

POINT-OCCURRENCE SELF-SIMILARITY IN CRACKLING-NOISE SYSTEMS AND IN OTHER COMPLEX SYSTEMS

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ABSTRACT. It has been recently found that a number of systems displaying crackling noise also show a remarkable behavior regarding the temporal occurrence of successive events versus their size: a scaling law for the probability distributions of waiting times as a function of a minimum size is fulfilled, signaling the existence on those systems of self-similarity in time-size. This property is also present in some non-crackling systems. Here, the uncommon character of the scaling law is illustrated with simple marked renewal processes, built by definition with no correlations. Whereas processes with a finite mean waiting time do not fulfill a scaling law in general and tend towards a Poisson process in the limit of very high sizes, processes without a finite mean tend to another class of distributions, characterized by double power-law waiting-time densities. This is somehow reminiscent of the generalized central limit theorem. A model with short-range correlations is not able to escape from the attraction of those limit distributions. A discussion on open problems in the modeling of these properties is provided.

1. INTRODUCTION

In the words of Sethna *et al.*, “crackling noise arises when a system responds to changing external conditions through discrete, impulsive events spanning a broad range of sizes” (1). Operationally, a broad range of sizes essentially means that the size s of the events fluctuates following a power-law distribution, $D(s) \propto 1/s^{1+\beta}$, where $D(s)$ is the probability density of s and $1 + \beta$ is the exponent. This is quite remarkable, as power laws signal the absence of characteristic scales, in this case of event sizes (2). It is also implicit that the driving that makes the external conditions change is relatively small and smooth, quite different from the resulting bursty response; therefore, crackling noise signals a highly nonlinear behavior.

Although these ideas have been developed within the physics of condensed matter (1), natural hazards show perhaps the largest number and best illustrations of crackling noise (3), including earthquakes (4; 5), landslides and rock avalanches (3; 6), volcanic eruptions (7), rainfall (8), hurricanes (9), solar flares (10; 11), the activity of the magnetosphere (12), and perhaps meteorite impacts (13) (provided

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that the Earth moves “slowly” through space). In other catastrophic phenomena, as forest fires (14; 15) or the extinctions of biological species (16), the events span a broad range of sizes but it is not clear if or in which cases they are power-law distributed.

Beyond the geosciences, notable examples of crackling noise arise in physiology and human affairs, like neuronal firings (17), epileptic seizures (18) and appearances of words in texts or speech (19) (if the “size of a word” is measured by its rarity, i.e., its position in a ranking of frequencies). Note that most examples of crackling noise arise in systems with a high degree of complexity, characterized by an enormous number of degrees of freedom that interact between them.

Crackling noise can be considered as the most important and apparent property of systems displaying self-organized criticality (SOC). This concept goes one step beyond, and proposes that the origin of the power-law distribution of sizes (i.e., the hallmark of crackling noise) is the existence of a critical point (analogous to those of equilibrium phase transitions, which are well-known to have scale-invariant properties) to which the dynamics of the system is attracted by means of a feedback mechanism that balances driving and dissipation (20; 21). The paradigmatic example is a sandpile over an open support to which grains are slowly added: when there are few grains, the pile is flat and grain dissipation at the border is low, then the pile grows; in contrast, when there are too many grains they easily travel through the system and border dissipation is large, so the slope of the pile decreases. At the end (in the attractor), the slope fluctuates around a critical value that balances the input and output of grains, and this state should have scale-invariant properties, i.e., power-law statistics, as in equilibrium critical phenomena. The behavior of real-world sandpiles is more diverse than what the SOC picture suggests, but that is a different story (22; 6). Then, SOC is a plausible physical mechanism for the emergence of crackling noise, but this does not preclude that other mechanisms could also lead to crackling noise (23).

In practice, although it is very simple to test if a system displays crackling noise (just measuring the size of the events and calculating their distribution, checking that the driving on the system is slow and smooth), it is not so easy to demonstrate the existence of SOC, as one would need to measure the relations between the internal variables of the system and their fluctuations, and they should behave in the same way as the equivalent ones in an equilibrium phase transition (24).

2. SCALING LAW FOR WAITING-TIME STATISTICS

Neither the above definition of crackling noise nor the usual studies of SOC pay too much attention on how the discrete and impulsive response events that the system develops occur in time. Is there a unique dynamical process that defines these behaviors? And in that case, is it periodic? Is it chaotic? Is it random? How does the dynamics reflect the scale-invariant properties of such systems? (25).

The case of earthquakes exemplifies our poor understanding of the dynamics of this kind of processes (25; 26). On the one hand, there is a widespread belief in the notion of *characteristic earthquakes*: the strongest events that a single fault segment is able to generate are always almost the same (same epicenter, same size, same focal mechanism) and should occur at regular times (27; 28). On the other hand, for extended regions, it is often assumed that mainshocks come in a total random way, i.e., from a Poisson process, and aftershocks follow a different process (29). Kagan has strongly argued against these simplistic views, showing evidence of time clustering in earthquake occurrence, just the behavior opposite to the characteristic-earthquake concept (30; 31).

Recently, it has been found that some of the systems mentioned above as prototypical of SOC or crackling noise display a remarkable temporal behavior. For such systems, let us consider the waiting time, also called recurrence time, inter-event time or inter-occurrence time; this is the time between consecutive events above a size threshold. So, we take into account only events whose size s verifies $s \geq c$, where c is the threshold value (but notice that for instance in the case of earthquakes one does not distinguish between foreshocks, mainshocks, and aftershocks). This defines a set of occurrence times, t_i^c , denoting the occurrence of the i -th event above c , from $i = 0$ to N_c . As each event is characterized by a unique occurrence time it is assumed that the duration of the event is very short in the scale of observation, and therefore the process can be described as a stochastic *point process* (32; 33). In addition, each event is also characterized by its size, so the process can be considered a *marked point process*, the size being a “mark” added to the time occurrence (we do not consider spatial degrees of freedom in this paper, but see Refs. (34; 35)). In any case, the waiting times for events with $s \geq c$ are obtained straightforwardly as $\tau_i^c = t_i^c - t_{i-1}^c$, with $i = 1, 2, \dots, N_c$.

The key element of analysis was introduced by Bak *et al.* (36), by means of a systematic study of the statistics of τ_i^c as a function of the size threshold c . Although the rise of c only eliminates some values of the occurrence times, leaving the rest unaltered (i.e., $t_i^c \rightarrow t_j^{c'}$, with $j \leq i$ and $c < c'$), the waiting times are changed in a more complicated way as they add in a variable number to give rise to the larger (or not) new waiting times. Usually, for the type of systems that have been studied so far, the waiting times show a large variability, and the best characterization on these processes is by means of the waiting-time probability density (37).

What has been found is that for many such systems, these probability densities verify a scaling law. If for events with $s \geq c$ we denote the waiting-time probability density by $D_c(\tau)$ and the mean waiting time by $\bar{\tau}_c$ ($\bar{\tau}_c \equiv \int_0^\infty \tau D_c(\tau) d\tau$), the scaling law can be written as

$$(1) \quad D_c(\tau) = F(\tau/\bar{\tau}_c)/\bar{\tau}_c,$$

where F is a scaling function, independent on c . This means that the shape of the distribution is independent on the scale given by $\bar{\tau}_c$ (which obviously is determined by the threshold c); in other words, when the waiting time is measured using as a unit its mean value, the results are independent on the value of c , which implies the existence of a self-similarity in the process. We will argue in the rest of this paper that this is quite a remarkable result in general, difficult to justify with the use of simple stochastic models.

In the case crackling noise or SOC systems, the mean waiting verifies $\bar{\tau}_c \propto 1/\int_c^\infty D(s)ds \propto c^\beta$, if $\beta > 0$, and substituting in the scaling law,

$$(2) \quad D_c(\tau) = \frac{1}{c^\beta} \tilde{F}\left(\frac{\tau}{c^\beta}\right),$$

where \tilde{F} is the scaling function F incorporating the factor of proportionality between $\bar{\tau}_c$ and c^β . Written in this form, the scaling law turns out to be a particular case of the condition of scale invariance for functions with two variables, τ and c (2). So, although for one variable the condition of scale invariance yields a power law (for instance, for s we have $D(s) \propto 1/s^{1+\beta}$), for two variables, like s and τ , scale invariance leads to Eq. (2), with \tilde{F} an undetermined function.

Crackling-noise or SOC systems showing this behavior include earthquakes (37; 38), fractures (39; 40), solar flares (11), literary texts (41), or some paradigmatic sandpile models, in contrast with previous belief (42; 43). But this property is shared by other systems for which its crackling-noise nature is in doubt, as printing requests in a computer network (44), forest fires (15) and tsunamis (45) (although the latter seem to be power-law distributed (46), certainly they are not slowly driven; rather, they are cracklingly driven by undersea earthquakes and landslides). Even systems that do not crackle, as diverse climate records (temperature, river levels, etc.) (47), or systems for which the crackling behavior is in the derivative of the response signal, as financial indices (48), verify a scaling law as Eq. (1) when the threshold is large enough that the events above it become extreme events. The corresponding scaling functions come in a variety of functional forms; are there any preferred types?

3. MODELS FOR TIME-SIZE SCALE INVARIANCE OF EVENT OCCURRENCE

3.1. Marked Poisson Process. Which is the meaning and depth of the scaling law (1)? Certainly, a *marked Poisson process* trivially fulfills it. This is a marked point process in which the occurrence times follow a Poisson process, and the sizes of the events (the “marks”) come from a random distribution independently on occurrence times and other sizes. Its simulation is very simple, with independent identical exponentially distributed waiting times and independent identical power-law distributed sizes (in the case of crackling-noise systems).

Indeed, a Poisson process is completely characterized by its rate, let us say, 10 events per hour. If we now raise the size threshold in such a way that half of the events are eliminated and half of them survive, this is equivalent to a *random thinning* of the events with a thinning probability equal to $1/2$ (in which any event has the same probability of being eliminated, independently of the rest), due to the fact that the sizes of the events are uncorrelated. So, we end with a Poisson process of rate equal to 5 events per hour.

It is well known how to show this more rigorously. Consider that events are removed from a marked Poisson process with a probability q , and are kept with probability $p = 1 - q$; then, the probability that the number of events N' that survive in a time interval of length Δ is equal to k is,

$$\Pr[N' = k] = \sum_{n=k}^{\infty} \Pr[N' = k|N = n] \Pr[N = n],$$

where \Pr denotes probability, “ $|$ ” conditional probability, N is the number of events in the interval prior to thinning, and n counts all the possible values of N . By hypothesis, the original process is Poisson of rate λ , so $\Pr[N = n] = e^{-\nu} \nu^n / n!$, with $\nu \equiv \lambda \Delta$, and by the uncorrelated nature of the process $\Pr[N' = k|N = n]$ is given by the binomial distribution,

$$\Pr[N' = k|N = n] = \binom{n}{k} p^k q^{n-k}.$$

Substituting both above,

$$\begin{aligned} \Pr[N' = k] &= e^{-\nu} \frac{p^k}{k!} \sum_{n=k}^{\infty} \frac{q^{n-k}}{(n-k)!} \nu^n = \frac{(p\nu)^k}{k!} e^{-\nu} \sum_{n=k}^{\infty} \frac{(q\nu)^{n-k}}{(n-k)!} \\ &= \frac{(p\nu)^k}{k!} e^{-\nu(1-q)} = e^{-p\nu} \frac{(p\nu)^k}{k!}, \end{aligned}$$

which defines another Poisson process of rate $p\lambda$. If we rescale the new rate $\lambda' = p\lambda$ as $\lambda' \rightarrow \lambda'/p$ we recover precisely the original Poisson distribution. Note that p is given by $p = \Pr[s \geq c' | s \geq c]$, and in our context it turns out that $p = (c/c')^\beta$.

So, could the trivial marked Poisson process explain the scaling law Eq. (1)? Certainly not, as none of the known examples mentioned above are characterized by an exponential scaling function. We should go beyond this trivial explanation.

3.2. Marked Renewal Process. The shapes of the scaling functions found for the real data mentioned above are rather diverse, including the gamma distribution, the stretched exponential, and the power law for large times. It seems necessary to incorporate this shape into the point process modeling those systems. The most straightforward way to do this is through a *renewal process*, which is characterized by independent identically distributed waiting times, following a specific distribution (32). If we add to this model independent identically distributed sizes we end

with a process that we may call *marked renewal process*. Note that there are no correlations whatsoever in this process, but there is a memory of the last event if the waiting-time distribution is not exponential.

The probability distribution of waiting times for events with $s \geq c'$ can be obtained from the probability distribution for those with $s \geq c$, if $c' \geq c$. The idea is the same as in the previous subsection but we will use the waiting-time representation rather than the count-number representation of the process. The same steps as in Ref. (49) will be followed, although the case here is simpler.

We start using the survivor function, $S_{c'}(\tau) \equiv \Pr[\text{waiting time} > \tau \text{ for events with } s \geq c'] = \int_{\tau}^{\infty} D_{c'}(\tau) d\tau$. If an event of size $s_0 \geq c'$ has taken place, the next one with $s \geq c'$ can happen in a variety of ways, depending on the number of events with $c \leq s < c'$ in between. So, we can write,

$$\begin{aligned} S_{c'}(\tau) &= \sum_{j=1}^{\infty} \Pr[\tau^{(j)} > \tau, s_1 < c', \dots, s_{j-1} < c', s_j \geq c'] \\ &= \sum_{j=1}^{\infty} \Pr[\tau^{(j)} > \tau | s_1 < c', \dots, s_{j-1} < c', s_j \geq c'] \\ &\quad \cdot \Pr[s_1 < c', \dots, s_{j-1} < c', s_j \geq c'] \\ &= \sum_{j=1}^{\infty} \Pr[\tau^{(j)} > \tau] \cdot \Pr[s_1 < c'] \cdots \Pr[s_{j-1} < c'] \cdot \Pr[s_j \geq c'] \end{aligned}$$

where \Pr denotes probability, “|” conditional probability, and the j -th return time is defined, for events with $s \geq c$, as $\tau_i^{(j)} = t_i^c - t_{i-j}^c$, that is, as the elapsed time between any event and the j -th event after it (naturally, the first return time is the waiting time). The conditions on $\Pr[\tau^{(j)} > \tau]$ are eliminated because waiting times are independent on sizes. As in the previous subsection $p \equiv \Pr[s \geq c' | s \geq c] = \Pr[s \geq c']$ and $q \equiv 1 - p = \Pr[s < c' | s \geq c] = \Pr[s < c']$ (the condition $s \geq c$ is always implicit, if it is not explicit). Therefore,

$$S_{c'}(\tau) = \sum_{j=1}^{\infty} p q^{j-1} \Pr[\tau^{(j)} > \tau].$$

If in this equation we derive with respect τ we obtain the probability densities of the return times; as the waiting times are considered independent on each other, we use that the j -th-return-time distribution is given by j convolutions of the first-return-time distribution (denoted by the symbol $*$) to get

$$\begin{aligned} (3) \quad D_{c'}(\tau) &= \sum_{j=1}^{\infty} p q^{j-1} [D_c(\tau)]^{*j} \\ &= p D_c(\tau) + q p D_c(\tau) * D_c(\tau) + q^2 p D_c(\tau) * D_c(\tau) * D_c(\tau) + \dots \end{aligned}$$

where the exponent $*j$ means that $D_c(\tau)$ is convoluted with itself j times. It is convenient to look at Eq. (3) in Laplace space, where things are simpler, then $D(s) \equiv \int_0^\infty e^{-s\tau} D(\tau) d\tau$, which is a (moment) generating function, and the convolutions turn into simple products,

$$(4) \quad D_{c'}(s) = pD_c(s) \sum_{j=1}^{\infty} q^{j-1} [D_c(s)]^{j-1} = pD_c(s) + qpD_c^2(s) + q^2pD_c^3(s) + \dots$$

As q and $D_c(s)$ are smaller than one (this is general for generating functions), the infinite sum can be performed, turning out that

$$(5) \quad D_{c'}(s) = \frac{pD_c(s)}{1 - qD_c(s)}.$$

Equation (5) describes the effect of rising the threshold on the waiting-time distribution. The next step is the scale transformation, which puts the distributions corresponding to c and c' on the same scale. We will obtain this by removing the effect of the decreasing of the rate, which, is proportional to p , so,

$$(6) \quad D_{c'}(\tau) \rightarrow p^{-1}D_{c'}(\tau/p),$$

and in Laplace space we get

$$(7) \quad D_{c'}(s) \rightarrow D_{c'}(ps).$$

Therefore, the combined effect of rising the threshold plus rescaling leads to a transformation \mathbb{T} that acts on the original distribution,

$$(8) \quad \mathbb{T}D_c(s) = \frac{pD_c(ps)}{1 - qD_c(ps)}.$$

We are very interested in the fixed points of this transformation, which are obtained by the solutions $D_c^*(s)$ of

$$\mathbb{T}D_c^*(s) = D_c^*(s),$$

where $*$ now means fixed point; The previous fixed-point equation is totally equivalent to the scaling law (1). Introducing the variable $\omega \equiv ps$ and substituting $p = \omega/s$ and $q = 1 - \omega/s$ in the fixed-point equation we get, separating variables,

$$(9) \quad \frac{1}{sD_c^*(s)} - \frac{1}{s} = \frac{1}{\omega D_c^*(\omega)} - \frac{1}{\omega} \equiv \frac{1}{\lambda};$$

where we have made both functions equal to an arbitrary constant due to the fact that p and s are independent variables and so s and ω are; then, the only way in which the equality could be fulfilled, for all s and ω , is that the function is a constant $1/\lambda$. The solution is then

$$(10) \quad D_c^*(s) = \frac{1}{1 + s/\lambda},$$

which is the Laplace transform of an exponential distribution,

$$(11) \quad D_c^*(\tau) = \lambda e^{-\lambda \tau}.$$

The dependence on c enters by means of λ , as $\lambda^{-1} = \bar{\tau}_c$. Note that this demonstration includes the one on the previous subsection, showing that the marked Poisson process displays a scaling law for the waiting-time distribution, but in this case we have achieved a more general result, as the marked Poisson process is the only marked renewal process which can fulfill such a scaling law (when the rescaling is done with the mean waiting time $\bar{\tau}_c$).

We can go one step beyond and demonstrate that the marked Poisson process is an attractor for the broad family of marked renewal processes for which the mean waiting time $\bar{\tau}_c$ is finite. The iterative application of transformation \mathbb{T} with a finite probability p is equivalent to the limit $p \rightarrow 0$ in Eq. (8). Expanding that equation up to first order in p , using $D_c(ps) = 1 - \bar{\tau}_c ps + \dots$, yields

$$\begin{aligned} \mathbb{T}D_c(s) &= \frac{pD_c(ps)}{1 - qD_c(ps)} = \frac{p(1 - \bar{\tau}_c sp + \dots)}{1 - (1-p)(1 - \bar{\tau}_c sp + \dots)} \\ &= p \frac{1 - \bar{\tau}_c sp + \dots}{1 - (1-p - \bar{\tau}_c sp + \dots)} = \frac{1 - \bar{\tau}_c sp + \dots}{1 + \bar{\tau}_c s + \dots} = \frac{1}{1 + \bar{\tau}_c s} + \dots \end{aligned}$$

which indeed corresponds to a Poisson process when $p \rightarrow 0$.

This result illustrates the strange particularity of the scaling relation (1): among the infinite number of probability distributions with a finite mean that can define a marked renewal process, only one type, the one with exponentially distributed waiting times, fulfills the scaling law. The results can be put in the language of the renormalization group. Indeed, the first part of the process, called thinning, where the threshold is raised from c to c' , corresponds to a decimation of events. This is analogous to the renormalization of the Ising model, where some portion of the spins are eliminated (50; 2). The second part of the process correspond to a change of scale in time, which is equivalent to the change of scale in real space renormalization. A third step, the renormalization of the field, is not necessary for the purposes of computing waiting-time statistics. So, the transformation \mathbb{T} can be considered a renormalization transformation, and we have seen how a renewal process (with a finite mean) renormalizes into the trivial Poisson fixed point. So, for all the renewal processes of this kind (except for a set of zero measure, given by the Poisson process) one expects a change under renormalization, and not scale invariance. This is one of the reasons why the existence of the scaling law (1) is so intriguing.

3.3. Marked Renewal Process without a Finite Mean Waiting Time. What happens for renewal processes whose waiting-time density does not have a finite mean? Obviously, a rate cannot be defined as the inverse of the mean, nevertheless,

still it is possible to follow an approach that makes sense. The first part of our transformation, in which the size threshold is increased from c to c' , does not change (5); however, the rescaling with the mean cannot be applied. We will use as a rescaling parameter $p = \Pr[s \geq c' | s \geq c]$, but in contrast to the previous case we will raise p to some power r in Eqs. (6) and (7). This is equivalent to seek for a scaling law of the form

$$D_c(\tau) = R_c^r F(R_c^r \tau),$$

where the rate R_c is understood as the number of events per unit time in the time window under consideration.

The transformation \top [Eq. (8)] then becomes

$$(12) \quad \top D_c(s) = \frac{p D_c(p^r s)}{1 - q D_c(p^r s)}.$$

In the same way as before, the fixed point equation leads to

$$(13) \quad \frac{1}{s^{1/r} D_c^*(s)} - \frac{1}{s^{1/r}} = \frac{1}{\omega^{1/r} D_c^*(\omega)} - \frac{1}{\omega^{1/r}} \equiv a,$$

whose solution is

$$(14) \quad D_c^*(s) = \frac{1}{1 + a s^\alpha},$$

with $\alpha \equiv 1/r$. Only for $0 \leq \alpha \leq 1$, i.e., $r \geq 1$, this function represents a probability distribution; this is so because for other values of α the expansion of $D_c^*(s)$ does not correspond to the expansion of a generating function. However, when $0 < \alpha < 1$ a finite mean does not exist.

Indeed, for small s , $D_c^*(s) = 1 - a s^\alpha$; this corresponds, if $0 < \alpha < 1$, to the Laplace transform of $D_c^*(\tau) = A/\tau^{1+\alpha}$, for large τ , with $a \equiv -A\Gamma(-\alpha)$ and $\Gamma(-\alpha)$ the gamma function of $-\alpha$ (51). On the other hand, the behavior for large s is $D_c^*(s) = 1/(a s^\alpha)$, and by means of a Tauberian theorem the limit behavior for small τ is $D_c^*(\tau) = 1/(a\Gamma(\alpha)\tau^{1-\alpha})$. Summarizing,

$$(15) \quad D_c^*(\tau) = \begin{cases} \frac{1}{a\Gamma(\alpha)\tau^{1-\alpha}}, & \text{for small } \tau, \\ \frac{a}{|\Gamma(-\alpha)|\tau^{1+\alpha}}, & \text{for large } \tau. \end{cases}$$

So, two power laws coexists, with exponents $1 - \alpha$ and $1 + \alpha$.

Which is the basin of attraction of the fixed-point distribution, $D_c^*(\tau)$? Let us consider $D_c(\tau) \simeq A/\tau^{1+\alpha}$ for large τ , with $0 < \alpha < 1$. Among the class of functions that do not have a finite mean, these are by far the most important. We already know that the Laplace transform of $D_c(\tau)$ behaves as $D_c(s) = 1 + A\Gamma(-\alpha)s^\alpha + \dots$, for $s \rightarrow 0$ and so, $D_c(p^r s) = 1 - a p^{r\alpha} s^\alpha + \dots$. Substituting into the new equation for $\top D_c(s)$, and taking into account that $p \rightarrow 0$,

$$\top D_c(s) = \frac{p D_c(p^r s)}{1 - q D_c(p^r s)} = \frac{p(1 - a s^\alpha p^{r\alpha} + \dots)}{1 - (1 - p)(1 - a s^\alpha p^{r\alpha} + \dots)} =$$

$$p \frac{1 - as^\alpha p^{r\alpha} + \dots}{1 - (1 - p - as^\alpha p^{r\alpha} + \dots)} = \frac{1 - as^\alpha p^{r\alpha} + \dots}{1 + as^\alpha p^{r\alpha-1} + \dots} = \frac{1}{1 + as^\alpha} + \dots$$

where we have used that $r \equiv 1/\alpha$, which, remember, means that the rescaling depends on the power-law exponent of the waiting-time density. Rescaling in this way ensures the existence of an attractor for the waiting-time densities that behave as a power law for long times, and this attractor is given by the fixed point (15).

Why does this counterintuitive rescaling provide the fulfillment of a scaling law? The reason is in the generalized central limit theorem. Imagine that $p = 1/2$, so we remove half of the events; in order to recover the original pattern (characterized by the same waiting-time probability density) we need to multiply the time interval under consideration not by $1/p = 2$ but by $1/p^{1/\alpha}$, which is larger than $1/p$ for $0 < \alpha < 1$. This is due to the fact that as the process evolves in time, longer waiting times appear, due to their power-law tail, being necessary to consider far longer time intervals.

Reference (52) puts numbers into this simple intuitive explanation. The total time interval up to the N -th event is $t_N = \sum_{n=1}^N \tau_n$. The largest waiting time $\tau_m(N)$ among the N values of the waiting time can be estimated as $N \int_{\tau_m(N)}^\infty D_c(\tau) d\tau \simeq 1$, this yields $\tau_c(N) \propto N^{1/\alpha}$ for large N , and this means that during the time interval of length t_N the process “does not see” the tail of $D_c(\tau)$ beyond $\tau_m(N)$ and one can effectively truncate the distribution at this value. Then, the “typical” value of t_N can be associated to the mean value of the truncated distribution, so $t_N \simeq N \langle \tau \rangle_{trunc} \propto N^{1/\alpha}$, and in this way the total time up to the N -th event scales in the same way as the largest waiting time.

Indeed, the generalization of the central limit theorem introduces rigor in this argument, see for instance Ref. (52). If the τ_n 's are power-law distributed, with a tail $D_c(\tau) = A/\tau^{1+\alpha}$, the standard central limit theorem does not hold for t_n and one has to apply its generalization. Rescaling t_N as $Z_N \equiv t_N/N^{1/\alpha}$ the theorem states that the variable Z_N has as a limit distribution, for $N \rightarrow \infty$, one of the so called Levy stable distributions (whose mathematical form is not relevant for our purposes).

Coming back to our problem, there is a particular value of α for which the exact inverse Laplace transform of the fixed-point distribution can be easily obtained. Indeed, if $\alpha = 1/2$, we get that

$$D_c^*(s) = \frac{1}{1 + a\sqrt{s}};$$

whose inverse Laplace transform turns out to be

$$(16) \quad D_c^*(\tau) = \frac{1}{a\sqrt{\pi\tau}} - \frac{e^{\tau/a^2}}{a^2} \operatorname{erfc}\left(\frac{\sqrt{\tau}}{a}\right),$$

where $\text{erfc}(x)$ is the complementary error function, $\text{erfc}(x) = 2\pi^{-1/2} \int_x^\infty e^{-x^2} dx$ (51). Its asymptotic expansion will be useful to learn how $D_c^*(\tau)$ behaves,

$$\text{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}x} \left(1 - \frac{1}{2x^2} + \dots \right),$$

this leads to

$$D_c^*(\tau) \simeq \frac{a}{2\sqrt{\pi}\tau^{3/2}} = \frac{A}{\tau^{3/2}},$$

for large τ , with $a = -A\Gamma(-1/2) = 2\sqrt{\pi}A$. This is indeed coincident with the tail of the original distribution. On the other hand, for small argument, $\text{erfc}(x) \simeq 1$ and then

$$D_c^*(\tau) \simeq \frac{1}{2A\pi\sqrt{\tau}},$$

for short times, in agreement with the previous result for a general value of α , Eq. (15).

We can test these results simulating the process for instance for

$$(17) \quad D_c(\tau) = \frac{\alpha}{\ell(1 + \tau/\ell)^{1+\alpha}}$$

for which $A = \alpha\ell^\alpha$. The results appear in Fig. 1. Figure 1 (a) shows this distribution, and the distributions that result after rising the size threshold, keeping 10%, 1%, etc. up to 0.01% of the events. Figure 1 (b) displays the same distributions but including the rescaling, which allows to see the complete effect of the transformation (12) for different values of p . It is clear how for $p = 0.01$ the distribution is very close to the expected fixed-point distribution (16), and the agreement improves for smaller p .

Can these results be useful for crackling-noise systems or other complex systems? Solar flares, when they are close to the minimum of the solar cycle (11), and also e-mail activity from individuals (53) and human-movement episodes (54) show power-law distributed waiting times, with exponents $1 + \alpha$ about 1.5, 1, and 1.8, respectively. However, there are no indications of a second power law with exponent $1 - \alpha$, as requested by our theoretical calculations.

Nevertheless, these are peculiar systems; first, in the case of solar flares, the thresholds that define the waiting-time distributions are not size thresholds but intensity thresholds (size is defined as the integral of intensity over time), and it is not clear how this change can modify the properties of the system. In addition, the scaling which defines the scaling law is done with the mean rate, as it would correspond to a distribution with a well-defined mean. Second, for human movements and e-mail activity, the scaling law is not achieved by means of a thinning of the process through the increasing of a threshold, rather, individuals with different rates are compared. And as a fourth example we could consider the BTW sandpile

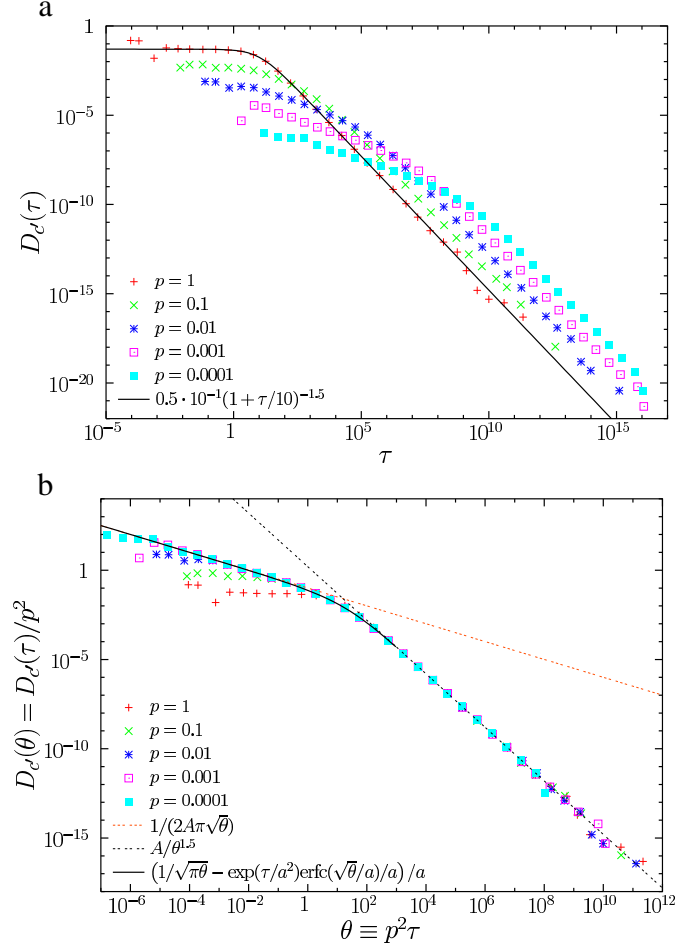


FIGURE 1. Illustration of the thinning plus rescaling transformation. A marked renewal process is simulated, with waiting-time distribution given by Eq. (17), using $\alpha = 0.5$ and $\ell = 10$. (a) The effect of thinning, which removes events with probability $1 - p$ is shown for different values of p . (b) The complete transformation \mathbb{T} , adding rescaling by p^2 to thinning, shows how the resulting distribution approaches the fixed-point solution, Eq. (16).

model, whose behavior (and the approach with which it has been studied) is very similar to that of solar flares, with a waiting-time exponent around 1.7 (42).

3.4. Processes with Short-Range Correlations. Reference (49) introduced a very simple point process in which each waiting time was correlated with the size of the previous event, in such a way that waiting times following larger events were drawn from a waiting-time density with a short characteristic time and waiting times after small events had a longer characteristic time. The transformation of the waiting time density when the threshold is increased and the time is rescaled verifies an equation which is a generalization of the previous cases,

$$\mathbb{T}D_c(s) = \frac{pD_{c\uparrow}(p^r s)}{1 - D_c(p^r s) + pD_{c\uparrow}(p^r s)}$$

where $D_{c\uparrow}(s)$ is the Laplace transform of $D_{c\uparrow}(\tau)$, which is the probability density of the waiting time that follows an event of size $c' > c$; more precisely $D_{c\uparrow}(\tau) \equiv D_c(\tau_i | s_{i-1} \geq c')$.

As in the last subsections, for $p \rightarrow 0$ we can write $D_c(s) = 1 - as^\alpha + \dots$ and $D_{c\uparrow}(s) = 1 - a's^{\alpha'} + \dots$, where a' , which gives the scale of the distribution, may depend on p , increasing as $p \rightarrow 0$. Substituting in the equation for the transformation \mathbb{T} ,

$$\begin{aligned} \mathbb{T}D_c(s) &= \frac{p(1 - a's^{\alpha'} p^{r\alpha'} + \dots)}{1 - (1 - as^\alpha p^{r\alpha} + \dots) + p(1 - a's^{\alpha'} p^{r\alpha} + \dots)} \\ &= p \frac{1 - a's^{\alpha'} p^{r\alpha'} + \dots}{as^\alpha p^{r\alpha} + \dots + p(1 - a's^{\alpha'} p^{r\alpha} + \dots)} \\ &= \frac{1 - a's^{\alpha'} p^{r\alpha'} + \dots}{1 + as^\alpha p^{r\alpha-1} + \dots} = \frac{1}{1 + as^\alpha} + \dots, \end{aligned}$$

with, as usual, $r \equiv 1/\alpha$; so, whatever the dependence of a' on p , the shape of $D_{c\uparrow}(\tau)$ is totally irrelevant to determine the asymptotic behavior of the process, provided that $a'p \rightarrow 0$ when $p \rightarrow 0$. This demonstrates that for the simple model we are considering, short-range correlations are not enough to escape from the attraction of the renewal fixed-point distributions. For the case of processes with a finite mean, a demonstration was already provided by Molchan, but the author believes the one here is more direct (55).

4. DISCUSSION

We have seen how, when a mean waiting time exists, a process without correlations cannot account for the appearance of a scaling law for waiting-time distributions (except in the trivial case of a marked Poisson process, which is not observed

in real systems). In the same way, when a process without correlations is characterized by the absence of a finite mean waiting-time, the theoretical results do not compare well with observations either. Therefore, correlations build the shape of the waiting-time distribution in crackling-noise as well as non-crackling noise systems (at least for the cases studied so far, see Sec. 2).

But short-range correlations do not seem enough to break the dominance of the trivial Poisson fixed point when a mean exists, or the double power-law distributions expected when the mean does not exist, as we have shown for the simple example introduced in Ref. (49). As in equilibrium critical phenomena (56), long-range correlations should be necessary in order to escape from the basin of attraction of the trivial fixed point. Further research is necessary regarding this issue, both from a fundamental point of view and with the goal of finding stochastic models of the systems displaying scaling laws.

A promising approach is that of Lennartz *et al.* (57), where a long-range correlated series of magnitudes is generated, associating each magnitude value to a discrete time (the authors have in mind earthquakes, but the results are more general). Then, only extreme events, i.e., events above a large magnitude threshold, are considered, and the corresponding waiting-time statistics is obtained, with a surprising agreement with observational earthquake data (37). In other words, starting with a delta distribution of waiting times for the initial process, renormalization leads to a nontrivial fixed point. It is an open question why this is so.

The fact that a nontrivial (nonexponential) scaling law for waiting-time distributions may exist has been criticized by Molchan (58) and Saichev and Sornette (59). Assuming that seismic occurrence is well described by the ETAS model, the latter authors were able to derive the form of the waiting-time probability density. Here we just mention that the ETAS (epidemic type aftershock sequence) model is the simplest modeling of the process of earthquake triggering that puts all earthquakes on the same footing: any earthquake triggers other events with a probability that is proportional to two main factors: the Omori law, which controls the decay in time of the seismic rate, and the productivity law, which links the rate with the magnitude of the triggering earthquake. In addition, the magnitudes of the resulting triggered earthquakes are drawn independently from the Gutenberg-Richter distribution. Despite its simplicity, the mathematical treatment of the ETAS model becomes an authentic *tour de force*. In any case, Saichev and Sornette get that $D_c(\tau)$ fulfills something like a scaling relation, Eq. (1), but with a scaling function which is not a only function of the rescaled waiting time,

$$F(x, \varepsilon) = \left[\frac{n\theta\varepsilon^\theta}{x^{1+\theta}} + \left(1 - n + \frac{n\varepsilon^\theta}{x^\theta} \right)^2 \right] \exp \left(-(1-n)x - \frac{n}{1-\theta} \varepsilon^\theta x^{1-\theta} \right).$$

where $1 + \theta$ is the exponent of the decay of the rate with time, given by the Omori law ($\theta \simeq 0$), $n < 1$ is the branching parameter, defining the number of events

triggered directly by any event, and $\varepsilon = \lambda C$, with C the small time constant that avoids divergence at zero time in the Omori law. So, in addition to the dependence on the rescaled waiting time $x = \lambda \tau$, the density depends on the rate λ through ε , in contrast with the idea of scale invariance.

The fact that the ETAS model does not fulfill an exact scaling law does not seem highly surprising, after all, due to the fact that the ETAS model itself is not fully self-similar (60; 61). At the end, for vanishing rate, $\varepsilon \rightarrow 0$, it turns out that $F(x)$ tends to an exponential distribution, and then the ETAS model renormalizes into the trivial Poisson process. It would be of the maximum interest to calculate if real self-similar models, as the Vere-Jones model (60; 61) or the Lippiello-Godano-de Arcangelis model (62) fulfill the scaling relation (1) and which is the corresponding scaling function. This is of course an unsolved problem.

As a final comment, let us mention that the models used in this paper are purely stochastic, or mathematical. Some readers, however, could ask for a more physical approach. For the author, the situation is analog to the study of diffusion processes using random-walk models (52): the outcome is robust and independent on small details about the nature of interactions (molecular collisions in one case and event-event triggering mechanisms in the other). Nevertheless, there are examples of crackling systems that have been successfully modeled on physical grounds, using for instance the random field Ising model (1), the so called ABBM model (63; 64), or models of dislocation dynamics in plastic deformation (65). It would be of the maximum interest to explore the time structure of events in those models. Regarding the geosciences, which are the systems we have in mind for this paper, the situation is more complicated, as the physics of those phenomena is still poor understood, and controlled experiments cannot be performed. It is a great challenge to find microscopic models of natural catastrophes that give rise to the self-organized structures that emerge in the long-run limit in those systems.

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