ON THE HOMOTOPY THEORY OF SPECTRAL CATEGORIES

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Abstract. Let Sp^{Σ} be the category of symmetric spectra, regarded as having the stable model structure. We prove that the category of small categories enriched over Sp^{Σ} admits a model category structure. The method of proof applies to other categories than symmetric spectra as well, as long as these categories are linked to the category of simplicial sets via a certain strong monoidal Quillen pair.

1. Introduction

Let Sp^{Σ} be the category of symmetric spectra. It admits a closed symmetric monoidal structure given by the smash product ([8], 2.2.10). A *spectral category* is a category enriched over Sp^{Σ} and a spectral functor between two spectral categories is a structure preserving map which satisfies natural associativity and unit conditions. We refer the reader to Kelly's book [10] for an introduction to enriched category theory in general. This determines the category $Sp^{\Sigma}\mathbf{Cat}$ of spectral categories. A spectral category can be large, but we shall only consider small (with respect to some fixed universe) ones.

The category Sp^{Σ} supports many Quillen model category structures, see e.g. [8] and [12], some of them being compatible in a certain sense with the smash product of symmetric spectra. For general purposes we shall work with the stable model structure, as defined in ([8], 3.4), whose weak equivalences are called stable equivalences.

The fact that the stable model structure is compatible with the smash product allows one to define a notion of "weak equivalence" of spectral categories, which is a natural generalisation of the notion of equivalence of spectral categories (in the sense of enriched category theory [10]). In detail, to every spectral category $\mathcal{A}$ one can associate a category $ho(\mathcal{A})$ enriched over the homotopy category $Ho(Sp^{\Sigma})$. A spectral functor $f\colon \mathcal{A} \to \mathcal{B}$ is said to be *Dwyer-Kan equivalence* if the functor $ho(f)\colon ho(\mathcal{A}) \to ho(\mathcal{B})$ is an equivalence of $Ho(Sp^{\Sigma})$-categories. This notion is the symmetric spectra analogue of the notion of (Dwyer-Kan) equivalence of simplicial categories [3]. The purpose of this note is to prove

Theorem 1.1. *The category $Sp^{\Sigma}\mathbf{Cat}$ admits a Quillen model category structure with the class of Dwyer-Kan equivalences as weak equivalences.*

The above statement is sloppy as one must specify the class of cofibrations or fibrations as well, but this is not important now. Theorem 1.1 is actually a corollary of a more general result which we prove as theorem 2.3. It will be apparent from theorem 2.3 that one can prove an analogue of theorem 1.1 for other model structures on symmetric spectra than the stable one. For example, one can consider on Sp^{Σ} the stable S-model structure [15].

The paper is organized as follows. In section 2 we set up the general frame, we recall the analogous model structure on simplicial categories [1] and we prove theorem 2.3. A technical result about pushouts of enriched categories, needed in the proof of theorem 2.3, is the content of section 3.

2. A general theorem

Let **Cat** be the category of small categories. It has a *natural* model structure in which a cofibration is a functor monic on objects, a weak equivalence is an equivalence of categories and a fibration is an isofibration [9]. The fibration weak equivalences are the equivalences surjective on objects.

Let $\mathcal{V}$ be a cofibrantly generated monoidal model category [13] with cofibrant unit I. The small $\mathcal{V}$-categories together with the $\mathcal{V}$-functors between them form a category written $\mathcal{V}\mathbf{Cat}$. If S is a set, we denote by $\mathcal{V}\mathbf{Cat}(S)$ the category of $\mathcal{V}$-categories with fixed set of objects S. The morphisms in $\mathcal{V}\mathbf{Cat}(S)$ are the $\mathcal{V}$-functors which are the identity on objects.

Let $\mathcal{M}$ be a class of maps of $\mathcal{V}$. Following [11], we say that a $\mathcal{V}$-functor $f\colon \mathcal{A} \to \mathcal{B}$ is *locally in* $\mathcal{M}$ if for each pair $x, y \in \mathcal{A}$ of objects, the map $f_{x,y}\colon \mathcal{A}(x, y) \to \mathcal{B}(f(x), f(y))$ is in $\mathcal{M}$. When $\mathcal{M}$ is the class of isomorphisms of $\mathcal{V}$, a $\mathcal{V}$-functor which is locally an isomorphism is called *full and faithful.*

We denote by $\mathfrak{W}$ (resp. $\mathfrak{Fib}$) the class of weak equivalences (resp. fibrations) of $\mathcal{V}$. We have a functor $[\_]_{\mathcal{V}}\colon \mathcal{V}\mathbf{Cat} \to \mathbf{Cat}$ obtained by change of base along the (symmetric monoidal) composite functor

$$\mathcal{V} \xrightarrow{\gamma} Ho(\mathcal{V}) \xrightarrow{Hom_{Ho(\mathcal{V})}(I,\_)} Set.$$

Definition 2.1. *Let $f\colon \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathcal{V}\mathbf{Cat}$.*

1. The morphism f is homotopy essentially surjective *if the induced functor $[f]_{\mathcal{V}}\colon [\mathcal{A}]_{\mathcal{V}} \to [\mathcal{B}]_{\mathcal{V}}$ is essentially surjective.*

2. The morphism f is a DK-equivalence *if it is homotopy essentially surjective and locally in $\mathfrak{W}$.*

3. The morphism f is a DK-fibration *if f is locally in $\mathfrak{Fib}$ and $[f]_{\mathcal{V}}$ is an isofibration.*

4. The morphism f is a trivial fibration *if it is a DK-equivalence and a DK–fibration.*

5. The morphism f is a cofibration *if it belongs to the saturated class generated by the map $u\colon \emptyset \to \mathcal{I}$, where $\emptyset$ is the initial $\mathcal{V}$-category and $\mathcal{I}$ is the $\mathcal{V}$-category with a single object $*$ and $\mathcal{I}(*,*) = I$, together with the maps*

$$\bar{2}_i\colon \bar{2}_A \to \bar{2}_B,$$

where i is a generating cofibration of $\mathcal{V}$. Here the $\mathcal{V}$-category $\bar{2}_A$ has objects 0 and 1, with $\bar{2}_A(0,0) = \bar{2}_A(1,1) = I$, $\bar{2}_A(0,1) = A$ and $\bar{2}_A(1,0) = \emptyset$.

It follows from the above definitions that (a) a $\mathcal{V}$-functor is a trivial fibration iff it is surjective on objects and locally a trivial fibration, and (b) a $\mathcal{V}$-functor is a cofibration iff it has the left lifting property with respect to the trivial fibrations.

$\mathcal{V}$ is said to be *DK-admissible* if the category $\mathcal{V}\mathbf{Cat}$ admits a Quillen model category structure with DK-equivalences as weak equivalences and trivial fibrations as above. One has the fundamental result

Theorem 2.2. [1] *Let $\mathbf{S}$ be the category of simplicial sets, regarded as having the classical model structure. Then $\mathbf{S}$ is DK-admissible. The fibrations are the DK-fibrations. A generating set of trivial cofibrations consists of*

(B1) $\{\bar{2}_j\}$, where j is a horn inclusion, and

(B2) inclusions $\mathcal{I} \xrightarrow{\delta_y} \mathcal{H}$, where $\{\mathcal{H}\}$ is a set of representatives for the isomorphism classes of simplicial categories on two objects which have countably many simplices in each function complex. Furthermore, each such $\mathcal{H}$ is required to be cofibrant and weakly contractible in $\mathbf{SCat}(\{x,y\})$. Here $\{x,y\}$ is the set with elements x and y and δ_y omits y.

With this we prove

Theorem 2.3. *Let $\mathcal{V}$ be a cofibrantly generated monoidal model category with cofibrant unit and satisfying the monoid axiom. Suppose furthermore that $\mathcal{V}$ satisfies the following technical condition (see [7], Thm. 2.1): if $\mathbf{I}$ (resp. $\mathbf{J}$) denote a generating set of cofibrations (resp. trivial cofibrations) of $\mathcal{V}$, then the domains of $\mathbf{I}$ (resp. $\mathbf{J}$) are required to be small relative to $\mathcal{V} \otimes \mathbf{I}$-cell (resp. $\mathcal{V} \otimes \mathbf{J}$-cell).*

Let

$$F\colon \mathbf{S} \rightleftarrows \mathcal{V}\colon G$$

be a Quillen pair such that F is strong symmetric monoidal and preserves the unit object. Then $\mathcal{V}$ is DK-admissible. The fibrations are the DK-fibrations. The model structure is right proper if the model structure on $\mathcal{V}$ is right proper.

Proof. The adjoint pair (F, G) induces adjoint pairs

$$F' : \mathbf{SCat} \rightleftarrows \mathcal{V}\mathbf{Cat} : G'$$

and

$$F : Ho(\mathbf{S}) \rightleftarrows Ho(\mathcal{V}) : RG.$$

The functor G' preserves the trivial fibrations and the functor F' preserves the DK-equivalences. The functor F is strong symmetric monoidal and preserves the unit object. We have a natural isomorphism of functors

$$\eta : [\_]_{\mathcal{V}} \cong [\_]_{\mathbf{S}} G' : \mathcal{V}\mathbf{Cat} \to \mathbf{Cat}$$

such that for all $\mathcal{A} \in \mathcal{V}\mathbf{Cat}$, $\eta_{\mathcal{A}}$ is the identity on objects.

To prove theorem 2.3 we shall use ([6], 2.1.19). We take in *loc. cit.*:

-the set I to be the set described in definition 2.1.5;

-the set J to be $F'(B2) \cup \{\bar{2}_j\}$, where j is a horn inclusion;

-the class W to be the class of DK-equivalences.

It is enough to prove that J-$cof \subset$ W and that W $\cap$ J-inj = I-inj. Notice that I-inj is the class of trivial fibrations, and that by definition we have W $\cap$ J-inj = I-inj. Thus, it remains to prove that if $\delta_y : \mathcal{I} \to \mathcal{H}$ is a map belonging to the set B2 in theorem 2.2 and $\mathcal{A}$ is any $\mathcal{V}$-category, then in the pushout diagram

$$\begin{array}{ccc} F'\mathcal{I} & \xrightarrow{x} & \mathcal{A} \\ {\scriptstyle F'\delta_y}\downarrow & & \downarrow \\ F'\mathcal{H} & \longrightarrow & \mathcal{B} \end{array}$$

the map $\mathcal{A} \to \mathcal{B}$ is a DK-equivalence. We factor the map δ_y as $\mathcal{I} \overset{(\delta_y)^u}{\longrightarrow} u^*\mathcal{H} \to \mathcal{H}$ where $u = Ob(\delta_y)$ and then we take consecutive pushouts:

$$\begin{array}{ccc} F'\mathcal{I} & \xrightarrow{x} & \mathcal{A} \\ {\scriptstyle F'\delta_y^u}\downarrow & & \downarrow{\scriptstyle j} \\ F'u^*\mathcal{H} & \longrightarrow & \mathcal{A}' \\ \downarrow & & \downarrow \\ F'\mathcal{H} & \longrightarrow & \mathcal{B}. \end{array}$$

Recall that $u^*\mathcal{H}$ has $\{x\}$ as set of objects and $u^*\mathcal{H}(x, x) = \mathcal{H}(x, x)$. By Lemma 2.4 the map $(\delta_y)^u$ is a trivial cofibration in the category of simplicial monoids, therefore the map j is a trivial cofibration in $\mathcal{V}\mathbf{Cat}(Ob(\mathcal{A}))$. By proposition 3.1 the map $\mathcal{A}' \to \mathcal{B}$ is a full and faithful inclusion, therefore the map $\mathcal{A} \to \mathcal{B}$ is a DK-equivalence. This finishes the proof that $\mathcal{V}$ is DK-admissible. Right properness is standard. □

Lemma 2.4. *Let $\mathcal{A}$ be a cofibrant simplicial category. Then for each $a \in Ob(\mathcal{A})$ the simplicial monoid $a^*\mathcal{A}$ is cofibrant (as a monoid).*

Proof. $a^*\mathcal{A}$ is the simplicial category with one object a and $a^*\mathcal{A}(a,a) = \mathcal{A}(a,a)$. Let $S = Ob(\mathcal{A})$. $\mathcal{A}$ is cofibrant iff it is cofibrant as an object of $\mathbf{SCat}(S)$. The cofibrant objects of $\mathbf{SCat}(S)$ are characterised in ([4], 7.6): they are the retracts of free simplicial categories. Therefore it suffices to prove that if $\mathcal{A}$ is a free simplicial category then $a^*\mathcal{A}$ is a free simplicial category for all $a \in S$. There is a full and faithful functor $\varphi\colon \mathbf{SCat} \to \mathbf{Cat}^{\Delta^{op}}$ given by $Ob(\varphi(\mathcal{A})_n) = Ob(\mathcal{A})$ for all $n \geq 0$ and $\varphi(\mathcal{A})_n(a,a') = \mathcal{A}(a,a')_n$. Recall ([4], 4.5) that $\mathcal{A}$ is a free simplicial category iff (i) for all $n \geq 0$ the category $\varphi(\mathcal{A})_n$ is a free category on a graph G_n, and (ii) for all epimorphisms $\alpha\colon [m] \to [n]$ of Δ, $\alpha^*\colon \varphi(\mathcal{A})_n \to \varphi(\mathcal{A})_m$ maps G_n to G_m.

Let $a \in S$. The category $\varphi(a^*\mathcal{A})_n$ is a full subcategory of $\varphi(\mathcal{A})_n$ with set of objects $\{a\}$, hence it is free as well. A set $G_n^{a^*\mathcal{A}}$ of generators can be described as follows. An element of $G_n^{a^*\mathcal{A}}$ is a path from a to a such that every arrow in the path belongs to G_n and there is at most one arrow in the path with source and target a. Since every epimorphism $\alpha\colon [m] \to [n]$ of Δ has a section, α^* maps $G_n^{a^*\mathcal{A}}$ to $G_m^{a^*\mathcal{A}}$. $\square$

Proof of Theorem 1.1. Let Σ be the symmetric groupoid ([8], 2.1.1). Let $\mathbf{S}_\bullet$ be the category of pointed simplicial sets and let $\wedge$ be the smash product of pointed simplicial sets. We let $\mathbf{S}_\bullet$ have the standard model structure; we shall denote it by $(\mathbf{S}_\bullet, \wedge, 1_+)$. Consider on $\mathbf{S}_\bullet^{\Sigma^{op}}$ the projective model structure: the fibrations and weak equivalences are defined pointwise. We have adjoint pairs

$$\mathbf{S} \underset{}{\overset{(\text{-})_+}{\rightleftarrows}} (\mathbf{S}_\bullet, \wedge, 1_+) \underset{\Gamma_0}{\overset{(\text{-})[0]}{\rightleftarrows}} (\mathbf{S}_\bullet^{\Sigma^{op}}, \boxtimes, \mathbf{1}) \underset{}{\overset{-\boxtimes S}{\rightleftarrows}} Sp^{\Sigma}_{pl} \underset{Id}{\overset{Id}{\rightleftarrows}} Sp^{\Sigma}$$

in which

$$A[0](n) = \begin{cases} A, & \text{if } n = 0 \\ \emptyset, & \text{if } n \neq 0, \end{cases}$$

$\boxtimes$ denotes the Day convolution product, $\mathbf{1}$ its unit, $\Gamma_0(X) = Hom_{\mathbf{S}_\bullet^{\Sigma^{op}}}(\mathbf{1}, X)$, S is the sphere spectrum and Sp^{Σ}_{pl} denotes the projective level model structure on symmetric spectra ([8], Thm. 5.1.2). The left adjoints are strong symmetric monoidal and preserve the unit. The composite adjunction is a Quillen pair.

3. On certain pushouts of enriched categories

Let $\mathcal{V}$ be a cocomplete closed symmetric monoidal category with tensor product $\otimes$ and unit I. We recall that a $\mathcal{V}$-graph is a $\mathcal{V}$-category without composition and unit maps. We shall denote by $\mathcal{V}$**Graph** the category of small $\mathcal{V}$-graphs.

We also recall (section 2) that a $\mathcal{V}$-functor which is locally an isomorphism is called *full and faithful*. The notion makes sense for morphisms of $\mathcal{V}$-graphs as well.

Proposition 3.1. *(*[5]*, Prop. 5.2) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three small $\mathcal{V}$-categories and let $i : \mathcal{A} \hookrightarrow \mathcal{B}$ be a full and faithful inclusion. Then in the pushout diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & \mathcal{B} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{C} & \xrightarrow{i'} & \mathcal{D} \end{array}$$

the map $i' \colon \mathcal{C} \to \mathcal{D}$ is a full and faithful inclusion.

Proof. We shall construct $\mathcal{D}$ explicitly, as was done in the proof of ([5], Prop. 5.2). We put $Ob(\mathcal{D}) = Ob(\mathcal{C}) \sqcup (Ob(\mathcal{B}) - Ob(\mathcal{A}))$ and $\mathcal{D}(p, q) = \mathcal{C}(p, q)$ if $p, q \in Ob(\mathcal{C})$. For $p \in Ob(\mathcal{C})$ and $q \in (Ob(\mathcal{B}) - Ob(\mathcal{A}))$ we define

$$\mathcal{D}(p, q) = \int^{x \in Ob(\mathcal{A})} \mathcal{B}(x, q) \otimes \mathcal{C}(p, f(x)).$$

For $p \in (Ob(\mathcal{B}) - Ob(\mathcal{A}))$ and $q \in Ob(\mathcal{C})$ we define

$$\mathcal{D}(p, q) = \int^{x \in Ob(\mathcal{A})} \mathcal{C}(f(x), q) \otimes \mathcal{B}(p, x).$$

For $p, q \in (Ob(\mathcal{B}) - Ob(\mathcal{A}))$ we define $\mathcal{D}(p, q)$ to be the pushout

$$\begin{array}{ccc} \int^{x \in Ob(\mathcal{A})} \mathcal{B}(x, q) \otimes \mathcal{B}(p, x) & \longrightarrow & \int^{x \in Ob(\mathcal{A})} \int^{y \in Ob(\mathcal{A})} \mathcal{B}(x, q) \otimes \mathcal{C}(f(y), f(x)) \otimes \mathcal{B}(p, y) \\ \downarrow & & \downarrow \\ \mathcal{B}(p, q) & \longrightarrow & \mathcal{D}(p, q). \end{array}$$

We shall describe a way to see that, with the above definition, $\mathcal{D}$ is indeed a $\mathcal{V}$-category.

Let $(\mathcal{B} - \mathcal{A})^+$ be the preorder with objects all finite subsets $S \subseteq Ob(\mathcal{B}) - Ob(\mathcal{A})$, ordered by inclusion. For $S \in (\mathcal{B} - \mathcal{A})^+$, let $\mathcal{A}_S$ be the full sub-$\mathcal{V}$-category of $\mathcal{B}$ with objects $Ob(\mathcal{A}) \cup S$. Then $\mathcal{B} = \lim\limits_{(\mathcal{B} - \mathcal{A})^+} \mathcal{A}_S$. On the other hand, a filtered colimit of full and faithful inclusions of $\mathcal{V}$-categories is a full and faithful inclusion. This is because the forgetful functor to $\mathcal{V}$**Graph** preserves filtered colimits ([11], Cor. 3.4) and a filtered colimit of full and faithful inclusions of $\mathcal{V}$-graphs is a full and faithful inclusion. Therefore one can assume from the beginning that $Ob(\mathcal{B}) = Ob(\mathcal{A}) \cup \{q\}$, where $q \notin Ob(\mathcal{A})$.

Case 1: f is full and faithful. In this case the pushout giving $\mathcal{D}(q,q)$ is simply $\mathcal{B}(q,q)$, all the other formulas remain unchanged. Then to show that $\mathcal{D}$ is a $\mathcal{V}$-category is straightforward.

Case 2: f is the identity on objects. The map i induces an adjoint pair

$$i_! : \mathcal{V}\mathbf{Cat}(Ob(\mathcal{A})) \rightleftarrows \mathcal{V}\mathbf{Cat}(Ob(\mathcal{B})) : i^*.$$

One has

$$i_!\mathcal{A}(a,a') = \begin{cases} \mathcal{A}(a,a'), & \text{if } a,a' \in Ob(\mathcal{A}), \\ \emptyset, & \text{otherwise}, \\ I, & \text{if } a = a' = q, \end{cases}$$

and i factors as $\mathcal{A} \to i_!\mathcal{A} \to \mathcal{B}$, where $i_!\mathcal{A} \to \mathcal{B}$ is the obvious map in $\mathcal{V}\mathbf{Cat}(Ob(\mathcal{B}))$. Then the original pushout can be computed using the diagram

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & i_!\mathcal{A} & \longrightarrow & \mathcal{B} \\ {\scriptstyle f}\downarrow & & {\scriptstyle i_!f}\downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & i_!\mathcal{C} & \longrightarrow & \mathcal{D} \end{array}$$

where the square on the right is a pushout in $\mathcal{V}\mathbf{Cat}(Ob(\mathcal{B}))$.

Recall [2] that the category $\mathcal{V}\mathbf{Graph}(Ob(\mathcal{B}))$ of $\mathcal{V}$-graphs with fixed set of objects $Ob(\mathcal{B})$ is a (nonsymmetric) monoidal category with monoidal product

$$X \square Y(a,b) = \coprod_{z \in Ob(\mathcal{B})} X(a,z) \otimes Y(z,b)$$

and unit

$$\mathcal{I}_{Ob(\mathcal{B})}(a,b) = \begin{cases} I, & \text{if } a = b \\ \emptyset, & \text{otherwise.} \end{cases}$$

The category $\mathcal{V}\mathbf{Cat}(Ob(\mathcal{B}))$ is precisely the category of monoids in $\mathcal{V}\mathbf{Graph}(Ob(\mathcal{B}))$ with respect to $-\square-$.

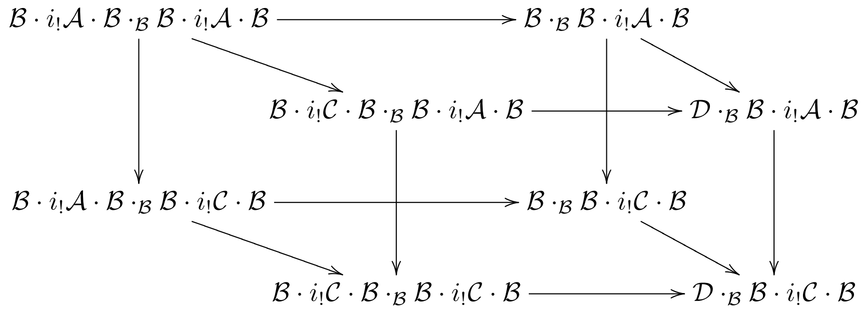
We claim that $\mathcal{D}$ can be calculated as the pushout in the category ${}_{\mathcal{B}}Mod_{\mathcal{B}}$ of $(\mathcal{B},\mathcal{B})$-bimodules of the diagram

$$\begin{array}{ccc} \mathcal{B}\square_{i_!\mathcal{A}}\mathcal{B} & \xrightarrow{\mathcal{B}\square_{i_!\mathcal{A}} i_! f \square_{i_!\mathcal{A}}\mathcal{B}} & \mathcal{B}\square_{i_!\mathcal{A}} i_!\mathcal{C} \square_{i_!\mathcal{A}}\mathcal{B} \\ \downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{B} & \longrightarrow & \mathcal{D}. \end{array}$$

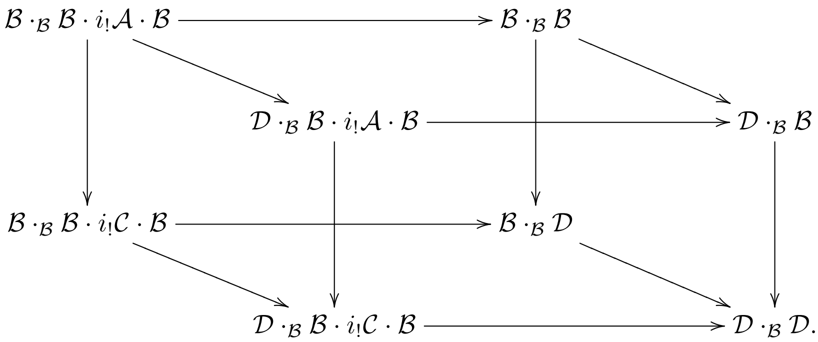
For this we have to show that $\mathcal{D}$ is a monoid in ${}_{\mathcal{B}}Mod_{\mathcal{B}}$. We first show that $\mathcal{B}\Box_{i_!\mathcal{A}}i_!\mathcal{C}\Box_{i_!\mathcal{A}}\mathcal{B}$ is a monoid in ${}_{\mathcal{B}}Mod_{\mathcal{B}}$. There is a canonical isomorphism

$$i_!\mathcal{C}\Box_{i_!\mathcal{A}}i_!\mathcal{C} \cong i_!\mathcal{C}\Box_{i_!\mathcal{A}}\mathcal{B}\Box_{i_!\mathcal{A}}i_!\mathcal{C}$$

of $(i_!\mathcal{A}, i_!\mathcal{A})$-bimodules which is best seen pointwise, using coends. This provides a multiplication for $\mathcal{B}\Box_{i_!\mathcal{A}}i_!\mathcal{C}\Box_{i_!\mathcal{A}}\mathcal{B}$ which is again best seen to be associative by working pointwise, using coends. To define a multiplication for $\mathcal{D}$ consider the cube diagrams



and



For space considerations we have suppressed tensors (always over $i_!\mathcal{A}$, unless explicitly indicated) from notation. The right face of the first cube is the same as the left face of the latter cube. Let PO_1 (resp. PO_2) be the pushout of the left (resp. right) face of the first cube diagram. Let PO_3 be the pushout of the right face of the second cube diagram. We have pushout digrams

$$\begin{array}{ccccc} PO_1 & \longrightarrow & PO_2 & \longrightarrow & PO_3 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B} & \longrightarrow & \mathcal{D}\cdot_{\mathcal{B}}\mathcal{B}\cdot i_!\mathcal{C}\cdot\mathcal{B} & \longrightarrow & \mathcal{D}\cdot_{\mathcal{B}}\mathcal{D} \end{array}$$

Using these pushouts and the fact that $\mathcal{B}\square_{i_!\mathcal{A}} i_!\mathcal{C}\square_{i_!\mathcal{A}}\mathcal{B}$ is a monoid one can define in a canonical way a map $\mu\colon \mathcal{D}\cdot_{\mathcal{B}}\mathcal{D}\to\mathcal{D}$. We omit the long verification that μ gives $\mathcal{D}$ the structure of a monoid. The map μ was constructed in such a way that m becomes a morphism of monoids. The fact that $\mathcal{D}$ has the universal property of the pushout in the category $\mathcal{V}\mathbf{Cat}(Ob(\mathcal{B}))$ follows from its definition.

Case 3: f is arbitrary. Let $u = Ob(f)$. We factor f as $\mathcal{A}\xrightarrow{f^u} u^*\mathcal{C}\to\mathcal{C}$, where f^u is a map in $\mathcal{V}\mathbf{Cat}(Ob(\mathcal{A}))$. One has $u^*\mathcal{C}(a,a') = \mathcal{C}(f(a), f(a'))$ $(a, a'\in Ob(\mathcal{A}))$ and $u^*\mathcal{C}\to\mathcal{C}$ is full and faithful. Take consecutive pushouts

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{i} & \mathcal{B} \\
{\scriptstyle f^u}\downarrow & & \downarrow \\
u^*\mathcal{C} & \longrightarrow & \mathcal{A}' \\
\downarrow & & \downarrow \\
\mathcal{C} & \longrightarrow & \mathcal{D}
\end{array}$$

and apply cases 2 and 1. □

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