COMPUTATION OF AN INTEGRAL BASIS OF QUARTIC NUMBER FIELD

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ABSTRACT. In this paper, for each prime integer p, a p-integral basis of a quartic number field K defined by an irreducible polynomial $P(X) = X^4+aX+b \in \mathbb{Z}[X]$ is given. The discriminant d_K of K and an integral basis of K are then obtained from its p-integral bases.

INTRODUCTION

Let K be a quartic number field defined by an irreducible polynomial $P(X) = X^4 + aX + b \in \mathbb{Z}[X]$, α a complex root of P, $\mathbb{Z}_K$ the ring of integers of K, d_K its discriminant and $ind(P) = [\mathbb{Z}_K : \mathbb{Z}[\alpha]]$ the index of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K$. It is well known that: $\triangle = N_{K/\mathbb{Q}}(P')(\alpha)$ and $\triangle = (ind(P))^2 d_K$, where $\triangle$ is the discriminant of P and we can assume that for every prime p, $v_p(a) \leq 2$ or $v_p(b) \leq 3$.
Let p be a prime integer. A p-integral basis of K is a set of integral elements $\{w_1, ..., w_4\}$ such that p does not divide the index $[\mathbb{Z}_K : \Lambda]$, where $\Lambda = \sum_{i=1}^{4} \mathbb{Z}w_i$. In that case, we said that Λ is a p-maximal order of K. A triangular p-integral basis of K is a p-integral basis of K $\{1, w_2, w_3, w_4\}$ such that $w_1 = \frac{\alpha+x_1}{p^{r_1}}$, $w_2 = \frac{\alpha^2+y_2\alpha+x_2}{p^{r_2}}$ and $w_3 = \frac{\alpha^3+z_3\alpha^2+y_3\alpha+x_3}{p^{r_3}}$. In Theorem 1.1, for every prime p, a triangular p-integral basis of K is given.

For every prime p and $(x, m) \in \mathbb{Z}^2$, denote $x_p = \frac{x}{p^{v_p(x)}}$ and $x[m]$: the remainder of the Euclidean division of x by m.

In this paper, for each prime integer p, a triangular p-integral basis of a quartic number field K is given. The discriminant d_K of K and a triangular integral basis of K are then obtained from its triangular p-integral bases. These results extend those of Alaca and Williams [1], where they did not achieved the case that: ($v_2(a) = 2$ and $b = 3[4]$). However, the methods are different, ours being based on Newton's polygon techniques. The results are complete without no exception.

Key words and phrases. p-integral basis, Newton polygon.

Newton polygon

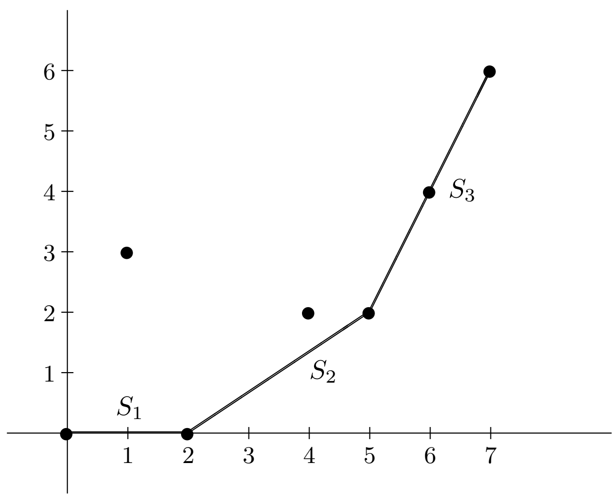
Let p be a prime integer such that p^2 divides $\triangle$ and $\phi(X)$ is an irreducible divisor of $P(X)$ modulo p. Set $m = deg\,(\phi(X))$ and let

$$P(X) = a_0(X)\phi(X)^t + a_1(X)\phi(X)^{t-1} + \cdots + a_t(X),$$

be the $\phi(X)$-adic development of $P(X)$ (every $a_i(X) \in \mathbb{Z}[X]$ and $\deg a_i(X) < m$). To any coefficient $a_i(X)$ we attach the integer $u_i = v_p(a_i(X))$ and the point of the plane $P_i = (i, u_i)$, if $u_i < \infty$.

The ϕ-Newton polygon of $P(X)$ is the lower convex envelope of the set of points $P_i = (i, u_i)$, $u_i < \infty$, in the cartesian plane. This (open) polygon is denoted by $N_\phi(P)$.

For instance, for a ϕ-development of degree 7 with $u_i = 0$, 3, 0, ∞, 2, 2, 4, 6 for $i = 0, 1, \dots, 7$, the polygon is let N be the $\phi(X)$-Newton polygon of $P(X)$.



The *length* $\ell(N_\phi(P))$ and the *height* $h(N_\phi(P))$ of the polygon are the respective lengths of the projection to the horizontal and vertical axis. Clearly, $\deg P(X) = m\ell(N_\phi(P)) + \deg a_0(X)$, where $m = deg\,\phi$.
The ϕ-Newton polygon is the union of different adjacent *sides* $S_1, \dots, S_t$ with increasing slope $\lambda_1 < \lambda_2 < \cdots < \lambda_t$. We shall write $N_\phi(P) = S_1 + \cdots + S_t$. The points joining two different sides are called the *vertexs* of the polygon. The polygon determined by the sides of positive slopes of $N_\phi(P)$ is called the *principal ϕ-polygon* of $P(X)$ and denoted by $N_\phi^+(P)$. The length and the height of $N_\phi^+(P)$ are the respective lengths of the projection to the horizontal and vertical axis.

For instance, the polygon of the figure has three sides S_1, S_2, S_3 with slopes $0 < 2/3 < 2$ and $N_\phi^+(P) = S_2 + S_3$. For every side S of the principal part $N_\phi^+(P)$, the *length* $\ell(S)$ and the *height* $h(S)$, of S, are the respective lengths of the projection to the horizontal and vertical axis. The *slope* of S is the quotient $h(S)/\ell(S)$. The positive integer $d(S) := \gcd(h(S), \ell(S))$ is called the *degree* of S. Denote $d := d(S)$ the degree of S, $h := h(S)/d$ and $e := \ell(S)/d$ positive coprime integers such that h/e is the slope of S. Let $s = \lfloor \frac{n}{m} \rfloor$ be the integral part of $\frac{n}{m}$, where $n = deg\,(P)$ and $m = deg\,(\phi)$. For every $1 \le j \le s$, let H_j be the length of the projection of $P_j(j, u_j)$ to the horizontal axis, h_j its integral part and $t_j = \text{red}\left(\dfrac{a_j(X)}{p^{h_j}}\right)$, where *red* is the canonical map defined on $\mathbb{Z}[X]$ by reduction modulo p. If $P_j \notin S$, then $t_j = 0$ and if $P_j \in S$, then $t_j \neq= 0$. If i is the abscissa of the initial point of S, let $P_S(Y)$ be the *residual polynomial* attached to S:

$$P_S(Y) := t_i Y^d + t_{i+e} Y^{d-1} + \cdots + t_{i+(d-1)e} Y + t_{i+de} \in \mathbb{F}_\phi[Y].$$

Let $ind_N(P) := \sum_{j=1}^{s} h_j$ the number of points with integer coordinates that lie below the polygon N, strictly above the horizontal axis and whose abscissas satisfy $1 \le j < l - 1$, where l is the length of N, $s = \lfloor \frac{n}{m} \rfloor$, $n = deg\,(P)$ and $m = deg\,(\phi)$.

Let $\bar{P}(X) = (\phi_1(X))^{l_1} P_2$ such that $\phi_1(X)$ does not divide $P_2(X)$ and $N_1^+ = S_1 + .. + S_s$ the principal part of $N_{\phi_1}(P)$. P is said to be ϕ_1-regular if for every $1 \le i \le s$, $P_{S_i}(Y)$ is square free. P is said to be p-regular if for every $1 \le i \le r$ and for every $1 \le j \le s_i$, $P_{S_j^i}(Y)$ is square free, where $\bar{P}(X) = \prod_{i=1}^{r} \phi_i(X))^{l_i}$ be the factorization of $\bar{P}(X)$ modulo p of irreducible polynomials and for every $1 \le i \le r$, $N_i = \oplus_j S_j^i$. The Theorem of index: $v_p(ind(P)) \ge \sum_{i=1}^{r} m_i ind_{N_i}(P)$, where $m_i = deg\,(\phi_i(X))$ for every i. With equality, if $P(X)$ is a p-regular polynomial. (cf. [4, p 326]).

1. p-integral basis of quartic number field defined by $X^4 + aX + b$

In this section, $K = \mathbb{Q}[\alpha]$, where α is a complex root of an irreducible trinomial $P(X) = X^4 + aX + b \in \mathbb{Z}[X]$ such that for every prime p, $v_p(a) \le 2$ or $v_p(b) \le 3$.

Lemma 1.1. *Let p be a prime integer and $w = \frac{t\alpha^3 + z\alpha^2 + y\alpha + x}{p^i} \in K$. Then $ch_w = X^4 + \frac{A_3}{p^i} X^3 + \frac{A_2}{p^{2i}} X^2 + \frac{A_1}{p^{3i}} X + \frac{A_0}{p^{4i}}$ is the characteristic polynomial of l_w the endomorphism of K defined by $l_w(x) = wx$, where*

$A_0 = x^4 + 3ax^2yz + 2bx^2z^2 - axy^3 - 4bxy^2z - 3ax^3t + by^4 + b^2z^4 + b^3t^4 + 3a^2x^2t^2 - 3a^2xyzt + a^2xz^3 - 5abxyt^2 + abxz^2t + 4b^2xzt^2 - a^3xt^3 + 4bx^2yt + 3aby^2zt + 2b^2y^2t^2 - abyz^3 - 4b^2yz^2t + a^2byt^3 - ab^2zt^3$,

$A_1 = -(4x^3 - 9ax^2t + 4bxz^2 + 8bxyt + 6axyz + 6a^2xt^2 - ay^3 - 4by^2z - 3a^2yzt + a^2z^3 - 5abyt^2 + abz^2t + 4b^2zt^2 - a^3t^3)$,

$A_2 = 6x^2 - 9axt + 3ayz + 4byt + 2bz^2 + 3a^2t^2$ *and* $A_3 = -4x + 3at$.

In particular, w is integral if and only if for every $1 \le j \le 3$, $\frac{A_j}{p^{ji}} \in \mathbb{Z}$.

The following theorem is an improvement and a specialization of the theorem of index on quartic number fields.

Theorem 1.2. *Let $P(X) = X^4 + mX^3 + nX^2 + aX + b \in \mathbb{Z}[X]$ be an irreducible polynomial such that for every prime p, $v_p(m) = 0$ or $v_p(n) \le 1$ or $v_p(a) \le 2$ or $v_p(b) \le 3$. Let p be a prime integer. If $P(X)$ is a p-regular polynomial, then we have the following:*

1. *If $\bar{P}(X)$ is square free, then $(1, \alpha, \alpha^2, \alpha^3)$ is a p-integral basis of $\mathbb{Z}_K$.*
2. *If $\bar{P}(X) = (\phi(X))^4$, where $\deg \phi = 1$, then $(1, \alpha, \frac{\alpha^2+a_3\alpha}{p^{h_2}}, \frac{\alpha^3+a_3\alpha^2+a_2\alpha}{p^{h_3}})$ is a p-integral basis of $\mathbb{Z}_K$, where $P(X) = \sum_{i=0}^4 a_i\phi^i$ is the ϕ_1-adic development of $P(X)$.*
3. *If $\bar{P}(X) = (\phi(X))^3 P_2$, where $\deg \phi = 1$ and $\phi(X)$ does not divide $P_2(X)$, then $(1, \alpha, \frac{\alpha^2+a_3\alpha}{p^{h_2}}, \frac{\alpha^3+a_3\alpha^2+a_2\alpha}{p^{h_3}})$ is a p-integral basis of $\mathbb{Z}_K$, where $P(X) = \sum_{i=0}^4 a_i\phi^i$ is the ϕ-adic development of $P(X)$.*
4. *If $\bar{P}(X) = (\phi(X))^2 P_2$, where $\deg \phi_1 = 1$, $\phi(X)$ does not divide $P_2(X)$ and $P_2(X)$ is square free, then $(1, \alpha, \frac{\alpha^2+a_3\alpha}{p^{h_2}}, \frac{\alpha^3+a_3\alpha^2+a_2\alpha}{p^{h_3}})$ is a p-integral basis of $\mathbb{Z}_K$, where $P(X) = \sum_{i=0}^4 a_i\phi^i$ is the ϕ-adic development of $P(X)$.*
5. *If $\bar{P}(X) = (\phi_1(X))^2(\phi_2(X))^2$, where $\deg \phi_i = 1$ and $\phi_1(X)$ does not divide $\phi_2(X)$, then for every i, let $P(X) = \sum_{j=0}^4 a_{i,j}\phi_i^j$ be the ϕ_i-adic development of $P(X)$, $w_i \frac{\alpha^3+a_{i,3}\alpha^2+a_{i,2}\alpha}{p^{h_3^i}}$ and $h_3^i \le h_3^j$. Then:*
   - (a) *If $h_3^i = 0$, then $(1, \alpha, \alpha^2, w_j)$ is a p-integral basis of $\mathbb{Z}_K$.*
   - (b) *If $h_3^i \ge 1$, then $(1, \alpha, w_i - p^{h_3^j - h_3^i} w_j, w_j)$ is a p-integral basis of $\mathbb{Z}_K$.*
6. *If $\bar{P}(X) = (\phi(X))^2$, where $\phi(X)$ is irreducible of degree 2, then $(1, \alpha, \frac{\phi(\alpha)}{p^h}, \frac{\alpha\phi(\alpha)}{p^h})$ is a p-integral basis of $\mathbb{Z}_K$, where $P(X) = \phi^2 + A(X)\phi + B(X)$ is the $\phi(X)$-adic development of $P(X)$ and h is the little of $(v_p(A(X)), \lfloor \frac{v_p(B(X))}{2} \rfloor)$.*

Proof.

1. Case 1. By Dedekind criterion, since $\bar{P}$ is square free, then $(1, \alpha, \alpha^2, \alpha^3)$ is a p-integral basis of $\mathbb{Z}_K$.
2. Cases 2, 3, 4. By theorem of index it suffices to show that every $w_i \in \mathbb{Z}_K$, where $w_2 = \frac{\alpha^2+a_3\alpha}{p^{h_2}}$ and $w_3 = \frac{\alpha^3+a_3\alpha^2+a_2\alpha}{p^{h_3}}$. Let $\phi(X) = X - x_0$. By replacing $P(X)$ by $P(X + x_0)$, we can assume that $x_0 = 0$, and then $w_2 = \frac{\alpha^2+m\theta}{p^{h_2}}$ and $w_3 = \frac{\alpha^3+m\alpha^2+n\alpha}{p^{h_3}}$. Let $Ch_{w_2}(X) = X^4 + \frac{2n}{p^{h_2}}X^3 + \frac{(n^2+2b+am)}{p^{2h_2}}X^2 + \frac{(amn+bm^2-a^2+2bn)}{p^{3h_2}}X + \frac{(bm^2n-abm+b^2)}{p^{4h_2}}$ and $Ch_{w_2}(X) = X^4 + \frac{3a}{p^{h_3}}X^3 +$

$\frac{(bn+3a^2)}{p^{2h_3}}X^2 + \frac{(a^3+2abn-b^2m)}{p^{3h_3}}X + \frac{(b^3+a^2bn-ab^2m)}{p^{4h_3}}$ be the respective characteristic polynomial of l_{w_2} and l_{w_3}, where l_w is the endomorphism of K defined by the multiplication by w. By definition of h_2, p^{h_2} divides n. Since $N_\phi(P)$ is convex, then $v_p(b) \geq 3h_2$ and $v_p(a) \geq 2h_2$. Thus, $Ch_{w_2} \in \mathbb{Z}[X]$, and then $w_2 \in \mathbb{Z}$. For w_3, by definition of h_3, p^{h_3} divides a. Since $N_\phi(P)$ is convex, then $v_p(b) \geq h_3+(h_3-h_1) \geq h_3+(h_3-h_2)$, and then $v_p(b)+v_p(n) \geq 2h_3$, $3v_p(b) \geq 4h_3$ and $2v_p(b)+v_p(m) \geq 3h_3$. Thus, $Ch_{w_3} \in \mathbb{Z}[X]$ and $w_3 \in \mathbb{Z}$.

(3) Case 5. As in the previous cases, every $w_k \in \mathbb{Z}_K$. By Hensel lemma, let $(P_1, P_2) \in \mathbb{Z}_p[X]^2$ such that $P_1P_2 = P$ and $\bar{P}_k = \phi_k^2$. Since $\bar{P}_1$ and $\bar{P}_2$ are cooprime, then $v_p(ind(P)) = v_p(ind(P_1)) + v_p(ind(P_2))$ (cf. MN). Since P is p-regular, then for every k, $v_p(ind(P_k)) = h_3^k$. Thus, If $h_3^i = 0$, then $(1, \alpha, \alpha^2, w_j)$ is a p-integral basis of $\mathbb{Z}_K$. Else, then for every k, let $\phi_k = X - x_k$. Then $a_{k,3} = 4x_k + m$. Since $x_1 \neq x_2$ modulo p, then $a_{1,3} \neq a_{2,3}$ modulo p, and then $w_i - p^{h_3^j - h_3^i}w_j = \frac{U(\alpha)}{p^{h_3^i}}$, where $U(X) \in \mathbb{Z}[X]$ of degree 2 such that the coefficient of X^2 is cooprime to p. Finally, $(1, \alpha, w_i - p^{h_3^j-h_3^i}w_j, w_j)$ is a p-integral basis of $\mathbb{Z}_K$.

(4) Case 6. By Theorem of index it suffices to show that every $\phi(\alpha) \in \mathbb{Z}_K$. Let $P(X) = \phi^2 + A(X)\phi + B(X)$ be the $\phi(X)$-adic development of $P(X)$ and $k = v_p(\phi(\alpha))$. Since $P(\alpha) = 0$, then $2v_p(\phi(\alpha)) \geq v_p(B(\alpha))$, and then $k \geq h$. Thus, $\frac{\phi(\alpha)}{p^h} \in \mathbb{Z}_K$.

The following Theorem gives us a triangular p-integral basis of K, $v_p(\triangle)$ and $v_p(d_K)$ for every prime integer p.

Theorem 1.3. *Let $p \geq 5$ be a prime integer. Under the above hypotheses, a p-integral (resp. a 2-integral, resp. a 3-integral) basis of $\mathbb{Z}_K$ is given in table A (resp. table B, B2*, B3* and B.3.2, resp. table C)*

Table A

case	$v_p(b)$	$v_p(a)$	$v_p(\triangle)$	*p-integral basis*	$v_p(d_K)$
A1	*3*	≥ 3	*9*	$(1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p^2})$	*3*
A2	≥ 3	*2*	*8*	$(1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p^2})$	*2*
A3	≥ 1	*0*	*0*	$(1, \alpha, \alpha^2, \alpha^3)$	*0*
A4	*2*	≥ 2	*6*	$((1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p})$	*2*
A5	*1*	≥ 1	*3*	$(1, \alpha, \alpha^2, \alpha^3)$	*3*
A6	≥ 2	*1*	*4*	$((1, \alpha, \alpha^2, \frac{\alpha^3}{p})$	*2*
A7	*0*	≥ 1	*0*	$(1, \alpha, \alpha^2, \alpha^3)$	*0*
A8	*0*	*0*	*?*	$(1, \alpha, \alpha^2, \frac{\alpha^3+t\alpha^2+t^2\alpha-3t^3}{p^m})$	$v_p(\triangle)[2]$

In case A8, $t \in \mathbb{Z}$ such that $3at + 4b = 0$ modulo p^{m+1}, $m = \lfloor \frac{v_p(\triangle}{2} \rfloor$ and $v_p(\triangle)[2]$ is the remainder of the Euclidean division of $v_p(\triangle)$ by 2.

Table B

case	*conditions*	$v_2(\triangle)$	*2-integral basis*	$v_2(d_K)$
B1	$v_2(b) \geq 3,\ v_2(a) = 2$	*8*	$(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{2^2})$	*2*
B2	$v_2(b) = 3, v_2(a) \geq 5$	*17*	$(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{2^2})$	*11*
B3	$v_2(b) = 3,\ v_2(a) = 4$	*16*	$(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{2^2})$	*10*
B4	$v_2(b) = 3,\ v_2(a) = 3$	*12*	$(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{2^2})$	*6*
B5	$b = 4 + 16B,\ a = 16A, A = B[2]$	*14*	$(1, \alpha, \frac{\alpha^2+2\alpha+2}{2^2}, \frac{\alpha^3+2\alpha^2+(2+4B)\alpha}{2^3})$	*4*
B6	$b = 4 + 16B,\ a = 16A, A \neq B[2]$	*14*	$(1, \alpha, \frac{\alpha^2+2\alpha+2}{2^2}, \frac{\alpha^3+2\alpha^2+2\alpha}{2^2})$	*6*
B7	$v_2(b) = 2,\ v_2(a) = 3$	*12*	$(1, \alpha, \frac{\alpha^2+2}{4}, \frac{\alpha^3+2\alpha}{4})$	*6*
B8	$v_2(b) = 2,\ v_2(a) = 2$	*8*	$(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{2})$	*4*
B9	$v_2(b) \geq 2,\ v_2(a) = 1$	*4*	$(1, \alpha, \alpha^2, \frac{\alpha^3}{2})$	*2*
B10	$v_2(b) = 1,\ v_2(a) \geq 3$	*11*	$(1, \alpha, \alpha^2, \alpha^3)$	*11*
B11	$v_2(b) = 1,\ v_2(a) = 2$	*8*	$(1, \alpha, \alpha^2, \alpha^3)$	*8*
B12	$v_2(b) = 1,\ v_2(a) = 1$	*4*	$(1, \alpha, \alpha^2, \alpha^3)$	*4*
B13	$v_2(a) = 0$	*0*	$(1, \alpha, \alpha^2, \alpha^3)$	*0*
B14	$v_2(a) \geq 3, b = 1\,[4]$	*8*	$(1, \alpha, \alpha^2, \alpha^3)$	*8*
B15	$v_2(a) \geq 3, b = 3\,[4]$	*8*	$(1, \alpha, \frac{\alpha^2+1}{2}, \frac{\alpha^3-\alpha^2+\alpha-1}{2})$	*4*
B16	$v_2(a) = 2, b = 1\,[4]$	*9*	$(1, \alpha, \alpha^2, \alpha^3)$	*9*
B17	$v_2(a) = 2, b = 7\,[8]$	*10*	$(1, \alpha, \frac{\alpha^2+1}{2}, \frac{\alpha^3-\alpha^2+\alpha-1}{2})$	*6*
B18	$v_2(a) = 1, b = 3\,[4]$	*4*	$(1, \alpha, \alpha^2, \alpha^3)$	*4*
B19	$v_2(a) = 1, b = 1\,[4]$	*4*	$(1, \alpha, \alpha^2, \frac{\alpha^3-\alpha^2+\alpha-1}{2})$	*2*
*B2**	$b = 3[4],\ v_2(a) = 2$	*	*cf table B2**	*

If $b = 3[4]$ and $v_2(a) = 2$, then let $A = 4 + a$, $B = 1 + a + b$. Consider $F(X) = P(X + 1) = X^4 + 4X^3 + 6X^2 + AX + B$ and $\theta = \alpha - 1$. Then

*Table B2**

conditions	$v_2(\triangle)$	*2-integral basis*	$v_2(d_K)$
$v_2(B)+1 \geq 2v_2(A)$	$5+2v_2(A)$	$(1, \alpha, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6\theta}{2^m}),\ m=v_2(A)$	3
$v_2(B)+1 < 2v_2(A),\ v_2(\triangle)=0[2]$ $v_2(B)=2m$	$8+v_2(B)$	$(1, \alpha, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6\theta}{2^m})$	6
$v_2(B)+1 < 2v_2(A),\ v_2(\triangle)=1[2]$ $v_2(B)=2m$	$5+2v_2(A)$	$(1, \alpha, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6\theta}{2^m})$	5
$v_2(B)+1 < 2v_2(A),\ v_2(B)=1[2]$	*	*cf Table B3**	*

Table B3:* $v_2(B)+1 < 2v_2(A),\ v_2(B)=2k+1$

conditions	*2-integral basis*
$2v_2(A) > v_2(B)+1, v_2(d)=1$	$(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6\theta}{2^{k+1}})$
$2v_2(A) > v_2(B)+1, v_2(d)\geq 2$	*go to* $B.3.2$

B.3.2: Assume that $v_2(d) \geq 2$*. Let* $t \in \mathbb{Z}$ *such that* $v_2(n_2t+A_2)=s$, $s_1 = L(\lfloor\frac{v_2(d)-1}{2}\rfloor, \lfloor\frac{k+2}{2}\rfloor)$ *and* $H(X)=F(X+2^kt)=X^4+m'X^3+n'X^2+A'X+B'$.

conditions	*2-integral basis*
$v_2(A)\geq k+3$	$s=1$, $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+(-3.2^kt+4)\theta^2+6\theta-6.2^kt}{2^{k+1}})$
$v_2(A)=k+2, v_2(d)\geq 3$	$s=s_1$ $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+(4-3.2^kt)\theta^2+(6-4.2^{k+1}t)\theta-6.2^kt}{2^{k+s_1+2}})$
$v_2(A)=k+2, v_2(d)=2, v_2(B')=2k+4$	$s=1$ $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+(4-3.2^kt)\theta^2+6\theta-6.2^kt}{2^{k+2}})$
$v_2(A)=k+2, v_2(d)=2, v_2(B')=2k+3$	*Replace* F *and* θ *by* H *and* $\theta-2^kt$

Table C

case	*conditions*	$v_3(\triangle)$	*3-integral basis*	$v_p(d_K)$
C1	$v_3(b)\ge 4,\ v_3(a)=2$	*11*	$(1,\alpha,\frac{\alpha^2}{3},\frac{\alpha^3}{3^2})$	*5*
C2	$v_3(b)\ge 4,\ v_3(a)=1$	*7*	$(1,\alpha,\alpha^2,\frac{\alpha^3}{3})$	*5*
C3	$v_3(b)\ge 4, v_3(a)=0, a^2\ne 1[9]$	*3*	$(1,\alpha,\alpha^2,\alpha^3)$	*3*
C4	$v_3(b)\ge 4, v_3(a)=0, a^2=1[9]$	*3*	$(1,\alpha,\alpha^2,\frac{\alpha^3-a\alpha^2+\alpha}{3})$	*1*
C5	$v_3(b)=3,\ v_3(a)\ge 2,$	*9*	$(1,\alpha,\frac{\alpha^2}{3},\frac{\alpha^3}{3^2})$	*3*
C6	$v_3(b)=3,\ v_3(a)=1$	*7*	$(1,\alpha,\alpha^2,\frac{\alpha^3}{3})$	*5*
C7	$v_3(b)=3,\ a^2=1[9]$	*3*	$(1,\alpha,\alpha^2,\frac{\alpha^3-a\alpha^2+\alpha}{3})$	*1*
C8	$v_3(b)=3,\ v_3(a)=0, a^2\ne 1[9]$	*3*	$(1,\alpha,\alpha^2,\alpha^3)$	*3*
C9	$v_3(b)=2,\ v_3(a)\ge 2,$	*6*	$((1,\alpha,\frac{\alpha^2}{3},\frac{\alpha^3}{3})$	*2*
C10	$v_3(b)=2,\ v_3(a)=1$	*6*	$(1,\alpha,\alpha^2,\frac{\alpha^3}{3})$	*4*
C11	$v_3(b)=2,\ a^2=1[9]$	*3*	$(1,\alpha,\alpha^2,\frac{\alpha^3-a\alpha^2+\alpha}{3})$	*1*
C12	$v_3(b)=2,\ v_3(a)=0,\ a^2\ne 1[9]$	*3*	$(1,\alpha,\alpha^2,\alpha^3)$	*3*
C13	$v_3(b)=1,\ v_3(a)\ge 1,$	*3*	$(1,\alpha,\alpha^2,\alpha^3)$	*3*
C14	$b=6[9],\ v_3(a)=0,\ a^2\ne 4[9],$	*3*	$(1,\alpha,\alpha^2,\alpha^3)$	*3*
C15	$b=6[9],\ a^2=4[9],$	*3*	$(1,\alpha,\alpha^2,\frac{\alpha^3-a\alpha^2+\alpha+a}{3})$	*1*
C16	$b=3[9],\ v_3(a)=0,\ a^2\ne 7[9],$	*4*	$(1,\alpha,\alpha^2,\alpha^3)$	*4*
C17	$b=3[9], a^2=7[9],\ a^4-a^2+b=0[27]$	≥ 6	$(1,\alpha,\frac{\alpha^2-1}{3},\frac{\alpha^3+z\alpha^2+y\alpha+x}{3^m})$	$v_3(\triangle)[2]$
C18	$b=3[9], a^2=7[9],\ v_3(a^4-a^2+b)=2$	5	$(1,\alpha,\alpha^2,\frac{\alpha^3-a\alpha^2+\alpha+a}{3})$	3
C19	$v_3(b)=0$	0	$(1,\alpha,\alpha^2,\alpha^3)$	0

In case C17, x,y,z are defined as follows: $4x=3a\,[3^m]$, $a^2y=16b^2\,[3^m]$, $az+4b_3=0[3^m]$ and $m=\lfloor\frac{v_2(\triangle)-2}{2}\rfloor$.

Proof. First, $\triangle=2^8b^3-3^3a^4$ and the proof is based on the Newton polygon. For every prime p, let $u_3=v_p(a)$, $u_4=v_p(b)$, $\bar{P}(X)$ the reduction of P modulo p, N the X-Newton polygon of P and N^+ its principal part.

(1) Case 1: $(v_p(b)=3$ and $v_p(a)\ge 2)$: A1, B2, B3, B4, C5.

$$\bar{P}(X)=X^4 \text{ and } ind_N(P)=3.$$

1) If $u_3\ge 3$, then $N=S$ is one side and $P_S(Y)=Y+\overline{b_p}$ is square free and $(1,\alpha,\frac{\alpha^2}{p},\frac{\alpha^3}{p^2})$ is a p-integral basis of $\mathbb{Z}_K$.

2) If $u_3 = 2$, then $N = S_1 + S_2$ with slopes respectively $1/3$ and 1. $P_{S_1}(Y) = Y + \overline{a_p}$ and $P_{S_2}(Y) = \overline{a_p}Y + \overline{b_p}$ are square free. Thus, $(1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p^2})$ is a p-integral basis of $\mathbb{Z}_K$.

(2) Case 2: ($v_p(b) \geq 2$ and $v_3(a) = 1$) : A6, B9, C2, C6, C10.

$N = S_1 + S_2$ with slopes respectively $1/3$ and 1. $P_{S_1}(Y)$ and $P_{S_2}(Y)$ are of degree 1. So, Thus, $(1, \alpha, \alpha^2, \frac{\alpha^3}{p})$ is a p-integral basis of $\mathbb{Z}_K$.

(3) Case 3: ($v_p(b) \geq 1$ and $v_3(a) = 0$): A3, B3, C3, C4, C7, C8, C11, C12, C14, C15, C16, C17,C18.
If $p \neq 3$, then $\bar{P}(X)$ is square free and then $v_p(ind(P)) = 0$.

For $p = 3$, let $F(X) = P(X - a) = X^4 - 4aX^3 + 6a^2X^2 + AX + B$, where $A = -a(4a^2 - 1)$ and $B = (a^4 - a^2 + b)$. Then $v_3(A) \geq 1$ and $v_3(B) \geq 1$. It follows that $v_3(ind(P)) = 0$ if and only if $v_3(B) = 1$, i.e., if ($a^2 = 1$ modulo 9 and $v_3(b) = 1$) or ($a^2 \neq 1$ modulo 9 and $v_3(b) \geq 2$, then $v_3(ind(P)) = 0$. Else, then $u_1 = 0$, $u_2 = 1$, $u_3 = v_3(A) \geq 1$ and $u_4 = v_3(B) \geq 2$.

(a) If $u_3 = 1$ or $u_4 = 2$, then $v_3(ind(P)) = 1$ and $(1, \alpha, \alpha^2, \frac{\alpha^3 - a\alpha^2 + \alpha + a}{3})$ is a 3-integral basis of $\mathbb{Z}_K$: ($b = 6$ and $a^2 = 4$ modulo 9) or ($b = 3$, $a^2 = 7$ modulo 9 and $v_3(a^4 - a^2 + b) = 2$).

(b) If $u_3 \geq 2$ and $u_4 \geq 3$:($b = 3$, $a^2 = 7$ modulo 9 and $a^4 - a^2 + b = 0$ modulo 27), then let $m = \lfloor \frac{v_2(\triangle)-2}{2} \rfloor$ and $(x, y, z) \in \mathbb{Z}_K$ defined by $4x = 3a\,[3^m]$, $a^2y = 16b_3^2\,[3^m]$, $az + 4b_3 = 0[3^m]$ as defined in Proof of B19 in [1]. Let $w = \frac{\alpha^3 + z\alpha^2 + y\alpha + x}{3^m}$; replacing the A_i as defined in Lemma 1.1, we have $A_3 = 0$ modulo 3^m, $A_2 = 0$ modulo 3^{2m}, $A_1 = 0$ modulo 3^{3m} and $A_0 = 0$ modulo 3^{4m}, and then $w \in \mathbb{Z}_K$. Finally, $(1, \alpha, \frac{\alpha^2 - 1}{3}, \frac{\alpha^3 + z\alpha^2 + y\alpha + x}{3^m})$ is a 3-integral basis of $\mathbb{Z}_K$.

(4) Case 4: ($v_p(b) = 2$ and $v_3(a) \geq 2$): A4, C9, B5, B6, B7, B8:

For $p \neq 2$, $N = S$ is one side and $P_S(Y) = Y^2 + \bar{b_p}$ is square free. Hence $v_p(ind(P)) = 2$
For $p = 2$, $N = S$ is one side and $P_S(Y) = (Y+1)^2$. Since $P(X)$ is not 2-regular, we will use a higher order. Let $t_1 = (X, 1/2, Y+1)$ and $\phi_2(X) = X^2+2$ as defined in[3], page 16 and let V_2 be the 2-adic valuation of 2^d-order as defined in[3], page 17. Then $V_2(\phi_2) = 2$, $V_2(X) = 1$ and for every $x \in \mathbb{Z}$, $V_2(x) = 2v_2(x)$. Let $P(X) = \phi_2^2(X) - 4\phi_2(X) + (4AX + 4b_2 + 4)$, $R_0 = V_2(\phi_2^2(X)) = 4$, $R_1 = V_2(4\phi_2(X)) = 6$ and $R_2 = V_2(4AX + 4b_2 + 4)$. From [3, Th 4.18, p:48], it follows that:

(a) If $v_2(a)=2$, then $R_2=5$ and $(1,\alpha,\frac{\alpha^2}{2},\frac{\alpha^3}{2})$ is a 2-integral basis of $\mathbb{Z}_K$.

(b) If $v_2(a)=3$, then $R_2=7$ and $(1,\alpha,\frac{\alpha^2}{2},\frac{\alpha^3+2\alpha}{4})$ is a 2-integral basis of $\mathbb{Z}_K$.

(c) If $b_2+1=0$ modulo 4 and $v_2(a)\geq 4$, then $R_2\geq 8$ and $(1,\alpha,\frac{\alpha^2+2}{4},\frac{\alpha^3+2\alpha}{4})$ is a 2-integral basis of $\mathbb{Z}_K$.

(d) If $b_2+1=2$ modulo 4 and $v_2(a)\geq 4$, then let $b=4+16B$, $a=16A$, $\phi_2(X)=X^2+2X+2t$ and $P(X)=\phi_2^2(X)-4(X+t-1)\phi_2(X)+8(t-1+2A)X+4(1+t^2+4B-2t)$. Then $R_0=4$ $R_1=7$. Since $1+t^2+4B-2t=2(1-t+2B)$ modulo 8, let $t\in\mathbb{Z}$ such that $1-t+2B=0$ modulo 4. It follows that: if $B=A$ modulo 2, then $R_2\geq 10$, $v_2(ind(P))=5$ and $(1,\alpha,\frac{\alpha^2+2\alpha+2}{4},\frac{\alpha^3+2\alpha^2+2(1+2B)\alpha}{2^3})$ is a 2-integral basis of $\mathbb{Z}_K$.

If $B\neq A$ modulo 2, then $R_2=9$, $v_2(ind(P))=4$ and $(1,\alpha,\frac{\alpha^2+2\alpha+2}{4},\frac{\alpha^3+2\alpha^2+2\alpha}{2^2})$ is a 2-integral basis of $\mathbb{Z}_K$.

(5) Case 5: $(v_p(b)=1$ and $v_3(a)\geq 1)$: A5, C13, B10,B11, B12.

$N=S$ is one side and $P_S(Y)=Y+\overline{b_p}$ is square free. Hence $v_p(ind(P))=0$. v

(6) Case 6: $v_p(b)\geq 3$ and $v_3(a)=2$: A2, C1, B1:

$N=S_1+S_2$ with slopes respectively $2/3$ and u_4-2. Since every $P_{S_i}(Y)$ is square free, then $v_p(ind(P))=3$.

(7) Case 7: $v_p(b)=0$ and $v_3(a)\geq 1$: A7, B14, B15,..., B19, B2*, B3* and C19.
If $p\neq 2$, then $\bar{P}(X)$ is square free.

For $p=2$, according to the Dedekind criterion, let $f(X)=\frac{P(X)-(X+1)^4}{2}=-2X^3-3X^2+\frac{a-4}{2}X+\frac{b-1}{2}$ and $f(-1)=1+\frac{a}{2}+\frac{b-1}{2}$. Thus,

If $(v_2(a)\geq 2$ and $b=1$ modulo 4) or $(v_2(a)=1$ and $b=3$ modulo 4), then $(1,\alpha,\alpha^2,\alpha^3)$ is a 2-integral basis of $\mathbb{Z}_K$.
Else, let $P(X+1)=X^4+4bX^3+6X^2+(4+a)X+(1+a+b)$. Then $u_1=2$, $u_2=1$ and

(a) If $v_2(a)=1$ and $b=1$ modulo 4, then $u_3=1$, $u_4\geq 2$ and $(1,\alpha,\alpha^2,\frac{\alpha^3-\alpha^2-\alpha-1}{2})$ is a 2-integral basis of $\mathbb{Z}_K$.

(b) If $(v_2(a)\geq 3$ and $b=3$ modulo 8) or $(v_2(a)=2$ and $b=7$ modulo 8), then $u_3=u_4=2$ and $N=S$ is one side such that $P_S(Y) = Y^2 + Y + 1$ is irreducible. Hence, $(1, \alpha, \frac{\alpha^2+1}{2}, \frac{\alpha^3-\alpha^2-\alpha-1}{2})$ is a 2-integral basis of $\mathbb{Z}_K$.

(c) If $v_2(a) = 2$ and $b = 3$ modulo 4, let $F(X) = P(X + 1) = X^4 + 4X^3 + 6X^2 + (4 + a)X + (1 + b + a) = X^4 + 4X^3 + 6X^2 + AX + B$, where $A = 4 + a$, $B = 1 + b + a$, $v_2(A) \geq 3$ and $v_2(B) \geq 3$. First, we have $\triangle = disc(F) = 256B^3 - 768B^2A + 768BA^2 + 176A^3 + 2304B^2 - 4608BA - 288A^2 - 27A^4 + 6912B$. Consequently,

(i) If $v_2(B) + 1 \geq 2v_2(A)$, then $v_2(\triangle) = 5 + 2v_2(A)$, $v_2(ind(P)) = 1 + v_2(A)$, $v_2(d_K) = 3$ and $(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3+4\alpha^2+6\alpha}{2^m})$ is a 2-integral basis of $\mathbb{Z}_K$, where $m = v_2(A)$.

(ii) If $v_2(B)$ is even and $v_2(B) + 1 < 2v_2(A)$, then

if $v_2(\triangle)$ is even, then $v_2(\triangle) = 8 + v_2(B)$, $v_2(ind(P)) = 1 + \lfloor \frac{v_2(B)}{2} \rfloor$, $v_2(d_K) = 6$ and $(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3+4\alpha^2+6\alpha)}{2^m})$ is a 2-integral basis of $\mathbb{Z}_K$, where $m = \lfloor \frac{v_2(B)}{2} \rfloor$.
If $v_2(\triangle)$ is odd, then $v_2(\triangle) = 5 + 2v_2(A)$, $v_2(ind(P)) = 1 + \lfloor \frac{v_2(B)}{2} \rfloor$, $(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3+4\alpha^2+6\alpha}{2^m})$ is a 2-integral basis of $\mathbb{Z}_K$, where $m = \lfloor \frac{v_2(B)}{2} \rfloor$. Since $v_2(\triangle) = 5 + 2v_2(A)$ and $v_2(B) + 1 < 2v_2(A)$, then $v_2(B) = 2v_2(A) - 2$. Thus, $v_2(d_K) = 3$.

(iii) $v_2(B)$ is odd and $v_2(B) + 1 < 2v_2(A)$. By the Theorem of the polygon $F(X) = H(X)G(X)$ in $\mathbb{Z}_2[X]$, where $H(X) = X^2 + rX + s$, $G(X) = X^2 + RX + S$,

$$\begin{cases} v_2(s) = 1, & v_2(S) = v_2(B) - 1, \\ v_2(r) \geq 1, & v_2(R) > \frac{v_2(B)-1}{2} \\ B = sR + rS, & A = Sr + Rs, \qquad 4 = r + R. \end{cases}$$

Since $4 = r + R$, $v_2(r) \geq 3$ and $v_2(r) = 2$. So, $v_2(disc(H)) = v_2(r^2 - 4s) = 3$, $v_2(Res(H, G)) = 2v_2(H(\theta)) = 2$, where θ is a root of $G(X)$. As $v_2(r) \geq 1$ and $v_2(s) = 1$, we have $H(X)$ is irreducible in $\mathbb{Z}_2[X]$. On the other hand, as $disc(F) = disc(H)\, disc(G)\, (Res(H, G))^2$,
$v_2(disc(F)) = v_2(disc(H)) + 2v_2(Res(H, G)) + v_2(disc(G)) = 7 + v_2(disc(G))$. Thus,

(A) If $G(X)$ is irreducible in $\mathbb{Z}_2[X]$, then from [4], $v_2(ind(F)) = 0 + 2 + v_2(ind(G))$. Let θ be a root of $G(X)$ and $u = \frac{\theta+x}{2^k} \in \mathbb{Q}_2[\theta]$. Since the characteristic polynomial of u is $Ch_u = X^2 - (\frac{2x-R}{2^k}X + \frac{(2x-R)^2+(disc(G))}{2^{2k+2}}$, where $disc(G) = 4S-$

R^2, then u is integral if and only if 2^{2k+2} divides $disc(G)$. Therefore, $v_2(ind(G)) = \lfloor \frac{v_2(disc(G))}{2} \rfloor - 1$. Thus, $v_2(ind(F)) = 2 + \lfloor \frac{v_2(disc(G))}{2} \rfloor - 1$, and then if $v_2(\triangle)$ is even, then $v_2(ind(F)) = \frac{v_2(disc(F))-6}{2}$ and $v_2(d_K)$=6. Else, then $v_2(ind(F)) = \frac{v_2(disc(F))-5}{2}$ and $v_2(d_K) = 5$.

(B) $G(X) = (X-\theta_1)(X-\theta_2)$ in $\mathbb{Z}_2[X]$, then $v_2(disc(G)) = 2v_2(\theta_1-\theta_2))$ and $v_2(disc(F)) = 7+2v_2((\theta_1-\theta_2))$. On the other hand, $v_2(ind(F)) = 2+v_2(Res(G_1,G_2) = 2+v_2((\theta_1-\theta_2))$. Hence $v_2(d_K) = 3$.

In these cases, $(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3+z\alpha^2+y\alpha^2+x}{2^m})$ is a 2-integral basis of $\mathbb{Z}_K$, where $m = \lfloor \frac{v_2(disc)-v_2(d_K)}{2} \rfloor - 1$, x, y and z are integers.

Since we can not compute the coefficients of $G(X)$ in $\mathbb{Q}_p$ neither to test if $G(X)$ is irreducible in $\mathbb{Q}_p[X]$, we must give, in $C.1$, a method which allows to compute the integers x, y and z independently of the knowledge of the irreducibility of $G(X)$.

(8) $C.1$: Let $F(X) = X^4+4X^3+6X^2+AX+B \in \mathbb{Z}[X]$, where $v_2(A) \geq 2$ and $v_2(B) \geq 3$. Let θ be a complex root of $F(X)$ and $d = A_2^2 - 3B_2$. It follows that:

(a) If $v_2(B)+1 \geq 2v_2(A)$, then $v_2(ind(F)) = r+1$ and $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6\theta}{2^r})$ is a 2-integral basis of $\mathbb{Z}_K$, where $r = v_2(B)$.

(b) If $v_2(B) \leq 2v_2(A)$ and $v_2(B)$ is even, then $v_2(ind(F)) = r+1$ and $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6\theta}{2^r})$ is a 2-integral basis of $\mathbb{Z}_K$, where $r = \lfloor \frac{v_2(B)}{2} \rfloor$.

(c) If $v_2(B) \leq 2v_2(A)$ and $v_2(B) = 2k+1$ is odd, then let $t \in \mathbb{Z}$ such that $v_2(3t+A_2) = s$, $H(X) = F(X+2^kt) = X^4+m_1X^3+n_1X^2+A_1X+B_1$, where $m_1 = 4+2^{k+2}t$, $n_1 = 6+3.2^{k+2}t+3.2^{2k+1}t^2$, $A_1 = A+3.2^{k+2}t+3.2^{2k+2}t^2+2^{3k+2}t^3$ and $B_1 = B+2^kAt+3.2^{2k+1}t^2+2^{3k+2}t^3+2^{4k}t^4$. Then $3^4B_1 = 2^{2k+1}(3^4B - 3^3A_2A + 3^3A_2^2 + 3^42^{3k+2}t^3 + 3^42^{4k}t^4 + 2^{2s}L) = -2^{2k+1}3^3d + 2^{3k+2}t^3 + 2^{4k}t^4 + 2^{2(s+k)+1}L$, where $L \in \mathbb{Z}$ is odd. Then $v_2(m_1) \geq 2$, $v_2(n_1) = 1$ and

(i) If $v_2(d) = 1$ or $k = 1$, then $v_2(B_1) = 2k+2$, $v_2(ind(F)) = k+2$, and $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6\theta}{2^{k+1}})$ is a 2-integral basis of $\mathbb{Z}_K$.

(ii) If $v_2(d) \geq 2$, $k \geq 2$ and $v_2(A) > k+2$, then $v_2(A_1) = k+2$ and $v_2(B_1) \geq 2k+3$. Hence $v_2(ind(F)) = k+3$

and $(1, \theta, \frac{\theta^2}{2}, w_3)$ is a 2-integral basis of $\mathbb{Z}_K$, where $w_3 = \frac{\theta^3+(4-3.2^k t)\theta^2+6\theta-3.2^{k+1}t}{2^{k+2}}$.

(iii) If $v_2(A) = k+2$, $k \geq 2$ and $v_2(d) \geq 3$, then for $s = L(\lfloor \frac{v_2(d)-1}{2} \rfloor, \lfloor \frac{k}{2} \rfloor)$, $v_2(A_1) = k+2+s$, $v_2(B_1) \geq 2(k+s+1)+1$. Hence $v_2(ind(F)) = k+2+s$ and $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+(4-3.2^k t)\theta^2+6\theta-3.2^{k+1}t}{2^{k+1+s}})$ is a 2-integral basis of $\mathbb{Z}_K$.

(iv) If $v_2(A) = k+2$ and $v_2(d) = 2$, then for $s = 1$, $v_2(A_1) = k+3$ and $v_2(B_1) \geq 2(k+1)+1$. Thus,

If $v_2(B_1) = 2(k+1)+1$, then replace F and θ by H and $\theta - 2^k t$ and resume with $C.1$.

If $v_2(B_1) = 2(k+2)$, then $v_2(ind(F)) = k+3$ and $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+(4-3.2^k t)\theta^2+6\theta-3.2^{k+1}t}{2^{k+2}})$ is a 2-integral basis of $\mathbb{Z}_K$.

If $v_2(B_1) \geq 2(k+2)+1$, then $v_2(ind(F)) = k+3$ and $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6t\theta}{2^{k+1}})$ is a 2-integral basis of $\mathbb{Z}_K$.

If $v_2(B_1) \geq 2(k+2)+1$, then $v_2(ind(F)) = k+4$ and $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+(4-3.2^k t)\theta^2+6\theta-3.2^{k+1}t}{2^{k+3}})$ is a 2-integral basis of $\mathbb{Z}_K$.

(9) Case 8: $v_p(ab) = 0$: A8.

If $p \in \{2, 3\}$, then $v_p(\triangle) = 0$, $(1, \alpha, \alpha^2, \alpha^3)$ is a p-integral basis of $\mathbb{Z}_K$ and $v_p(d_K) = 0$.
Let $p \geq 5$. If $v_p(disc(2^8 b^3 - 3^3 a^4)) \leq 1$, then $(1, \alpha, \alpha^2, \alpha^3)$ is a p-integral basis of $\mathbb{Z}_K$ and $v_p(d_K) = 0$.

Else, since $3a \neq 0$ modulo p, let $t \in \mathbb{Z}$ such that $3at + 4b = 0$ modulo p^s, where $s = m+1$ and $m = \lfloor \frac{v_p(\triangle)}{2} \rfloor$ ($3at + 4b = p^s L$). Then $(3a)^3 P'(t) = -\triangle + 3.4^3 b^2 p^s L$ modulo p^{2s}. Thus, $v_p(P'(t)) \geq s$. Moreover, $(3a)^4 P(t) = b\triangle - p^s \triangle$ modulo p^{2s}. Thus, $v_p(P(t)) = v_p(\triangle)$. Let $P(X+t) = X^4 + 4tX^3 + 6t^2 X^2 + P'(t)X + P(t)$. Since $6t^2 \neq 0$ modulo p, then $N = S_0 + S_1$ with slopes respectively 0 and $\frac{v_p(\triangle)}{2}$ and $P_{S_1}(Y)$ is square free. Hence, $v_p(ind(P)) = \lfloor \frac{v_p(\triangle)}{2} \rfloor$, $\frac{\theta^3+4t\theta^2+6t^2\theta}{p^m} \in \mathbb{Z}_K$, where $\theta = \alpha - t$. Thus, $(1, \alpha, \alpha^2, \frac{\alpha^3+t\alpha^2+t^2\alpha-3t^3}{p^m})$ is a p-integral basis of z_K and $v_p(d_K) = v_p(\triangle)$ modulo 2.

(10) Case 9: B13, C19. Since $v_p(\triangle) = 0$, $(1, \alpha, \alpha^2, \alpha^3)$ is a p-integral basis of $\mathbb{Z}_K$ and $v_p(d_K) = 0$.

2. An integral basis of a quartic number field defined by $X^4 + aX + b$

Remarks 2.1. (1) *Let p be a prime integer such that p^2 divides $\triangle$. For every $1 \le i \le 3$, let $w_{i,p} = \frac{L_i^p(\alpha)}{p^{r_{i,p}}}$, where $L_i^p(X) \in \mathbb{Z}[X]$ is a monic polynomial of degree i such that $\mathcal{F} = (1, w_{1,p}, w_{2,p}, w_{3,p})$ is a triangular p-integral basis of K. Ten $r_{1,p} \le r_{2,p} \le r_{3,p}$, $v_p(\triangle) = r_1 + r_2 + r_3$ and $v_p(d_K) = v_p(\triangle) - 2(r_1 + r_2 + r_3)$.*

(2) *Let $p_1, \cdots, p_r$ be the primes such that every p_i^2 divides $\triangle$. For every $1 \le i \le 3$, denote $d_i = \prod_{j=1}^{r} p_j^{r_{ij}}$, where for every j, $w_{i,j} = \frac{L_i^{p_j}(\alpha)}{p^{r_{ij}}}$ and $(1, w_{1,j}, w_{2,j}, w_{3,j})$ is a p_j-integral basis of K. Then $1 \mid d_1 \mid d_2 \mid d_3$ are the elementary divisors of $\mathbb{Z}_K/\mathbb{Z}[\alpha]$. In particular, d_3 is the conductor of the order $\mathbb{Z}[\alpha]$ and $d_1 d_2 d_3 = \mp \operatorname{ind}(P)$.*

(3) *We can always assume that a triangular p-integral basis has the property: if $r_i = r_{i+1}$, then we can take $w_{i+1} = \alpha w_i$.*

One can recover a triangular integral basis from different triangular p-integral basis for all p as follows:

Proposition 2.2. *Let $p_1,\ldots,p_s$ the prime integers such that p^2 divides $\triangle$ and 1, d_1, d_2 and d_3 the elementary divisors of the abelian group $\mathbb{Z}_K/\mathbb{Z}[\alpha]$. For every j, let $\mathcal{F}_j = (1, w_{1,j}, w_{2,j}, w_{3,j})$ be a triangular p_j-integral basis of K, i.e., $w_{i,j} = \frac{L_i^j(\alpha)}{p_j^{r_{ij}}}$ such that every $L_i^j(X)$ is a monic polynomial of $\mathbb{Z}[X]$ of degree i. Then $\mathcal{B} = (1, w_1, w_2, w_3)$ is a triangular integral basis of K, where every $w_i = \frac{L_i(\alpha)}{d_i}$, $L_i(X) = L_i^j(X)$ modulo $p_j^{r_{ij}}$.*

Proof. Since $ind(P) = d_1 d_2 d_3$, we need only to check that every $w_i \in \mathbb{Z}_K$. Let $1 \le i \le 3$. Since for every i the integers $(\frac{d_i}{p_j^{r_{ij}}})_{1 \le j \le s}$ are pairwise coprime, there exist integers $t_1, \ldots, t_s$ such that $\sum_{j=1}^{s} t_j \frac{d_i}{p_j^{r_{ij}}} = 1$. Hence, $\frac{L_i(\alpha)}{d_i} = \sum_{j=1}^{s} t_j \dfrac{L_i(\alpha)}{p_j^{r_{ij}}} \in \mathbb{Z}_K$, because all $\frac{L_i(\alpha)}{p_j^{r_{ij}}} \in \mathbb{Z}_K$.

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