ON THE TRACE OF AN ENDOFUNCTOR OF A SMALL CATEGORY

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Abstract. The trace of a square matrix can be defined by a universal property which, appropriately generalized yields the concept of "trace of an endofunctor of a small category". We review the basic definitions of this general concept and give a new construction, the "pretrace category", which allows us to obtain the trace of an endofunctor of a small category as the set of connected components of its pretrace. We show that this pretrace construction determines a finite-product preserving endofunctor of the category of small categories, and we deduce from this that the trace inherits any finite-product algebraic structure that the original category may have. We apply our results to several examples from Representation Theory obtaining a new (indirect) proof of the fact that two finite dimensional linear representations of a finite group are isomorphic if and only if they have the same character.

1. The Pretrace of a small Category

1.1. **Trace or "dimension" of a small category.** The trace or "dimension" of a small category $\mathcal{C}$ is defined as the coend of its hom functor $\mathcal{C}(-,-)\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$:

$$\mathsf{Trc}(\mathcal{C}) = \int^{A} \mathcal{C}(A, A). \tag{1}$$

Thus, $\mathsf{Trc}(\mathcal{C})$ is a set which comes equipped with a canonical map $\mathrm{tr}_A\colon \mathcal{C}(A,A) \to \mathsf{Trc}(\mathcal{C})$ for each object A of $\mathcal{C}$ in such a way that all these maps are "compatible" in the sense that for every map $f\colon B \to A$ in $\mathcal{C}$, the following square is commutative:

Date: November 5, 2008.

$$\begin{array}{ccc}
\mathcal{C}(A,B) & \xrightarrow{\mathcal{C}(f,1_B)} & \mathcal{C}(B,B) \\
{\scriptstyle \mathcal{C}(1_A,f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{tr}_B} \\
\mathcal{C}(A,A) & \xrightarrow[\mathrm{tr}_A]{} & \mathsf{Trc}(\mathcal{C})
\end{array}$$

In other words: For every $g\colon A \to B$,

$$\mathrm{tr}_A(fg) = \mathrm{tr}_B(gf), \tag{2}$$

Furthermore these data are "universal" in an appropriate and obvious way.

Definition (1) suggests that $\mathsf{Trc}(\mathcal{C})$ should be something like the set of all endomorphisms of $\mathcal{C}$ modulo an equivalence relation which identifies every pair of endomorphisms $A^{\circlearrowleft u}$ and $B^{\circlearrowleft v}$ for which there is a pair of maps, $f : A \rightleftarrows B : g$, such that $u = gf$ and $v = fg$. Unfortunately, this does not define in general an equivalence relation on the set of endomorphisms of $\mathcal{C}$, although it does if $\mathcal{C}$ is a groupoid:

Proposition 1. *If $\mathcal{C}$ is a small groupoid, the binary relation defined on the endomorphisms of $\mathcal{C}$ by*

$u \sim v$ if and only if there exist maps $f : A \rightleftarrows B : g$, such that $u = gf$ and $v = fg$

is an equivalence relation.

Proof. Reflexivity, $u \sim u$, is proved by the pair $1_A : A \rightleftarrows B : u$. Symmetry is formally evident. Finally, if $u \sim v$ is proved by the pair $f\colon A \rightleftarrows B\colon g$, and $v \sim w$, is proved by the pair $h\colon B \rightleftarrows C\colon k$, then $u \sim w$ is proved by the pair $hf : A \rightleftarrows C : gh^{-1} = f^{-1}k$ □

1.2. **The dual concept of trace is "center".** The dual concept of the trace of a small category $\mathcal{C}$ is the *center* of $\mathcal{C}$, which is defined as the *end* of its hom functor:

$$\mathsf{Cen}(\mathcal{C}) = \int_A \mathcal{C}(A,A). \tag{3}$$

It is easy to prove that for any small category $\mathcal{C}$ the center of $\mathcal{C}$ is the set of natural transformations from the identity functor of $\mathcal{C}$ to itself.[1] In the case that $\mathcal{C}$ is a monoid (a category with only one object) then $\mathsf{Cen}(\mathcal{C})$ is the set of elements (arrows) which commute with all the elements of $\mathcal{C}$, that is, the usual notion of center.

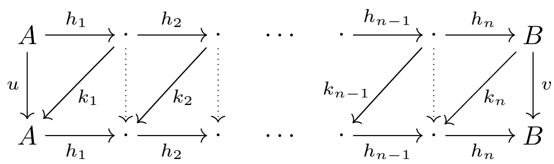
[1]MacLane, *Categories for the Working Mathematicien*, ch.**IX, 5**.

1.3. Even if $\mathcal{C}$ is not a groupoid, the binary relation of Proposition 1 is reflexive and symmetric. Therefore, it generates an equivalence relation which is the smallest equivalence relation containing it. This equivalence relation identifies two endomorphisms u, v if and only if there is a finite sequence $(h_1, k_1), \dots, (h_n, k_n)$ of pairs of maps such that

$$k_1 h_1 = u,\ h_n k_n = v, \quad \text{and} \quad h_{i-1} k_{i-1} = k_i h_i, \quad \text{for} \quad 2 \le i \le n, \tag{4}$$

or, equivalently, such that the following diagram is commutative:

(5)



Definition 1. *An arrow* $r \colon A \to B$ *in a category* $\mathcal{C}$ *will be called a* trace arrow from u to v *if there is a finite sequence* $(h_1, k_1), \dots, (h_n, k_n)$ *of pairs of maps such that* $r = h_n \cdots h_1$ *and equations* (4) *are satisfied. Note that in this situation the composite* $s = k_1 \cdots k_n$ *is a trace arrow from* v *to* u*.*

Clearly, in any commutative diagram like (5) all vertical arrows are endomorphisms of $\mathcal{C}$ having the same trace.

1.4. The endomorphisms and trace arrows of a small category $\mathcal{C}$ are respectively the objects and arrows of a small category $\widetilde{\mathrm{Tr}}(\mathcal{C})$ which will be called the *pretrace category* of $\mathcal{C}$. Clearly, "there is a trace arrow from u to v" is an equivalence relation on the set of endomorphisms of $\mathcal{C}$ and (the elements of) the corresponding equivalence classes are (the objects of) the connected components of $\widetilde{\mathrm{Tr}}(\mathcal{C})$. We have:

Proposition 2. *If* $\mathcal{C}$ *is a small category then the trace of* $\mathcal{C}$ *is the set of connected components of the pretrace category* $\widetilde{\mathrm{Tr}}(\mathcal{C})$*:*

$$\mathsf{Trc}(\mathcal{C}) = \pi_0\big(\widetilde{\mathrm{Tr}}(\mathcal{C})\big).$$

Proof. There are obvious maps $\tau_A \colon \mathcal{C}(A, A) \to \pi_0\big(\widetilde{\mathrm{Tr}}(\mathcal{C})\big)$ (taking an endo u of A to the connected component of $A^{\circlearrowleft u}$ in $\widetilde{\mathrm{Tr}}(\mathcal{C})$) which are obviously "compatible". This implies that there is a unique map $h \colon \mathsf{Trc}(\mathcal{C}) \to \pi_0\big(\widetilde{\mathrm{Tr}}(\mathcal{C})\big)$ such that for all $A \in \mathcal{C}$, $h\,\mathrm{tr}_A = \tau_A$. We just need to prove that this map has an inverse $k \colon \pi_0\big(\widetilde{\mathrm{Tr}}(\mathcal{C})\big) \to \mathsf{Trc}(\mathcal{C})$. Define the map $\tilde{k} \colon \mathrm{obj}\big(\widetilde{\mathrm{Tr}}(\mathcal{C})\big) \to \mathsf{Trc}(\mathcal{C})$ on the set of objects of $\widetilde{\mathrm{Tr}}(\mathcal{C})$ by $\tilde{k}(A^{\circlearrowleft u}) = \mathrm{tr}_A(u)$. It is easy to see that $\tilde{k}$ is constant on connected components and hence it determines a map $k \colon \pi_0\big(\widetilde{\mathrm{Tr}}(\mathcal{C})\big) \to \mathsf{Trc}(\mathcal{C})$ which is easily seen to be inverse to h. □

1.5. Every arrow $r\colon A^{\circlearrowleft u} \to B^{\circlearrowleft v}$ in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ should be regarded as a "reason" to identify the traces of u and v (or as a "proof" of $\mathrm{tr}_A(u) = \mathrm{tr}_B(v)$). The same must be said of any sequence $(h_1, k_1), \ldots, (h_n, k_n)$ verifying (4). Such a sequence of length n will be called a n-step proof that u and v have the same trace, or a proof of length n that r is a trace arrow from u to v.

Evidently, for any sequence $(h_1, k_1), \ldots, (h_n, k_n)$ of pairs of arrows of $\mathcal{C}$ verifying (4) each map h_i and k_i is a (one-step) trace arrow in $\mathcal{C}$.

1.6. If there is no reason to identify the traces of two endomorphisms, their traces are different. For example, if $\mathcal{C}$ is a commutative monoid (seen as a one-object category), any one-step proof of $\mathrm{tr}_A(u) = \mathrm{tr}_B(v)$ will imply $u = v$ (if $u = kh$ and $v = hk$ then, on account of the commutativity, $u = v$). This implies that no two different elements can have the same trace and therefore the trace of a commutative monoid is the set of its elements (arrows). It turns out that in this case of commutative monoids, the trace coincides with the center. Recall (Paragraph 1.2) that the center of a small category $\mathcal{C}$ is the set of natural transformations from the identity functor of $\mathcal{C}$ to itself: $\mathsf{Cen}(\mathcal{C}) = \mathsf{Nat}(1_{\mathcal{C}}, 1_{\mathcal{C}})$. Evidently this is a monoid and even a *commutative monoid*. If $\mathcal{C}$ is already a commutative monoid then $\mathsf{Cen}(\mathcal{C}) = \mathcal{C}$. Thus:

Proposition 3. *If $\mathcal{C}$ is a commutative monoid, then*

$$\mathsf{Trc}(\mathcal{C}) = \{\textit{elements (arrows) of } \mathcal{C}\} = \mathsf{Cen}(\mathcal{C}).$$

Furthermore:

Proposition 4. *If the small category $\mathcal{C}$ is discrete, then $\widetilde{\mathrm{Tr}}(\mathcal{C}) = \mathsf{Trc}(\mathcal{C}) = \mathcal{C}$.*

Proposition 5. *If the small category $\mathcal{P}$ is a poset, then $\widetilde{\mathrm{Tr}}(\mathcal{P}) = \mathsf{Iso}(\mathcal{P})$ and*

$$\mathsf{Trc}(\mathcal{P}) = \{\textit{Classes of isomorphic objects}\}.$$

(In a poset the only endomorphisms are the identities. Thus, all trace arrows are isomorphisms.)

1.7. The forgetful functor. Identities and compositions in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ are those of $\mathcal{C}$, so that we have an obvious forgetful functor

$$\epsilon_{\mathcal{C}}\colon \widetilde{\mathrm{Tr}}(\mathcal{C}) \longrightarrow \mathcal{C} \tag{6}$$

given by $\epsilon_{\mathcal{C}}(A^{\circlearrowleft u}) = A$ and $\epsilon_{\mathcal{C}}\big(A^{\circlearrowleft u} \xrightarrow{r} B^{\circlearrowleft v}\big) = r$.

2. Trace via cyclic nerve

We see in this section a second construction of the trace set of a small category based on the calculation of the cyclic nerve of the category.

2.1. The cyclic nerve of a category. If the simplicial category $\mathbf{\Delta}$ is enlarged by adjoining in each dimension one extra degeneracy $s_n\colon [n] \to [n{+}1]$ subject to the condition $(d_0 s_n)^n = \mathrm{id}$, one obtains a category $C\mathbf{\Delta}$ whose presheaves are the cyclic sets and which plays in cyclic (co)homology the same role that $\mathbf{\Delta}$ plays in regular (co)homology. As objects of $C\mathbf{\Delta}$, the objects and arrows of $\mathbf{\Delta}$ cannot be thought of as ordinals and order preserving maps. The intuition behind "the point" $[0]$, "the arrow" $[1]$, "the triangle" $[2]$, etc. breaks down in $C\mathbf{\Delta}$ and has to be substituted by a different picture of the objects of $C\mathbf{\Delta}$. In order to reflect this change of picture we shall denote the objects of $\mathbf{\Delta}$, when viewed as objects of $C\mathbf{\Delta}$, as $[n]_{\mathrm{c}}$. To each category $\mathcal{C}$ we can associate a "cyclic nerve", which is the cyclic set $C\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ defined by $[n]_{\mathrm{c}} \mapsto \mathsf{Func}([n]_{\mathrm{c}}, \mathcal{C})$.

$$N_{\mathrm{c}}\colon \mathbf{Cat} \to \mathbf{Set}^{C\mathbf{\Delta}^{\mathrm{op}}}, \quad N_{\mathrm{c}}(\mathcal{C}) = \mathsf{Func}(-, \mathcal{C}).$$

2.2. Trace and cyclic nerve. A functor $[0]_{\mathrm{c}} \to \mathcal{C}$ is the same as an endomorphism in $\mathcal{C}$, therefore the 0-cells of the cyclic nerve are the endomorphisms of $\mathcal{C}$, that is, the objects of the pretrace $\widetilde{\mathrm{Tr}}(\mathcal{C})$. A functor $r\colon [1]_{\mathrm{c}} \to \mathcal{C}$ is completely determined by a pair of arrows in $\mathcal{C}$, $f\colon A \rightleftarrows B\colon g$, such that $d_0(r) = gf$ and $d_1(r) = fg$; in other words, r is determined by a length 1 trace arrow in $\mathcal{C}$, or simply by an arrow in $\widetilde{\mathrm{Tr}}(\mathcal{C})$. Thus, the 1-cells of the cyclic nerve are the the arrows of $\widetilde{\mathrm{Tr}}(\mathcal{C})$. We have:

Proposition 6. *The trace set of a small category $\mathcal{C}$ is the set of connected components of its cyclic nerve of $\mathcal{C}$:* $\mathsf{Trc}(\mathcal{C}) = \pi_0\big(N_{\mathrm{c}}(\mathcal{C})\big)$.

3. The Set-Trace of Finite Dimensional Vector Spaces

3.1. In this section we address the question of what is the trace set of the category of finite dimensional k-vector spaces for an arbitrary field k. Thus, our objective is to determine the coend:

$$\mathsf{Trc}^{\mathrm{set}}\big(k\text{-}\mathbf{Vect}_{\mathrm{f.d.}}\big) = \int^{V} |\hom(V,V)|$$

where hom is the (enriched) hom functor of finite dimensional vector spaces, $\hom\colon k\text{-}\mathbf{Vect}_{\mathrm{f.d.}}{}^{\mathrm{op}} \times k\text{-}\mathbf{Vect}_{\mathrm{f.d.}} \to k\text{-}\mathbf{Vect}$, and $|-|$ means "underlying set". In other words, we are asking the following question: Given two arbitrary rectangular matrices A and B with the appropriate sizes so that the two products AB and BA exist, what do the two matrices AB and BA have in common?

3.2. It is a known theorem in Linear Algebra that if A and B are arbitrary rectangular matrices of appropriate sizes, AB and BA have the same invariants. That is: for every non-negative integer p, the trace of the matrix of p-minors of AB is equal to the trace of the corresponding matrix of p-minors of BA. These invariants are (up to a factor of ± 1) the coefficients of the corresponding characteristic polynomial.

Proposition 7. *Let A and B be matrices of respective orders $m \times n$ and $n \times m$. If $m \geq n$, then the characteristic polynomials of AB and BA, $p_{AB}(\lambda) = \det(AB - \lambda I)$ and $p_{BA}(\lambda) = \det(BA - \lambda I)$, are related by:*

$$p_{AB}(\lambda) = (-\lambda)^{m-n} p_{BA}(\lambda)$$

3.3. It is tempting to conjecture that "all that the matrices AB and BA have in common is their reduced characteristic polynomial", where by the "reduced polynomial of $p(x)$" is meant the monic polynomial obtained when dividing $p(x)$ by the highest degree monomial dividing $p(x)$. (In other words, the reduced polynomial of $p(x)$ is $p^*(x) = p(x)/ax^\alpha$ where α is the multiplicity of zero as a root of $p(x)$ ($\alpha = 0$ if $p(0) \neq 0$) and a is the leading coefficient of $p(x)$.) The following example shows that the said tempting conjecture is not true.

Counterexample 1. Let H and K be the matrices

$$H = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad K = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The characteristic polynomials of these are:

$$p_H(x) = (-x)(x-1)^2 \, ; \quad p_K(x) = (x-1)^2 \, ,$$

which have the same reduced form. However, there are no matrices A, B such that $H = AB$ and $K = BA$ as this would imply $H^2 = H$, which is not true. In fact, it is not hard to prove that there exist no matrices $H_1, K_1, \ldots, H_m, K_m$ verifying

$$K_1 H_1 = A, \; H_m K_m = B, \quad \text{and} \quad H_{i-1} K_{i-1} = K_i H_i, \quad \text{for} \quad 2 \leq i \leq m \, ,$$

for this would imply that $H^{n+1} = H^n$ which is impossible:

$$H^n = \begin{pmatrix} 1 & 0 & 1 \\ n & 1 & n \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, although H and K have the same reduced characteristic polynomial, yet they do not have the same set trace.

As we will see, the problem with H and K is that they do not have the same reduced *minimal polynomial*, their minimal polynomials being

$$m_H(x) = x(x-1)^2\,, \quad m_K(x) = x-1\,.$$

3.4. The question of what do two matrices AB and BA have in common was considered by Harley Flanders in a 1951 paper (see [2]) The following proposition is an immediate consequence of Theorem 2 in that paper:

For any endomorphism $f \in \hom(V,V)$ of a finite dimensional vector space, let $m_f(x)$, be its minimal polynomial, and let K_f be the kernel of the endomorphism $m_f^*(f)$, so that f restricts to an endomorphism $f'\colon K_f \to K_f$.

Proposition 8. *Let $f \in \hom(V,V)$ and $g \in \hom(W,W)$ be endomorphisms of finite dimensional vector spaces. If there exist linear maps h, k such that $f = kh$ and $g = hk$ then $m_f^*(x) = m_g^*(x)$ and there is an isomorphism $\varphi\colon \ker\big(m_f^*(f)\big) \to \ker\big(m_g^*(g)\big)$ such that*

$$\begin{array}{ccc} K_f & \xrightarrow{\varphi} & K_g \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ K_f & \xrightarrow[\varphi]{} & K_g \end{array} \tag{7}$$

Conversely, if there exists an isomorphism φ verifying (7) *then $m_f^*(x) = m_g^*(x)$ and there exists a finite sequence $(h_1,k_1),\ldots,(h_n,k_n)$ of pairs of linear maps such that*

$$k_1h_1 = f,\; h_nk_n = g, \quad \textit{and} \quad h_{i-1}k_{i-1} = k_ih_i, \quad \textit{for} \quad 2 \le i \le n.$$

It follows from this that the **Set**-trace of a linear endomorphism f of a finite dimensional vector space is precisely the set of reduced invariant factors of f. This set is completely determined by the structure of the powers of those irreducible factors of the minimal polynomial $m(x)$ of f which are not equal to x. The **Set**-trace of the category of finite dimensional k-vector spaces is a set T^S that encodes such structure for all possible endomorphisms. There are two equivalent ways of describing T^S. The first one is as the set of finite chains $p_1(x)\,|\,p_2(x)\,|\,\cdots\,|\,p_n(x)$ of normal polynomials (i.e. such that $p_i(0) = 1$). Using this description we have:

Theorem 9. *The* **Set**-*trace of the category of finite dimensional k-vector spaces is the set T^S of finite chains $p_1(x)\,|\,p_2(x)\,|\,\cdots\,|\,p_n(x)$ of polynomials in $k[x]$ such that $p_i(0) = 1$. For a given endomorphism $f\colon V \to V$ of a finite dimensional k-vector space, the trace function $\mathrm{tr}_V^{set}\colon \hom(V,V) \to T^S$ can be defined as $\mathrm{tr}_V^{set}(f) =$ chain of normalized reduced invariant factors of f.*

For example, let $p_f(x)$ be the reduced characteristic polynomial of $f\colon V \to V$. If $p_f(x)$ has degree 1, then $\mathrm{tr}_V^{\mathrm{set}}(f) = p_f(x)/p_f(0)$. If $p_f(x)$ has degree 0, then $\mathrm{tr}_V^{\mathrm{set}}(f) = 1$.

Another description of T^S is as the set of all possible shapes of normal forms of matrices. More specifically:

Let $\mathcal{P}$ denote the set of all partitions of natural numbers and let $\mathcal{F}$ denote the set of all finite subsets of $\mathcal{P}$. There is a function

$$\Sigma\colon \mathcal{F} \to \mathcal{P}$$

assigning to each finite set of partitions $\sigma = \{\sigma^{(1)}, \dots, \sigma^{(n)}\}$ the partition obtained by adding up all elements in each partition $\sigma^{(j)}$; that is: $\Sigma(\sigma) = \{\sum_i \sigma_i^{(1)}, \dots, \sum_i \sigma_i^{(n)}\}$.

On the other hand, there is a function $\mathrm{mult}\colon k[x] \to \mathcal{P}$ assigning to each polynomial $p(x)$ the partition of the degree of $p(x)$ consisting of the degrees of the irreducible factors of $p(x)$ multiplied by their multiplicities. For example, if $k = \mathbf{R}$ then $\mathrm{mult}\big((x^2+1)(x-1)^3\big) = \{2, 3\}$. Obviously, if k is algebraically closed all irreducible factors of $p(x)$ have degree 1 and $\mathrm{mult}\big(p(x)\big)$ is just the set of multiplicities of the different roots of $p(x)$.

An element of T^S can be described as a pair formed by a monic polynomial $p(x)$ with nonzero independent term (a reduced characteristic polynomial of a linear endomorphism) and, for each irreducible factor $p_i(x)$ of $p(x)$, a partition of the multiplicity n_i of $p_i(x)$ in $p(x)$. Since among the constant multiples of $p(x)$ there is exactly one with constant term equal to 1, if we denote $k[x]^* = \{p(x) \in k[x] \mid p(0) = 1\}$, we can say that T^S is the subset of $k[x]^* \times \mathcal{F}$ formed by all those pairs $\langle p, \sigma\rangle$ such that $\mathrm{mult}(p) = \Sigma(\sigma)$

$$T^S = \big\{\langle p, \sigma\rangle \in k[x]^* \times \mathcal{F} \mid \mathrm{mult}(p) = \Sigma(\sigma)\big\}. \tag{8}$$

Using this description of T^S we have:

Theorem 10. *The* **Set***-trace of the category of finite dimensional k-vector spaces is the set T^S defined by (8). On a given endomorphism $f\colon V \to V$ whose reduced characteristic polynomial is $p^*(x)$ and whose reduced minimal polynomial has a decomposition into irreducible factors as $m^*(x) = m_1(x)^{n_1} \cdots m_k(x)^{n_k}$, the trace function $\mathrm{tr}_V\colon \hom(V,V) \to T^S$ is defined by $\mathrm{tr}_V(f) = \langle \bar{p}(x), \sigma\rangle$ where $\bar{p}(x) = p^*(x)/p^*(0)$, and σ is the set of partitions $\{r^{(1)}, \dots, r^{(k)}\}$ where for each i, $r^{(i)} = \{r_1^{(i)}, \dots, r_{n_i}^{(i)}\}$ is the partition of $d_{n_i}^{(i)} = \dim\big(\ker(m_i(f)^{n_i})\big)$ defined as $r_j^{(i)} = d_j^{(i)} - d_{j-1}^{(i)}$, $(j = 1, \dots, n_i)$, with $d_j^{(i)} = \dim\big(\ker(m_i(f)^j)\big)$ for $j = 0, \dots, n_i$.*

For example, the identity map of an n-dimensional vector space has reduced characteristic polynomial $(1-x)^n$ and minimal polynomial $(x-1)^1$. Thus, the set trace of this identity is the pair $\langle (1-x)^n, \{n\}\rangle$.

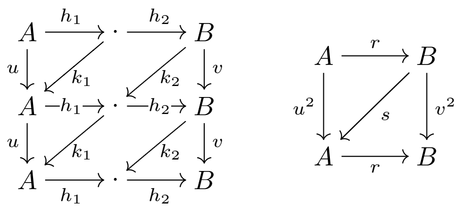
4. Properties of The Traces of Endomorphisms

4.1. One has to be careful when trying to apply the results of this section to the traces of matrices. In order to do that it is necessary to interpret "trace" of a matrix as **Set**-trace (see Section 3).

We begin with a useful lemma:

Lemma 11 (The Reduction Lemma)**.** *In any small category $\mathcal{C}$ if r is a trace arrow of length n from u to v, then r is a trace arrow of length 1 from u^n to v^n.*

Proof. Let's begin with the case $n = 2$. If (h_1, k_1), (h_2, k_2) is a proof that r is a trace arrow from u to v (that is: $r = h_2h_1$, $k_1h_1 = u$, $h_2k_2 = v$, and $h_1k_1 = k_2h_2$), then we can put $s = k_1k_2$ and we have



$rs = (h_2h_1)(k_1k_2) = h_2k_2h_2k_2 = v^2$ and $sr(k_1k_2)(h_2h_1) = k_1h_1k_1h_1 = u^2$. Essentially the same proof is valid for a trace arrow of arbitrary length n; we just need to paste n copies of a diagram like (5). If $(h_1, k_1), \ldots, (h_n, k_n)$ is a n-step proof that $\mathrm{tr}(u) = \mathrm{tr}(v)$ then putting $r = h_n \cdots h_1$ and $s = k_1 \cdots k_n$, we get $rs = (h_n \cdots h_1)(k_1 \cdots k_n) = (h_n \cdots h_2)(k_2h_2)(k_2 \cdots k_n) = \cdots = h_n(k_nh_n) \cdots (k_nh_n)k_n = v^n$ and similarly that $sr = (k_1 \cdots k_n)(h_n \cdots h_1) = u^n$. □

This has the following immediate consequence:

Proposition 12. *In any small category $\mathcal{C}$ all trace arrows from one idempotent to another are of length 1.*

It follows that the identities of non-isomorphic objects have different traces.

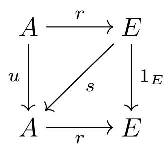
Proposition 13. *In any small category $\mathcal{C}$ a map is a trace arrow from one identity to another if and only if it is an isomorphism.*

Proof. Obviously isomorphisms are trace arrows between the corresponding identities. Conversely, by the previous lemma all trace arrows between identities are of length 1, hence isomorphisms. □

As a consequence of this we have:

Proposition 14. *In any small category $\mathcal{C}$ two identities have the same trace if and only if their corresponding objects are isomorphic.*

4.2. Let us now consider the question of what endomorphisms $A^{\circlearrowleft u}$ in $\mathcal{C}$ have the same trace as an identity map. First, we remark that a one-step proof that u has the same trace as the identity map of E is the same as a proof that u is a split idempotent with object of fixed points E:



Thus, every split idempotent in $\mathcal{C}$ has the same trace as the identity of its fixed points. With a similar reasoning to the proof of Lemma 11 it is easy to show that the converse also holds, so that:

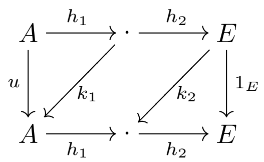
Proposition 15. *In any small category $\mathcal{C}$ an idempotent has the same trace as an identity if and only if it splits (in which case it has the same trace as the identity of its object of fixed points).*

Proof. By hypothesis there is a trace arrow from the given idempotent to the given identity. By Proposition 12 this trace arrow is of length 1. Therefore this trace arrow is a splitting of the given idempotent. ☐

As a consequence of this we have the following generalization of Proposition 14:

Proposition 16. *In any small category $\mathcal{C}$ two split idempotents have the same trace if and only if their objects of fixed points are isomorphic.*

4.3. Let now u be an arbitrary endomorphism and let us suppose that there is a 2-step proof that $\mathrm{tr}_A(u) = \mathrm{tr}_E(1_E)$



It follows that, if we put $v = h_1 k_1$, then v is an idempotent and $u^2 = (k_1 h_1)(k_1 h_1) = k_1 v h_1$ and $u^3 = k_1 v^2 h_1 = k_1 v h_1 = u^2$. This calculation can be easily generalized to yield a proof of the following:

Proposition 17. *If v is an endomorphism in $\mathcal{C}$ such that $v^n = v^{n+1}$ and u is an endomorphism in $\mathcal{C}$ such that there is a k-step proof that* $\mathrm{tr}(u) = \mathrm{tr}(v)$ *then, setting $m = n + k$, we have $u^m = u^{m+1}$.*

Note that the particular case $n = 0$, $k = 1$ captures the case of a split idempotent discussed above. The particular case $n = 0$ (v is an identity) gives a necessary condition for a map to have the same trace as an identity:

Corollary 18. *If u is an endomorphism in $\mathcal{C}$ having the same trace as an identity then the sequence of powers of u stops (becomes constant).*

Corollary 19. *If u is a root of unity in $\mathcal{C}$ having the same trace as an identity then u is an identity.*

4.4. When does the converse of Corollary 18 hold? If $\mathcal{C}$ has splitting of idempotents (i.e. it is Cauchy complete) then, by the previous remarks, every idempotent has the same trace as an identity map. Hence, for an endomorphism $A^{\circlearrowleft u}$ whose sequence of powers stops ($u^n = u^{n+1}$), since u^n is an idempotent, we have $\mathrm{tr}_A(u^n) = \mathrm{tr}_E(1_E)$ where E is the object of fixed points of u^n.

A better result can be obtained if we assume a stronger condition on $\mathcal{C}$. If $\mathcal{C}$ admits *epi-mono factorizations* then we can restrict any endomorphism to its image and get an endomorphism of the image which has the same trace as the given one:

$$\begin{array}{ccc} A & \xrightarrow{r} & \mathsf{Im}(u) \\ {\scriptstyle u}\downarrow & \swarrow{\scriptstyle s} & \downarrow{\scriptstyle v} \\ A & \xrightarrow[r]{} & \mathsf{Im}(u) \end{array} \qquad u = sr,\ v = rs,\ \mathrm{tr}(u) = \mathrm{tr}(v).$$

If the original endomorphism satisfies $u^{n+1} = u^n$ then $sv^n r = sv^{n-1}r$, which implies (since s is mono and r epi) $v^n = v^{n-1}$. We can continue doing image factorizations so that in each step the exponents get reduced by one unit until we get an idempotent and a splitting for it. The final result is the following converse of Corollary 18:

Proposition 20. *Let's assume that $\mathcal{C}$ has epi-mono factorizations. If u is an endomorphism in $\mathcal{C}$ such that the sequence of powers of u stops, then u has the same trace as an identity.*

A more complete statement is:

Proposition 21. *Let's assume that $\mathcal{C}$ has epi-mono factorizations. If u is an endomorphism in $\mathcal{C}$ such that the sequence of powers of u stops, then all powers of u have the same trace. If $u^{n+1} = u^n$, this trace is the trace of the identity of the object of fixed points of u^n.*

5. The Trace Set of a Groupoid

5.1. Conjugate endomorphisms of a category have the same trace: If h is an isomorphism in $\mathcal{C}$ and u and v are two endomorphisms in $\mathcal{C}$ related by $v = huh^{-1}$, then, evidently, $\mathrm{tr}(v) = \mathrm{tr}(huh^{-1}) = \mathrm{tr}(uh^{-1}h) = \mathrm{tr}(u)$. The converse (endos with same trace are conjugate) is true if $\mathcal{C}$ is a groupoid. In fact we have:

Proposition 22. *If $\mathcal{C}$ is a groupoid then:*

(1) *An arrow in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ from $A^{\circlearrowleft u}$ to $B^{\circlearrowleft v}$ is the same as a commutative square of the form:*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow[h]{} & B \end{array}$$

(2) *$\widetilde{\mathrm{Tr}}(\mathcal{C})$ is a groupoid.*
(3) *Two endomorphisms of $\mathcal{C}$ have the same trace if and only if they are conjugate of each other: $v = huh^{-1}$.*
(4) *The trace of $\mathcal{C}$ is the set of conjugacy classes of $\mathcal{C}$.*

In this case it is easy to give a direct proof of Proposition 22 item 4. We leave it as an exercise for the reader.

5.2. When $\mathcal{C}$ is a groupoid, the forgetful functor $\epsilon_{\mathcal{C}} \colon \widetilde{\mathrm{Tr}}(\mathcal{C}) \to \mathcal{C}$ is actually a fibration of categories. This is basically due to the fact that the lifting of maps of $\mathcal{C}$ to $\widetilde{\mathrm{Tr}}(\mathcal{C})$ is unique once the codomain has been chosen:

Proposition 23. *If $\mathcal{C}$ is a groupoid then the forgetful functor $\epsilon_{\mathcal{C}} \colon \widetilde{\mathrm{Tr}}(\mathcal{C}) \to \mathcal{C}$ taking every endomorphism of $\mathcal{C}$ to its underlying object is a fibration of categories (groupoids).*

Proof. Let $f \colon A \to B$ be a map in $\mathcal{C}$ and v an endomorphism of B. Then obviously f is a map in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ from $A^{\circlearrowleft u}$ to $B^{\circlearrowleft v}$ where $u = f^{-1}vf$, and $\epsilon_{\mathcal{C}}(f) = f$. We just need to prove that f, regarded as a map in $\widetilde{\mathrm{Tr}}(\mathcal{C})$, is cartesian. Let $g \colon C^{\circlearrowleft w} \to B^{\circlearrowleft v}$ be a map in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ (so that $w = g^{-1}vg$). Any map $r \colon C \to A$ verifying $fr = g$ uniquely determines a map $C^{\circlearrowleft w} \to A^{\circlearrowleft u}$ over r, and this map is r itself since $rw = r(g^{-1}vg) = r(fr)^{-1}v(fr) = f^{-1}vfr = ur$. □

5.3. For every object A of $\mathcal{C}$ the fiber of $\epsilon_{\mathcal{C}}$ over A is the discrete category whose objects are the endomorphisms of A. We can regard the fiber functor as a functor $\mathsf{End} \colon \mathcal{C} \to \mathbf{Set}$. As a corollary of Proposition

22 (item *4.*) and Proposition 23 we have that the set of connected components of the Grothendieck construction on this fiber functor is the set of conjugacy classes of the groupoid $\mathcal{C}$.

Corollary 24. *If $\mathcal{C}$ is a groupoid, then $\widetilde{\mathrm{Tr}}(\mathcal{C})$ is equivalent to the Grothendieck construction applied to the "fiber of $\epsilon_{\mathcal{C}}$" functor* $\mathsf{End}\colon \mathcal{C} \to \mathbf{Set}$*, so that*

$$\{\textit{Conjugacy classes of } \mathcal{C}\} = \mathsf{Trc}(\mathcal{C}) \simeq \pi_0\big(\textstyle\int_{\mathcal{C}} \mathsf{End}\big).$$

6. The Pretrace Comonad

6.1. The pretrace construction is functorial:

Proposition 25. *The pretrace of a category is the objects function of a functor* $\widetilde{\mathrm{Tr}}\colon \mathbf{Cat} \to \mathbf{Cat}$ *and the functor "trace of a category",* $\mathsf{Trc}\colon \mathbf{Cat} \to \mathbf{Set}$*, is the composite* $\pi_0\widetilde{\mathrm{Tr}}$*:*

$$\begin{array}{ccccc} & & \mathbf{Cat} & & \\ & {\scriptstyle\widetilde{\mathrm{Tr}}}\nearrow & & \searrow{\scriptstyle\pi_0} & \\ \mathbf{Cat} & & \xrightarrow[\mathsf{Trc}]{\hspace{4em}} & & \mathbf{Set}. \end{array}$$

Proof. The definition of $\widetilde{\mathrm{Tr}}$ on arrows is as follows: If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, $\widetilde{\mathrm{Tr}}(F)$ takes $A^{\circlearrowleft u}$ to $F(A)^{\circlearrowleft F(u)}$ and an arrow r in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ to $F(r)$, which is easily checked to be an arrow in $\widetilde{\mathrm{Tr}}(\mathcal{D})$. The functoriality is evident. □

6.2. The pretrace functor $\widetilde{\mathrm{Tr}}$ restricted to groupoids is a 2-functor $\widetilde{\mathrm{Tr}}\colon \mathbf{Gpd} \to \mathbf{Gpd}$; if $\eta\colon F \to G$ is a natural transformation between functors of groupoids, then putting $\widetilde{\mathrm{Tr}}(\eta)_{(A,u)} = F(\eta_A)$ we get a natural transformation, $\widetilde{\mathrm{Tr}}(\eta)$, from $\widetilde{\mathrm{Tr}}(F)$ to $\widetilde{\mathrm{Tr}}(G)$.

6.3. The commutativity of

$$\begin{array}{ccc} A & \xrightarrow{u} & A \\ {\scriptstyle u}\downarrow & \swarrow{\scriptstyle 1_A} & \downarrow{\scriptstyle u} \\ A & \xrightarrow[u]{} & A \end{array}$$

shows that for any endomorphism $A^{\circlearrowleft u}$ of $\mathcal{C}$, u is an arrow from $A^{\circlearrowleft u}$ to $A^{\circlearrowleft u}$ in $\widetilde{\mathrm{Tr}}(\mathcal{C})$, that is, an endomorphism in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ and therefore an object in $\widetilde{\mathrm{Tr}}(\widetilde{\mathrm{Tr}}(\mathcal{C}))$. This object will be denoted $\delta_{\mathcal{C}}(u)$. If $r\colon A^{\circlearrowleft u} \to B^{\circlearrowleft v}$ is an arrow in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ then any proof of it, $(h_1, k_1), \ldots, (h_n, k_n)$ gives rise to a proof that

r is actually a map in $\widetilde{\mathrm{Tr}}\big(\widetilde{\mathrm{Tr}}(\mathcal{C})\big)$ from $\delta_{\mathcal{C}}(u)$ to $\delta_{\mathcal{C}}(v)$. In fact, it is easy to prove that $\delta_{\mathcal{C}}$ is the objects function of a functor

$$\delta_{\mathcal{C}} \colon \widetilde{\mathrm{Tr}}(\mathcal{C}) \longrightarrow \widetilde{\mathrm{Tr}}\big(\widetilde{\mathrm{Tr}}(\mathcal{C})\big)$$

such that on arrows $\delta_{\mathcal{C}}(r) = r$. Furthermore, it is easy to prove the following:

Proposition 26. *Each of the functors $\epsilon_{\mathcal{C}}$ and $\delta_{\mathcal{C}}$ depend naturally on $\mathcal{C}$ and we actually have natural transformations*

$$\epsilon \colon \widetilde{\mathrm{Tr}} \to 1, \quad \delta \colon \widetilde{\mathrm{Tr}} \to \widetilde{\mathrm{Tr}}^{2}$$

which are the counit and comultiplication of a comonad structure on $\widetilde{\mathrm{Tr}}$.

6.4. What are the coalgebras for this comonad? A coalgebra in this case is a category $\mathcal{C}$ together with a structure functor $\Sigma \colon \mathcal{C} \to \widetilde{\mathrm{Tr}}(\mathcal{C})$ satisfying identity and associative laws. Before discussing these coalgebras it will be useful to address this other question: What is a functor $\Sigma \colon \mathcal{C} \to \widetilde{\mathrm{Tr}}(\mathcal{C})$?, or, more generally: What is a functor into a pretrace category $\Sigma \colon \mathcal{C} \to \widetilde{\mathrm{Tr}}(\mathcal{D})$?

7. Functors into a Pretrace Category

7.1. Obviously, a functor $\Sigma \colon \mathcal{C} \to \widetilde{\mathrm{Tr}}(\mathcal{D})$ determines a functor $S \colon \mathcal{C} \to \mathcal{D}$ as the composite $\epsilon_{\mathcal{D}}\Sigma$. Furthermore, it also determines for each object A of $\mathcal{C}$ an endomorphism $\sigma_A \colon S(A) \to S(A)$ so that on objects $\Sigma(A) = S(A)^{\circlearrowleft \sigma_A}$ and on arrows $\Sigma(r) = S(r)$. The fact that for each arrow $r \colon A \to B$ in $\mathcal{C}$, $S(r)$ is an arrow in $\widetilde{\mathrm{Tr}}(\mathcal{D})$ from $S(A)^{\circlearrowleft \sigma_A}$ to $S(B)^{\circlearrowleft \sigma_B}$ implies that σ is a natural transformation from S to S. Thus, a functor $\Sigma \colon \mathcal{C} \to \widetilde{\mathrm{Tr}}(\mathcal{D})$ determines a functor $S \colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $\sigma \colon S \to S$, that is, an object in $\widetilde{\mathrm{Tr}}\big(\mathcal{D}^{\mathcal{C}}\big)$. The converse falls short of being true: Any object $S^{\circlearrowleft \sigma} \in \widetilde{\mathrm{Tr}}\big(\mathcal{D}^{\mathcal{C}}\big)$ determines a functor $\Sigma \colon \mathcal{C} \to \widetilde{\mathrm{Tr}}(\mathcal{D})$ provided that for every arrow $r \colon A \to B$ in $\mathcal{C}$, $S(r)$ is a trace arrow from σ_A to σ_B. In other words, there is an injective but not necessarily surjective function

$$\mathrm{obj}\big(\widetilde{\mathrm{Tr}}(\mathcal{D})^{\mathcal{C}}\big) \longrightarrow \mathrm{obj}\big(\,\widetilde{\mathrm{Tr}}\big(\mathcal{D}^{\mathcal{C}}\big)\,\big). \tag{9}$$

Proposition 27. *For any small category $\mathcal{C}$, each functor $\Sigma \colon \mathcal{C} \to \widetilde{\mathrm{Tr}}(\mathcal{D})$ determines an object $S^{\circlearrowleft \sigma} \in \widetilde{\mathrm{Tr}}\big(\mathcal{D}^{\mathcal{C}}\big)$ where $S = \epsilon_{\mathcal{C}}\Sigma$ and σ is determined by the fact that for every object $A \in \mathcal{C}$, $\Sigma(A) = S(A)^{\circlearrowleft \sigma_A}$. This correspondence determines an injective function* (9) *from the objects of $\widetilde{\mathrm{Tr}}(\mathcal{D})^{\mathcal{C}}$ to those of $\widetilde{\mathrm{Tr}}(\mathcal{D}^{\mathcal{C}})$.*

7.2. Could the above function (9) be the objects function of a functor $\widetilde{\mathrm{Tr}}(\mathcal{D})^{\mathcal{C}} \to \widetilde{\mathrm{Tr}}\big(\mathcal{D}^{\mathcal{C}}\big)$? This seems unlikely because for an arrow in $\mathcal{D}^{\mathcal{C}}$ (natural transformation) to be a trace arrow is not sufficient that each component be a trace arrow in $\mathcal{D}$. The hom sets in $\widetilde{\mathrm{Tr}}(\mathcal{D})^{\mathcal{C}}$ are "bigger" than those in $\widetilde{\mathrm{Tr}}\big(\mathcal{D}^{\mathcal{C}}\big)$. What is obviously true is the reverse, that is: for any two functors Σ, Φ in $\widetilde{\mathrm{Tr}}(\mathcal{D})^{\mathcal{C}}$ we have an inclusion of hom-sets:

$$\widetilde{\mathrm{Tr}}\big(\mathcal{D}^{\mathcal{C}}\big)\big(\Sigma,\Phi\big) \ \subset\ \widetilde{\mathrm{Tr}}(\mathcal{D})^{\mathcal{C}}\big(\Sigma,\Phi\big)$$

(each component of a trace arrow in $\mathcal{D}^{\mathcal{C}}$ is a trace arrow in $\mathcal{D}$). Thus, the full subcategory of $\widetilde{\mathrm{Tr}}\big(\mathcal{D}^{\mathcal{C}}\big)$ determined by the objects of $\widetilde{\mathrm{Tr}}(\mathcal{D})^{\mathcal{C}}$ is a subcategory of $\widetilde{\mathrm{Tr}}\big(\mathcal{D}\big)^{\mathcal{C}}$, this inclusion being bijective on objects.

Proposition 28. *If we regard each object of $\widetilde{\mathrm{Tr}}(\ d)^{\mathcal{C}}$ as an object of $\widetilde{\mathrm{Tr}}(\mathcal{D}^{\mathcal{C}})$ and denote $\mathcal{E}_{\mathcal{C}\mathcal{D}}$ the full subcategory of $\widetilde{\mathrm{Tr}}(\mathcal{D}^{\mathcal{C}})$ determined by the objects of $\widetilde{\mathrm{Tr}}(\mathcal{D})^{\mathcal{C}}$, we have a faithful functor*

$$\beta_{\mathcal{C}\mathcal{D}}\colon \mathcal{E}_{\mathcal{C}\mathcal{D}} \longrightarrow \widetilde{\mathrm{Tr}}(\mathcal{D})^{\mathcal{C}}$$

which is bijective on objects.

7.3. In the case that $\mathcal{C}$ is a groupoid, the above function (9) is a bijection and therefore $\mathcal{E}_{\mathcal{C}\mathcal{D}} = \widetilde{\mathrm{Tr}}(\mathcal{D}^{\mathcal{C}})$.

Proposition 29. *If $\mathcal{C}$ is a groupoid, the forgetful functor $\epsilon_{\mathcal{D}^{\mathcal{C}}}\colon \widetilde{\mathrm{Tr}}\big(\mathcal{D}^{\mathcal{C}}\big) \to \mathcal{D}^{\mathcal{C}}$ factors through the functor $\epsilon^*_{\mathcal{D}}\colon \widetilde{\mathrm{Tr}}(\mathcal{D})^{\mathcal{C}} \to \mathcal{D}^{\mathcal{C}}$ via a faithful functor which is bijective on objects:*

$$\begin{array}{ccc} & \widetilde{\mathrm{Tr}}(\mathcal{D})^{\mathcal{C}} & \\ {\scriptstyle \beta_{\mathcal{C}\mathcal{D}}}\nearrow^{\text{b.o.}} & & \searrow{\scriptstyle \epsilon^*_{\mathcal{D}}} \\ \widetilde{\mathrm{Tr}}\big(\mathcal{D}^{\mathcal{C}}\big) & \xrightarrow[\epsilon_{\mathcal{D}^{\mathcal{C}}}]{} & \mathcal{D}^{\mathcal{C}}. \end{array}$$

7.4. If $\mathcal{C}$ is a groupoid $\mathsf{Cen}(\mathcal{C})$ is an abelian group which has a canonical map to $\widetilde{\mathrm{Tr}}(\mathcal{C})^{\mathcal{C}}$:

$$\mathsf{Cen}(\mathcal{C}) \longrightarrow \mathrm{obj}\big(\widetilde{\mathrm{Tr}}(\mathcal{C}^{\mathcal{C}})\big).$$

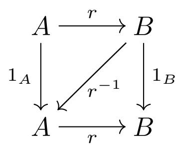
The neutral element of the abelian group, $\mu = \mathrm{id}_{1_{\mathcal{C}}}$, is mapped to the special functor

$$i_{\mathcal{C}} = (1_{\mathcal{C}}, \mathrm{id}_{1_{\mathcal{C}}})\colon \mathcal{C} \to \widetilde{\mathrm{Tr}}(\mathcal{C})$$

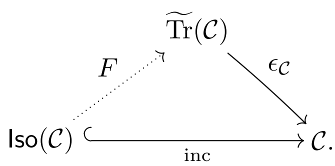
for which $\epsilon_{\mathcal{C}} i_{\mathcal{C}}$ is the identity of $\mathcal{C}$. For an object $A \in \mathcal{C}$, $i_{\mathcal{C}}(A) = A^{\circlearrowleft 1_A}$, and for an arrow r in $\mathcal{C}$, $i_{\mathcal{C}}(r) = r$. This functor $i_{\mathcal{C}}$, section of $\epsilon_{\mathcal{C}}$, is full and faithful.

7.5. If $\mathcal{C}$ is a groupoid, then the set of connected components of $\mathcal{C}$ is a retract of the set of traces of $\mathcal{C}$: $\pi_0(\mathcal{C}) \hookrightarrow \mathsf{Trc}(\mathcal{C})$, the retraction being the map induced by $\epsilon_{\mathcal{C}}$ on connected components. The idempotent endomap of the trace set $\mathsf{Trc}(\mathcal{C})$ determined by this section-retraction pair is the map assigning to each conjugacy class of $\mathcal{C}$ the corresponding conjugacy class of identities. If $\mathcal{C}$ is a group, this idempotent is the constant "conjugacy class of the identity".

7.6. According to Proposition 13, if $\mathcal{C}$ is any small category, for an arrow $r\colon A \to B$ in $\mathcal{C}$ to be an arrow in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ from 1_A to 1_B it is necessary and sufficient that it be an isomorphism:



This implies that there is a full and faithful functor, F, from the category of isomorphisms of $\mathcal{C}$, $\mathsf{Iso}(\mathcal{C})$, to $\widetilde{\mathrm{Tr}}(\mathcal{C})$ defined on objects as $F(A) = 1_A$ and on arrows as $F(r) = r$. Thus, the inclusion functor of $\mathsf{Iso}(\mathcal{C})$ into $\mathcal{C}$ factors through the forgetful functor $\epsilon_{\mathcal{C}}$:



Obviously, if $\mathcal{C}$ is a groupoid then F is a section of $\epsilon_{\mathcal{C}}$. This functor F is nothing but the composite of the functor $i_{\mathsf{Iso}(\mathcal{C})}\colon \mathsf{Iso}(\mathcal{C}) \to \widetilde{\mathrm{Tr}}\big(\mathsf{Iso}(\mathcal{C})\big)$ with the inclusion functor $\widetilde{\mathrm{Tr}}(\mathrm{inc})\colon \widetilde{\mathrm{Tr}}\big(\mathsf{Iso}(\mathcal{C})\big) \longrightarrow \widetilde{\mathrm{Tr}}(\mathcal{C})$

7.7. If $\Sigma\colon \mathcal{C} \to \widetilde{\mathrm{Tr}}(\mathcal{C})$ is a functor given by $\Sigma = (1_{\mathcal{C}}, \mu)$ and $r\colon A \to B$ is any arrow in $\mathcal{C}$, then $r(= \Sigma(r))$ is a trace arrow from μ_A to μ_B and therefore $\mathrm{tr}_A(\mu_A) = \mathrm{tr}_B(\mu_B)$. It follows that if A and B are in the same connected component of $\mathcal{C}$, so that there is a path $A \xrightarrow{r_1} E_1 \xleftarrow{r_2} E_2 \to \cdots \leftarrow B$, then $\mathrm{tr}_A(\mu_A) = \mathrm{tr}_B(\mu_B)$. Thus, any two components of μ, e.g. μ_A and μ_B, have the same trace if and only if A and B are in the same connected component of $\mathcal{C}$. This implies that the function $\mathrm{tr}(\mu)\colon \mathrm{obj}(\mathcal{C}) \to \mathsf{Trc}(\mathcal{C})$ defined by $\mathrm{tr}(\mu)(A) = \mathrm{tr}_A(\mu_A)$ is defined on the set of connected components of $\mathcal{C}$ so that we have a map:

$$\mathrm{tr}(\mu)\colon \pi_0(\mathcal{C}) \to \mathsf{Trc}(\mathcal{C}).$$

It is clear that this map is precisely the map obtained by applying π_0 to the functor $\Sigma\colon \mathcal{C} \to \widetilde{\mathrm{Tr}}(\mathcal{C})$:

Proposition 30. *Any functor $\Sigma \in \widetilde{\mathrm{Tr}}(\mathcal{C})^{\mathcal{C}}$ induces a map $\pi_0(\Sigma)\colon \pi_0(\mathcal{C}) \to \mathsf{Trc}(\mathcal{C})$. If $\Sigma = (S, \sigma)$ and we put $\mathrm{tr}_\Sigma = \pi_0(\Sigma)$ then this map is given by:*

$$\mathrm{tr}_\Sigma([A]) = \mathrm{tr}_{S(A)}(\sigma_A),$$

where the square bracket means "connected component".

8. Coalgebras for $\widetilde{\mathrm{Tr}}$

8.1. A coalgebra for $\widetilde{\mathrm{Tr}}$ is a small category $\mathcal{C}$ together with a structure functor $\Sigma\colon \mathcal{C} \to \widetilde{\mathrm{Tr}}(\mathcal{C})$ satisfying the identity and associative laws:

$$\begin{array}{ccc} \mathcal{C} & & \\ {\scriptstyle\Sigma}\downarrow & \searrow^{1_{\mathcal{C}}} & \\ \widetilde{\mathrm{Tr}}(\mathcal{C}) & \xrightarrow[\epsilon_{\mathcal{C}}]{} & \mathcal{C}, \end{array} \qquad \begin{array}{ccc} \mathcal{C} & \xrightarrow{\Sigma} & \widetilde{\mathrm{Tr}}(\mathcal{C}) \\ {\scriptstyle\Sigma}\downarrow & & \downarrow{\scriptstyle\delta_{\mathcal{C}}} \\ \widetilde{\mathrm{Tr}}(\mathcal{C}) & \xrightarrow[\widetilde{\mathrm{Tr}}(\Sigma)]{} & \widetilde{\mathrm{Tr}}\big(\widetilde{\mathrm{Tr}}(\mathcal{C})\big). \end{array}$$

As we saw before, Σ is essentially an object $S^{\circlearrowleft\sigma} \in \widetilde{\mathrm{Tr}}\big(\mathcal{C}^{\mathcal{C}}\big)$. The identity law says that S is the identity of $\mathcal{C}$, so that we are left with σ, an endo natural transformation of the identity of $\mathcal{C}$ (that is, an element of the center of $\mathcal{C}$). The associative law in this case follows from the identity law and therefore does not impose any further condition on σ.

Proposition 31. *A $\widetilde{\mathrm{Tr}}$-coalgebra is a pair $(\mathcal{C}, \mu)$ where $\mathcal{C}$ is a small category and μ is an element of the center of $\mathcal{C}$ such that every arrow $r\colon A \to B$ in $\mathcal{C}$ is a trace arrow from μ_A to μ_B. If $(\mathcal{C}, \mu)$ and $(\mathcal{D}, \nu)$ are $\widetilde{\mathrm{Tr}}$-coalgebras, a $\widetilde{\mathrm{Tr}}$-coalgebra homomorphism from $(\mathcal{C}, \mu)$ to $(\mathcal{D}, \nu)$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ that preserves the chosen element in the center: $F(\mu_A) = \nu_{FA}$.*

In view of Proposition 13 we have:

Proposition 32. *A small category $\mathcal{C}$ is a groupoid, if and only if $(\mathcal{C}, \mathrm{id}_{1_\mathcal{C}})$ is a $\widetilde{\mathrm{Tr}}$-coalgebra.*

Proof. The previous proposition implies that for a groupoid $\mathcal{C}$, $(\mathcal{C}, \mathrm{id}_{1_\mathcal{C}})$ is a $\widetilde{\mathrm{Tr}}$-coalgebra. Conversely: if $(\mathcal{C}, \mathrm{id}_{1_\mathcal{C}})$ is a $\widetilde{\mathrm{Tr}}$-coalgebra then every arrow $r : A \to B$ in $\mathcal{C}$ is a trace arrow from 1_A to 1_B and, by Proposition 13 this implies that r is an isomorphism. Thus $\mathcal{C}$ is a groupoid. □

Proposition 33. *If $\mathcal{C}$ is a groupoid, every element of the center of $\mathcal{C}$ determines a $\widetilde{\mathrm{Tr}}$-coalgebra structure on $\mathcal{C}$ (and vice-versa), so that the set of $\widetilde{\mathrm{Tr}}$-coalgebra structures on $\mathcal{C}$ is $\mathsf{Cen}(\mathcal{C})$.*

9. Algebraic Structures

9.1. In this section we are concerned with the question of what extra structure may the trace set of a category have (beyond being a set) on account of the special structure $\mathcal{C}$ may have. In other words, does some of the structure of $\mathcal{C}$ get inherited by $\mathsf{Trc}(\mathcal{C})$? We have an example of this in Proposition 3, which implies that if $\mathcal{C}$ is a commutative monoid so is $\mathsf{Trc}(\mathcal{C})$.

9.2. Suppose $\mathcal{C}$ has a monoidal structure. Does this induce some sort of algebraic structure on its trace, $\mathsf{Trc}(\mathcal{C})$? In order to focus on this question let's suppose that we have a binary operation defined on $\mathcal{C}$, that is, a functor $\mathcal{C} \times \mathcal{C} \xrightarrow{*} \mathcal{C}$. Under what conditions does this induce a functor $\mathsf{Trc}(\mathcal{C}) \times \mathsf{Trc}(\mathcal{C}) \to \mathsf{Trc}(\mathcal{C})$? One obvious way in which this can occur is if $*$ induces a binary operation on $\widetilde{\mathrm{Tr}}(\mathcal{C})$. In that case we would automatically get an operation on $\mathsf{Trc}(\mathcal{C})$ because the connected component functor, π_0, is product-preserving and therefore any bifunctor induces a bifunctor on connected components.

So, the question is whether $*$ induces a functor

$$\widetilde{\mathrm{Tr}}(\mathcal{C}) \times \widetilde{\mathrm{Tr}}(\mathcal{C}) \xrightarrow{\tilde{*}} \widetilde{\mathrm{Tr}}(\mathcal{C}).$$

9.3. If one tries to define this by letting $*$ act componentwise, one finds that there is no problem at the level of objects: $\big(A^{\circlearrowright u}\big)\tilde{*}\big(B^{\circlearrowright v}\big) = (A * B)^{\circlearrowright u*v}$; but if we try to do the same at the level of arrows then for any two given arrows $r : A^{\circlearrowright u} \to A'^{\circlearrowright u'}$ and $s \colon B^{\circlearrowright v} \to B'^{\circlearrowright v'}$ in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ the question arises of whether the natural candidate, namely the map $r * s \colon A * B \to A' * B'$, is actually a map in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ from $(A * B)^{\circlearrowright u*v}$ to $(A' * B')^{\circlearrowright u'*v'}$.

The answer to this question is positive and the proof of it not very complicated. It is easy to see that $r * 1_B$ is a map in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ from $(A * B)^{\circlearrowright u*v}$ to $(A' * B)^{\circlearrowright u'*v}$. Similarly $1_{A'} * s$ is a map in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ from $(A' * B)^{\circlearrowright u'*v}$ to $(A' * B')^{\circlearrowright u'*v'}$. Therefore we do have a composite map from $(A * B)^{\circlearrowright u*v}$ to $(A' * B')^{\circlearrowright u'*v'}$. Functoriality of $*$ implies the middle-two interchange law, from which it follows that this composite map is precisely $r * s$:

$$(1_{A'} * s) \circ (r * 1_B) = (1_{A'} \circ r) * (s \circ 1_B) = r * s.$$

9.4. It is clear that the above reasoning goes through for any operation of any finite product theory and that the induced operations would satisfy the same axioms as long as they are equational axioms. We can therefore state:

Conjecture 34. *For any Lawvere theory T if $\mathcal{C}$ is a small category with a structure of T-algebra in* **Cat***, then $\mathsf{Trc}(\mathcal{C})$ is a T-algebra in* **Set**.

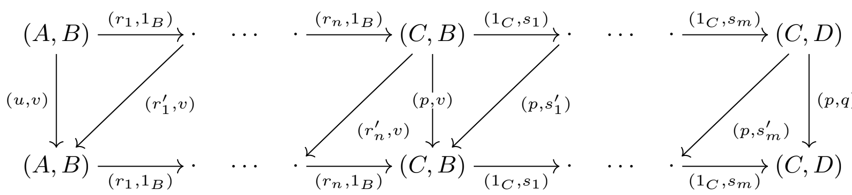
This immediately follows from the following:

Theorem 35. *The trace functor* $\mathsf{Trc}\colon \mathbf{Cat} \to \mathbf{Set}$ *preserves finite products.*

Proof. Since π_0 preserves finite products and $\mathsf{Trc} = \pi_0 \circ \widetilde{\mathrm{Tr}}$ (see Proposition 25), it is sufficient to prove that $\widetilde{\mathrm{Tr}}$ preserves finite products. It is a simple exercise to check that $\widetilde{\mathrm{Tr}}$ preserves the terminal category $\mathbf{1}$ (see Proposition 4). It only remains to prove that $\widetilde{\mathrm{Tr}}$ preserves binary products, which amounts to prove that, for any small categories $\mathcal{A}$, $\mathcal{B}$, the canonical functor

$$\widetilde{\mathrm{Tr}}(\mathcal{A} \times \mathcal{B}) \longrightarrow \widetilde{\mathrm{Tr}}(\mathcal{A}) \times \widetilde{\mathrm{Tr}}(\mathcal{B})$$

has an inverse. Given an arrow (r, s) in $\widetilde{\mathrm{Tr}}(\mathcal{A}) \times \widetilde{\mathrm{Tr}}(\mathcal{B})$ from $\big(A^{\circlearrowleft u}, B^{\circlearrowleft v}\big)$ to $\big(C^{\circlearrowleft p}, D^{\circlearrowleft q}\big)$, we need to prove that (r, s) is an arrow in $\widetilde{\mathrm{Tr}}(\mathcal{A} \times \mathcal{B})$ from $(A, B)^{\circlearrowleft (u,v)}$ to $(C, D)^{\circlearrowleft (p,q)}$. By hypothesis r is an arrow in $\widetilde{\mathrm{Tr}}(\mathcal{A})$ and s is an arrow in $\widetilde{\mathrm{Tr}}(\mathcal{B})$. Let $(r_1, r'_1), \ldots, (r_n, r'_n)$ be a proof of the former and let $(s_1, s'_1), \ldots, (s_m, s'_m)$ be a proof of the latter. Doing a trick similar to the one used in Paragraph 9.3 we can obtain a commutative diagram in $\mathcal{A} \times \mathcal{B}$:

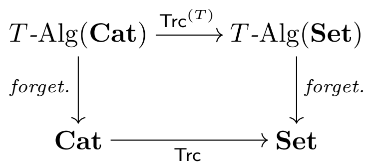


where

$$(1_C, s_m) \circ \cdots \circ (1_C, s_1) \circ (r_n, 1_B) \circ \cdots \circ (r_1, 1_B) = (r_n \cdots r_1, s_m \cdots s_1) = (r, s)$$

and therefore this proves that (r, s) is an arrow in $\widetilde{\mathrm{Tr}}(\mathcal{A} \times \mathcal{B})$. □

Corollary 36. *For any Lawvere theory T the trace functor* $\mathsf{Trc}\colon \mathbf{Cat} \to \mathbf{Set}$ *induces a functor* $\mathsf{Trc}^{(T)}\colon T\text{-Alg}(\mathbf{Cat}) \to T\text{-Alg}(\mathbf{Set})$ *and we have a commutative square:*



Corollary 37. *If $\mathcal{C}$ is a small monoidal category then* $\mathsf{Trc}(\mathcal{C})$ *is a monoid and so is* $\mathsf{Trc}\big(\mathcal{C}^{\mathcal{D}}\big)$ *for any small category $\mathcal{D}$.*

9.5. For any small category $\mathcal{C}$, the functor category $\mathcal{C}^{\mathcal{C}}$ is an internal monoid in $\mathbf{Cat}$. Therefore, we have:

Proposition 38. *For any small category $\mathcal{C}$,* $\mathsf{Trc}(\mathcal{C}^{\mathcal{C}})$ *is a monoid.*

10. Trace Sets of General Endoprofunctors

10.1. The construction of the category $\widetilde{\mathrm{Tr}}(\mathcal{C})$ can be generalized for any endoprofunctor $\varphi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ to obtain a category $\widetilde{\mathrm{Tr}}(\varphi)$ such that

$$\mathsf{Trc}(\varphi) = \pi_0\big(\widetilde{\mathrm{Tr}}(\varphi)\big).$$

The definition of $\widetilde{\mathrm{Tr}}(\varphi)$ is as follows:

(1) The objects of $\widetilde{\mathrm{Tr}}(\varphi)$ are the pairs (A, u) where A is an object of $\mathcal{C}$ and u is an element $u\colon 1 \to \varphi(A, A)$.
(2) An arrow from (A, u) to (B, v) in $\widetilde{\mathrm{Tr}}(\varphi)$ is an arrow $r\colon A \to B$ in $\mathcal{C}$ such that there is a finite chain

$$A \xrightarrow{h_1} E_1 \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} E_{n-1} \xrightarrow{h_n} B$$

in $\mathcal{C}$ such that $h_n \cdots h_1 = r$ and there is a cone in $\mathbf{Set}$ from 1 to the diagram

$$\varphi(A,A) \xleftarrow{h_{1*}} \varphi(E_1,A) \xrightarrow{h_1{}^*} \varphi(E_1,E_1) \xleftarrow{h_{2*}} \cdots \xrightarrow{h_n{}^*} \varphi(B,B) \tag{10}$$

(where for any arrow h in $\mathcal{C}$, $h_* = \varphi(h, 1)$ and $h^* = \varphi(1, h)$) such that the map in this cone from 1 to $\varphi(A, A)$ is u and the map $1 \to \varphi(B, B)$ is v.
(3) Identities and composition are those of $\mathcal{C}$.

With this definition it is immediate to show that:

Proposition 39. *Identities in $\mathcal{C}$ are arrows in $\widetilde{\mathrm{Tr}}(\varphi)$ and the composite in $\mathcal{C}$ of two composable arrows in $\widetilde{\mathrm{Tr}}(\varphi)$ is an arrow in $\widetilde{\mathrm{Tr}}(\varphi)$. Hence the above definition gives us a category $\widetilde{\mathrm{Tr}}(\varphi)$.*

Theorem 40. *For any endoprofunctor $\varphi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ the trace set of φ is the set of connected components of the category $\widetilde{\mathrm{Tr}}(\varphi)$:*

$$\mathsf{Trc}(\varphi) = \pi_0\big(\widetilde{\mathrm{Tr}}(\varphi)\big).$$

Proof. This is essentially the same as the proof of Proposition 2. □

Proposition 41. *For any endoprofunctor $\varphi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ there is a forgetful functor*

$$\epsilon_\varphi\colon \widetilde{\mathrm{Tr}}(\varphi) \to \mathcal{C}.$$

10.2. If φ is and endoprofunctor of a groupoid $\mathcal{C}$, for any given arrow $r\colon (A, u) \to (B, v)$ in $\widetilde{\mathrm{Tr}}(\varphi)$ each of the maps in diagram (10) is a bijection because for any map h in $\mathcal{C}$, $(h^{-1})_* = (h_*)^{-1}$ and $(h^{-1})^* = (h^*)^{-1}$. It follows that the cone over (10) is unique if it exists and it exists if and only if $v = h_n^*(h_{n*})^{-1} \cdots h_1^*(h_{1*})^{-1}\, u$, that is, if and only if $v = \varphi(r^{-1}, r)(u)$.

Definition 2. *If φ is and endoprofunctor of a category $\mathcal{C}$, we will say that an element $u \in \varphi(A, A)$ is a* φ-conjugate *(or simply a conjugate if φ is understood) of an element $v \in \varphi(B, B)$, if an invertible map $r\colon A \to B$ exists in $\mathcal{C}$ such that $\varphi(r^{-1}, r)(u) = v$. (Note that this particularizes to the usual concept when φ is the hom-set functor of a groupoid.) Each map $r\colon A \to B$ in $\mathcal{C}$ induces a map*

$$\varphi(r^{-1}, r)\colon \varphi(A, A) \to \varphi(B, B)$$

which will be called φ-conjugation by r*.*

It is clear that for any isomorphism r, φ-conjugation by r is a bijective map whose inverse is φ-conjugation by r^{-1}. A result similar to Proposition 22 holds:

Proposition 42. *If φ is an endoprofunctor of a groupoid $\mathcal{C}$, then:*

(1) *An arrow $r\colon (A, u) \to (B, v)$ in $\widetilde{\mathrm{Tr}}(\varphi)$ is just an arrow $r\colon A \to B$ such that $r^*(v) = r_*(u)$, that is, such that the following diagram is commutative:*

$$\begin{array}{ccc} 1 & \xrightarrow{\;u\;} & \varphi(A, A) \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle r_* = \varphi(1, r)} \\ \varphi(B, B) & \xrightarrow[\;r^*\;]{} & \varphi(A, B) \end{array}$$

(2) $\widetilde{\mathrm{Tr}}(\varphi)$ *is a groupoid.*
(3) *Two elements $u \in \varphi(A, A)$ and $v \in \varphi(B, B)$ have the same trace if and only if they are φ-conjugate of each other: $v = \varphi(h^{-1}, h)(u)$.*
(4) *The trace of φ is the set of φ-conjugacy classes of $\mathcal{C}$.*

10.3. Can the construction of our old functor $\widetilde{\mathrm{Tr}}\colon \mathbf{Cat} \to \mathbf{Cat}$ be extended to a functor such that to each endoprofunctor $\varphi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ it assigns the category $\widetilde{\mathrm{Tr}}(\varphi)$? We need to clarify what the domain of such a functor would be. The objects of the domain should be all the endoprofunctors of categories, that is, pairs $(\mathcal{C}, \varphi)$. What would an arrow $(\mathcal{C}, \varphi) \to (\mathcal{C}', \varphi')$ be?

10.4. It is a known general fact that given profunctors $\varphi\colon \mathcal{C} \to \mathcal{D}$ and $\psi\colon \mathcal{D} \to \mathcal{C}$, the following "generalized trace property" holds:

$$\mathsf{Trc}_{\mathcal{C}}(\psi\varphi) = \mathsf{Trc}_{\mathcal{D}}(\varphi\psi).$$

this establishes a suggestive analogy: $\mathsf{Trc}_{\mathcal{C}}$ is to the category of small categories and profunctors as tr_A (for A an object in $\mathcal{C}$) is to the category $\mathcal{C}$. The analogy can be stretched: We have defined a category $\widetilde{\mathrm{Tr}}(\mathcal{C})$ whose objects are endomorphisms in $\mathcal{C}$. We now want to define a category whose objects are endomorphisms in the category of small categories and profunctors. The arrows in $\widetilde{\mathrm{Tr}}(\mathcal{C})$ are arrows in $\mathcal{C}$ with a special property... Could the appropriate arrows we are looking for be profunctors with a special property?

In order to answer this question, let's consider this other one: Given endoprofunctors $\varphi\colon \mathcal{C} \to \mathcal{C}$ and $\psi\colon \mathcal{D} \to \mathcal{D}$, what condition must a functor $F\colon \mathcal{C} \to \mathcal{D}$ satisfy so that it induces a functor from $\widetilde{\mathrm{Tr}}(\varphi)$ to $\widetilde{\mathrm{Tr}}(\psi)$?

10.5. Clearly, a functor $F\colon \mathcal{C} \to \mathcal{D}$ would induce an object function $(A, u) \mapsto (F(A), u')$ from $\mathrm{obj}\big(\widetilde{\mathrm{Tr}}(\varphi)\big)$ to $\mathrm{obj}\big(\widetilde{\mathrm{Tr}}(\psi)\big)$ if we had a map $\lambda_A\colon \varphi(A, A) \to \psi\big(F(A), F(A)\big)$ so that we can put $u' = \lambda_A(u)$. But this would not be sufficient for an arrows function. It would be necessary to have, for each pair (A, B) of objects of $\mathcal{C}$, a map $\lambda_{A,B}\colon \varphi(A, B) \to \psi\big(F(A), F(B)\big)$ which is natural in (A, B). In short: we need a natural transformation from φ to $\psi \circ F \times F$.

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{\quad\varphi\quad} & \mathbf{Set}. \\ & {\scriptstyle F \times F} \searrow \quad \lambda \Downarrow \quad \nearrow {\scriptstyle \psi} & \\ & \mathcal{D}^{\mathrm{op}} \times \mathcal{D} & \end{array}$$

It is clear that taking as objects the endoprofunctors of small categories and as arrows from $(\mathcal{C}, \varphi)$ to $(\mathcal{D}, \psi)$ the pairs (F, λ) where F is a functor $F\colon \mathcal{C} \to \mathcal{D}$ and λ is a natural transformation $\lambda\colon \varphi \to \psi \circ (F \times F)$ we get a category. We shall denote $\boldsymbol{\mathcal{T}}$ the category so defined.

Proposition 43. *There is a functor* $\widetilde{\mathrm{Tr}}'\colon \boldsymbol{\mathcal{T}} \to \mathbf{Cat}$ *taking* $(\mathcal{C}, \varphi)$ *to* $\widetilde{\mathrm{Tr}}(\varphi)$ *and there is an inclusion* $\mathbf{Cat} \hookrightarrow \boldsymbol{\mathcal{T}}$ *taking* $\mathcal{C}$ *to* $(\mathcal{C}, 1_{\mathcal{C}})$ *such that the following diagram of functors is commutative:*

$$\begin{array}{ccc} & \boldsymbol{\mathcal{T}} & \\ {\scriptstyle \mathrm{inc}} \nearrow & & \searrow {\scriptstyle \widetilde{\mathrm{Tr}}'} \\ \mathbf{Cat} & \xrightarrow[\widetilde{\mathrm{Tr}}]{\qquad\qquad} & \mathbf{Cat}. \end{array}$$

10.6. There is an obvious forgetful functor $U\colon \boldsymbol{\mathcal{T}} \longrightarrow \mathbf{Cat}$ and a natural transformation $\epsilon\colon \widetilde{\mathrm{Tr}}' \to U$ whose component at an object $(\mathcal{C}, \varphi)$ of $\boldsymbol{\mathcal{T}}$ is the forgetful functor $\epsilon_\varphi\colon \widetilde{\mathrm{Tr}}(\varphi) \to \mathcal{C}$ of Proposition 41. Furthermore, the composite

$$\mathbf{Cat} \overset{\mathrm{inc}}{\hookrightarrow} \boldsymbol{\mathcal{T}} \underset{U}{\overset{\widetilde{\mathrm{Tr}}'}{\rightrightarrows}} \mathbf{Cat} \qquad (\epsilon \Downarrow)$$

is the counit of the comonad structure of $\widetilde{\mathrm{Tr}}$ seen in Proposition 26.

11. **Set**-Trace of Endofunctors

11.1. An intermediate situation between the case of the trace of (the identity functor of) a category and that of a general endoprofunctor is the case of the trace of an endofunctor of a small category. Given an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$ of a small category $\mathcal{C}$ there are two ways to regard it as a profunctor, that is, it determines two endoprofunctors of $\mathcal{C}$, namely:

$$\varphi^R(A,B) = \mathcal{C}(A, F(B))\,, \quad \text{and} \quad \varphi^L(A,B) = \mathcal{C}(F(A), B). \tag{11}$$

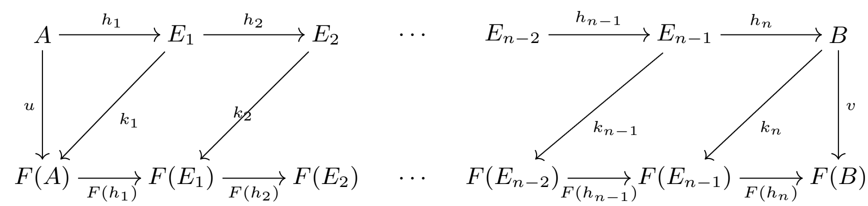
Let us consider the first one. The general construction of the pretrace category $\widetilde{\mathrm{Tr}}^R(F) = \widetilde{\mathrm{Tr}}(\varphi^R)$ is particularized in this case to the following construction: The objects of $\widetilde{\mathrm{Tr}}^R(F)$ are pairs (A,u) where A is an object of $\mathcal{C}$ and u is a map $A \xrightarrow{u} F(A)$. A $\mathcal{C}$-map $r\colon A \to B$ is a (right) F-trace arrow from u to v if and only if there is a finite chain

$$A = E_0 \xrightarrow{h_1} E_1 \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} E_{n-1} \xrightarrow{h_n} E_n = B$$

in $\mathcal{C}$ such that $h_n \cdots h_1 = r$ and for each $i = 1, \ldots, n$ there is a map $k_i\colon E_i \to F(E_{i-1})$ verifying

$$k_1 h_1 = u,\ F(h_n)k_n = v, \quad \text{and} \quad h_{i-1}k_{i-1} = k_i h_i, \quad \text{for} \quad 2 \le i \le n, \tag{12}$$

or, equivalently, such that the following diagram is commutative:



11.2. Any natural transformation $\mu\colon F \to G$ between two endofunctors of $\mathcal{C}$ induces a functor

$$\tilde{\mu}\colon \widetilde{\mathrm{Tr}}^R(F) \to \widetilde{\mathrm{Tr}}^R(G) \tag{13}$$

taking an object $(A,u) \in \widetilde{\mathrm{Tr}}^R(F)$ with $u\colon A \to F(A)$, to the object $(A, \mu_A u)$ of $\widetilde{\mathrm{Tr}}^R(G)$. It is a simple exercise to show that if $r\colon A \to B$ is a (right) F-trace arrow from u to v then r is also a (right) G-trace arrow from $\mu_A u$ to $\mu_B v$ so that $\tilde{\mu}$ is indeed a functor.

11.3. If the endofunctor F of $\mathcal{C}$ has a comonad structure, let's say with counit $\epsilon\colon F \to 1_{\mathcal{C}}$ and comultiplication $\delta\colon F \to F^2$, then every F-coalgebra determines an object in $\widetilde{\mathrm{Tr}}^R(F)$ and therefore it makes sense to ask what is the trace of an F-coalgebra. To investigate this question, let's suppose that A and B are F-coalgebras with respective structure maps u, v, and that $r\colon A \to B$ is a 1-step F-trace arrow from u to v, that is

$$\begin{array}{ccc} A & \xrightarrow{\;r\;} & B \\ {\scriptstyle u}\Big\downarrow & \swarrow{\scriptstyle k} & \Big\downarrow{\scriptstyle v} \\ F(A) & \xrightarrow[F(r)]{} & F(B). \end{array}$$

We see that r is a coalgebra morphism, but even more: composing k with the counit at A we get a map $s = \epsilon_A k\colon B \to A$ which:

(1) is inverse to r ($sr = \epsilon_A kr = \epsilon_A u = 1_A$ and $rs = r\epsilon_A k = \epsilon_B F(r)k = \epsilon_B v = 1_B$);
(2) It determines k (since $us = krs = k$); and
(3) It is a coalgebra morphism ($F(s)v = F(s)F(r)k = F(sr)k = k = us$).

The situation is similar to that of Proposition 13. In fact, a very similar proof gives us the following:

Lemma 44. *Let F be an endofunctor of a small category $\mathcal{C}$ and $\mu\colon F \to 1_{\mathcal{C}}$ a natural transformation. If $u\colon A \to F(A)$ and $v\colon B \to F(B)$ are such that $\mu_A u = 1_A$ and $\mu_B v = 1_B$ then any (right) F-trace arrow from u to v is an isomorphism from A to B.*

Alternatively, this can be proved by applying the induced functor $\tilde{\mu}$ (13) to the F-trace arrow to get a trace arrow in $\mathcal{C}$ between two identities and then applying Proposition 13.

It is now immediate to deduce from Lemma 44 that any two coalgebras with the same trace are isomorphic. A simple clculation shows that isomorphic coalgebras have the same trace (if $r\colon (A,u) \to (B,v)$ is an iso, $u = (ur^{-1})r$ and $F(r)(ur^{-1}) = (F(r)u)r^{-1} = (vr)r^{-1} = v$). Thus, we have:

Proposition 45. *Given a comonad $\langle F, \epsilon, \delta\rangle$ on a small category $\mathcal{C}$, the following hold:*

(1) *Every right trace arrow between F-coalgebras has length 1 and it is an isomorphism.*
(2) *Two F-coalgebras have the same right F-trace if and only if they are isomorphic.*

(3) *The full subcategory of* $\widetilde{\mathrm{Tr}}^R(F)$ *determined by all the* F*-coalgebras is the groupoid* $\mathsf{Iso}(F\text{-}\mathsf{CoAlg})$ *of isomorphisms of* F*-coalgebras.*

By duality (and using the second induced profunctor in (11)) we also have:

Proposition 46. *Given a monad on a small category* $\mathcal{C}$*, the following hold:*

(1) *Every left trace arrow between* F*-algebras has length 1 and it is an isomorphism.*
(2) *Two* F*-algebras have the same left* F*-trace if and only if they are isomorphic.*
(3) *The full subcategory of* $\widetilde{\mathrm{Tr}}^L(F)$ *determined by all the* F*-algebras is the groupoid* $\mathsf{Iso}(F\text{-}\mathsf{Alg})$ *of isomorphisms of* F*-algebras.*

This result is in part a generalization of a known theorem in Representation Theory.

11.4. Observe that the multiplication/comultiplication played no role in Propositions 45 and 46. Hence they apply more generally than in the case of monads/comonads. In general, for any natural transformation $\mu\colon 1_{\mathcal{C}} \to F$ we can define a category of "μ-retractions" whose objects are pairs (A,u) such that $u\mu_A = 1_A$ and arrows $(A,u) \to (B,v)$ are those $A \xrightarrow{r} B$ such that $ru = vF(r)$. Then, two μ-retractions are isomorphic if and only if they have the same left F-trace.

11.5. Suppose now that we have two functors $F, U\colon \mathcal{C} \to \mathcal{C}$ and that F is left adjoint to U, $F \dashv U$. Then it is clear that:

$$\mathsf{Trc}^L(F) = \int^A \mathcal{C}\big(F(A),A\big) = \int^A \mathcal{C}\big(A,U(A)\big) = \mathsf{Trc}^R(U).$$

In fact, in this situation there is an isomorphism of categories between the pretrace categories $\widetilde{\mathrm{Tr}}^L(F)$ and $\widetilde{\mathrm{Tr}}^R(U)$:

Proposition 47. *In the situation described above, let* $\varphi_{AB}\colon \mathcal{C}\big(F(A),B\big) \longrightarrow \mathcal{C}\big(A,U(B)\big)$ *be tha canonical isomorphisms of the adjunction* $F \dashv U$*. Then* φ *induces an isomorphism of categories,*

$$\hat{\varphi}\colon \widetilde{\mathrm{Tr}}^L(F) \to \widetilde{\mathrm{Tr}}^R(U)$$

which takes each object $(A,u) \in \widetilde{\mathrm{Tr}}^L(F)$ *to* $\hat{\varphi}(A,u) = \big(A, \varphi_{AA}(u)\big)$ *and each left trace arrow* $r\colon (A,u) \to (B,v)$ *to* $\hat{\varphi}(r) = r$*.*

Proof. If $(h_1,k_1),\ldots,(h_n,k_n)$ is a proof that r is a left trace arrow from (A,u) to (B,v) then $(h_1,\varphi(k_1)),\ldots,(h_n,\varphi(k_n))$ is a proof that r is a right trace arrow from $(A,\varphi_{AA}(u))$ to $(B,\varphi_{BB}(v))$. □

12. An Example from Representation Theory

12.1. Let k be a field and G a finite group, and let's consider the endofunctor of k-$\mathbf{Vect}_{\text{f.d.}}$ $F =$ "tensoring with the group algebra $k[G]$":

$$F\colon k\text{-}\mathbf{Vect}_{\text{f.d.}} \longrightarrow k\text{-}\mathbf{Vect}_{\text{f.d.}}, \quad F(V) = k[G] \otimes V. \tag{14}$$

This functor is a monad whose algebras are the finite dimensional k-linear representations of G. Since

$$\hom\big(k[G] \otimes V, V\big) \simeq \hom\big(k[G], \hom(V, V)\big), \tag{15}$$

an F-algebra (that is, a finite dimensional k-linear representation of G) can be regarded as a special k-linear map $\rho\colon k[G] \to \hom(V, V)$, "special" meaning "$k$-algebra homomorphism." Note that, since the elements of the finite group G form a basis of the group algebra $k[G]$, giving a k-linear map $k[G] \to \hom(V, V)$ is equivalent to giving a set map $G \to \hom(V, V)$. In this section, unless indicated otherwise, we shall regard any linear representation V_ρ of the group G on V as a k-linear map $\rho\colon k[G] \to \hom(V, V)$ or even as a set map $G \to \hom(V, V)$.

12.2. The character of a linear representation V_ρ of the group G is the class function $\mathrm{tr}_V \circ \rho$ where ρ is regarded as a group homomorphism $G \to \mathrm{Aut}(V)$. A classic theorem in Representation Theory asserts that two finite dimensional k-linear representations of a finite group G have the same character if and only if they are isomorphic. In view of Proposition 46 this suggests that the character of a representation may be its trace from the point of view of that proposition. Thus, it is reasonable to conjecture that the trace of "the representations monad functor" (14), $\mathsf{Trc}(F)$, contains all characters of G-representations and that the canonical F-trace maps $\mathrm{tr}_V^F\colon \hom(F(V), V) \to \mathsf{Trc}(F)$ assign to a linear map $\rho\colon F(V) \to V$ that happens to be an F-algebra (G-representation), the character of V_ρ. So, we make the following conjecture:

Conjecture 48. *If k is a field, G a finite group, and F the representations monad functor* (14)*, then the characters of the finite dimensional k-linear representations of G are the linear F-traces of these representations regarded as F-algebras.*

Proposition 49. *If G is a finite group and k is a field then the (set) trace, in the sense of the previous section, of the representations monad functor* (14)*, $F = k[G] \otimes (-)$, is the set $\mathbf{Set}(G, T^S)$ of maps from G to the set trace of finite dimensional vector spaces; that is,*

$$\mathsf{Trc}(k[G] \otimes (-)) = \mathbf{Set}(G, T^S) \quad \textit{or} \quad \int^V \mathbf{Set}\big(G, \hom(V, V)\big) = \mathbf{Set}(G, T^S).$$

For any finite dimensional k-linear space V, the corresponding trace function $\operatorname{tr}^{\mathrm{set}}_{FV}$ assigns to a linear map $k[G] \otimes V \to V$ —regarded as a map $\rho\colon G \to \hom(V,V)$—, the function $\operatorname{tr}^{\mathrm{set}}_V \circ \rho$, that is,

(16)

$$\operatorname{tr}^{\mathrm{set}}_{FV} = \big(\operatorname{tr}^{\mathrm{set}}_V\big)^G \colon \mathbf{Set}\big(G, \hom(V,V)\big) \to \mathbf{Set}(G, T^S)\,, \quad \operatorname{tr}^{\mathrm{set}}_{FV}(\rho) = \operatorname{tr}^{\mathrm{set}}_V \circ \rho.$$

Proof. Given any linear map $f\colon W \to V$ we can apply the functor $\mathbf{Set}\big(G, -\big)$ to the following commutative square:

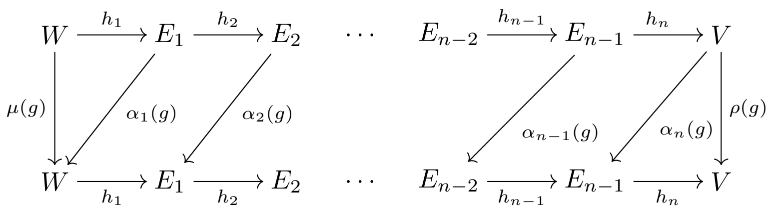
$$\begin{array}{ccc} \hom(V,W) & \xrightarrow{\hom(1_V,f)} & \hom(V,V) \\ {\scriptstyle \hom(f,1_W)}\downarrow & & \downarrow{\scriptstyle \operatorname{tr}^{\mathrm{set}}_V} \\ \hom(W,W) & \xrightarrow[\operatorname{tr}^{\mathrm{set}}_W]{} & T^S \end{array}$$

to obtain a commutative square:

$$\begin{array}{ccc} \hom\big(k[G], \hom(V,W)\big) & \xrightarrow{\hom(1_V,f)^*} & \hom\big(k[G], \hom(V,V)\big) \\ {\scriptstyle \hom(f,1_W)^*}\downarrow & & \downarrow{\scriptstyle \operatorname{tr}^{\mathrm{set}}_{FV}} \\ \hom\big(k[G], \hom(W,W)\big) & \xrightarrow[\operatorname{tr}^{\mathrm{set}}_{FW}]{} & \mathbf{Set}(G, T^S) \end{array}$$

where for each linear map $\rho\colon k[G] \to \hom(V,V)$ we have $\operatorname{tr}^{\mathrm{set}}_{FV}(\rho) = \operatorname{tr}^{\mathrm{set}}_V \circ \rho$ (and similarly for $\operatorname{tr}^{\mathrm{set}}_{FW}$). This shows that (16) indeed defines a compatible family. Let us now show that it has the required universal property. If T is a set and $\{\tau_V\colon \hom\big(k[G], \hom(V,V)\big) \to T\}_V$ is a compatible family of maps, we must show that there exists a unique map $L\colon \mathbf{Set}(G, T^S) \to T$ such that for every map $\chi \in \mathbf{Set}(G, T^S)$, if $\chi = \operatorname{tr}^{\mathrm{set}}_V \circ \rho$ then $L(\chi) = \tau_V(\rho)$. Obviously, if such a map exists it is unique since it is determined by the condition it must satisfy. It only remains to prove the existence. Let $\chi\colon G \to T^S$ be any map; it is clear that there exists at least one linear map $\rho\colon k[G] \to \hom(V,V)$ such that $\chi = \operatorname{tr}^{\mathrm{set}}_V \circ \rho$. We define $L(\chi)$ by choosing one such linear map ρ and setting $L(\chi) = \tau_V(\rho)$. Suppose that $\mu\colon k[G] \to \hom(W,W)$ is another linear map such that $\chi = \operatorname{tr}^{\mathrm{set}}_W \circ \mu$. We must show that $\tau_V(\rho) = \tau_V(\mu)$. For each group element $g \in G$ the linear endomorphisms $\rho(g)\colon V \to V$ and $\mu(g)\colon W \to W$ have the same set trace. Therefore, for each group element $g \in G$ there is a linear map $\gamma^{(g)}\colon W \to V$ which is a trace arrow from $\mu(g)$ to $\rho(g)$. There is no loss of generality in assuming that all $\gamma^{(g)}$ have "proofs" of the same length, say n. Since G is finite we can rearrange the maps involved in $\gamma^{(g)}$ so that for every $g \in G$

we have a commutative diagram of the form:



This shows that for each $g \in G$, we have linear maps $\nu_i(g)\colon E_i \to E_i$ and that we can define linear maps

$$\alpha_i \in \hom\big(k[G], \hom(E_i, E_{i-1})\big), \quad \nu_i \in \hom\big(k[G], \hom(E_i, E_i)\big)$$

such that

$$\begin{array}{lll} \hom(h_1, 1_W)(\alpha_1) = \mu, & \hom(1_{E_1}, h_1)(\alpha_1) = \nu_1, & \text{implying: } \tau_W(\mu) = \tau_{E_1}(\nu_1) \\ \hom(h_2, 1_{E_1})(\alpha_2) = \nu_1, & \hom(1_{E_2}, h_2)(\alpha_2) = \nu_2, & \text{implying: } \tau_{E_1}(\nu_1) = \tau_{E_2}(\nu_2) \\ \vdots & \vdots & \vdots \\ \hom(h_n, 1_{E_{n-1}})(\alpha_n) = \nu_{n-1}, & \hom(1_V, h_n)(\alpha_n) = \rho, & \text{implying: } \tau_{E_{n-1}}(\nu_{n-1}) = \tau_V(\rho). \end{array}$$

It follows that $\tau_W(\mu) = \tau_V(\rho)$ and therefore L is well defined. □

12.3. In this paragraph we show that the "linear" trace of the representations monad functor (14) is the vector space $k^G = \hom(k[G], k) = k[G]^*$ and therefore prove Conjecture 48.

As before, G is a fixed finite group. If V is a finite dimensional vector space we can use a base $\{e_1, \ldots, e_n\}$ of V to define a base $\{\varphi_{ijg} | 1 \le i, j \le n,\, g \in G\}$ of the vector space $\mathbf{Set}\big(G, \hom(V, V)\big)$ as:

$$\varphi_{ijg_0}(g) = \delta_{gg_0} E_{ij}.$$

Lemma 50. *For every $i, j \in \{1, \ldots, n\}$ and $g, g_0 \in G$,*

$$\mathrm{tr}_k^{\mathrm{set}}\big(\mathrm{tr}_V^{\mathrm{lin}}(\varphi_{ijg_0}(g))\big) = \mathrm{tr}_V^{\mathrm{set}}\big(\varphi_{ijg_0}(g)\big)$$

Proof.

$$\begin{aligned} \mathrm{tr}_k^{\mathrm{set}}\big(\mathrm{tr}_V^{\mathrm{lin}}(\varphi_{ijg_0}(g))\big) &= \mathrm{tr}_k^{\mathrm{set}}\big(\mathrm{tr}_V^{\mathrm{lin}}(\delta_{gg_0} E_{ij})\big) \\ &= \mathrm{tr}_k^{\mathrm{set}}(\delta_{gg_0}\delta_{ij}) \\ &= \mathrm{tr}_V^{\mathrm{set}}\big(\delta_{gg_0} E_{ij}\big) \\ &= \mathrm{tr}_V^{\mathrm{set}}\big(\varphi_{ijg_0}(g)\big). \end{aligned}$$

□

Proposition 51. *Let F, as before, denote the representations monad functor* (14)*. The coend of*

$$\hom\big(F(-),-\big)\colon k\text{-}\mathbf{Vect}_{f.d.}{}^{\mathrm{op}}\times k\text{-}\mathbf{Vect}_{f.d.}\to k\text{-}\mathbf{Vect}$$

is the vector space k^G with canonical maps

$$\mathrm{tr}^{\mathrm{lin}}_{FV}\colon \hom\big(F(V),V\big)\simeq \mathbf{Set}\big(G,\hom(V,V)\big)\longrightarrow k^G$$

given by

$$\mathrm{tr}^{\mathrm{lin}}_{FV}=\mathbf{Set}\big(G,\mathrm{tr}^{\mathrm{set}}_{FV}\big)$$

Proof. Suppose that T is a vector space and that to every finite dimensional vector space V there is associated a linear map $\tau_V\colon \mathbf{Set}\big(G,\hom(V,V)\big)\to T$ so that the family $\{\tau_V\}_V$ is compatible. Then there is a unique map $\beta\colon T^S\to T$ such that for every V, $\tau_V=\beta\circ\mathrm{tr}^{\mathrm{set}}_{FV}$. Define $\alpha=\beta\circ\Gamma^G$ where $\Gamma=\mathrm{tr}^{\mathrm{set}}_k$. Let now V be a finite dimensional vector space and let's choose a base $\{e_1,\dots,e_n\}$ in it so that we get the base φ_{ijg} of $\mathbf{Set}\big(G,\hom(V,V)\big)$ indicated above.

Let g_0 be an element of G and $i,j\in\{1,\dots,n\}$. It will be sufficient to prove that the maps $\alpha\circ\mathrm{tr}^{\mathrm{lin}}_{FV}=\beta\circ\Gamma^G\circ\mathrm{tr}^{\mathrm{lin}}_{FV}$ and τ_V agree on φ_{ijg_0} and for that it is sufficient to show that the maps $\Gamma^G\circ\mathrm{tr}^{\mathrm{lin}}_{FV}$ and $\mathrm{tr}^{\mathrm{set}}_{FV}$ agree on φ_{ijg_0}. But for every $g\in G$,

$$\begin{aligned}\big(\Gamma^G\circ\mathrm{tr}^{\mathrm{lin}}_{FV}\big)(\varphi_{ijg_0})(g)&=\Gamma\big(\mathrm{tr}^{\mathrm{lin}}_V(\varphi_{ijg_0}(g))\big)\\&=\mathrm{tr}^{\mathrm{set}}_V\big(\varphi_{ijg_0}(g)\big)\\&=\big(\mathrm{tr}^{\mathrm{set}}_V\big)^G\big(\varphi_{ijg_0}\big)(g)\\&=\mathrm{tr}^{\mathrm{set}}_{FV}(\varphi_{ijg_0})(g).\end{aligned}$$

therefore

$$\Gamma^G\circ\mathrm{tr}^{\mathrm{lin}}_{FV}(\varphi_{ijg_0})=\mathrm{tr}^{\mathrm{set}}_{FV}(\varphi_{ijg_0}).$$ □

Thus, Proposition 46 has the following immediate corollary:

Corollary 52. *If k is any field, two finite dimensional k-linear representations of a finite group G are isomorphic if and only if they have the same trace.*

Acknowledgements

The foregoing ideas and observations on the concept of the trace set of an endofunctor evolved from discussions with A. Kock while attending a workshop on Differentiable Stacks at the CRM, Barcelona, in April 2008. After trying to put a little order on the initial thoughts, they further developed during a research stay at the CRM in May 2008. All the results of Section 12 are the result of conversations with Manuel Bullejos. I want to express my thanks to Carles Casacuberta for his encouragement to come for a stay to the CRM, to Joaquin Bruna for his kind invitation and to the staff of the CRM for their help and hospitality.

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