t}, \ldots, F_{n,t}$,
- the (non tropical) Y-variables $Y_{1,t}, \ldots, Y_{n,t}$.

Here the tropical Y-variables are monomials in the indeterminates $y_1, \ldots y_n$. At $t = t_0$, we have $y_{i,t} = y_i$, $1 \leq i \leq n$. If t is linked to t' by an edge labelled i, then

$$y_{j,t'} = \begin{cases} y_{i,t}^{-1} & \text{if } j = i \\ y_{j,t} y_{i,t}^{[b_{ij}]_+} & \text{if } j \neq i\ , \end{cases}$$

where b_{ij} is the antisymmetric matrix associated with Q_t and, for an integer a, we write $[a]_+$ for $\max(a, 0)$. Notice that $[b_{ij}]_+$ is the number of arrows from i to j in Q_t.

The F-polynomials lie in $\mathbb{Z}[y_1, \ldots, y_n]$. At $t = t_0$, they all equal 1. If t is linked to t' by an edge labeled i, then

$$F_{j,t'} = F_{j,t} \quad \text{if } j \neq i\ ,$$

$$F_{i,t'} = \frac{1}{F_{i,t}} \Big(\prod_{c_{lj} > 0} y_l^{c_{lj}} \prod_{j=1}^{n} F_{j,t}^{[b_{ij}]_+} + \prod_{c_{lj} < 0} y_l^{-c_{lj}} \prod_{j=1}^{n} F_{j,t}^{[-b_{ij}]_+} \Big)\ ,$$

where the c_{lj} are the exponents in the tropical Y-variables $y_{j,t} = \prod_{l=1}^{n} y_l^{c_{lj}}$ and $B = (b_{ij})$ is the antisymmetric matrix associated with Q_t.

Finally, the non tropical Y-variables $Y_{j,t}$ lie in the field $\mathbb{Q}(y_1, \ldots, y_n)$. At $t = t_0$, we have $Y_{j,t} = y_j$, $1 \leq j \leq n$, and if t and t' are linked by an edge labeled i, then

$$Y_{j,t'} = \begin{cases} Y_{i,t}^{-1} & \text{if } j = i \\ Y_{j,t} Y_{i,t}^{[b_{ij}]_+} (Y_{i,t} + 1)^{-b_{ij}} & \text{if } j \neq i. \end{cases} \tag{7.6.1}$$

The *principal extension* $\widetilde{Q}$ *of* Q is the quiver obtained from Q by adding, for each vertex i of Q, a new vertex i' and a new arrow $i' \to i$. We assume that we are given a 2-Calabi-Yau realization $(\widetilde{\mathcal{C}}, \widetilde{T})$ of $\widetilde{Q}$. Let $\mathcal{U} \subset \widetilde{\mathcal{C}}$ be the full subcategory of the objects U such that $\mathsf{Ext}^1(T_{i'}, U) = 0$ for each new vertex i'. Then, according to the theorem on Calabi-Yau reductions (*cf.* section 7.2), the quotient category

$$\mathcal{U}/\langle T_{i'} | i \in Q_0 \rangle$$

together with the image of $\widetilde{T}$ is a 2-Calabi-Yau realization of the quiver Q. We assume that $(\mathcal{C}, T)$ is this realization. Now let

$$t_0 \xrightarrow{\ i_1\ } t_1 \xrightarrow{\ i_2\ } \cdots \xrightarrow{\ i_N\ } t_N$$

be a path in $\mathbb{T}_n$ and suppose that for each $1 \leq i \leq N$, the quiver of the endomorphism algebra of T_{t_i} does not have loops nor 2-cycles. Let $t = t_N$ and let $T'_1, \ldots, T'_n$ be the indecomposable summands of $T' = T_t$. Recall from part b) of theorem 7.9 that the $[T'_l]$, $1 \leq l \leq n$, form a basis of the group $K_0(\mathcal{T})$, where $\mathcal{T}$ is the full subcategory of $\mathcal{C}$ formed by the direct summands of finite direct sums of copies of T.

Theorem 7.13. a) *The exchange matrix $B_t = (b_{ij})$ associated with t is the antisymmetric matrix associated with the quiver of the endomorphism algebra of T_t.*

b) *We have $y_{j,t} = \prod_{l=1}^{n} y_l^{c_{lj}}$, $1 \leq j \leq n$, where c_{lj} is defined by*

$$[T_j] = \sum_{l=1}^{n} c_{lj} [T'_l].$$

c) *We have $F_{j,t} = F'_{T'_j}$, $1 \leq j \leq n$, where $F'_{T'_j}$ is defined by equation 7.5.2.*

d) *We have*

$$Y_{j,t} = y_{j,t} \prod_{i=1}^{n} F_{i,t}^{b_{ij}}.$$

Here part a) follows by induction from theorem 7.6, part b) is easily proved by induction, part c) is Theorem 5.3 of [58] and part d) is Proposition 3.12 of [56].

7.7. **2-CY categories from algebras of global dimension 2.** The results of the preceding sections only become interesting if we are able to construct 2-Calabi-Yau realizations for large classes of quivers. In the case of a quiver without oriented cycles, this problem is solved by the cluster category. The results of Geiss-Leclerc-Schröer provide another large class of quivers admitting 2-Calabi-Yau realizations. Here, we will exhibit a construction which generalizes both cluster categories of quivers without oriented cycles and the stable categories of preprojective algebras of Dynkin diagrams.

Let k be an algebraically closed field and A a finite-dimensional k-algebra of global dimension ≤ 2. For example, A can be the path algebra of a finite quiver without oriented cycles. Let $\mathcal{D}_A$ be the bounded derived category of the category $\mathsf{mod}\, A$ of k-finite-dimensional right A-modules. It admits a Serre functor, namely the total derived functor of the tensor product $? \otimes_A DA$ with the k-dual bimodule of A considered as a bimodule over itself. We can form the orbit category

$$\mathcal{D}_A/(S^{-1}\Sigma^2)^{\mathbb{Z}}.$$

This is a k-linear category endowed with a suspension functor (induced by Σ) but in general it is no longer triangulated. Nevertheless, one can show that it embeds fully faithfully into a 'smallest triangulated overcategory' [91]. We denote this overcategory by $\mathcal{C}_A$ and call it the *generalized cluster category of A*.

Theorem 7.14 (Amiot [1]). *If the functor*

$$\mathsf{Tor}_2^A(?, DA) \colon \mathsf{mod}\, A \to \mathsf{mod}\, A$$

is nilpotent, then $\mathcal{C}_A$ is Hom-finite and 2-Calabi-Yau. Moreover, the image T of A in $\mathcal{C}_A$ is a cluster-tilting object. The quiver of its endomorphism algebra is obtained from that of A by adding, for each pair of vertices (i, j), a number of arrows equal to

$$\dim \mathsf{Tor}_2^A(S_j, S_i^{op})$$

from i to j, where S_j is the simple right module associated with j and S_i^{op} the simple left module associated with i.

We consider two classes of examples obtained from this theorem: First, let Δ be a simply laced Dynkin diagram and $k\vec{\Delta}$ the path algebra of a quiver with underlying graph Δ. Let A the *Auslander algebra of $k\vec{\Delta}$, i.e.* the endomorphism algebra of the direct sum of a system of representatives of the indecomposable B-modules modulo isomorphism. Then it is not hard to check that the assumptions of the theorem hold. The quiver of A is simply the quiver of the category of indecomposables of $k\vec{\Delta}$ and the minimal relations correspond to its meshes. For example, if we choose $\Delta = \vec{A}_4$ with the linear orientation, the quiver of the endomorphism algebra of the image T of A in $\mathcal{C}_A$ is the quiver obtained from Geiss-Leclerc-Schröer's construction at the end of section 7.1.

As a second class of examples, we consider an algebra A which is the tensor product $kQ \otimes_k kQ'$ of two path algebras of quivers Q and Q' without oriented cycles. Such an algebra is clearly of global dimension ≤ 2. Let us assume that Q and Q' are moreover Dynkin quivers. Then it is not hard to check that the functor $\mathsf{Tor}_2(?, DA)$ is indeed nilpotent. Thus, the theorem applies. Another elementary exercise in homological algebra shows that the space $\mathsf{Tor}_2^A(S_{j,j'} \otimes S_{i,i'}^{op})$ is at most one-dimensional and that it is non zero if and only if there is an arrow $i \to j$ in Q and an arrow $i' \to j'$ in Q'. This immediately yields the shape of the quiver of the endomorphism algebra of the image T of A in $\mathcal{C}_A$: It is the *tensor product* $Q \otimes Q'$ obtained from the product $Q \times Q'$ by adding an arrow $(j, j') \to (i, i')$ for each pair of arrows $i \to j$ of Q and $i' \to j'$ of Q'. For example, for suitable orientations of A_4 and D_5, we obtain the quiver of figure 4. Notice that if we perform mutations at the six vertices of the form (i, i'), where i is a sink of $Q = \vec{A}_4$ and i' a source of $Q' = \vec{D}_5$ (they are marked by $\bullet$), we obtain the quiver of figure 5 related to the periodicity conjecture for (A_4, D_5), *cf.* below.

8. Application: The periodicity conjecture

Let Δ and Δ' be two Dynkin diagrams with vertex sets I and I'. Let A and A' be the incidence matrices of Δ and Δ', *i.e.* if C is the Cartan matrix of Δ and J the identity matrix of the same format, then $A = 2J - C$. Let h and h' be the Coxeter numbers of Δ and Δ'.

The *Y-system of algebraic equations* associated with the pair of Dynkin diagrams (Δ, Δ') is a system of countably many recurrence relations in the variables $Y_{i,i',t}$, where (i, i') is a vertex of $\Delta \times \Delta'$ and t an integer. The system reads as follows:

$$Y_{i,i',t-1}Y_{i,i',t+1} = \frac{\prod_{j \in I}(1 + Y_{j,i',t})^{a_{ij}}}{\prod_{j' \in I'}(1 + Y_{i,j',t}^{-1})^{a'_{i'j'}}}. \tag{8.0.1}$$

Periodicity Conjecture 8.1. *All solutions to this system are periodic of period dividing* $2(h + h')$.

Here is an algebraic reformulation: Let K be the fraction field of the ring of integer polynomials in the variables $Y_{ii'}$, where i runs through the vertices of Δ and i' through those of Δ'. Define an automorphism φ of K by

$$\varphi(Y_{ii'}) = Y_{ii'}^{-1} \prod_j (1 + Y_{ji'})^{a_{ij}} \prod_{j'} (1 + Y_{ij'}^{-1})^{-a'_{i'j'}}. \tag{8.0.2}$$

Then, as in [55], it is not hard to check that the periodicity conjecture holds iff the automorphism φ is periodic of period dividing $h + h'$.

The conjecture was formulated

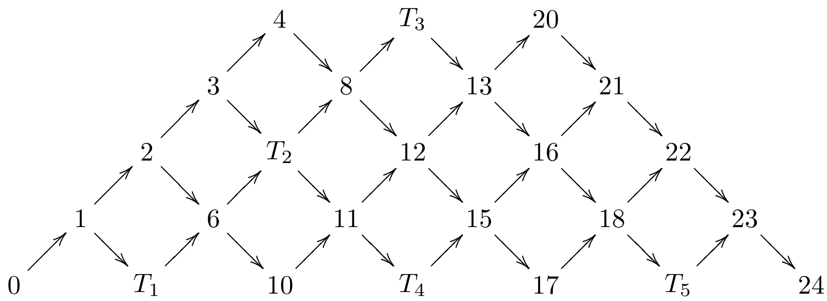


FIGURE 3. A cluster-tilting set in A_5

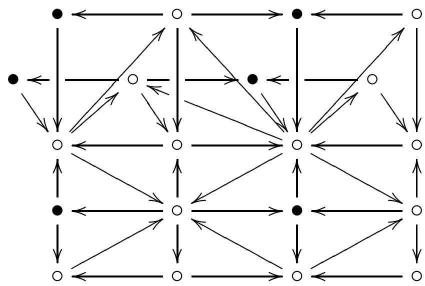


FIGURE 4. The quiver $\vec{A}_4 \otimes \vec{D}_5$

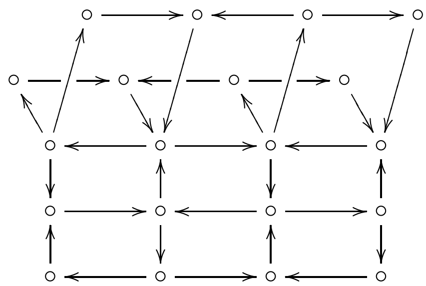


FIGURE 5. The quiver $\vec{A}_4 \square \vec{D}_5$

- by Al. B. Zamolodchikov for (Δ, A_1), where Δ is simply laced [126, (12)];
- by Kuniba-Nakanishi for (Δ, A_n), where Δ is not necessarily simply laced [101, (2a)], see also Kuniba-Nakanishi-Suzuki [102, B.6];
- by Gliozzi-Tateo for (Δ, Δ'), both simply laced (or of type T) [73].

It was proved

- for (A_n, A_1) by Frenkel-Szenes [57] (who produced explicit solutions) and by Gliozzi-Tateo [74] (via volumes of threefolds computed using triangulations),
- by Fomin-Zelevinsky [55] for (Δ, A_1), where Δ is not necessarily simply laced (via the cluster approach and a computer check for the exceptional types),
- by Volkov for (A_n, A_m), cf. also Henriques [79], by exhibiting explicit solutions using cross ratios,
- by Hernandez-Leclerc for (A_n, A_1) using representations of quantum affine algebras (which yield formulas for solutions in terms of q-characters). They expect to treat (A_n, Δ) similarly, *cf.* [80].

Theorem 8.2. *The periodicity conjecture 8.1 is true.*

Let us sketch a proof based on 2-Calabi-Yau categories: First, using the folding technique of Fomin-Zelevinsky's [55] one reduces the conjecture to the case where both Δ and Δ' are simply laced. Thus, from now on, we assume that Δ and Δ' are simply laced.

First step: We choose quivers Q and Q' whose underlying graphs are Δ and Δ'. We assume that Q and Q' are *alternating*, *i.e.* that each vertex is either a source or a sink. For example, we can consider the following quivers

$$\begin{tikzcd} \vec{A}_4 : & 1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd} ,$$

$$\begin{tikzcd} & & & & 4 \arrow[dl] \\ \vec{D}_5 : & 1 & 2 \arrow[l] \arrow[r] & 3 & \\ & & & & 5. \arrow[ul] \end{tikzcd}$$

We define the *square product* $Q \square Q'$ to be the quiver obtained from $Q \times Q'$ by reversing all the arrows in the full subquivers of the form $\{i\} \times Q'$ and $Q \times \{i'\}$, where i is a sink of Q and i' a source of Q'. The square product of the above quivers $\vec{A}_4$ and $\vec{D}_5$ is depicted in figure 5. The *initial Y-seed* associated with $Q \square Q'$ is the pair $\mathbf{y}_0$ formed by the quiver $Q \square Q'$ and the family of variables $Y_{i,i'}$, $(i, i') \in I \times I'$. We can apply mutations to it using the quiver mutation rule and the mutation rule for (non tropical) Y-variables given in equation 7.6.1.

A general construction. Let R be a quiver and v a sequence of vertices $v_1, \ldots, v_N$ of R. We assume that the composed mutation

$$\mu_v = \mu_{v_N} \ldots \mu_{v_2} \mu_{v_1}$$

transforms R into itself. Then clearly the same holds for the inverse sequence

$$\mu_v^{-1} = \mu_{v_1} \mu_{v_2} \ldots \mu_{v_N}.$$

Now the *restricted Y-system* associated with R and μ_v is the sequence of Y-seeds obtained from the initial Y-seed $\mathbf{y}_0$ associated with R by applying all integral powers of μ_v. Thus this system is given by a sequence of seeds $\mathbf{y}_t$, $t \in \mathbb{Z}$, such that $\mathbf{y}_0$ is the initial Y-seed associated with R and, for all $t \in \mathbb{Z}$, $\mathbf{y}_{t+1}$ is obtained from $\mathbf{y}_t$ by the sequence of mutations μ_v.

Second step. For two elements σ, σ' of $\{+,-\}$ define the following composed mutation of $Q\Box Q'$:

$$\mu_{\sigma,\sigma'} = \prod_{\varepsilon(i)=\sigma, \varepsilon(i')=\sigma'} \mu_{(i,i')}.$$

Notice that there are no arrows between any two vertices of the index set so that the order in the product does not matter. Then it is easy to check that $\mu_{+,+}\mu_{-,-}$ and $\mu_{-,+}\mu_{+,-}$ both transform $Q\Box Q'$ into $(Q\Box Q')^{op}$ and vice versa. Thus the composed sequence of mutations

$$\mu_{\Box} = \mu_{-,-}\mu_{+,+}\mu_{-,+}\mu_{+,-}$$

transforms $Q\Box Q'$ into itself. We define the *Y-system* $\mathbf{y}_{\Box}$ associated with $Q\Box Q'$ to be the restricted Y-system associated with $Q\Box Q'$ and $\mu_{\Box}$. It is easy to check that in this system, the variables occurring at stage t are obtained from the initial variables by applying the automorphism φ^t, where φ is defined in equation 8.0.2. Thus, it suffices to prove that the system $\mathbf{y}_{\Box}$ is periodic of period dividing $h+h'$.

Third step. One checks easily that we have

$$\mu_{+-}(Q\Box Q') = Q \otimes Q',$$

where the tensor product $Q \otimes Q'$ is defined at the end of section 7.7. For the above quivers $\vec{A}_4$ and $\vec{D}_5$, the tensor product is depicted in figure 4. Therefore, the periodicity of the restricted Y-system associated with $Q\Box Q'$ and $\mu_{\Box}$ is equivalent to that of the restricted Y-system associated with $Q \otimes Q'$ and

$$\mu_{\otimes} = \mu_{+,-}\mu_{-,-}\mu_{+,+}\mu_{-,+}.$$

Fourth step. As we have seen in section 7.7, the quiver $Q \otimes Q'$ admits a 2-Calabi-Yau realization given by the cluster category $\mathcal{C}_{kQ\otimes kQ'}$ associated with the tensor product of the path algebras $kQ \otimes kQ'$. Let T be the initial cluster tilting object. By theorem 7.3, we can define its iterated mutations

$$T_t = \mu_\otimes^t(T)$$

for all $t \in \mathbb{Z}$. Now we use the

Proposition 8.3. *None of the endomorphism quivers occurring in the sequence of mutated cluster-tilting objects joining T to T_t contains loops nor 2-cycles.*

It is not hard to see that $(\mathcal{C}_{kQ\otimes kQ'}, T)$ is the Calabi-Yau reduction of a 2-Calabi-Yau realization of the principal extension of $Q\Box Q'$. Therefore, it follows from theorem 7.13 that the quiver of the endomorphism algebra of T_t is $Q \otimes Q'$ and that the Y-variables in the Y-seed $\mathbf{y}_{\Box,t}$ can be expressed in terms of the triangulated category $\mathcal{C}_{kQ\otimes kQ'}$ and the objects T and T_t. Thus, it suffices to show that T_t is isomorphic to T whenever t is an integer multiple of $h + h'$.

Fifth step. Let $\tau\otimes\mathbf{1}$ denoted the auto-equivalence of the bounded derived category of $kQ \otimes kQ'$ given by the total left derived functor of the tensor product with the bimodule complex $(\Sigma^{-1}D(kQ)) \otimes kQ'$, where D is duality over k. It induces an autoequivalence of $\mathcal{C}_{kQ\otimes kQ'}$ which we still denote by $\tau \otimes \mathbf{1}$. Define Φ to be the autoequivalence $\tau^{-1} \otimes \mathbf{1}$ of $\mathcal{C}_{kQ\otimes kQ'}$.

Proposition 8.4. *For each integer t, the image $\Phi^t(T)$ is isomorphic to T_t.*

The proposition is proved by showing that the indices of the two objects are equal. This suffices by theorem 7.9. Now we conclude thanks to the following categorical periodicity result:

Proposition 8.5. *The power $\Phi^{h+h'}$ is isomorphic to the identity functor.*

Let us sketch the proof of this proposition: With the natural abuse of notation, the Serre functor S of the bounded derived category of $kQ\otimes kQ'$ is given by the 'tensor product' $S\otimes S$ of the Serre functors for kQ and kQ'. In the generalized cluster category, the Serre functor becomes isomorphic to the square of the suspension functor. So we have

$$S = S \otimes S = \Sigma^2 = \Sigma \otimes \Sigma$$

as autoequivalences of $\mathcal{C}_{kQ\otimes kQ'}$. Now recall from equation 5.5.4 that $\tau = \Sigma^{-1}S$. Thus we get the isomorphism

$$\tau \otimes \tau = \mathbf{1}$$

of autoequivalences of $\mathcal{C}_{kQ\otimes kQ'}$. So we have

$$\tau^{-1} \otimes \mathbf{1} = \mathbf{1} \otimes \tau.$$

Now we compute $\Phi^{h+h'}$ by using the left hand side for the first h factors and the right hand side for the last h' factors:

$$\Phi^{h+h'} = (\tau^{-1} \otimes \mathbf{1})^h (\mathbf{1} \otimes \tau)^{h'}.$$

But we know from equation 5.5.3 that $\tau^{-h} = \Sigma^2$. So we find

$$\Phi^{h+h'} = (\Sigma^2 \otimes \mathbf{1})(\mathbf{1} \otimes \Sigma^{-2}) = \Sigma^2\Sigma^{-2} = \mathbf{1}$$

as required.

9. Quiver mutation and derived equivalence

Let Q be a finite quiver and i a *source* of Q, *i.e.* no arrows have target i. Then the mutation Q' of Q at i is simply obtained by reversing all the arrows starting at i. In this case, the categories of representations of Q and Q' are related by the Bernstein-Gelfand-Ponomarev reflection functors [18] and these induce equivalences in the derived categories [75]. We would like to present a similar categorical interpretation for mutation at arbitrary vertices. For this, we use recent work by Derksen-Weyman-Zelevinsky [40] and a construction due to Ginzburg [72]. We first recall the classical reflection functors in a form which generalizes well.

9.1. **A reminder on reflection functors.** We keep the above notations. In the sequel, k is a field and kQ is the path algebra of Q over k. For each vertex j of Q, we write e_j for the lazy path at j, the idempotent of kQ associated with j. We write $P_j = e_j kQ$ for the corresponding indecomposable right kQ-module, $\mathsf{Mod}\, kQ$ for the category of (all) right kQ-modules and $\mathcal{D}(kQ)$ for its derived category.

The module categories over kQ and kQ' are linked by a pair of adjoint functors [18]

$$\begin{tikzcd} \mathsf{Mod}\, kQ' \arrow[d, shift right, "F_0"'] \\ \mathsf{Mod}\, kQ. \arrow[u, shift right, "G_0"'] \end{tikzcd}$$

The right adjoint G_0 takes a representation V of Q^{op} to the representation V' with $V'_j = V_j$ for $j \neq i$ and where V'_i is the kernel of the map

$$\bigoplus_{\substack{\text{arrows of } Q \\ i \to j}} V_j \longrightarrow V_i$$

whose components are the images under V of the arrows $i \to j$.

Then the left derived functor of F_0 and the right derived functor of G_0 are quasi-inverse equivalences [75]

$$\begin{tikzcd} \mathcal{D}kQ' \arrow[d, shift right, "F"'] \\ \mathcal{D}kQ. \arrow[u, shift right, "G"'] \end{tikzcd}$$

The functor F sends the indecomposable projective P'_j, $j \neq i$, to P_j and the indecomposable projective P'_i to the cone over the morphism

$$P_i \to \bigoplus_{\substack{\text{arrows of } Q \\ i \to j}} P_j .$$

In fact, the sum of the images of all the P'_j has a natural structure of complex of kQ'-kQ-bimodules and F can also be described as the derived tensor product over kQ' with this complex of bimodules.

The example of the following quiver Q

$$\begin{tikzcd} & 1 \arrow[dr, "a"] & \\ 2 \arrow[ur, "b"] & & 3 \arrow[ll, "c"] \end{tikzcd}$$

and its mutation Q' at 1

$$Q' : 2 \leftarrow 1 \leftarrow 3$$

shows that if we mutate at a vertex which is neither a sink nor a source, then the derived categories of representations of Q and Q' are not equivalent in general. The cluster-tilted algebra associated with the mutation of Q' at 1 (*cf.* section 5.8) is the quotient of kQ by the ideal generated by ab, bc and ca. It is of infinite global dimension and therefore not derived equivalent to kQ, either.

Clearly, in order to understand mutation from a representation-theoretic point of view, more structure is needed. Now quiver mutation has been independently invented and investigated in the physics literature, *cf.* for example equation (12.2) on page 70 in [27] (I thank to S. Fomin for this reference). In the physics context, a crucial role is played by the so-called superpotentials, *cf.* for example [17]. This lead Derksen-Weyman-Zelevinsky to their systematic study of quivers with potentials and their mutations in [40]. We now sketch their main result.

9.2. **Mutation of quivers with potentials.** Let Q be a finite quiver. Let $\widehat{kQ}$ be the *completed path algebra*, *i.e.* the completion of the path algebra at the ideal generated by the arrows of Q. Thus, $\widehat{kQ}$ is a topological algebra and the paths of Q form a topologial basis so that the underlying vector space of $\widehat{kQ}$ is

$$\prod_{p \text{ path}} kp.$$

The *continuous Hochschild homology* of $\widehat{kQ}$ is the vector space HH_0 obtained as the quotient of $\widehat{kQ}$ by the closure of the subspace generated by all commutators. It admits a topological basis formed by the *cycles* of Q, *i.e.* the orbits of paths $p = (i|\alpha_n|\ldots|\alpha_1|i)$ of length $n \geq 0$ with identical source and target under the action of the cyclic group of order n. In particular, the space HH_0 is a product of copies of k indexed by the vertices if Q does not have oriented cycles. For each arrow a of Q, the *cyclic derivative with respect to* a is the unique linear map

$$\partial_a \colon \mathsf{HH}_0 \to \widehat{kQ}$$

which takes the class of a path p to the sum

$$\sum_{p=uav} vu$$

taken over all decompositions of p as a concatenation of paths u, a, v, where u and v are of length ≥ 0. A *potential* on Q is an element W of HH_0 whose expansion in the basis of cycles does not involve cycles of length 0.

Now assume that Q does not have loops or 2-cycles.

Theorem 9.1 (Derksen-Weyman-Zelevinsky [40])**.** *The mutation operation*

$$Q \mapsto \mu_i(Q)$$

admits a good extension to quivers with potentials

$$(Q, W) \mapsto \mu_i(Q, W) = (Q', W')\ ,$$

i.e. *the quiver Q' is isomorphic to $\mu_i(Q)$ if W is generic.*

Here, 'generic' means that W avoids a certain countable union of hypersurfaces in the space of potentials. The main ingredient of the proof of this theorem is the construction of a 'minimal model', which, in a different language, was also obtained in [84] and [98].

If W is not generic, then $\mu_i(Q)$ is not necessarily isomorphic to Q' but still isomorphic to its 2-reduction, *i.e.* the quiver obtained from Q' by removing the arrows of a maximal set of pairwise disjoint 2-cycles. For example, the mutation of the quiver

(9.2.1)
$$Q : \begin{tikzcd} & 1 \arrow[dr, "a"] & \\ 2 \arrow[ur, "b"] & & 3 \arrow[ll, "c"] \end{tikzcd}$$

with the potential $W = abc$ at the vertex 1 is the quiver with potential

$$Q' : 2 \leftarrow 1 \leftarrow 3\,, \qquad W = 0.$$

On the other hand, the mutation of the above cyclic quiver Q with the potential $W = abcabc$ at the vertex 1 is the quiver

$$\begin{tikzcd} & 1 \arrow[dl, "b'"'] & \\ 2 \arrow[rr, shift left, "e"] & & 3 \arrow[ul, "a'"'] \arrow[ll, shift left, "c"] \end{tikzcd}$$

with the potential $ecec + eb'a'$.

Two quivers with potentials (Q, W) and (Q', W') are *right equivalent* if there is an isomorphism $\varphi \colon \widehat{kQ} \to \widehat{kQ'}$ taking W to W'. It is shown in [40] (*cf.* also [128]) that the mutation μ_i induces an involution on the set of right equivalence classes of quivers with potentials, where the quiver does not have loops and does not have a 2-cycle passing through i and the potential does not involve cycles of length ≤ 2.

9.3. **Derived equivalence of Ginzburg dg algebras.** We will associate a derived equivalence of differential graded (=dg) algebras with each mutation of a quiver with potential. The dg algebra in question is the algebra $\Gamma = \Gamma(Q, W)$ associated by Ginzburg with an arbitrary quiver Q with potential W, *cf.* section 4.3 of [72]. The underlying graded algebra of $\Gamma(Q, W)$ is the (completed) graded path algebra of a *graded quiver*, *i.e.* a quiver where each arrow has an associated integer degree. We first describe this graded quiver $\widetilde{Q}$: It has the same vertices as Q. Its arrows are

- the arrows of Q (they all have degree 0),
- an arrow $a^* \colon j \to i$ of degree -1 for each arrow $a \colon i \to j$ of Q,
- loops $t_i \colon i \to i$ of degree -2 associated with each vertex i of Q.

We now consider the completion $\Gamma = \widehat{k\widetilde{Q}}$ (formed in the category of graded algebras) and endow it with the unique continuous differential of degree 1 such that on the generators we have:

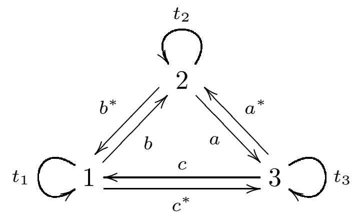
- $da = 0$ for each arrow a of Q,
- $d(a^*) = \partial_a W$ for each arrow a of Q,
- $d(t_i) = e_i(\sum_a [a, a^*])e_i$ for each vertex i of Q, where e_i is the idempotent associated with i and the sum runs over the set of arrows of Q.

One checks that we do have $d^2 = 0$ and that the homology in degree 0 of Γ is the *Jacobi algebra* as defined in [40]:

$$\mathcal{P}(Q, W) = \widehat{kQ}/(\partial_\alpha W \mid \alpha \in Q_1).$$

Let us consider two typical examples: Let Q be the quiver with one vertex and three loops labeled X, Y and Z. Let $W = XYZ - XZY$. Then the cyclic derivatives of W yield the commutativity relations between X, Y and Z and the Jacobi algebra is canonically isomorphic to the power series algebra $k[[X, Y, Z]]$. It is not hard to check that in this example, the homology of Γ is concentrated in degree 0 so that we have a quasi-isomorphism $\Gamma \to \mathcal{P}(Q, W)$. Using theorem 5.3.1 of Ginzburg's [72], one can show that this is the case if and only if the full subcategory of the derived category of the Jacobi algebra formed by the complexes whose homology is of finite total dimension is 3-Calabi-Yau as a triangulated category (*cf.* also below).

As a second example, we consider the Ginzburg dg algebra associated with the cyclic quiver 9.2.1 with the potential $W = abc$. Here is the graded quiver $\widetilde{Q}$:



The differential is given by

$$d(a^*) = bc\,,\ d(b^*) = ca\,,\ d(c^*) = ab\,,\ d(t_1) = cc^* - b^*b\,,\ \ldots .$$

It is not hard to show that for each $i \leq 0$, we have a canonical isomorphism

$$H^i(\Gamma) = \mathcal{C}_{\vec{A}_3}(T, \Sigma^i T)$$

where $\vec{A}_3$ is the quiver $1 \to 2 \to 3$, $\mathcal{C}_{\vec{A}_3}$ its cluster category and T the sum of the images of the modules P_1, P_3 and P_3/P_2. Since Σ is an autoequivalence of finite order of the cluster category $\mathcal{C}_{\vec{A}_3}$, we see that Γ has non vanishing homology in infinitely many degrees < 0. In particular, Γ is not quasi-isomorphic to the Jacobi algebra.

Let us denote by $\mathcal{D}\Gamma$ the derived category of Γ. Its objects are the differential $\mathbb{Z}$-graded right Γ-modules and its morphisms obtained from morphisms of dg Γ-modules (homogeneous of degree 0 and which commute with the differential) by formally inverting all quasi-isomorphisms, *cf.* [90]. Let us denote by $\mathsf{per}\,\Gamma$ the *perfect derived category*, *i.e.* the full subcategory of $\mathcal{D}\Gamma$ which is the closure under shifts, extensions and passage to direct factors of the free right Γ-module Γ_Γ. Finally, we denote by $\mathcal{D}^b(\Gamma)$ the *bounded derived category*, *i.e.* the full subcategory of $\mathcal{D}\Gamma$ formed by the dg modules whose homology is of finite total dimension (!). We recall that the objects of $\mathsf{per}(\Gamma)$ can be intrinsically characterized as the *compact* ones, *i.e.* those whose associated covariant Hom-functor commutes with arbitrary coproducts. The objects M of the bounded derived category are characterized by the fact that $\mathsf{Hom}(P, M)$ is finite-dimensional for each object P of $\mathsf{per}(\Gamma)$.

The following facts will be shown in [87]. Notice that they hold for arbitrary Q and W.

1) The dg algebra Γ is *homologically smooth*, *i.e.* it is perfect as an object of the derived category of $\Gamma^e = \Gamma\widehat{\otimes}\Gamma^{op}$. This implies that the bounded derived category is contained in the perfect derived category, *cf. e.g.* [89].

2) The dg algebra Γ is 3-*Calabi-Yau as a bimodule*, *i.e.* there is an isomorphism in the derived category of Γ^e

$$\mathsf{RHom}_{\Gamma^e}(\Gamma, \Gamma^e) \xrightarrow{\sim} \Gamma[-3].$$

As a consequence, the bounded derived category $\mathcal{D}^b(\Gamma)$ is 3-Calabi-Yau as a triangulated category, *cf. e.g.* lemma 4.1 of [87].

3) The triangulated category $\mathcal{D}(\Gamma)$ admits a t-structure whose left aisle $\mathcal{D}_{\leq 0}$ consists of the dg modules M such that $H^i(M) = 0$ for all $i > 0$. This t-structure induces a t-structure on $\mathcal{D}^b(\Gamma)$ whose heart is equivalent to the category of finite-dimensional (and hence nilpotent) modules over the Jacobi algebra.

The *generalized cluster category* [2] is the idempotent completion $\mathcal{C}_{(Q,W)}$ of the quotient

$$\mathsf{per}(\Gamma)/\mathcal{D}^b(\Gamma).$$

The name is justified by the fact that if Q is a quiver without oriented cycles (and so $W = 0$), then $\mathcal{C}_{(Q,W)}$ is triangle equivalent to $\mathcal{C}_Q$. In general, the category $\mathcal{C}_{(Q,W)}$ has infinite-dimensional Hom-spaces. However, if $H^0(\Gamma)$ is finite-dimensional, then $\mathcal{C}_{(Q,W)}$ is a 2-Calabi-Yau category with cluster-tilting object $T = \Gamma$, in the sense of section 7.1, *cf.* [2] (the present version of [2] uses non completed Ginzburg algebras; the complete case will be included in a future version).

From now on, we suppose that W does not involve cycles of length ≤ 1. Then the simple $\widetilde{Q}$-modules S_i associated with the vertices of Q yield a basis of the Grothendieck group $K_0(\mathcal{D}^b(\Gamma))$. Thanks to the Euler form

$$\langle P, M\rangle = \chi(\mathsf{RHom}(P, M)) \ , \ P \in \mathsf{per}(\Gamma) \ , \ M \in \mathcal{D}^b(\Gamma) \ ,$$

this Grothendieck group is dual to $K_0(\mathsf{per}(\Gamma))$ and the dg modules $P_i = e_i\Gamma$ form the basis dual to the S_i. The Euler form also induces an antisymmetric bilinear form on $K_0(\mathcal{D}^b(\Gamma)$ ('Poisson form'). If W does not involve cycles of length ≤ 2, then the quiver of the Jacobi algebra is Q and the matrix of the form on $\mathcal{D}^b(\Gamma)$ in the basis of the S_i is given by

$$\langle S_i, S_j\rangle = |\{\text{arrows } j \to i \text{ of } Q\}| - |\{\text{arrows } i \to j \text{ of } Q\}.$$

The dual of this form is given by the map ('symplectic form')

$$K_0(k) \to K_0(\mathsf{per}(\Gamma\widehat{\otimes}\Gamma^{op}) = K_0(\mathsf{per}(\Gamma)) \otimes_{\mathbb{Z}} K_0(\mathsf{per}(\Gamma))$$

associated with the k-Γ^e-bimodule Γ.

Now suppose that W does not involve cycles of length ≤ 2 and that Q does not have loops nor 2-cycles. Then each simple S_i is a (3-)spherical object in the 3-Calabi-Yau category $\mathcal{D}^b(\Gamma)$, *i.e.* we have an isomorphism of graded algebras

$$\mathsf{Ext}^*(S_i, S_i) \xrightarrow{\sim} H^*(S^3, k).$$

Moreover, the quiver Q encodes the dimensions of the Ext^1-groups between the S_i: We have

$$\dim \mathsf{Ext}^1(S_i, S_j) = |\{\text{arrows } j \to i \text{ in } Q\}|.$$

In particular, for $i \neq j$, we have either $\mathsf{Ext}^1(S_i, S_j) = 0$ or $\mathsf{Ext}^1(S_j, S_i) = 0$ because Q does not contain 2-cycles. It follows that the S_i, $i \in Q_0$, form a *spherical collection* in $\mathcal{D}^b(\Gamma)$ in the sense of [99]. Conversely, each spherical collection in an (A_∞-) 3-Calabi-Yau category in the sense of [99] can be obtained in this way from the Ginzburg algebra associated to a quiver with potential.

The categories we have considered so far are summed up in the exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{D}^b(\Gamma) \longrightarrow \mathsf{per}(\Gamma) \longrightarrow \mathcal{C}_{(Q,W)} \longrightarrow 0\,. \tag{9.3.1}$$

Notice that the left hand category is 3-Calabi-Yau, the middle one does not have a Serre functor and the right hand one is 2-Calabi-Yau if the Jacobi algebra is finite-dimensional.

We keep the last hypotheses: Q does not have loops or 2-cycles and W does not involve cycles of length ≤ 2. Let i be a vertex of Q and Γ' the Ginzburg algebra of the mutated quiver with potential $\mu_i(Q, W)$. The following theorem improves on Vitória's [123], *cf.* also [108] [17].

Theorem 9.2 ([95])**.** *There is an equivalence of derived categories*

$$F\colon \mathcal{D}(\Gamma') \to \mathcal{D}(\Gamma)$$

which takes the dg modules P'_j to P_j for $j \neq i$ and P'_i to the cone on the morphism

$$P_i \longrightarrow \bigoplus_{\substack{\text{arrows}\\ i\to j}} P_j.$$

It induces triangle equivalences $\mathsf{per}(\Gamma') \xrightarrow{\sim} \mathsf{per}(\Gamma)$ and $\mathcal{D}^b(\Gamma') \to \mathcal{D}^b(\Gamma)$.

In fact, the functor F is given by the left derived functor of the tensor product by a suitable Γ'-Γ-bimodule.

By transport of the canonical t-structure on $\mathcal{D}(\Gamma')$, we obtain new t-structures on $\mathcal{D}(\Gamma)$ and $\mathcal{D}^b(\Gamma)$. They are related to the canonical ones by a tilt (in the sense of [77]) at the simple object corresponding to the vertex i. By iterating mutation (as far as possible), one obtains many new t-structures on $\mathcal{D}^b(\Gamma)$.

In the context of Iyama-Reiten's [82], the dg algebras Γ and Γ' have their homologies concentrated in degree 0 and the equivalence of the theorem is given by the tilting module constructed in [loc. cit.].

9.4. **A geometric illustration.** To illustrate the sequence 9.3.1, let us consider the example of the quiver

$$Q:\ \begin{tikzcd} & 2 \arrow[dr, shift left=1.5] \arrow[dr] \arrow[dr, shift right=1.5] & \\ 0 \arrow[ur, shift left=1.5] \arrow[ur] \arrow[ur, shift right=1.5] & & 1 \arrow[ll, shift left=1.5] \arrow[ll] \arrow[ll, shift right=1.5] \end{tikzcd}$$

where the arrows going out from i are labeled x_i, y_i, z_i, $0 \le i \le 2$, endowed with the potential

$$W = \sum_{i=0}^{2} (x_i y_i z_i - x_i z_i y_i).$$

Let $p\colon \omega \to \mathbb{P}^2$ be the canonical bundle on $\mathbb{P}^2$. Then we have a triangle equivalence

$$\mathcal{D}^b(\mathsf{coh}(\omega)) \xrightarrow{\sim} \mathsf{per}(\Gamma)$$

which sends $p^*(O(-i))$ to the dg module P_i, $0 \le i \le 2$. Under this equivalence, the subcategory $\mathcal{D}^b\Gamma$ corresponds to the subcategory $\mathcal{D}^b_Z(\mathsf{coh}(\omega))$ of complexes of coherent sheaves whose homology is supported on the zero section Z of the bundle ω. This subcategory is indeed Calabi-Yau of dimension 3. Bridgeland has studied its t-structures [22] by linking them to t-structures and mutations (in the sense of Rudakov's school) in the derived category of coherent sheaves on the projective plane. In this example, the Ginzburg algebra has its homology concentrated in degree 0. So it is quasi-isomorphic to the Jacobi algebra. The Jacobi algebra is infinite-dimensional so that the generalized cluster category $\mathcal{C}_{(Q,W)}$ is not Hom-finite. The category $\mathcal{C}_{(Q,W)}$ identifies with the quotient

$$\mathcal{D}^b(\mathsf{coh}(\omega))/\mathcal{D}^b_Z(\mathsf{coh}(\omega))$$

and thus with the bounded derived category $\mathcal{D}^b(\mathsf{coh}(\omega \setminus Z))$ of coherent sheaves on the complement of the zero section of ω. In order to understand why this category is 'close to being' Calabi-Yau of dimension 2, we consider the scheme C obtained from ω by contracting the zero section Z to a point P_0. The projection $\omega \to C$ induces an isomorphism from $\omega \setminus Z$ onto $C \setminus \{P_0\}$ and so we have an equivalence

$$\mathcal{C}_{(Q,W)} \xrightarrow{\sim} \mathcal{D}^b(\mathsf{coh}(\omega \setminus Z)) \xrightarrow{\sim} \mathcal{D}^b(\mathsf{coh}(C \setminus \{P_0\})).$$

Let $\widehat{C}$ denote the completion of C at the singular point P_0. Then $\widehat{C}$ is of dimension 3 and has P_0 as its unique closed point. Thus the subscheme $\widehat{C} \setminus \{P_0\}$ is of dimension 2. The induced functor

$$\mathcal{C}_{(Q,W)} \xrightarrow{\sim} \mathcal{D}^b(\mathsf{coh}(C \setminus \{P_0\})) \to \mathcal{D}^b(\mathsf{coh}(\widehat{C} \setminus \{P_0\}))$$

yields a 'completion' of $\mathcal{C}_{(Q,W)}$ which is of 'dimension 2' and Calabi-Yau in a generalized sense, *cf.* [36].

9.5. **Ginzburg algebras from algebras of global dimension 2.** Let us link the cluster categories obtained from algebras of global dimension 2 in section 7.7 to the generalized cluster categories $\mathcal{C}_{(Q,W)}$ constructed above.

Let A be an algebra given as the quotient kQ'/I of the path algebra of a finite quiver Q' by an ideal I contained in the square of the ideal J generated by the arrows of Q'. Assume that A is of global dimension 2 (but not necessarily of finite dimension over k). We construct a quiver with potential (Q,W) as follows: Let R be the union over all pairs of vertices (i,j) of a set of representatives of the vectors belonging to a basis of

$$\mathsf{Tor}_2^A(S_j, S_i^{op}) = e_j(I/(IJ+JI))e_i.$$

We think of these representatives as 'minimal relations' from i to j, *cf.* [19]. For each such representative r, let ρ_r be a new arrow from j to i. We define Q to be obtained from Q' by adding all the arrows ρ_r and the potential is given by

$$W = \sum_{r\in R} r\rho_r.$$

Recall that a *tilting module* over an algebra B is a B-module T such that the total derived functor of the tensor product by T over the endomorphism algebra $\mathsf{End}_B(T)$ is an equivalence

$$\mathcal{D}(\mathsf{End}_B(T)) \xrightarrow{\sim} \mathcal{D}(B).$$

The second assertion of part a) of the following theorem generalizes a result by Assem-Brüstle-Schiffler [3].

Theorem 9.3 ([87]). a) *The category $\mathcal{C}_{(Q,W)}$ is triangle equivalent to the cluster category $\mathcal{C}_A$. This equivalence takes Γ to the image $\pi(A)$ of A in $\mathcal{C}_A$ and thus induces an isomorphism from the Jacobi algebra $\mathcal{P}(Q,W)$ onto the endomorphism algebra of the image of A in $\mathcal{C}_A$.*

b) *If T is a tilting module over kQ'' for a quiver without oriented cycles Q'' and A is the endomorphism algebra of T, then $\mathcal{C}_{(Q,W)}$ is triangle equivalent to $\mathcal{C}_{Q''}$.*

9.6. **Cluster-tilting objects, spherical collections, decorated representations.** We consider the setup of theorem 9.2. The equivalence F induces equivalences $\mathsf{per}(\Gamma') \to \mathsf{per}(\Gamma)$ and $\mathcal{D}^b(\Gamma') \to \mathcal{D}^b(\Gamma)$. The last equivalence takes S'_i to ΣS_i and, for $j \neq i$, the module S'_j to S_j if $\mathsf{Ext}^1(S_j, S_i) = 0$ and, more generally, to the middle term FS'_j of the universal extension

$$S_i^d \to FS'_j \to S_j \to \Sigma S_i^d \ ,$$

where the components of the third morphism form a basis of $\mathsf{Ext}^1(S_j, S_i)$. This means that the images FS'_j, $j \in Q_0$, form the *left mutated spherical collection* in $\mathcal{D}^b(\Gamma)$ in the sense of [99].

If $H^0(\Gamma)$ is finite-dimensional so that $\mathcal{C}_{(Q,W)}$ is a 2-Calabi-Yau category with cluster-tilting object, then the images of the FP_j, $j \in Q_0$, form the mutated cluster-tilting object, as it follows from the description in lemma 7.5.

Now suppose that $H^0(\Gamma)$ is finite-dimensional. Then we can establish a connection with decorated representations and their mutations in the sense of [40]: Recall from [loc. cit.] that a decorated representation of (Q, W) is given by a finite-dimensional (hence nilpotent) module M over the Jacobi algebra and a collection of vector spaces V_j indexed by the vertices j of Q. Given an object L of $\mathcal{C}_{(Q,W)}$, we put $M = \mathcal{C}(\Gamma, L)$ and, for each vertex j, we choose a vector space V_j of maximal dimension such that the triangle

$$T_1 \to T_0 \to L \to \Sigma T_1 \qquad (9.6.1)$$

of lemma 7.8 admits a direct factor

$$V_j \otimes P_j \to 0 \to \Sigma V_j \otimes P_j \to \Sigma V_j \otimes P_j .$$

Let us write GL for the decorated representation thus constructed. The assignment $L \mapsto GL$ defines a bijection between isomorphism classes of objects L in $\mathcal{C}_{(Q,W)}$ and right equivalence classes of decorated representations. It is compatible with mutations: If (M', V') is the image GL' of an object L' of $\mathcal{C}_{(Q',W')}$, then the mutation (Q, W, M, V) of (Q', W', M', V') at i in the sense of [40] is right equivalent to (Q, W, M'', V'') where $(M'', V'') = GFL'$ and F is the equivalence of theorem 9.2.

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Université Paris Diderot – Paris 7
UFR de Mathématiques, Institut de Mathématiques de Jussieu
UMR 7586 du CNRS
Case 7012
2, place Jussieu
75251 Paris Cedex 05
France

E-mail address: `keller@math.jussieu.fr`