A MODEL CATEGORY STRUCTURE ON THE CATEGORY OF SIMPLICIAL MULTICATEGORIES

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ABSTRACT. We establish a Quillen model structure on simplicial (symmetric) multicategories. It extends the model structure on simplicial categories due to J. Bergner [2]. We observe that our technique of proof enables us to prove a similar result for (symmetric) multicategories enriched over other monoidal model categories than simplicial sets. Examples include small categories, simplicial abelian groups and compactly generated Hausdorff spaces.

1. INTRODUCTION

A multicategory can be thought of as a generalisation of the notion of category, to the amount that an arrow is allowed to have a source (or input) consisting of a (possibly empty) string of objects, whereas the target (or output) remains a single object. Composition of arrows is performed by inserting the output of an arrow into (one of) the input(s) of the other. Then a multifunctor is a structure preserving map between multicategories. For example, every multicategory has an underlying category obtained by considering only those arrows with source consisting of strings of length one (or, one input). At the same time, a multicategory can be thought of as an "operad with many objects", an operad itself being precisely a multicategory with only one object. By allowing the symmetric groups to act on the various strings of objects of a multicategory, and consequently require that composition of arrows to be compatible with this actions in a certain natural way, one obtains the concept of symmetric multicategory. We refer the reader to [13], [5] and [1] for the precise definitions, history and examples.

As there is a notion of category enriched over a symmetric monoidal category other that the category of sets, the same happens with multicategories. In this paper we mainly consider symmetric multicategories enriched over simplicial sets, simply called simplicial multicategories. Moreover, we shall assume that all simplicial multicategories in sight are small, that is, they have a set of objects rather than a class.

We are interested in doing homotopy theory in the category of simplicial categories and simplicial multifunctors between them. For this we have to decide which class of arrows to invert, in other words we have to choose a class of weak equivalences. It turns out that the right notion of weak equivalence of simplicial multicategories is what we call here *multi DK-equivalence*. "DK" is for Dwyer and Kan. This notion has been previously defined in ([5], Def. 12.1), and is the obvious extension of the notion of Dwyer-Kan equivalence of simplicial categories [2]. We recall below the definition.

Every simplicial multicategory has an underlying simplicial category (we recall that this means a category enriched over simplicial sets), and this association is functorial. To every simplicial category $\mathcal{C}$ one can associate a genuine category $\pi_0\mathcal{C}$, the category of connected components of $\mathcal{C}$. The objects of $\pi_0\mathcal{C}$ are the objects of $\mathcal{C}$ and the hom set $\pi_0\mathcal{C}(x, y)$ is $\pi_0(\mathcal{C}(x, y))$. Now, a simplicial multifunctor $f\colon \mathrm{M} \to \mathrm{N}$ is a multi DK-equivalence if $\pi_0 f$ is essentially surjective and for every $k \geq 0$ and every $(k+1)$-tuple $(a_1, \ldots, a_k; b)$ of objects of M, the map $\mathrm{M}_k(a_1, \ldots, a_k; b) \to \mathrm{N}_k(f(a_1), \ldots, f(a_k); f(b))$ is a weak homotopy equivalence. Our first main result is

Theorem. (Theorem 4.5) *The category of simplicial multicategories admits a Quillen model category structure with multi DK-equivalences as weak equivalences and fibrations defined in 4.3.(2).*

We call this model structure the *Dwyer-Kan model structure* on simplicial multicategories. To prove this theorem we use the similar model structure on simplicial categories due to J. Bergner [2] and a very primitive form of the transfer principle, together with a modification of some parts of Bergner's original argument. We point out that the technique used to prove the above theorem applies to other categories than simplicial sets as well. Precisely, our second main result (theorem 5.2) establishes the analogous Dwyer-Kan model structure on the category of small (symmetric) multicategories enriched over certain monoidal model categories $\mathcal{V}$. We treat the cases when $\mathcal{V}$ is the category of small categories, simplicial abelian groups and compactly generated Hausdorff spaces. In the case of small categories, we thus extend the work of S. Lack [11] (see also [12]). As expected, the Dwyer-Kan model structure on simplicial categories and on small (symmetric) multicategories enriched over compactly generated Hausdorff spaces will be Quillen equivalent.

We also introduce another model structure on simplicial multicategories, the *fibred* model structure (section 6). The weak equivalences are the multi DK-equivalences bijective on objects and the cofibrations strictly contain the cofibrations of the Dwyer-Kan model structure. Although it seems to be standard, we believe that the fibred model structure deserves to be better known. One can try to obtain it by using a classical result of A. Roig ([14], Thm. 5.1) on the interplay between Grothendieck bifibrations and model category structures. While doing that we noticed that Roig's theorem is not correct as stated. We have appended a reworking of Roig's main result which sheds some light into the Dwyer-Kan model structure as well. For example, it gives an explicit description of the class of cofibrations.

The paper is organised as follows. In sections 2 and 3 we review the notions and results from enriched (multi)category theory that we use. Primarily for simplicity, we have chosen to work in full generality in both sections, in the sense that our (multi)categories are enriched over an arbitrary closed symmetric monoidal category. Section 4 contains the proof of the above theorem. In section 5 we prove the existence of a Dwyer-Kan model structure on symmetric multicategories enriched over other categories of interest than simplicial sets, cf. above. Section 6 contains a recast of Roig's theorem alluded to above in terms of weak factorisation systems. In section 7 we introduce the fibred model structure on small $\mathcal{V}$-categories, for a fairly general monoidal model category $\mathcal{V}$, and on simplicial multicategories. Section 8 recalls a test for lifting Quillen equivalences along adjoint pairs.

2. Review of $\mathcal{V}$-graphs and $\mathcal{V}$-categories

2.1. Let $\mathcal{V}$ be a complete and cocomplete closed symmetric monoidal category with unit I. The small $\mathcal{V}$-categories together with the $\mathcal{V}$-functors between them form a category written $\mathcal{V}\mathbf{Cat}$. It is a closed symmetric monoidal category with unit the $\mathcal{V}$-category $\mathcal{I}$ with a single object $*$ and $\mathcal{I}(*,*) = I$. We denote by $\mathcal{V}\mathbf{Graph}$ the category of small $\mathcal{V}$-graphs. A $\mathcal{V}$-graph is a $\mathcal{V}$-category without composition and unit maps. We denote by Ob the functor sending a $\mathcal{V}$-category (or a $\mathcal{V}$-graph) to its set of objects. The functor Ob is a Grothendieck bifibration. There is a free-forgetful (fibred) adjunction

$$\begin{array}{ccc} \mathcal{F} : \mathcal{V}\mathbf{Graph} & \rightleftarrows & \mathcal{V}\mathbf{Cat} : \mathcal{U} \\ & {\scriptstyle Ob} \searrow \quad \swarrow {\scriptstyle Ob} & \\ & Set & \end{array} \qquad (1)$$

where Set is the category of sets. We write $\mathcal{V}\mathbf{Graph}(S)$ (resp. $\mathcal{V}\mathbf{Cat}(S)$) for the fibre category over a set S. The category $\mathcal{V}\mathbf{Graph}(S)$ is a (nonsymmetric) monoidal category with monoidal product

$$X \Box_S Y(a,b) = \coprod_{c \in S} X(a,c) \otimes Y(c,b)$$

and unit

$$\mathcal{I}_S(a,b) = \begin{cases} I, & \text{if } a = b \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $\mathcal{V}\mathbf{Cat}(S)$ is precisely the category of monoids in $\mathcal{V}\mathbf{Graph}(S)$ with respect to $-\Box_S-$.

2.2. Let $\mathcal{M}$ be a class of maps of $\mathcal{V}$. Following [10], we say that a $\mathcal{V}$-functor $f\colon \mathcal{A} \to \mathcal{B}$ is *locally in* $\mathcal{M}$ if for each pair $x, y \in \mathcal{A}$ of objects, the map $f_{x,y}\colon \mathcal{A}(x,y) \to \mathcal{B}(f(x), f(y))$ is in $\mathcal{M}$. This definition makes also sense for morphisms of $\mathcal{V}$-graphs. When $\mathcal{M}$ is the class of isomorphisms of $\mathcal{V}$, a $\mathcal{V}$-functor which is locally an isomorphism is called *full and faithful.*

2.3. Let $f\colon \mathcal{A} \to \mathcal{B}$ be a $\mathcal{V}$-functor and let $u = Ob(f)$. Then (6.1) f factors as $\mathcal{A} \xrightarrow{f^u} u^*\mathcal{B} \to \mathcal{B}$, where f^u is a map in $\mathcal{V}\mathbf{Cat}(Ob(\mathcal{A}))$. One has $u^*\mathcal{B}(a,a') = \mathcal{B}(f(a), f(a'))$ and $u^*\mathcal{B} \to \mathcal{B}$ is full and faithful.

2.4. We denote by ! the $\mathcal{V}$-graph with a single object $*$ and $!(*,*) = \emptyset$. There is a unique arrow $u\colon \emptyset_{graph} \to !$, where $\emptyset_{graph}$ is the initial $\mathcal{V}$-graph.

For an object A of $\mathcal{V}$, we denote by $(2,A)$ (or $\begin{pmatrix} \emptyset & A \\ \emptyset & \emptyset \end{pmatrix}$) the $\mathcal{V}$-graph with two objects 0 and 1 and with $(2,A)(0,0) = (2,A)(1,1) = (2,A)(1,0) = \emptyset$ and $(2,A)(0,1) = A$. This defines a colimit preserving functor $(2,\_)\colon \mathcal{V} \to \mathcal{V}\mathbf{Graph}$.

2.5. Let $\mathbb{C}$ be a small category. We endow $\mathcal{V}^{\mathbb{C}}$ with the pointwise monoidal product. We have a full and faithful functor

$$\varphi : \mathcal{V}^{\mathbb{C}}\mathbf{Cat} \longrightarrow (\mathcal{V}\mathbf{Cat})^{\mathbb{C}},$$

given by $Ob(\varphi(\mathcal{A})(j)) = Ob(\mathcal{A})$ for all $j \in \mathbb{C}$ and $\varphi(\mathcal{A})(j)(a,b) = \mathcal{A}(a,b)(j)$ for all $a, b \in Ob(\mathcal{A})$. The category $\mathcal{V}^{\mathbb{C}}\mathbf{Cat}$ is coreflective in $(\mathcal{V}\mathbf{Cat})^{\mathbb{C}}$, that is, the functor φ has a right adjoint G, defined as follows. For $\mathcal{A} \in (\mathcal{V}\mathbf{Cat})^{\mathbb{C}}$, we put $Ob(G(\mathcal{A})) = \varprojlim Ob(\mathcal{A}(i))$ and $G(\mathcal{A})((a_i),(b_i))(j) = \mathcal{A}(j)(a_j, b_j)$.

3. Review of symmetric $\mathcal{V}$-multigraphs and symmetric $\mathcal{V}$-multicategories

3.1. Let $\mathcal{V}$ be a complete and cocomplete closed symmetric monoidal category. For the notions of *symmetric* $\mathcal{V}$*-multicategory* and *symmetric* $\mathcal{V}$*-multifunctor* we refer the reader to ([13], 2.2.21) and ([5], 2.1, 2.2). If M is a symmetric $\mathcal{V}$-multicategory, $k \geq 0$ is an integer and $(a_1, \ldots, a_k; b)$ is a $(k+1)$-tuple of objects, we shall denote by $\mathrm{M}_k(a_1, \ldots, a_k; b)$ the $\mathcal{V}$-object of "k-morphisms", cf. ([5], 2.1(2)). When $k = 0$, the $\mathcal{V}$-object of 0-morphisms is denoted by $\mathrm{M}(\ ; b)$.

The small symmetric $\mathcal{V}$-multicategories together with the symmetric $\mathcal{V}$-multifunctors between them form a category written $\mathcal{V}\mathbf{SymMulticat}$. It is a symmetric monoidal category with tensor product defined pointwise. Precisely, if $\mathrm{M}, \mathrm{N} \in \mathcal{V}\mathbf{SymMulticat}$ then $\mathrm{M} \otimes \mathrm{N}$ has $Ob(\mathrm{M}) \times Ob(\mathrm{N})$ as set of objects and

$$(\mathrm{M}\otimes\mathrm{N})_k((a_1, a'_1), \ldots, (a_k, a'_k); (b, b')) = \mathrm{M}_k(a_1, \ldots, a_k; b) \otimes \mathrm{N}_k(a'_1, \ldots, a'_k; b').$$

The unit Com of this tensor product has a single object $*$ and $Com_k(*, \ldots, *; *) = I$.

When $\mathcal{V}$ is the category Set of sets, symmetric Set-multicategories will be simply referred to as *multicategories*, and the category will be denoted by **SymMulticat**.

A *symmetric $\mathcal{V}$-multigraph* is by definition a symmetric $\mathcal{V}$-multicategory without composition and unit maps. We shall write $\mathcal{V}\mathbf{SymMultigraph}$ for the category of symmetric $\mathcal{V}$-multigraphs with the evident notion of arrow. When $\mathcal{V} = Set$, the category is denoted by **SymMultigraph**.

We denote by Ob the functor sending a symmetric $\mathcal{V}$-multicategory (or a symmetric $\mathcal{V}$-multigraph) to its set of objects. The functor Ob is a Grothendieck bifibration. There is a free-forgetful (fibred) adjunction

$$\begin{array}{ccc} \mathcal{F}_{Multi} : \mathcal{V}\mathbf{SymMultigraph} & \rightleftarrows & \mathcal{V}\mathbf{SymMulticat} : \mathcal{U}_{Multi} \\ & {\scriptstyle Ob} \searrow \quad \swarrow {\scriptstyle Ob} & \\ & Set & \end{array} \qquad (2)$$

We write $\mathcal{V}\mathbf{SymMulticat}(S)$ for the fibre category over a set S. $\mathcal{V}$-categories and symmetric $\mathcal{V}$-multicategories can be related by the (fibred) adjunction

$$\begin{array}{ccc} E : \mathcal{V}\mathbf{Cat} & \rightleftarrows & \mathcal{V}\mathbf{SymMulticat} : (\_)_1 \\ & {\scriptstyle Ob} \searrow \quad \swarrow {\scriptstyle Ob} & \\ & Set & \end{array} \qquad (3)$$

where

$$(E\mathcal{A})_n(a_1, \ldots, a_n; b) = \begin{cases} \mathcal{A}(a_1, b) & \text{if } n = 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

and $\mathrm{M}_1(a, b) = \mathrm{M}_1(a; b)$. The functor E is full and faithful.

3.2. Let $\mathcal{M}$ be a class of maps of $\mathcal{V}$. We say that a symmetric $\mathcal{V}$-multifunctor $f \colon \mathrm{M} \to \mathrm{N}$ is *locally in* $\mathcal{M}$ if for each integer $k \geq 0$ and each $(k+1)$-tuple of objects $(a_1, \ldots, a_k; b)$, the map $f \colon \mathrm{M}_k(a_1, \ldots, a_k; b) \to \mathrm{N}_k(f(a_1), \ldots, f(a_k); f(b))$ is in $\mathcal{M}$. When $\mathcal{M}$ is the class of isomorphisms of $\mathcal{V}$, a symmetric $\mathcal{V}$-multifunctor which is locally an isomorphism is called *full and faithful*.

3.3. We recall that a *$\mathcal{V}$-multigraph* M consists of a set of objects $Ob(\mathrm{M})$ together with an object $\mathrm{M}_k(a_1, \ldots, a_k; b)$ of $\mathcal{V}$ assigned to each integer $k \geq 0$ and each $(k+1)$-tuple of objects$(a_1, \ldots, a_k; b)$. We write $\mathcal{V}$**Multigraph** for the resulting category. In the case when $\mathcal{V} = Set$, this category is denoted by **Multigraph** and its objects will be called *multigraphs*.

The forgetful functor from symmetric $\mathcal{V}$-multigraphs to $\mathcal{V}$-multigraphs has a left adjoint Sym defined by

$$(Sym\mathrm{M})_k(a_1, \ldots, a_k; b) = \coprod_{\sigma \in \Sigma_k} \mathrm{M}_k(a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(k)}; b),$$

where Σ_k is the symmetric group on k elements.

3.4. For each integer $k \geq 0$ we denote by $\underline{k+1}$ the set $\{1, 2, \ldots, k, *\}$, where $* \notin \{1, 2, \ldots, k\}$. We have a functor

$$(\underline{k+1}, \ ) \colon \mathcal{V} \to \mathcal{V}\mathbf{Multigraph}$$

given by $(\underline{k+1}, A)_n(a_1, \ldots, a_n; b) = \emptyset$ unless $n = k$ and $a_i = i$ and $b = *$, in which case we define it to be A. To give a map of $\mathcal{V}$-multigraphs $(\underline{k+1}, A) \to \mathrm{M}$ is to give a map $A \to \mathrm{M}_k(a_1, \ldots, a_k; b)$.

3.5. Let $\mathbb{C}$ be a small category. We endow $\mathcal{V}^{\mathbb{C}}$ with the pointwise monoidal product. We have a full and faithful functor

$$\varphi' : \mathcal{V}^{\mathbb{C}}\mathbf{SymMulticat} \longrightarrow (\mathcal{V}\mathbf{SymMulticat})^{\mathbb{C}},$$

given by $Ob(\varphi'(\mathrm{M})(j)) = Ob(\mathrm{M})$ for all $j \in \mathbb{C}$ and $\varphi'(\mathrm{M})(j)_k(a_1, \ldots, a_k; b) = \mathrm{M}_k(a_1, \ldots, a_k; b)(j)$ for each $(k+1)$-tuple of objects $(a_1, \ldots, a_k; b)$. The category $\mathcal{V}^{\mathbb{C}}\mathbf{SymMulticat}$ is coreflective in $(\mathcal{V}\mathbf{SymMulticat})^{\mathbb{C}}$. The right adjoint to φ, denoted by G', can be defined as follows. For $\mathrm{M} \in (\mathcal{V}\mathbf{SymMulticat})^{\mathbb{C}}$, we put $Ob(G'(\mathrm{M})) = \underleftarrow{\lim} Ob(\mathrm{M}(i))$ and

$$G'(\mathrm{M})_k((a_i^1), \ldots, (a_i^k); (b_i))(j) = \mathrm{M}(j)_k(a_j^1, \ldots, a_j^k; b_j).$$

We have a commutative square of adjunctions

$$\begin{array}{ccc} \mathcal{V}^{\mathbb{C}}\mathbf{Cat} & \underset{(\_)_1}{\overset{E}{\rightleftarrows}} & \mathcal{V}^{\mathbb{C}}\mathbf{SymMulticat} \\ G \uparrow \downarrow \varphi & & G' \uparrow \downarrow \varphi' \\ (\mathcal{V}\mathbf{Cat})^{\mathbb{C}} & \underset{(\_)_1^{\mathbb{C}}}{\overset{E^{\mathbb{C}}}{\rightleftarrows}} & (\mathcal{V}\mathbf{SymMulticat})^{\mathbb{C}}. \end{array} \qquad (4)$$

4. The Dwyer-Kan model structure on SSymMulticat

In this section we prove the theorem stated in the introduction. We first recall the analogous (Dwyer-Kan) model structure on simplicial categories.

We denote by **Cat** the category of small categories. It has a *natural* model structure in which a cofibration is a functor monic on objects, a weak equivalence is an equivalence of categories and a fibration is an isofibration [9]. The fibration weak equivalences are the equivalences surjective on objects. Let **S** be the category of simplicial sets, regarded as having the classical model structure.

Let $\pi_0 \colon \mathbf{S} \to Set$ be the set of connected components functor. By change of base it induces a functor $\pi_0 \colon \mathbf{SCat} \to \mathbf{Cat}$ which is the identity on objects.

Definition 4.1. (Dwyer and Kan) *Let $f \colon \mathcal{A} \to \mathcal{B}$ be a morphism in* **SCat**.

1. The morphism f is homotopy essentially surjective *if the induced functor $\pi_0 f \colon \pi_0 \mathcal{A} \to \pi_0 \mathcal{V}$ is essentially surjective.*

2. The morphism f is a DK-equivalence *if it is homotopy essentially surjective and locally a weak homotopy equivalence.*

3. The morphism f is a DK-fibration *if f is locally a Kan fibration and $\pi_0 f$ is an isofibration.*

4. The morphism f is a trivial fibration *if it is a DK-equivalence and a DK–fibration.*

Remark. A morphism is a trivial fibration iff it is surjective on objects and locally a trivial fibration. The class of simplicial functors having the left lifting property with respect to the trivial fibrations can be explicitly described (7.2).

Theorem 4.2. [2] *The category* **SCat** *of simplicial categories admits a cofibrantly generated model structure in which the weak equivalences are the DK-equivalences and the fibrations are the DK-fibrations. A generating set of trivial cofibrations consists of*

(B1) $\{\mathcal{F}(2,X) \xrightarrow{\mathcal{F}(2,j)} \mathcal{F}(2,Y)\}$, where j is a horn inclusion, and

(B2) inclusions $\mathcal{I} \xrightarrow{\delta_y} \mathcal{H}$, where $\{\mathcal{H}\}$ is a set of representatives for the isomorphism classes of simplicial categories on two objects which have countably many simplices in each function complex. Furthermore, each such $\mathcal{H}$ is required to be cofibrant and weakly contractible in $\mathbf{SCat}(\{x,y\})$. Here $\{x,y\}$ is the set with elements x and y and δ_y omits y.

Definition 4.3. *Let $f \colon \mathrm{M} \to \mathrm{M}'$ be a morphism in* **SSymMulticat**.

1. The morphism f is a multi DK-equivalence *if it is locally a weak homotopy equivalence and f_1 is homotopy essentially surjective.*

2. The morphism f is a multi DK-fibration *if f is locally a Kan fibration and $\pi_0 f_1$ is an isofibration.*
3. The morphism f is a trivial fibration *if it is a multi DK-equivalence and a multi DK-fibration.*

Remark. A morphism is a trivial fibration iff it is surjective on objects and locally a trivial fibration. The class of simplicial multifunctors having the left lifting property with respect to the trivial fibrations can be explicitly described (7.9).

Lemma 4.4. *The functor E (3.1(3)) sends DK-equivalences to multi DK-equivalences. The functor $(\_)_1$ preserves trivial fibrations.*

Our main result is

Theorem 4.5. *The category* **SSymMulticat** *admits a cofibrantly generated model structure in which the weak equivalences are the multi DK-equivalences and the fibrations are the multi DK-fibrations. The model structure is right proper.*

Proof. We shall use ([7], Thm. 11.3.1). We take in *loc. cit.*:

-the set I to be $E(\emptyset \to \mathcal{I}) \cup \{\mathcal{F}_{Multi}Sym(\underline{k+1}, i)\}_{k\geq 0}$, where i is a generating cofibration of $\mathbf{S}$;
-the set J to be $E(B2) \cup \{\mathcal{F}_{Multi}Sym(\underline{k+1}, j)\}_{k\geq 0}$, where j is a horn inclusion;
- the class W to be the class of multi-equivalences.

It is enough to prove that $\mathrm{J}-cof \subset \mathrm{W}$ and that $\mathrm{W} \cap \mathrm{J}-inj = \mathrm{I}-inj$. Notice that $\mathrm{I}-inj$ is the class of trivial fibrations, and that by definition we have $\mathrm{W} \cap \mathrm{J}-inj = \mathrm{I}-inj$. The next four lemmas complete the proof of the existence of the model structure. Right properness is standard. $\square$

Lemma 4.6. *Let $\delta_y\colon \mathcal{I} \to \mathcal{H}$ be a map belonging to the set B2 from theorem 4.2.*
Then in the pushout diagram

$$\begin{array}{ccc} E\mathcal{I} & \xrightarrow{x} & \mathrm{M} \\ {\scriptstyle E\delta_y}\downarrow & & \downarrow \\ E\mathcal{H} & \longrightarrow & \mathrm{N} \end{array}$$

the map $\mathrm{M} \to \mathrm{N}$ *is a multi-equivalence.*

Proof. We factor the map δ_y as $\mathcal{I} \xrightarrow{(\delta_y)^u} u^*\mathcal{H} \to \mathcal{H}$ where $u = Ob(\delta_y)$ and then we take consecutive pushouts:

$$\begin{array}{ccc} E\mathcal{I} & \xrightarrow{x} & \mathrm{M} \\ {\scriptstyle E\delta_y^u}\downarrow & & \downarrow{\scriptstyle j} \\ Eu^*\mathcal{H} & \longrightarrow & \mathrm{M}' \\ \downarrow & & \downarrow \\ E\mathcal{H} & \longrightarrow & \mathrm{N}. \end{array}$$

By lemma 4.7 the map $(\delta_y)^u$ is a trivial cofibration in the category of simplicial monoids, therefore the map j is a trivial cofibration in **SSymMulticat**$(Ob(\mathrm{M}))$. We conclude by an application of the adjunction 3.5(4) to the bottom pushout diagram above, together with ([6], Prop. 5.2) and lemma 4.8. □

Lemma 4.7. *Let $\mathcal{A}$ be a cofibrant simplicial category. Then for each $a \in Ob(\mathcal{A})$ the simplicial monoid $a^*\mathcal{A}$ (2.3) is cofibrant (as a monoid).*

Proof. Let $S = Ob(\mathcal{A})$. $\mathcal{A}$ is cofibrant iff it is cofibrant as an object of **SCat**(S). The cofibrant objects of **SCat**(S) are characterised in ([4], 7.6): they are the retracts of free simplicial categories. Therefore it suffices to prove that if $\mathcal{A}$ is a free simplicial category then $a^*\mathcal{A}$ is a free simplicial category for all $a \in S$. Recall ([4], 4.5) that $\mathcal{A}$ is a free simplicial category iff (i) for all $n \geq 0$ the category $\varphi(\mathcal{A})_n$ (2.5) is a free category on a graph G_n, and (ii) for all epimorphisms $\alpha\colon [m] \to [n]$ of Δ, $\alpha^*\colon \varphi(\mathcal{A})_n \to \varphi(\mathcal{A})_m$ maps G_n into G_m.

Let $a \in S$. The category $\varphi(a^*\mathcal{A})_n$ is a full subcategory of $\varphi(\mathcal{A})_n$ with object set $\{a\}$, hence it is free as well. A set $G_n^{a^*\mathcal{A}}$ of generators can be described as follows. An element of $G_n^{a^*\mathcal{A}}$ is a path from a to a such that every arrow in the path belongs to G_n and there is at most one arrow in the path with source and target a. Since every epimorphism $\alpha\colon [m] \to [n]$ of Δ has a section, α^* maps $G_n^{a^*\mathcal{A}}$ into $G_m^{a^*\mathcal{A}}$. □

Lemma 4.8. *Let A and B be two small categories and let $i : A \hookrightarrow B$ be a full and faithful inclusion. Let* M *be a multicategory. Then in the pushout diagram*

$$\begin{array}{ccc} EA & \xrightarrow[Ei]{} & EB \\ \downarrow & & \downarrow \\ \mathrm{M} & \longrightarrow & \mathrm{N} \end{array}$$

the map $\mathrm{M} \to \mathrm{N}$ *is a full and faithful inclusion.*

Proof. Let $(B-A)^+$ be the preorder with objects all finite subsets $S \subseteq Ob(B) - Ob(A)$, ordered by inclusion. For $S \in (B-A)^+$, let A_S be the full subcategory of B with objects $Ob(B) \cup S$. Then $B = \lim\limits_{(B-A)^+} A_S$. On the other hand, a filtered colimit of full and faithful inclusions of multicategories is a full and faithful inclusion. This is because the forgetful functor from **SymMulticat** to **Multigraph** preserves filtered colimits and a filtered colimit of full and faithful inclusions of multigraphs is a full and faithful inclusion. Therefore one can assume that $Ob(B) = Ob(A) \cup \{q\}$, where $q \notin Ob(A)$. Furthermore, by ([6], Prop. 5.2) it is enough to consider the following situation. M is a multicategory, $i : \mathrm{M}_1 \hookrightarrow B$ is a full and faithful inclusion with $Ob(B) = Ob(\mathrm{M}) \sqcup \{q\}$, and the pushout diagram is

$$\begin{array}{ccc} E\mathrm{M}_1 & \xrightarrow{Ei} & EB \\ {\scriptstyle \epsilon_M}\downarrow & & \downarrow \\ \mathrm{M} & \longrightarrow & \mathrm{N}, \end{array}$$

where ϵ_{M} is the counit of the adjunction 3.1(3) (with $\mathcal{V} = Set$). But this follows by taking $\mathcal{V} = Set$ in the next lemma. □

Lemma 4.9. *Let $\mathcal{V}$ be a cocomplete closed symmetric monoidal category. Let* M *be a small symmetric $\mathcal{V}$-multicategory, B a small $\mathcal{V}$-category with $Ob(B) = Ob(\mathrm{M}) \sqcup \{q\}$ and $i \colon \mathrm{M}_1 \hookrightarrow B$ a full and faithful inclusion. Then in the pushout diagram*

$$\begin{array}{ccc} E\mathrm{M}_1 & \xrightarrow{Ei} & EB \\ {\scriptstyle \epsilon_M}\downarrow & & \downarrow \\ \mathrm{M} & \longrightarrow & \mathrm{N} \end{array}$$

the map $\mathrm{M} \to \mathrm{N}$ is a full and faithful inclusion. Here ϵ_{M} is the counit of the adjunction 3.1(3).

Proof. Let $\otimes$ be the tensor product of $\mathcal{V}$. We shall explicitly describe the $\mathcal{V}$-objects of k-morphisms of N. For $k \geq 0$ and $(a_1, \dots, a_k; a)$ a $(k+1)$-tuple of objects with $a \in \mathrm{M}$ and $a_i \in \mathrm{M}$ $(i = \overline{1,k})$, we put $\mathrm{N}_k(a_1, \dots, a_k; a) = \mathrm{M}_k(a_1, \dots, a_k; a)$. Then we set $\mathrm{N}(\ ; q) = \int^{x \in Ob(\mathrm{M})} B(x, q) \otimes M(\ ; x)$ and

$\mathrm{N}_k(a_1, \dots, a_k; q) = \int^{x \in Ob(\mathrm{M})} B(x, q) \otimes M_k(a_1, \dots, a_k; x)$ if $a_i \in Ob(\mathrm{M})$ $(i = \overline{1,k})$.

Next, let $(a_1, \dots, a_k)$ be a k-tuple of objects of M. For each $1 \leq s \leq k$ let $\{i_1, \dots, i_s\}$ be a (nonempty, apriori unordered, the elements can't repeat) subset of $\{1, \dots, k\}$. We denote by $(a_1, \dots, a_k)^{q_{i_1, \dots, i_s}}$ the k-tuple of objects of B obtained by inserting q in the k-tuple $(a_1,\ldots,a_k)$ at the spot i_j $(1\le j\le s)$. For each $1\le j\le s$ and $x_{i_j}\in \mathrm{M}$ we denote by $(a_1,\ldots,a_k)^{\{x_{i_1},\ldots,x_{i_s}\}}$ the k-tuple of objects of M obtained by inserting x_{i_j} in the k-tuple $(a_1,\ldots,a_k)$ at the spot i_j. We put

$$\begin{aligned}&\mathrm{N}_k((a_1,\ldots,a_k)^{q_{i_1,\ldots,i_s}};a)=\\&\qquad\int^{x_1\in Ob(\mathrm{M})}\cdots\int^{x_s\in Ob(\mathrm{M})}\mathrm{M}_k((a_1,\ldots,a_k)^{\{x_{i_1},\ldots,x_{i_s}\}};a)\\&\qquad\qquad\otimes B(q,x_{i_1})\otimes\cdots\otimes B(q,x_{i_s}),\end{aligned}$$

if $a\in Ob(\mathrm{M})$, and

$$\begin{aligned}&\mathrm{N}_k((a_1,\ldots,a_k)^{q_{i_1,\ldots,i_s}};q)=\\&\qquad=\int^{x\in Ob(\mathrm{M})}\int^{x_{i_1}\in Ob(\mathrm{M})}\cdots\int^{x_{i_s}\in Ob(\mathrm{M})}B(x,q)\otimes\\&\qquad\mathrm{M}_k((a_1,\ldots,a_k)^{\{x_{i_1},\ldots,x_{i_s}\}};x)\otimes B(q,x_{i_1})\otimes\cdots\otimes B(q,x_{i_s}).\end{aligned}$$

This completes the definition of the $\mathcal{V}$-objects of k-morphisms of N. To prove that N is a symmetric $\mathcal{V}$-multicategory is long and tedious. Once this is proved, the fact that it has the desired universal property follows. □

5. The Dwyer-Kan model structure on $\mathcal{V}$**SymMulticat**, for certain $\mathcal{V}'s$

Using the same method of proof as for theorem 4.5, one can prove a similar result for other categories that simplicial sets. The precise statements follow after some definitions. We shall only sketch the proofs.

Let $\mathcal{V}$ be a monoidal model category [15] with cofibrant unit I. We denote by $\mathfrak{W}$ (resp. $\mathfrak{Fib}$) the class of weak equivalences (resp. fibrations) of $\mathcal{V}$. We have a functor $[\_]_{\mathcal{V}}\colon \mathcal{V}\mathbf{Cat}\to\mathbf{Cat}$ obtained by change of base along the (symmetric monoidal) composite functor

$$\mathcal{V}\xrightarrow{\gamma}Ho(\mathcal{V})\xrightarrow{Hom_{Ho(\mathcal{V})}(I,\_)}Set.$$

Definition 5.1. *Let $f\colon \mathrm{M}\to\mathrm{M}'$ be a morphism in $\mathcal{V}$**SymMulticat**.*

1. The morphism f is a multi DK-equivalence *if it is locally in $\mathfrak{W}$ and $[f_1]_{\mathcal{V}}$ is essentially surjective.*

2. The morphism f is a multi DK-fibration *if f is locally in $\mathfrak{Fib}$ and $[f_1]_{\mathcal{V}}$ is an isofibration.*

3. The morphism f is a trivial fibration *if it is a multi DK-equivalence and a multi DK-fibration.*

One can check that a morphism is a trivial fibration iff it is surjective on objects and locally a trivial fibration.

Theorem 5.2. *Let $\mathcal{V}$ be one of the categories* **Cat** *(with the natural model structure),* **SAb** *(simplicial abelian groups, with the Quillen model structure) or* **CGHaus** *(compactly generated Hausdorff spaces, with the Quillen model structure). Then the category* $\mathcal{V}$**SymMulticat** *admits a cofibrantly generated model structure in which the weak equivalences are the multi DK-equivalences and the fibrations are the multi DK-fibrations. The model structure is right proper.*

Proof. We first treat the case $\mathcal{V} \in \{\mathbf{SAb}, \mathbf{CGHaus}\}$. There is a Quillen pair

$$F \colon \mathbf{S} \rightleftarrows \mathcal{V} : G$$

such that F is strong symmetric monoidal and preserves the unit object. We have a commutative square of adjunctions

$$\begin{array}{ccc} \mathbf{SCat} & \underset{(\_)_1}{\overset{E}{\rightleftarrows}} & \mathbf{SSymMulticat} \\ G' \uparrow \downarrow F' & & G' \uparrow \downarrow F' \\ \mathcal{V}\mathbf{Cat} & \underset{(\_)_1}{\overset{E}{\rightleftarrows}} & \mathcal{V}\mathbf{SymMulticat}. \end{array}$$

Use again ([7], Thm. 11.3.1) together with ([17], Prop. 3.1) and lemma 4.9. The model categories **SSymMulticat** and **CGHausSymMulticat** are Quillen equivalent.

For $\mathcal{V} = \mathbf{Cat}$ follow again the proof of theorem 4.5, using now theorem 5.3 instead of theorem 4.2, lemma 5.4 instead of lemma 4.7, together with ([17], Prop. 3.1) and lemma 4.9. □

Theorem 5.3. *(*[11],[12]*) The category* $\mathbf{CatCat}(:= \mathbf{2Cat})$ *of 2-categories admits a cofibrantly generated model structure in which the weak equivalences are the DK-equivalences and the fibrations are the DK-fibrations. A generating set of trivial cofibrations consists of*

(L1) $\{\mathcal{F}(2, j)\}$*, where j is the single generating trivial cofibration of* **Cat** *(*[11]*, example 1.1), and*

(L2) the map $\mathcal{I} \overset{\delta_y}{\rightarrow} E'$*, where E' is the 2-category on two objects x and y described in the last paragraph of (*[12]*, page 197) and δ_y picks out the object x; a formal description of E' is as follows: let G be the graph*

$$x \underset{v}{\overset{u}{\rightleftarrows}} y$$

and let $\iota\colon Set \to \mathbf{Cat}$ *be the indiscrete-category functor. Then* ι *induces a functor* $\iota_2\colon \mathbf{Cat} \to \mathbf{2Cat}$ *and* E' *is* ι_2 *applied to the free category on the graph* G.

Lemma 5.4. *Let* $\mathcal{I} \xrightarrow{(\delta_y)^u} u^*E' \to E'$ *be the factorisation (2.3) of the map* δ_y *in theorem 5.3. Then* $u^*E'(x,x)$ *is cofibrant as a monoid in* $\mathbf{Cat}$.

Proof. One shows that the map $1 \to u^*E'(x,x)$ has the left lifting property with respect to the trivial fibrations of monoids in $\mathbf{Cat}$. □

6. Appendix 1: Bifibrations and weak factorisation systems

This appendix is a reworking of A. Roig's main result ([14], Thm. 5.1). His theorem is false as stated, cf. 7.3. We shall give here the correct statement. Our proof follows closely Roig's, but we could not follow entirely his argument.

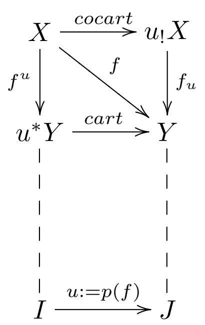
6.1. Let $p\colon \mathbb{E} \to \mathbb{B}$ be a Grothendieck fibration. If I is an object of $\mathbb{B}$, we write $\mathbb{E}_I$ for the fibre category over I. Every map $f\colon X \to Y$ of $\mathbb{E}$ can be decomposed as

$$X \xrightarrow{f^u} u^*(Y) \to Y,$$

where $u = p(f)$ and $u^*(Y) \to Y$ is cartesian over u.

Let $(\mathcal{M},\mathcal{N})$ be a weak factorisation system (w.f.s. abbreviated) on $\mathbb{B}$. If $\mathcal{M}'$ is the inverse image of $\mathcal{M}$ by the functor p and $\mathcal{N}'$ is the class of cartesian lifts of the maps in $\mathcal{N}$, then $(\mathcal{M}',\mathcal{N}')$ is a w.f.s. on $\mathbb{E}$. There is a dual statement for Grothendieck opfibrations.

6.2. Let $p\colon \mathbb{E} \to \mathbb{B}$ be a (Grothendieck) bifibration. Then every map $f\colon X \to Y$ of $\mathbb{E}$ can be decomposed in two ways:



For every morphism $u\colon I \to J$ of $\mathbb{B}$, one can choose an adjoint pair

$$u_!\colon \mathbb{E}_I \rightleftarrows \mathbb{E}_J : u^*.$$

Suppose that

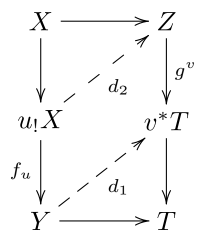
(i) the base category $\mathbb{B}$ has a w.f.s. $(\mathcal{A}, \mathcal{B})$;

(ii) for each object I of $\mathbb{B}$, the fibre category $\mathbb{E}_I$ has a w.f.s. $(\mathcal{A}_I, \mathcal{B}_I)$;

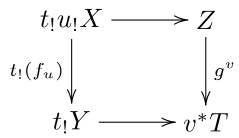
(iii) for every morphism $u\colon I \to J$ of $\mathbb{B}$, we have $u^*(\mathcal{B}_J) \subseteq \mathcal{B}_I$.

Let $\mathcal{A}^p$ be the class of maps f of $\mathbb{E}$ such that $f_u \in \mathcal{A}_{p(cod(f))}$ and $p(f) \in \mathcal{A}$.
Let $\mathcal{B}^p$ be the class of maps f of $\mathbb{E}$ such that $f^u \in \mathcal{B}_{p(dom(f))}$ and $p(f) \in \mathcal{B}$.
Then $(\mathcal{A}^p, \mathcal{B}^p)$ is a w.f.s. on $\mathbb{E}$.

Proof. First, the classes $\mathcal{A}^p$ and $\mathcal{B}^p$ are closed under retracts. This is shown in ([14], section 6). Second, every map f in $\mathbb{E}$ can be factored as $f = gh$, where $h \in \mathcal{A}^p$ and $g \in \mathcal{B}^p$ (cf. *loc.cit*). Third, every commutative diagram



in $\mathbb{E}$ with f in $\mathcal{A}^p$ and g in $\mathcal{B}^p$ has a diagonal filler. Indeed, since $v^*T \to T$ is a cartesian lift of a map in $\mathcal{B}$, there is by 6.1 a diagonal filler d_1. The resulting diagram has a diagonal filler d_2 again by 6.1 since p is also a Grothendieck opfibration. The image of the above diagram in $\mathbb{B}$ has a diagonal filler $t\colon p(Y) \to p(Z)$. By (iii), the diagram



has a diagonal filler which precomposed with $Y \to t_!Y$ gives the desired diagonal filler for the initial diagram. Fourth, we conclude by the usual "retract argument". □

6.3. Let $p\colon \mathbb{E} \to \mathbb{B}$ be a bifibration. We replace condition (i) in 6.2 by (i') the base category $\mathbb{B}$ has a model structure $(\mathfrak{Cof}, \mathfrak{W}, \mathfrak{Fib})$.

We call a map f of $\mathbb{E}$ a *weak equivalence* if $p(f)$ is a weak equivalence in $\mathbb{B}$; a *cofibration* if $f_u \in \mathcal{A}_{p(cod(f))}$ and $p(f) \in \mathfrak{Cof}$; a *fibration* if $f^u \in \mathcal{B}_{p(dom(f))}$ and $p(f) \in \mathfrak{Fib}$.

Then, these classes of maps provide $\mathbb{E}$ with a model structure. Therefore, one has two model structures on $\mathbb{E}$ if each fibre category $\mathbb{E}_I$ has a model structure for which the adjoint pair $(u_!, u^*)$ is a Quillen pair.

Proof. Immediate from 6.2. □

6.4. Let $p\colon \mathbb{E} \to \mathbb{B}$ be a bifibration satisfying condition 5.3(i') above together with

(ii') for each object I of $\mathbb{B}$, the fibre category $\mathbb{E}_I$ admits a model structure $(\mathfrak{Cof}_I, \mathfrak{W}_I, \mathfrak{Fib}_I)$; and

(iii') for every morphism $u\colon I \to J$ of $\mathbb{B}$, the adjoint pair $(u_!, u^*)$ is a Quillen pair.

We define a map f in $\mathbb{E}$ to be a *weak equivalence* if $f^u \in \mathfrak{W}_I$ and $p(f)$ is a weak equivalence in $\mathbb{B}$; a *cofibration* if $f_u \in \mathfrak{Cof}_J$ and $p(f) \in \mathfrak{Cof}$; a *fibration* if $f^u \in \mathfrak{Fib}_I$ and $p(f) \in \mathfrak{Fib}$.

Theorem. [14] *Under the hypotheses (i'), (ii') and (iii'), the three classes of maps defined above give $\mathbb{E}$ a model structure provided that $\mathbb{E}$ is complete and cocomplete and*

(a) *for $u = p(f)$ a weak equivalence of $\mathbb{B}$, the functor u^* preserves and reflects weak equivalences*;

(b) *for $u = p(f)$ a trivial cofibration of $\mathbb{B}$, the unit of the adjoint pair $(u_!, u^*)$ is a weak equivalence.*

Proof. Part (a) implies the three-for-two property of weak equivalences. Let f be a map of $\mathbb{E}$ with $u = p(f)$ a trivial cofibration of $\mathbb{B}$. Then f_u is a cofibration and f^u is a weak equivalence iff f_u is a trivial cofibration, as one can see using (a) and (b). Hence we may conclude by 6.2. □

Remark. (i) An object X of $\mathbb{E}$ is cofibrant (resp. fibrant) in this model structure iff $p(X)$ is cofibrant (resp. fibrant) in $\mathbb{B}$ and X is cofibrant (resp. fibrant) as an object of $\mathbb{E}_{p(X)}$, cf. [14].

(ii) The model structure on $\mathbb{E}$ is right proper if the base category and all fibre categories are right proper. The model structure on $\mathbb{E}$ is left proper if

• the base category and all fibre categories are left proper;

•• for $u = p(f)$ a weak equivalence in $\mathbb{B}$, the unit of the adjoint pair $(u_!, u^*)$ is a weak equivalence;

• • • for $u = p(f)$ a cofibration in $\mathbb{B}$, the functor $u_!$ preserves weak equivalences.

6.5. Model category structures on some comma categories. (a) Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which preserves cofibrations and trivial cofibrations. Recall that the comma category $(F \downarrow id_{\mathcal{D}})$ has objects the triples (A, f, X), where $f\colon FA \to X$ is an arrow of $\mathcal{D}$. It may happen that $(F \downarrow id_{\mathcal{D}})$ is not (co)complete, but we shall neglect this issue since in practice it can be checked case by case. The map $pr_1\colon (F \downarrow id_{\mathcal{D}}) \to \mathcal{C}$, $pr_1((A,f,X)) = A$, is a bifibration and theorem 6.4 can be applied to it. In this case a map

$$(u,v)\colon (A,f,X) \to (A',f',X')$$

of $(F \downarrow id_{\mathcal{D}})$ is a fibration (resp. weak equivalence) if u and v are fibrations (resp. weak equivalences). The map (u,v) is a cofibration if u and the canonical map $X \underset{FA}{\sqcup} FA' \to X'$ are cofibrations in the respective categories.

If, moreover, F sends trivial fibrations to weak equivalences, then the full subcategory of $(F \downarrow id_{\mathcal{D}})$ whose objects are triples (A,f,X) for which f is a weak equivalence of $\mathcal{D}$, satisfies the factorisation axioms required in the definition of a model category.

There are similar considerations for the comma category $(id_{\mathcal{D}} \downarrow G)$, where now the functor $G\colon \mathcal{C} \to \mathcal{D}$ preserves fibrations and trivial fibrations.

(b) Let $\mathcal{C}, \mathcal{D}$, $\mathbb{B}$ be three model categories and let $\mathcal{C} \overset{F}{\to} \mathcal{D} \overset{F'}{\leftarrow} \mathbb{B}$ be a diagram of functors in which F is a left Quillen functor having a right adjoint G. Let $\mathbb{E}$ be the comma category $(F \downarrow F')$; an object of $\mathbb{E}$ is a triple (A,f,X) where $f\colon FA \to F'X$ is an arrow of $\mathcal{D}$. Again we neglect the issue of (co)completeness of $\mathbb{E}$. The map $pr_3\colon \mathbb{E} \to \mathbb{B}$, $pr_3((A,f,X)) = X$, is a bifibration and the dual of theorem 6.4 (theorem 6.4 is not selfdual) can be applied to it. In this case a map

$$(u,v)\colon (A,f,X) \to (A',f',X')$$

of $\mathbb{E}$ is a cofibration (resp. weak equivalence) if u and v are cofibrations (resp. weak equivalences) in the respective categories.

Particular case. Let $\mathbb{B}$ be a model category, I a set and $\mathcal{E}_i$ $(i \in I)$ a family of model categories. Suppose that for each $i \in I$ we are given Quillen pairs $F_i\colon \mathcal{E}_i \rightleftarrows \mathbb{B} : G_i$. Let $\mathbb{E}$ be the comma category $(\prod_{i\in I} F_i \downarrow d)$, where $d\colon \mathbb{B} \to \prod_{i\in I} \mathbb{B}$ is the diagonal functor. An object of $\mathbb{E}$ is a family $(X_i, u_i, b)_i$ where $u_i\colon F_i(X_i) \to b$. An arrow $(f_i, g)\colon (X_i,u_i,b)_i \to (Y_i,v_i,b')_i$ of $\mathbb{E}$ consists of arrows $f_i : X_i \to Y_i$ and $g\colon b \to b'$ making the obvious diagram commute. We denote by p the map pr_3; one has $p((X_i,u_i,b)) = b$. The (existence of the) model structure on $\mathbb{E}$ generalises and at the same time gives a conceptual explanation of the fibre product of model categories considered by B. Toën ([18], page 599).

7. Appendix 2: The fibred model structure on $\mathcal{V}$**Cat** and **SSymMulticat**

This appendix is mostly an application of theorem 6.4. In the first part we derive and study the fibred model structure on $\mathcal{V}$**Cat**, for a fairly general $\mathcal{V}$. In the second part we deal with the fibred model structure on

SSymMulticat, where **S** is the category of simplicial sets regarded as having the classical model structure. Other monoidal model categories than simplicial sets can be considered too. We denote by *Set* the category of sets. We shall use the notations and terminology given in sections 2 and 3.

7.1. Let $\mathcal{V}$ be a closed symmetric monoidal category with enough limits and colimits. We let $\mathcal{V}\mathbf{Cat}(S)$ have the w.f.s. $(all\ maps, isomorphisms)$ and *Set* have the w.f.s. $(monos, epis)$. We obtain from 6.2 a w.f.s. on $\mathcal{V}\mathbf{Cat}$ in which the left class is the class of $\mathcal{V}$-functors monic on objects and the right class is the class of $\mathcal{V}$-functors surjective on objects and fully faithful.

7.2. Let $\mathcal{V}$ be a cofibrantly generated monoidal model category [15] with cofibrant unit I and satisfying the monoid axiom. For technical reasons, we impose on $\mathcal{V}$ the following condition (see [8], Thm. 2.1). Let $\mathbf{I}$ (resp. $\mathbf{J}$) be a generating set of cofibrations (resp. trivial cofibrations). Then the domains of $\mathbf{I}$ (resp. $\mathbf{J}$) are small relative to $\mathcal{V}\otimes\mathbf{I}$-cell (resp. $\mathcal{V}\otimes\mathbf{J}$-cell).

Consider on *Set* the w.f.s. $(monos, epis)$ and on the fibre category $\mathcal{V}\mathbf{Cat}(S)$ over S the w.f.s. $(cofibrations, trivial\ fibrations)$. We obtain from 6.2 a w.f.s. on $\mathcal{V}\mathbf{Cat}$ in which the left class is the class of $\mathcal{V}$-functors monic on objects and with f_u a cofibration in $\mathcal{V}\mathbf{Cat}(T)$; the right class is the class of $\mathcal{V}$-functors surjective on objects and locally a trivial fibration.

7.3. *A counterexample to Roig's theorem (*[14]*, Thm. 5.1).* Let $\mathcal{V}$ be as in 7.2. We let *Set* have model structure in which all maps are weak equivalences, the cofibrations are the monos and the fibrations are the epis. Then the class of weak equivalences in $\mathcal{V}\mathbf{Cat}$ as defined in [14] does not satisfy the three-for-two property: in the sequence

$$\mathcal{I}\to\mathcal{F}(2,\emptyset)=\begin{pmatrix} I & \emptyset \\ \emptyset & I \end{pmatrix}\longrightarrow\mathcal{F}(2,1)=\begin{pmatrix} I & 1 \\ \emptyset & I \end{pmatrix}$$

the first map and the composite are weak equivalences, but the map on the right is not necessarily locally a weak equivalence.

7.4. Let $\mathcal{V}$ be as in 7.2. Consider on *Set* the minimal model structure, in which the weak equivalences are the isomorphisms and all maps are cofibrations as well as fibrations. One can check that the conditions of theorem 6.4 are satisfied, hence $\mathcal{V}\mathbf{Cat}$ has a model structure, referred to as the *fibred model structure.* A $\mathcal{V}$-category $\mathcal{A}$ is cofibrant (resp. fibrant) iff $\mathcal{A}$ is cofibrant (resp. fibrant) as an object of $\mathcal{V}\mathbf{Cat}(Ob(\mathcal{A}))$.

7.5. There is a fibred model structure on $\mathcal{V}\mathbf{Graph}$ as well. Now, a $\mathcal{V}$-graph $\mathcal{X}$ is cofibrant (resp. fibrant) iff $\mathcal{X}$ is locally cofibrant (resp. locally fibrant). This model structure is cofibrantly generated: the generating cofibrations are $(2,i)$ (i generating cofibration of $\mathcal{V}$), $u\colon \emptyset_{graph} \to !$ and $! \sqcup ! \to !$; the generating trivial cofibrations are $(2,j)$ (j generating trivial cofibration of $\mathcal{V}$).

Using the fibred model structure on $\mathcal{V}$**Graph**, we give below another proof of the fibred model structure on $\mathcal{V}$**Cat**, showing that it is cofibrantly generated.

We shall use the transfer principle for the adjunction 2.1(1). The functor $\mathcal{U}$ preserves filtered colimits by ([10], Cor. 3.4). We shall describe the pushout in $\mathcal{V}$**Cat** of a diagram

$$\mathcal{A} \leftarrow \mathcal{F}(2,A) \stackrel{\mathcal{F}(2,j)}{\longrightarrow} \mathcal{F}(2,B),$$

where j is a generating trivial cofibration of $\mathcal{V}$.

Let

$$\begin{array}{ccc} (2,A) & \xrightarrow{(2,j)} & (2,B) \\ \downarrow & & \downarrow \\ \mathcal{U}(\mathcal{A}) & \longrightarrow & \mathcal{X} \end{array}$$

be the pushout in $\mathcal{V}$**Graph** of the adjoint transposed diagram. The $\mathcal{V}$-graph $\mathcal{X}$ has the same set of objects as $\mathcal{A}$ and the map $\mathcal{U}(\mathcal{A}) \to \mathcal{X}$ is the identity on objects. To give a map $(2,A) \to \mathcal{U}(\mathcal{A})$ is to give two objects a and a' of $\mathcal{A}$ and a map $A \to \mathcal{A}(a,a')$. We have

$$\mathcal{X}(x,y) = \begin{cases} B \sqcup_A \mathcal{A}(a,a'), & \text{if } (x,y) = (a,a') \\ \mathcal{A}(a,a'), & \text{otherwise.} \end{cases}$$

Then the pushout of the original diagram is the pushout of the diagram

$$\mathcal{A} \stackrel{\epsilon_{\mathcal{A}}}{\longleftarrow} \mathcal{F}\mathcal{U}(\mathcal{A}) \to \mathcal{F}(\mathcal{X}).$$

Since the inclusion functor from $\mathcal{V}$-categories with fixed set of objects to $\mathcal{V}$-categories preserves colimits, this can be viewed as the pushout in the category of $\mathcal{V}$-categories with fixed set of objects $Ob(\mathcal{A})$. Now one may use ([16], Prop. 6.3(1)).

7.6. Let $\mathcal{A}$ be a $\mathcal{V}$-category. Then $Hom_{Ho(\mathcal{V}\mathbf{Cat})}(\mathcal{I},\mathcal{A})$ is naturally isomorphic to $Ob(\mathcal{A})$.

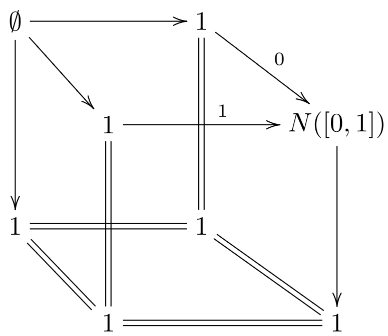
7.7. In general, the fibred model structure on $\mathcal{V}$**Cat** is not compatible with the monoidal product, that is, the pushout-product axiom does not hold. Indeed, take $\mathcal{V}$ to be the category **S** of simplicial sets. Let $[0,1]$ be the "free-living isomorphism" in **Cat** and let N be the nerve functor to simplicial sets. Consider the map $\mathcal{F}(2, N([0,1])) \to \mathcal{F}(2,1)$, which is, in matrix representation,

$$\begin{pmatrix} 1 & N([0,1]) \\ \emptyset & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ \emptyset & 1 \end{pmatrix}.$$

This map is a fibration weak equivalence, but the exponential transpose

$$\mathcal{F}(2, N([0,1]))^{\mathcal{F}(2,1)} \longrightarrow \mathcal{F}(2,1)^{\mathcal{F}(2,1)}$$

is not locally in a trivial fibration: in the diagram of simplicial sets



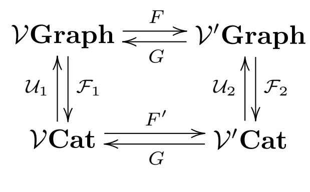
with top and bottom squares pullbacks, the map $\emptyset \to 1$ is not a fibration and/or a weak equivalence. However, the tensor product of weak equivalences between two locally cofibrant $\mathcal{V}$-categories is a weak equivalence.

7.8 Change of base category. Let $\mathcal{V}$ and $\mathcal{V}'$ be two categories as in 7.2.

1. *Let $F\colon \mathcal{V} \rightleftarrows \mathcal{V}' : G$ be a monoidal adjunction and suppose that F preserves the unit object. If the adjoint pair (F, G) is a Quillen pair (resp. Quillen equivalence), then so is the induced adjunction between $\mathcal{V}$***Cat*** *and* $\mathcal{V}'$***Cat***.

2. *Let $F\colon \mathcal{V} \rightleftarrows \mathcal{V}' : G$ be a weak monoidal Quillen pair* ([16], Def. 3.6) *(resp. Quillen equivalence). Then the induced adjunction between $\mathcal{V}$***Cat*** *and* $\mathcal{V}'$***Cat*** *is a Quillen pair (resp. Quillen equivalence).*

Proof. Part 1 is clear. For part 2, we have a commutative square of adjunctions



in which the top horizontal adjunction is a Quillen equivalence. We wish to apply the lifting criterion from appendix 3. We are left to show that the natural transformation $F\mathcal{U}_1 \Rightarrow \mathcal{U}_2 F'$ is a weak equivalence on cofibrant objects. Let $\mathcal{A}$ be a cofibrant $\mathcal{V}$-category and let $S = Ob(\mathcal{A})$. We may restrict ourselves to showing that in the commutative square of adjunctions

$$\begin{array}{ccc} \mathcal{V}Graph(S) & \underset{G}{\overset{F}{\rightleftarrows}} & \mathcal{V}'Graph(S) \\ {\scriptstyle \mathcal{U}_1}\uparrow\downarrow{\scriptstyle \mathcal{F}_1} & & {\scriptstyle \mathcal{U}_2}\uparrow\downarrow{\scriptstyle \mathcal{F}_2} \\ \mathcal{V}Cat(S) & \underset{G}{\overset{F'}{\rightleftarrows}} & \mathcal{V}'Cat(S) \end{array}$$

the natural transformation $F\mathcal{U}_1 \Rightarrow \mathcal{U}_2 F'$ is a weak equivalence on cofibrant objects. But this follows from ([16], Prop. 6.4.(1)). □

Remark. Let $\mathcal{V}$ be as in 7.2. The composite adjunction

$$\mathbf{S} \underset{}{\overset{\pi_0}{\rightleftarrows}} Set \underset{Hom_{\mathcal{V}}(I,\_)}{\rightleftarrows} \mathcal{V}$$

induces an adjunction

$$\mathbf{SCat} \rightleftarrows \mathcal{V}\mathbf{Cat}.$$

Thus, $\mathcal{V}\mathbf{Cat}$ is enriched, tensored and cotensored over simplicial categories. The above adjoint pair is a Quillen pair since the unit I of $\mathcal{V}$ is cofibrant.

7.9. Let S be a set. By ([1], Thm. 2.1) the fibre category $\mathbf{SSymMulticat}(S)$ has a model structure. Letting Set have the w.f.s. $(monos, epis)$ and $\mathcal{V}\mathbf{SymMulticat}(S)$ the w.f.s. $(cofibrations, trivial\ fibrations)$ we obtain from 6.2 a w.f.s. on $\mathbf{SSymMulticat}$ in which the left class is the class of symmetric $\mathbf{S}$-multifunctors monic on objects and with f_u a cofibration in $\mathbf{SSymmMulticat}(T)$; the right class is the class of symmetric $\mathbf{S}$-multifunctors surjective on objects and locally a trivial fibration.

7.10. We let Set have the minimal model structure (7.4). By ([1], Thm. 2.1) and theorem 6.4 we obtain the fibred model structure on $\mathbf{SSymmMulticat}$. The adjoint pair 3.1(3) is a Quillen pair between the fibred model structures on $\mathbf{SCat}$ and $\mathbf{SSymmMulticat}$.

8. Appendix 3: Lifting of Quillen equivalences

Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen equivalence between model categories. Let $F' : \mathcal{C}' \rightleftarrows \mathcal{D}' : G'$ be an adjunction between two categories having initial and terminal objects. Suppose that these adjoint pairs are connected via two adjunctions $\mathcal{F}_1 \colon \mathcal{C} \rightleftarrows \mathcal{C}' \colon \mathcal{U}_1$ and $\mathcal{F}_2 \colon \mathcal{D} \rightleftarrows \mathcal{D}' \colon \mathcal{U}_2$ such that $\mathcal{U}_1 G' \simeq G\mathcal{U}_2$. The picture is

$$\begin{array}{ccc} \mathcal{C} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathcal{D} \\ \mathcal{U}_1 \Big\uparrow \Big\downarrow \mathcal{F}_1 & & \mathcal{U}_2 \Big\uparrow \Big\downarrow \mathcal{F}_2 \\ \mathcal{C}' & \underset{G'}{\overset{F'}{\rightleftarrows}} & \mathcal{D}'. \end{array}$$

Define a map f of $\mathcal{C}'$ to be a *weak equivalence* (resp. *fibration*) if $\mathcal{U}_1(f)$ is a weak equivalence (resp. fibration) in $\mathcal{C}$. The class of *cofibrations* of $\mathcal{C}'$ is by definition the class of maps having the right lifting property with respect to fibration weak equivalences. Similarly, one defines the weak equivalences, fibrations and cofibrations of $\mathcal{D}'$.

Furthermore, we assume that $\mathcal{U}_i$ $(i = 1, 2)$ preserves the cofibrant objects and that the induced natural transformation $F\mathcal{U}_1 \Rightarrow \mathcal{U}_2 F'$ is a weak equivalence on cofibrant objects. Then for any cofibrant object X of $\mathcal{C}'$ and any fibrant object Y of $\mathcal{D}'$, a map $F'(X) \to Y$ is a weak equivalence iff its adjoint transpose $X \to G'(Y)$ is such.

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