THE SIMULTANEOUS CONJUGACY PROBLEM IN GROUPS OF PIECEWISE LINEAR FUNCTIONS

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Abstract. Guba and Sapir asked, in their joint paper [8], if the simultaneous conjugacy problem was solvable in Diagram Groups or, at least, for Thompson's group F. We give an elementary proof for the solution of the latter question. This relies purely on the description of F as the group of piecewise linear orientation-preserving homeomorphisms of the unit. The techniques we develop allow us also to solve the ordinary conjugacy problem as well, and we can compute roots and centralizers. Moreover, these techniques can be generalized to solve the same questions in larger groups of piecewise-linear homeomorphisms.

1. Introduction

Richard Thompson's group F can be defined by the following presentation:

$$F = \langle x_0, x_1, x_2, \dots \mid x_n x_k = x_k x_{n+1}, \ \forall\, k < n \rangle.$$

This group was introduced and studied by Thompson in the 1960s. The standard introduction to F is [5]. The group F can be regarded as a subgroup of the group of piecewise linear self-homeomorphisms of the unit interval and this is the point of view that we will adopt throughout the paper, and that we will introduce in detail in Section 2.

We say that a group G has *solvable ordinary conjugacy problem* if there is an algorithm such that, given any two elements $y, z \in G$, we can determine whether there is, or not, a $g \in G$ such that $g^{-1}yg = z$. Similarly, for a fixed $k \in \mathbb{N}$, we say that the group G has *solvable k-simultaneous conjugacy problem* if there is an algorithm such that, given any two k-tuples of elements in G, $(y_1, \dots, y_k), (z_1, \dots, z_k)$, can determine whether there is, or not, a $g \in G$ such that $g^{-1}y_i g = z_i$ for all $i = 1, \dots, k$. For both these problems, we say that there is an *effective solution* if the algorithm produces such an element g, in addition to proving its existence.

This problem was studied before for various classes of groups. The k-simultaneous conjugacy problem has been proved to be solvable for the matrix groups

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$\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z})$ by Sarkisyan in 1979 in [12] and independently by Grunewald and Segal in 1980 in [7]. In 1984 Scott constructed examples of finitely presented infinite simple groups that have an unsolvable conjugacy problem in [13]. In their 2005 paper [3] Bridson and Howie constructed examples of finitely presented groups where the ordinary conjugacy problem is solvable, but the k-simultaneous conjugacy problem is unsolvable for every $k \geq 2$.

Theorem A. *Thompson's group F has a solvable k-simultaneous conjugacy problem, for every $k \in \mathbb{N}$. There is an algorithm which produces an effective solution.*

As an application of the proof of Theorem we have the following corollaries, ((1) and (2) appear in [9] by different techniques):

Theorem B. *(1) $C_F(x) \cong F^m \times \mathbb{Z}^n$, for some numbers $0 \leq m \leq n+1$.*
(2) $x \in F$ has a finite number of roots, which can be effectively computed.
(3) The centralizer of any finitely generated subgroup $A \subset F$ decomposes as athe direct product of the groups C_i, where each C_i is either trivial, infinite cyclic or isomorphic to F.

The ordinary conjugacy problem for F was addressed by Guba and Sapir [9], who solved it for general diagram groups in 1997, observing that F itself is a diagram group. Their solution, for general diagram groups, amounted to an algorithm which had the same complexity as the isomorphism problem of planar graphs. This last problem was solved in linear time in 1974 by Hopcroft and Wong [10], thus proving the Guba and Sapir solution of the conjugacy problem for diagram groups optimal. We mention here relevant related work: in 2001 Brin and Squier in [4] produced a criterion for describing conjugacy classes in $\mathrm{PL}_+(I)$, the group of all piecewise-linear orientation preserving self-homeomorphisms of the unit interval with only finitely many breakpoints, that contains F as a proper subgroup. In 2007 Gill and Short [6] extended this criterion to work in F, thus finding another way to characterize conjugacy classes from a piecewise linear point of view. Using a different approach close to Guba and Sapir's original solution, in 2007 Belk and Matucci [2] produced a unified solution of the conjugacy problem for all Thompson's groups F, T and V.

In 1999, Guba and Sapir [8] posed the question of whether or not the simultaneous conjugacy problem was solvable for diagram groups. Even though some of the results of the present paper are already known, we include our proof of them to show how everything can be deduced by our tools. In addition to that, with similar techniques we can prove that the same result holds for larger groups of piecewise linear homeomorphisms.

The paper is organized as follows. In Section 2 we will define the groups $\mathrm{PL}_{S,G}(I)$ that generalize Thompson's group F and give an outline of the solution of simultaneous conjugacy problem. In Section 3 we show how to build an approximate conjugator which makes the fixed point set of y and z coincide. In Section 4 we introduce the main algorithm to create candidate conjugators.

In Section 5 we compute centralizers and roots and obtain the solution of the ordinary conjugacy problem as a corollary. In Section 6 we describe how to reduce the simultaneous conjugacy problem to a special instance of the ordinary one, thus solving it for Thompson's group F. In Section 7, we generalize the previous machinery to the groups $\mathrm{PL}_{S,G}(I)$. In Section 8 we show interesting examples where the simultaneous conjugacy problem can be solved.

2. The idea of the argument

In this section we describe the groups that we will study and outline the steps of our proof. This section is intended to give a quick overview of the results that we will prove in the later sections.

2.1. **Notations.** We introduce here the notation that will be used across the paper. Let $I = [0,1]$ be the unit interval. We define $\mathrm{PL}_+(I)$ to be the group of piecewise linear orientation-preserving homeomorphisms of unit interval into itself, with finitely many breakpoints such that slopes are positive real numbers. The product of two elements is given by the composition of functions.

We can impose additional the requirements on the breakpoints and the slopes to define subgroups of $\mathrm{PL}_+(I)$. Let S be a subring of $\mathbb{R}$, let $U(S)$ denote the group of invertible elements of S and let G be a subgroup of $U(S) \cap \mathbb{R}_+$. We define $\mathrm{PL}_{S,G}(I)$ to be the group of piecewise linear orientation-preserving homeomorphisms from the unit interval into itself, with only a finite number of breakpoints and such that

- all breakpoints are in the subring S,
- all slopes are in the subgroup G,

the product of two elements is given by the composition of functions. If $G = U(S) \cap \mathbb{R}_+$ we write $\mathrm{PL}_S(I)$, instead of $\mathrm{PL}_{S,G}(I)$. If $S = \mathbb{R}$, then $\mathrm{PL}_S(I) = \mathrm{PL}_+(I)$. For the special case $S = \mathbb{Z}\left[\frac{1}{2}\right]$, we denote the group $\mathrm{PL}_{\mathbb{Z}\left[\frac{1}{2}\right]}(I)$ by $\mathrm{PL}_2(I)$. The group $\mathrm{PL}_2(I)$ is also known as *Thompson's group F* and it is isomorphic to the group F defined in the introduction (see [5] for a proof). We observe that in order to make some calculations possible inside the ring S and its quotients, we need to ask for some requirements to be satisfied by S from the computability standpoint. These will be clearly stated in Remark 7.4 and will be assumed throughout this paper.

To attack the ordinary and the simultaneous conjugacy problems, we will split the study into that of some families of functions inside $\mathrm{PL}_+(I)$. The reduction to these subfamilies will come from the study of the fixed point subset of the interval I for a function f.

Remark 2.1. We would like to define the group $\mathrm{PL}_{S,G}(J)$, where J is any interval contained in I. There are two natural ways to define it:

(1) The group of restrictions of functions in $\mathrm{PL}_{S,G}(I)$ fixing the endpoints of J:

$$\mathrm{PL}_{S,G}^{\mathrm{Rest}}(J) := \{f|_J \mid f \in \mathrm{PL}_{S,G}(I), f(\eta) = \eta, f(\zeta) = \zeta\}$$

It is not clear that $\mathrm{PL}_{S,G}^{\mathrm{Rest}}(J)$ can be regarded as a subgroup of $\mathrm{PL}_{S,G}(I)$.

(2) The group of functions of $\mathrm{PL}_{S,G}(I)$ which fix the endpoints of J and are the identity on $I \setminus J$:

$$\mathrm{PL}_{S,G}^{\mathrm{Fix}=I\setminus J}(J) := \{f \in \mathrm{PL}_{S,G}(I) \mid f(t) = t, \forall t \in I \setminus J\}.$$

We observe that $\mathrm{PL}_{S,G}^{\mathrm{Fix}=I\setminus J}(J)$ is clearly a subgroup of $\mathrm{PL}_{S,G}(I)$.

In the case where the endpoints of J are contained in S, it is easy to check that the two definitions coincide (i.e. $\mathrm{PL}_{S,G}^{\mathrm{Fix}=I\setminus J}(J) \cong \mathrm{PL}_{S,G}^{\mathrm{Rest}}(J)$) and thus the group $\mathrm{PL}_{S,G}^{\mathrm{Rest}}(J)$ can be regarded as a subgroup of $\mathrm{PL}_{S,G}(I)$. However, if one of the two endpoints is not in S, the group $\mathrm{PL}_{S,G}^{\mathrm{Fix}=I\setminus J}(J)$ is not finitely generated: if $J = [\eta, \zeta]$ and $\zeta \notin S$ then, given any finite set of functions $f_1, \ldots, f_k$, there exists an interval $[\mu, \zeta]$, for some $\eta < \mu < \zeta$, where they all coincide with the identity map; this happens because ζ is not a breakpoint for any of them and so $\langle f_1, \ldots, f_k \rangle \lneq \mathrm{PL}_{S,G}^{\mathrm{Fix}=I\setminus J}(J)$. Since in this paper we will always assume both endpoints to be in S, this difference will not matter and any of the two descriptions can be adopted to define $\mathrm{PL}_{S,G}(J)$.

We state the following interesting question:

Question 2.2. Let S be a finitely generated subring of $\mathbb{R}$ and G be a be finitely generated subgroup of $U(S)$. Let J be an interval contained in $[0, 1]$, with no assumption about the endpoints of J. Is the group $\mathrm{PL}_{S,G}^{\mathrm{Rest}}(J)$ defined in Remark 2.1 finitely generated?

Remark 2.3. Throughout the paper we will always assume the interval J to have endpoints in S. For the special case $S = \mathbb{Z}\left[\frac{1}{2}\right]$, it is straightforward to verify that $\mathrm{PL}_2(J) \cong \mathrm{PL}_2(I)$, though we will not use this fact. We observe that the analogue fact may not be true for the groups $\mathrm{PL}_{S,G}(I)$ (see Remark 7.5).

For a function $f \in \mathrm{PL}_{S,G}(J)$ we define the fixed point set on the interval J as

$$\mathrm{Fix}_J(f) := \{t \in J \mid f(t) = t\},$$

to simplify the notation will often drop the subscript J. The motivation for introducing this subset is easily explained — if $y, z \in \mathrm{PL}_+(J)$ are conjugate through $g \in \mathrm{PL}_+(I)$ and $t \in (\eta, \zeta)$ is such that $y(t) = t$ then $z(g^{-1}(t)) = (g^{-1}yg)(g^{-1}(t)) = g^{-1}(t)$, that is, if y has a fixed point then z must have a fixed point.

Definition 2.4. We define $\mathrm{PL}_{S,G}^{<}(J)$ (and respectively. $\mathrm{PL}_{S,G}^{>}(J)$) to be the set of all functions in $\mathrm{PL}_{S,G}(J)$ with graph below the diagonal (respectively, above the diagonal). A function is $x \in \mathrm{PL}_{S,G}(J)$ is called *one-bump function* if either $x \in \mathrm{PL}^{>}_{S,G}(J)$ or $x \in \mathrm{PL}^{<}_{S,G}(J)$.

Given a function $f \in \mathrm{PL}_{S,G}(I)$ and a number $0 < t_0 < 1$ fixed by f, it is not always true that $t_0 \in S$. The example in figure 1 shows a function in $\mathrm{PL}_2(I)$ with a non-dyadic rational fixed point. In order to avoid working in intervals J where the endpoints may not be in S, we introduce a new definition of boundary which deals with this situation: for a subset $X \subseteq [0,1]$, we define

$$\partial_S X := \partial X \cap S$$

where ∂X denotes the usual topological boundary of X inside $\mathbb{R}$. With this definition, the set $\partial X \setminus \partial_S X$ becomes the set of isolated points of X that are not in S. For the special case $S = \mathbb{Z}\left[\frac{1}{2}\right]$ we write $\partial_2 X$.

Definition 2.5. We define $\mathrm{PL}^0_{S,G}(J) \subseteq \mathrm{PL}_{S,G}(J)$, the set of functions $f \in \mathrm{PL}_{S,G}(J)$ such that the set $\mathrm{Fix}(f)$ does not contain elements of S other than the endpoints of J, i.e., $\mathrm{Fix}(f)$ is discrete and $\partial_S \mathrm{Fix}(f) = \partial_S J$.

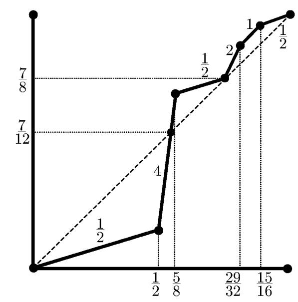

FIGURE 1. A function in $\mathrm{PL}_2(I)$ with a non-dyadic fixed point.

2.2. **Outline of the strategy.** We will begin our investigation with the special case of Thompson's group $\mathrm{PL}_2(I)$ (in Sections 3 through 6. Most of the techniques that we develop for this case will extend to the general case of the groups $\mathrm{PL}_{S,G}(I)$ as it will be shown in Section 7. For this reason, we will now give the general outline our strategy in $\mathrm{PL}_{S,G}(I)$.

Step 1. *Find a $g \in \mathrm{PL}_{S,G}(I)$ such that* $\mathrm{Fix}(y) = g(\mathrm{Fix}(z))$. The set $\mathrm{Fix}(x)$ consists of a disjoint union of a finite number of closed intervals and isolated points, because every $x \in \mathrm{PL}_{S,G}(I)$ has only finitely many breakpoints. As mentioned before, if $g^{-1}yg = z$, then $\mathrm{Fix}(y) = g(\mathrm{Fix}(z))$. Thus, as a first step we need to know if, given y and z, there exists a $g \in \mathrm{PL}_{S,G}(I)$ such that $\mathrm{Fix}(g^{-1}yg) = g(\mathrm{Fix}(y)) = \mathrm{Fix}(z)$. In Section 3 we show whether or not there exists an approximate conjugator g_* such that $\mathrm{Fix}(g_*^{-1}yg_*) = \mathrm{Fix}(z)$. We then study the conjugacy problem problem for $g_*^{-1}yg_*$ and z.

Step 2. *If* $\mathrm{Fix}(y) = \mathrm{Fix}(z)$*, then* $\partial_S\mathrm{Fix}(y) = \partial_S\mathrm{Fix}(y) = \{\alpha_1, \dots, \alpha_n\}$ *and we look for conjugators in* $\mathrm{PL}_{S,G}([\alpha_i, \alpha_{i+1}])$ *of the restrictions of* y *and* z *to* $[\alpha_i, \alpha_{i+1}]$*.* We reduce to study the problem on smaller intervals. If $y = z = id$ on the interval $[\alpha_i, \alpha_{i+1}]$ there is nothing to prove, otherwise y and z are one-bump functions. This case will be dealt with through a procedure called the "stair algorithm" that we provide in Subsection 4.2.

Step 3. *Compute the intersection of centralizers of elements and derive a solution to the conjugacy problem.* Finding centralizers g of an element y is equivalent to find all elements g such that $g^{-1}yg = y$. Using similar techniques we can also classify the structure of intersection of centralizers, which will be useful for the last step. Since the set of all conjugators for y and z is given by a particular conjugator times an element in the centralizer of y, Step 2 and Step 3 give us a solution to the conjugacy problem.

Step 4. *Reduce the simultaneous conjugacy problem to a "restricted" conjugacy problem.* It can be seen that the simulatenous conjugacy problem is equivalent to solving the conjugacy problem for two elements y and z with the restriction that the conjugator g must lie in the intersection of centralizers of some elements $x_1, \dots, x_k$. In Section 6 we will show how to build such a conjugator, if it exists follwing steps 1 and 2.

3. Moving fixed points

In this Section we carry out the first step of the outline described in Section 2.

Theorem 3.1. *Given* $y, z \in \mathrm{PL}_2(I)$*, we can determine if there is (or not) a* $g \in \mathrm{PL}_2(I)$ *such that* $g(\mathrm{Fix}(y)) = \mathrm{Fix}(g^{-1}yg) = \mathrm{Fix}(z)$*. If such an element exists, it can be constructed.*

To start off, we need a tool to decide if this can be proved for the boundaries of the fixed point sets. In other words, we need to decide if it is possible to make $\partial\mathrm{Fix}(y)$ coincide with $\partial\mathrm{Fix}(z)$ (see figure 2). The first step is to see how, given two rational numbers α and β, we can find a $g \in \mathrm{PL}_2(I)$ with $g(\alpha) = \beta$. The next two results are well known:

The following Lemma 3.2 is well known and a proof it can be found in [5].

Lemma 3.2. *If* $0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$ *and* $0 = y_0 < y_1 < y_2 < \ldots < y_n = 1$ *are two partitions of* $[0, 1]$ *consisting of dyadic rational numbers, then we can build an* $f \in F$*, such that* $f(x_i) = y_i$*.*

Lemma 3.3 (Extension of Partial Maps)**.** *Suppose* $I_1, \dots, I_k \subseteq [0, 1]$ *is a family of disjoint compact intervals* $I_i = [a_i, b_i]$*, with* $b_i < a_{i+1}$ *for all* $i = 1, \dots, k$ *and* $a_i, b_i \in \mathbb{Z}[\frac{1}{2}]$*. Let* $J_1, \dots, J_k \subseteq [0, 1]$*, with* $J_i = [c_i, d_i]$*, be another family of intervals with the same property. Suppose that* $g_i : I_i \to J_i$ *is a piecewise-linear function with a finite number of breakpoints, occurring at dyadic rational points,*

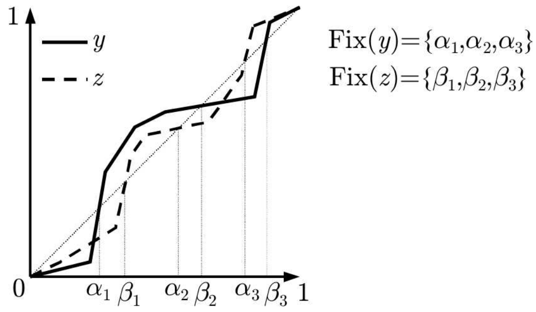

FIGURE 2. An example with $\partial\mathrm{Fix}(y) \neq \partial\mathrm{Fix}(z)$.

and such that all slopes are integral powers of 2. *Then there exists an* $\widetilde{g} \in \mathrm{PL}_2(I)$ *such that* $\widetilde{g}|_{I_i} = g_i$.

Proof. By our hypotheses we have that $0 < a_1 < b_1 < \ldots < a_k < b_k < 1$ and $0 < c_1 < d_1 < \ldots < c_k < d_k < 1$ are two partitions of $[0,1]$ with the same number of points. By the previous Lemma, there exists an $h \in \mathrm{PL}_2(I)$ with $h(a_i) = c_i$ and $h(b_i) = d_i$. Define

$$\widetilde{g}(t) := \begin{cases} h(t) & t \notin I_1 \cup \ldots \cup I_k \\ g_i(t) & t \in I_i \end{cases}$$

This function satisfies the extension condition. □

We observe that this proof is constructive and produces easily an element of F seen as a piecewise-linear function.

Proposition 3.4. *Let* $\alpha, \beta \in \mathbb{Q} \cap (0,1)$. *Then there is a* $g \in \mathrm{PL}_2(I)$ *such that* $g(\alpha) = \beta$ *if and only if*

$$\alpha = \frac{2^t m}{n}, \quad \beta = \frac{2^k u}{n},$$

with $t, k \in \mathbb{Z}$, m, n, u *odd integers,* $(m,n) = (u,n) = 1$, *and the following holds*

$$u \equiv 2^R m \pmod{n} \tag{1}$$

for some $R \in \mathbb{Z}$. *Moreover, if such element* g *exists, it can be constructed.*

Proof. Suppose that there is $g \in \mathrm{PL}_2(I)$ such that $g(\alpha) = \beta$. If α is a dyadic rational then β is also a dyadic rational and the conclusion of the lemma holds. Otherwise $g(t) = 2^r t + 2^s w$ inside a small open neighborhood of α, for some $r, s, w \in \mathbb{Z}$. Let $\alpha = \frac{2^t m}{n}, \beta = \frac{2^k u}{v}$, for some $t, k \in \mathbb{Z}$, $(m,n) = (u,n) = 1$, m, n, u, v odd. Then

$$\frac{2^k u}{v} = \beta = g(\alpha) = 2^r \frac{2^t m}{n} + 2^s w = \frac{2^{r+t} m + 2^s wn}{n}.$$

Now the numerator of $\frac{2^{r+t}m+2^s w}{n}$ and n may not be coprime any more, in which case we may cancel the common part and get a new odd part n' of the denominator of the right hand side. Moreover we have $v|n$. Applying the same argument for g^{-1} we have that $n|v$, i.e., $v = n$. Thus, if there is a g carrying α to β, then

$$u = 2^{r+t-k}m + 2^{s-k}wn.$$

Now we can rename $R := r + t - k$ so that the equation becomes

$$u \equiv 2^R m \pmod{n}.$$

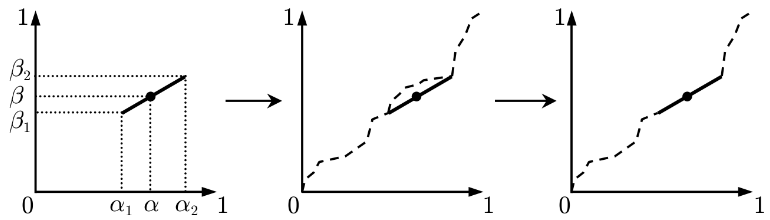

FIGURE 3. How to build a $g \in \mathrm{PL}_2(I)$, with $g(\alpha) = \beta$.

Conversely, suppose u satisfies (1). Then we can find r, s, w such that, by going backwards in the "only if" argument, there is a small open interval $(\gamma, \delta) \subset [0, 1]$ containing α and a function $g(t) = 2^r t + 2^s w$, with $g(\alpha) = \beta$ and we can choose γ, δ so that they are dyadic rationals and $g(\gamma), g(\delta) \in I$. Now we just apply the extension Lemma 3.3 and extend g to the whole interval $[0, 1]$ (see figure 3). $\square$

Example 3.5. Let $\alpha = \frac{1}{17}$, $\beta = \frac{13}{17}$ and $\gamma = \frac{3}{17}$. It is easy to see that we can find a $g \in \mathrm{PL}_2(I)$ with $g(\alpha) = \beta$, but there is no $h \in \mathrm{PL}_2(I)$ with $h(\alpha) = \gamma$.

Corollary 3.6. *Given $\alpha, \beta \in \mathbb{Q} \cap (0, 1)$ there is an algorithm to determine whether or not there is a $g \in \mathrm{PL}_2(I)$ such that $g(\alpha) = \beta$. Moreover, if such an element g exists, it can be constructed.*

We now state the same results for a finite number of points. Its proof uses the extension Lemma 3.3 on a number of disjoint intervals, one around each point.

Corollary 3.7. *Let $0 < \alpha_1 < \ldots < \alpha_r < 1$ and $0 < \beta_1 < \ldots < \beta_r < 1$ be two rational partitions of $[0, 1]$. There exists a $g \in \mathrm{PL}_2(I)$ with $g(\alpha_i) = \beta_i$ if and only if there are $g_1, \ldots, g_r \in \mathrm{PL}_2(I)$ such that $g_i(\alpha_i) = \beta_i$. Moreover, if such an element g exists, it can be constructed.*

Proof of Theorem 3.1. Using the previous Lemma we can determine whether or not we can make $\partial\mathrm{Fix}(y)$ and $\partial\mathrm{Fix}(z)$ coincide. First we have to check if $\#\partial\mathrm{Fix}(y) = \#\partial\mathrm{Fix}(z)$. Then we use the previous Corollary to find a $g \in \mathrm{PL}_2(I)$, with $g(\partial\mathrm{Fix}(y)) = \partial\mathrm{Fix}(z)$, if it exists. Let $\widehat{y} := g^{-1}yg$. Now we just have to check if the sets where the graphs of the two functions $\widehat{y}$ and z intersect the diagonal are the same. In fact, we know that the boundary points of these sets are the same, so it is enough to check whether $\mathrm{Fix}(\widehat{y})$ contains the same intervals as $\mathrm{Fix}(z)$. $\square$

4. The Stair Algorithm

In this Section we carry out the second step of the strategy described in Section 2 by restricting our study to a square where the given functions have "no relevant" intersection with the diagonal, and showing how to build possible candidates for conjugator

4.1. **The Linearity Boxes.** The very first thing to check, if y and z are to be conjugate through a $g \in \mathrm{PL}_2(J)$, is whether they can be made to coincide in neighborhoods of the endpoints of $J = [\eta, \zeta]$. This subsection and the following one will deal with functions in $\mathrm{PL}_+(J)$: we will reuse them in the discussion on $\mathrm{PL}_{S,G}(I)$. We start by making the following observation: the map $\mathrm{PL}_+(J) \to \mathbb{R}_+$ which sends a function f to $f'(\eta^+)$ is a group homomorphism.

Lemma 4.1. *Given three functions $y, z, g \in \mathrm{PL}_+(J)$ such that $g^{-1}yg = z$, there exist $\alpha, \beta \in (\eta, \zeta)$ such that $y(t) = z(t)$, for all $t \in [\eta, \alpha] \cup [\beta, \zeta]$ (refer to figure 4).*

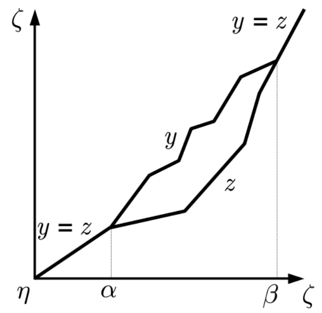

Figure 4. y and z coincide around the endpoints.

Proof. We prove the Lemma for the first interval. Let $\varepsilon > 0$ be a number small enough that

$$\begin{aligned} g(t) - \eta &= a(t-\eta), && \text{for } t \in [\eta, \eta+\varepsilon], \\ y(t) - \eta &= b(t-\eta), && \text{for } t \in [\eta, g(\eta+\varepsilon)], \\ g^{-1}(t) - \eta &= a^{-1}(t-\eta), && \text{for } t \in [\eta, yg(\eta+\varepsilon)] \end{aligned}$$

for some $a, b > 0$. Let $\alpha = \min\{\eta+\varepsilon, g(\eta+\varepsilon), yg(\eta+\varepsilon)\}$. Then, for $t \in [\eta, \alpha]$, we have

$$z(t) = g^{-1}yg(t) - \eta = a^{-1}ba(t-\eta) = b(t-\eta) = y(t).$$

The second interval is found in the same way, after recentering the axis at the point (ζ, ζ). $\square$

If two functions coincide at the beginning and at the end, then a candidate conjugator g will have to be linear in certain particular "boxes", which depend only on y and z.

Lemma 4.2 (Initial Box)**.** *Suppose $y, z, g \in \mathrm{PL}_+(J)$ and $g^{-1}yg = z$. Let $\alpha > 0$ and $y'(\eta^+) = z'(\eta^+) = c > 1$ satisfy*

$$y(t) - \eta = z(t) - \eta = c(t - \eta) \text{ for } t \in [\eta, \eta + \alpha].$$

Then the graph of g is linear inside the square $[\eta, \eta + \alpha] \times [\eta, \eta + \alpha]$, i.e., the graph of g is linear in some neighborhood of the point (η, η) in $J \times J$ depending only on y and z (see figure 5).

Proof. We can rewrite the conclusion of this lemma, by saying that, if we define

$$\varepsilon = \sup\{r \mid g \text{ is linear on } [\eta, \eta + r]\},$$

then $\eta + \varepsilon \geq \min\{g^{-1}(\eta + \alpha), \eta + \alpha\}$. Assume the contrary, let $\varepsilon < \alpha$ and $\eta + \varepsilon < g^{-1}(\eta + \alpha)$ and write $g(t) - \eta = \gamma(t - \eta)$ for $t \in [\eta, \eta + \varepsilon]$, for some constant $\gamma > 0$. Let $0 \leq \sigma < 1$ be any number. Since $\sigma < 1$ and $\varepsilon < \alpha$, we have $\eta + \sigma\varepsilon < \eta + \alpha$ and so y is linear around $\eta + \sigma\varepsilon$:

$$g(y(\eta + \sigma\varepsilon)) = g(\eta + c\sigma\varepsilon).$$

On the other hand, since $\eta + \varepsilon < g^{-1}(\eta + \alpha)$, it follows that $g(\eta + \sigma\varepsilon) < g(\eta + \varepsilon) < \eta + \alpha$ and so z is linear around the point $g(\eta + \sigma\varepsilon) = \eta + \gamma\sigma\varepsilon$:

$$z(g(\eta + \sigma\varepsilon)) = z(\eta + \gamma\sigma\varepsilon) = \eta + c\gamma\sigma\varepsilon.$$

Since $gy = zg$, we can equate the previous two equations and write $g(\eta + c\sigma\varepsilon) = \eta + \gamma c\sigma\varepsilon$, for any number $0 \leq \sigma < 1$. If we choose $1/c < \sigma < 1$, we see that g must be linear on the interval $[0, c\sigma\varepsilon]$, where $c\sigma\varepsilon > \varepsilon$. This is a contradiction to the definition of ε. $\square$

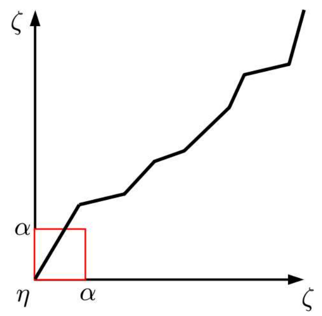

FIGURE 5. Initial linearity box.

Notice that the square neighborhood depends only on y and z. We observe that the Lemma also holds when $z'(\eta^+) = y'(\eta^+) = c < 1$ and the proof is given by applying the previous proof to the homeomorphisms y^{-1}, z^{-1}. Thus we can remove any requirement on the initial slopes of y and z. Note that the Initial Box Lemma has an analogue for the points close to ζ:

Remark 4.3 (Final Box)**.** Let $y, z, g \in \mathrm{PL}_+(J)$. Suppose $(g^{-1}yg)(t) = y(t)$, for all $t \in J$. If there exist $\beta, c \in (0,1)$ such that $y(t) = z(t) = c\cdot(t-\zeta)+\zeta$ on $[\beta,\zeta]$, then the graph of g is linear inside the square $[\beta,\zeta]\times[\beta,\zeta]$.

4.2. **The Stair Algorithm for** $\mathrm{PL}_+^<(J)$**.** This subsection will deal with the main construction of this paper. We show how, under certain hypotheses, if there is a conjugator, then it is unique. On the other hand, we give a construction of such a conjugator, if it exists. Given two elements y, z the set of their conjugators is a coset of the centralizer of one of them, thus it makes sense to start by deriving properties of centralizers.

Lemma 4.4. *Let $z \in \mathrm{PL}_+(J)$. Suppose there exist $\eta \le \lambda \le \mu \le \zeta$ such that $z(t) \le \lambda$, for every $t \in [\eta,\mu]$. Suppose further that $g \in \mathrm{PL}_+(I)$ is such that*

(i) *$g(t) = t$, for all $t \in [\eta,\lambda]$ and*
(ii) *$g^{-1}zg(t) = z(t)$, for all $t \in [\eta,\mu]$.*

Then $g(t) = t$, for all $t \in [\eta,\mu]$.

Proof. Suppose, by contradiction, that there exist points $\lambda \le \theta_1 < \theta_2 \le \mu$ such that $g(t) = t$, for all $t \in [\eta,\theta_1]$ and $g(t) \neq t$ and g is linear, for $t \in (\theta_1,\theta_2]$. Recenter the axes in the point (θ_1,θ_1) through $T = t-\theta_1$ and $Z = z-\theta_1$. Then $g(t) = \alpha t$, for $t \in [0,\theta_2-\theta_1]$, for some positive $\alpha \neq 1$ and $z(t) = \beta t - \gamma$, for $t \in [0,\varepsilon]$, for $\beta,\gamma \in \mathbb{R}, \varepsilon > 0$ suitable numbers. Observe that now $-\theta_1 \le z(t) \le z(\theta_2-\theta_1) \le \lambda-\theta_1 \le 0$ and that due to the recentering $g(t) = t$ on $[-\theta_1, 0]$. For any $0 < t < \min\{\theta_2-\theta_1, \varepsilon, \varepsilon/\alpha\}$ the following equalities hold:

$$\beta t - \gamma = z(t) = gz(t) = zg(t) = z(\alpha t) = \alpha\beta t - \gamma,$$

and so this implies $\beta t = \alpha\beta t$, hence $\alpha = 1$. Contradiction. □

Corollary 4.5. *Let $z \in \mathrm{PL}_+^<(J)$ and $g \in \mathrm{PL}_+(J)$ be such that*

(i) *$g'(\eta^+) = 1$,*
(ii) *$g^{-1}zg(t) = z(t)$, for all $t \in J$.*

Then $g(t) = t$, for all $t \in J$.

Proof. Since $g'(\eta^+) = 1$, we have $g(t) = t$ in an open neighborhood of η. Suppose, to set a contradiction, that $g(t_0) \neq t_0$, for some $t_0 \in (\eta,\zeta)$. Let λ be the first point after which $g(t) \neq t$. It is obvious that $\eta < \lambda < \zeta$. Thus $z(\lambda) < \lambda$ and we let $\mu = z^{-1}(\lambda) > \lambda$. So we have that $z(t) \le \lambda$ on $[0,\mu]$, $g(t) = t$ on $[\eta,\lambda]$ and $g^{-1}zg = z$ on I. By the previous Lemma, $g(t) = t$ on $[\eta,\mu]$, with $\mu > \lambda$. Contradiction. □

Lemma 4.6. *Let $z \in \mathrm{PL}_+^<(J)$. Let $C_{\mathrm{PL}_+(J)}(z)$ be the centralizer of z in $\mathrm{PL}_+(J)$. Define the map*

$$\begin{array}{rccc} \varphi_z: & C_{\mathrm{PL}_+(J)}(z) & \longrightarrow & \mathbb{R}_+ \\ & g & \longmapsto & g'(\eta^+). \end{array}$$

Then φ_z is an injective group homomorphism.

Proof. Let $y \in \mathrm{PL}_+^<(J)$ and suppose that there exists two elements $g_1, g_2 \in C_{\mathrm{PL}_+(J)}(y)$ such that $\varphi_y(g_1) = \varphi_y(g_2)$, then $g_1^{-1}g_2$ has a slope 1 near η and by the previous Lemma is equal to the identity. Therefore $g_1 = g_2$, which proves the injectivity. Clearly this is a group homomorphism. $\square$

This Lemma implies the following:

Lemma 4.7. *Let $y, z \in \mathrm{PL}_+^<(J)$, let $C_{\mathrm{PL}_+(J)}(y,z) = \{g \in \mathrm{PL}_+(J) \mid y^g = z\}$ be the set of all conjugators and let λ be in the interior of J. We define the following two maps*

$$\begin{array}{rrcl} \varphi_{y,z}: & C_{\mathrm{PL}_+(J)}(y,z) & \longrightarrow & \mathbb{R}_+ \\ & g & \longmapsto & g'(\eta^+) \end{array}$$

$$\begin{array}{rrcl} \psi_{y,z,\lambda}: & C_{\mathrm{PL}_+(J)}(y,z) & \longrightarrow & J \\ & g & \longmapsto & g(\lambda). \end{array}$$

Then

(i) $\varphi_{y,z}$ *is an injective map.*
(ii) *There is a map $\rho_\lambda : J \to \mathbb{R}_+$ such that the following diagram commutes:*

$$\begin{tikzcd} C_{\mathrm{PL}_+(J)}(y,z) \arrow[r, "\varphi_{y,z}"] \arrow[dr, "\psi_{y,z,\lambda}"'] & \mathbb{R}_+ \\ & J \arrow[u, "\rho_\lambda"'] \end{tikzcd}$$

(iii) $\psi_{y,z,\lambda}$ *is injective.*

Proof. (i) is an immediate corollary of Lemma 4.6. (ii) Without loss of generality we can assume that the initial slopes of y, z are the same (otherwise the set $C_{\mathrm{PL}_+(J)}(y,z)$ is obviously empty and any map will do). We define the map $\rho_\lambda : J \to \mathbb{R}_+$ as

$$\rho_\lambda(\mu) = \lim_{n\to\infty} \frac{y^n(\mu) - \eta}{z^n(\lambda) - \eta}$$

The above limit exists, because the sequence stabilizes under these assumptions.

To prove that the diagram commutes we define $\mu = g(\lambda)$ and observe that $y^n(\mu) \underset{n\to\infty}{\longrightarrow} \eta$ and $z^n(\lambda) \underset{n\to\infty}{\longrightarrow} \eta$. By hypothesis $y(\mu) = g(z(\lambda))$ so that $g(z^n(\lambda)) = y^n(\mu)$, for every $n \in \mathbb{Z}$. Since g fixes η we have

$$g(t) = g'(\eta^+)(t - \eta) + \eta \text{ on a small interval } [\eta, \eta + \varepsilon],$$

where ε depends on g. Let $N = N(g) \in \mathbb{N}$ be large enough, so that the numbers $y^N(\lambda), z^N(\lambda) \in (\eta, \eta + \varepsilon)$. This implies that, for any $n \geq N$

$$y^n(\mu) = g(z^n(\lambda)) = g'(\eta^+)(z^n(\lambda) - \eta) + \eta$$

and so then

$$\varphi_{y,z}(g) = g'(\eta^+) = \frac{y^n(\mu) - \eta}{z^n(\lambda) - \eta} = \rho_\lambda(\psi_{y,z,\lambda}(g)).$$

(iii) Since $\varphi_{y,z} = \rho_\lambda \psi_{y,z,\lambda}$ is injective by part (i), then $\psi_{y,z,\lambda}$ is also injective. $\square$

Our strategy will be to construct a "section" of the map $\varphi_{y,z}$, if it exists. Then as a consequence we will build a "section" of the map $\psi_{y,z,\lambda}$ too. The main tool of this subsection is the **Stair Algorithm**. This procedure builds a conjugator (if it exists) with a given fixed initial slope. The idea of the algorithm is the following. In order for y and z to be conjugate, they must have the same initial slope; by the initial linearity box Lemma this determines uniquely the first piece of a possible conjugator. Then we "walk up the first step of the stair", with the Identification Trick, that is basically identifying y and z inside a rectangle next to the linearity box, by taking a suitable product of y and z as a conjugator. Then we repeat and walk up more rectangles, until we "reach the door" (represented by the final linearity box) and this happens when a rectangle that we are building crosses the final linearity box.

Lemma 4.8 (Identification Trick)**.** *Let $y, z \in \mathrm{PL}_+^<(J)$ and let $\alpha \in (\eta, \zeta)$ be such that $y(t) = z(t)$ for $t \in [\eta, \alpha]$. Then there exists a $g \in \mathrm{PL}_+(I)$ such that $z(t) = y^g(t)$ for $t \in [\eta, z^{-1}(\alpha)]$ and $g(t) = t$ in $[\eta, \alpha]$. The element g is uniquely defined up to the point $z^{-1}(\alpha)$. If $y, z \in \mathrm{PL}_2^<(J)$ then g can be chosen in $\mathrm{PL}_2(J)$ (see figure 6).*

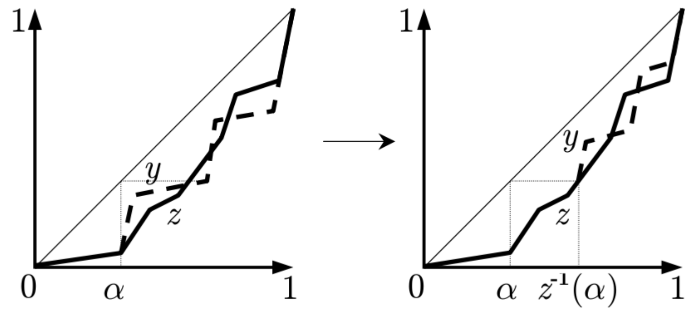

FIGURE 6. The identification trick

Proof. If such g exists then we have that, for $t \in [\eta, z^{-1}(\alpha)]$

$$y(g(t)) = g(z(t)) = z(t)$$

since $z(t) \leq \alpha$ in $[\eta, z^{-1}(\alpha)]$. Thus, for $t \in [\alpha, z^{-1}(\alpha)]$, we have that $g(t) = y^{-1}z(t)$. Now, that we have derived this necessary condition, we are ready to prove that such a g exists. Now define

$$g(t) := \begin{cases} t & t \in [\eta, \alpha] \\ y^{-1}z(t) & t \in [\alpha, z^{-1}(\alpha)] \end{cases}$$

and extend it to J as a line from the point $(z^{-1}(\alpha), y^{-1}(\alpha))$ to (ζ, ζ). If $y, z \in \mathrm{PL}_2(J)$ then we extend g to J, through the extension Lemma. A direct computation verifies that $y^g(t) = z(t)$ for $t \in z^{-1}(\alpha)$. □

Proposition 4.9 (Stair Algorithm for $\mathrm{PL}_+^<(J)$)**.** *Let $y, z \in \mathrm{PL}_+^<(J)$ and let q be a fixed positive real number. We can decide whether or not there is a $g \in \mathrm{PL}_+(I)$ with initial slope $g'(\eta^+) = q$ such that $y^g = z$. If g exists, it is unique and can be constructed.*

Proof. Assume $y \neq z$ and, up to taking inverses, suppose $0 < g'(\eta^+) = q < 1$. Let $[\eta, \alpha]^2$ the initial linearity box and $[\beta, \zeta]^2$ the final one. Then, for y and z to be conjugate we must have that g has is linear in $[\eta, \alpha]^2$ and in $[\beta, \zeta]^2$. Since $q < 1$ we must have g linear on the interval $[\eta, \alpha]$ and so we define:

$$g_0(t) := q(t - \eta) + \eta \qquad t \in [\eta, \alpha]$$

and extend it to the whole J. Now take the function $y_1 = g_0^{-1} y g_0$, which is still below the diagonal. Our goal now is to see if y_1 and z are conjugate. What is different now is that the new conjugator we will try to build is the identity on $[\eta, \alpha]$, where we already know that the functions y_1 and z coincide. We use the Identification Trick under the diagonal and build

$$g_1(t) := \begin{cases} t & t \in [\eta, \alpha] \\ y_1^{-1} z(t) & t \in [\alpha, z^{-1}(\alpha)] \end{cases}$$

then extending it to J. Again, we want to see we can find a conjugator of y_2 and z such that it is the identity on $[\eta, z^{-1}(\alpha)]$. Thus if we iterate this process and we build a sequence $g_2, y_3, g_3 \ldots, y_r, g_r, \ldots$. By construction, we always have that g_r is the identity on $[\eta, z^{-r}(\alpha)]$ and that $y_r(t) = z(t)$ for all $t \in [\eta . z^{-r}(\alpha)]$. We apply Lemma 4.11 and choose the smallest integer r so that

$$\min\{z^{-r}(\alpha), y^{-r}(\eta + q(\alpha - \eta))\} > \beta$$

and notice that this r depends *only* on y, z and q. Observe now that the Identification Trick tells us that, if the function g of the statement exists, it must coincide with the function $h(t) := g_0 \ldots g_r(t)$, for $t \in [\eta, z^{-r}(\alpha)]$. If we prove that the part of the graph of h on the interval $[z^{-r}(\alpha), 1]$ is inside the final box, then we can build g by extending it linearly up to the point (ζ, ζ). Recall that, by construction $g_{i-1} y_i^{-1} = y_{i-1}^{-1} g_{i-1}$ and $g_i(z^{-i}(\alpha)) = y_i^{-1}(z^{-i+1}(\alpha))$, for all $i = 1, \ldots, r$. Then

$$\begin{aligned} h(z^{-r}(\alpha)) &= g_0 \ldots g_{r-2} y_{r-1}^{-1} g_{r-1}(z^{-r+1}(\alpha)) = \\ &= g_0 \ldots g_{r-2} y_{r-1}^{-2}(z^{-r+2}(\alpha)) = \ldots = \\ &= y^{-r} g_0(\alpha) = y^{-r}(\eta + q(\alpha - \eta)) > \beta. \end{aligned}$$

Since $z^{-r}(\alpha) > \beta$ by our choice of r then $(z^{-r}(\alpha), h(z^{-r}(\alpha))) \in [\beta, \zeta]^2$ and therefore we can define g by extending it linearly in the last segment, i.e. joining $(z^{-r}(\alpha), h(z^{-r}(\alpha)))$ with $(1, 1)$.

If the function h is not linear on $[\beta, z^{-r}(\alpha)]$, then there is no conjugator for y and z. Otherwise, we have to check whether $g^{-1}yg = z$ and we are done. To prove the uniqueness of g, we just apply Lemma 4.7. □

Lemma 4.10. *Let $y, z \in \mathrm{PL}_+^<(J)$, $g \in \mathrm{PL}_+(J)$ and $n \in \mathbb{N}$. Then $g^{-1}yg = z$ if and only if $g^{-1}y^n g = z^n$.*

Proof. The "only if" part is obvious. The "if" part follows from the injectivity of φ_x of Lemma 4.6 since $g^{-1}yg$ and z both centralize the element $g^{-1}y^n g = z^n$. □

Lemma 4.11. *Let $J = [\eta, \zeta]$ be a compact interval, let $y, z \in \mathrm{PL}_+^<(J)$ and $g \in \mathrm{PL}_+(J)$ be such that $g^{-1}yg = z$. Suppose moreover that $[\eta, \alpha] \times [\eta, \alpha]$ is the initial linearity box and $[\beta, \zeta] \times [\beta, \zeta]$ is the final one for y and z. For every positive real number q there is a $k \in \mathbb{N}$ such that $y^k(\beta) < \eta + q(\alpha - \eta)$, $z^k(\beta) < \alpha$. Moreover y^k and z^k are still conjugate through g, so g must still be linear in the same linearity boxes of y and z.*

Proof. Since $y(\beta) < \beta$ and $y \in \mathrm{PL}_+^<(J)$ then $y^n(\beta) \underset{n\to\infty}{\longrightarrow} \eta$. Similarly this is true for $\{z^n(\beta)\}$ and so we can pick any number $r \in \mathbb{N}$ big enough to satisfy the statement. Moreover, we have $g^{-1}y^k g = (g^{-1}yg)^k = z^k$. Finally we observe that the linearity box of y^r and z^r is smaller than that of y and z, but that we already know that g has to be linear on $[\eta, \alpha]$ and on $[\beta, \zeta]$. □

The stair algorithm can also be proved in a slightly different way. We can apply Lemma 4.11 at the beginning and work with y^r and z^r instead of y and z. This gives a proof which concludes in two steps, although it yields the same complexity for a machine which has to compute immediately the powers y^r and z^r.

"Short" Proof of Proposition 4.9. Assume the same setting of the Proposition 4.9. We choose r to be the smallest number satisfying Lemma 4.11, so that

$$\min\{z^{-r}(\alpha), y^{-r}(\eta + q(\alpha - \eta))\} > \beta.$$

We now find a conjugator between y^r and z^r. We notice that the linearity boxes of y^r and z^r are again given by $[\eta, \alpha]^2$ and $[\beta, \zeta]^2$. With our assumption on r, the algorithm will need only two steps to end. Define g_0 as before and then define $y_1 = g_0^{-1}\widehat{y}g_0$. We then define an approximate conjugator g_1 for y_1^r and z^r as in the previous proof. Now we observe that the map $g_0 g_1$ is a conjugator for y^r and z^r up to the point $z^{-r}(\alpha)$ and that it enters the final linearity box, as in the previous proof. Now we extend it by linearity and we check whether this is a conjugator for y^r and z^r. If it is, it is the unique one. Finally, Lemma 4.10 tells us that g is a conjugator for y^r and z^r if and only if is is for y and z and so we are done. □

Remark 4.12. By the uniqueness of the conjugator (Lemma 4.7) we remark that both the proofs of the stair algorithm do not depend on the choice of g_0. The only requirements on g_0 are that it must be linear in the initial box and $g_0'(\eta^+) = q$.

Corollary 4.13 (Explicit Conjugator)**.** *Let $y, z \in \mathrm{PL}_+^<(J)$, let $[\eta, \alpha]$ be the initial linearity box and let q be a positive real number. There is an $r \in \mathbb{N}$ such that the unique candidate conjugator with initial slope $q < 1$ is given by*

$$g(t) = y^{-r} g_0 z^r(t) \qquad \forall t \in [\eta, z^{-r}(\alpha)]$$

and linear otherwise, where g_0 is any map in $\mathrm{PL}_+(J)$ which is linear in the initial box and such that $g_0'(\eta^+) = q$.

Proof. We run the short stair algorithm and let $g = g_0 g_1$ be defined as above. By the short proof of the stair algorithm and the previous Remark, we have $g = g_0 g_1 = y^{-1} g_0 g_1 z$ on $[\eta, z^{-r}(\alpha)]$ for some r. Therefore

$$g(t) = y^{-r} g_0 g_1 z^{-r}(t) = y^{-r} g_0 z^r(t) \qquad \forall t \in [\eta, z^{-r}(\alpha)]$$

and it is linear on $[z^{-r}(\alpha), \zeta]$. $\square$

Corollary 4.14. *Let $y, z \in \mathrm{PL}_+^<(J)$, and let λ be in the interior of J. The map*

$$\begin{array}{rrcl} \psi_{y,z,\lambda}: & C_{\mathrm{PL}_+(J)}(y,z) & \longrightarrow & J \\ & g & \longmapsto & g(\lambda) \end{array}$$

admits a section, *i.e. if $\psi_{y,z,\lambda}(g) = \mu \in J$, then g is unique and can be constructed.*

Remark 4.15. Suppose $y, z \in \mathrm{PL}_+^<(J) \cup \mathrm{PL}_+^>(J)$, then in order to be conjugate, they will have to be both in $\mathrm{PL}_+^<(J)$ or both in $\mathrm{PL}_+^>(J)$, because by Lemma 4.1 they will have to coincide in a small interval $[\eta, \alpha]$. Moreover, $g^{-1} y g = z$ if and only if $g^{-1} y^{-1} g = z^{-1}$, and so, up to working with y^{-1}, z^{-1}, we may reduce to studying the case where they are both in $\mathrm{PL}_+^<(J)$.

Remark 4.16 (Backwards Stair Algorithm)**.** The stair algorithm for the group $\mathrm{PL}_+^<(J)$ can be reversed. This is to say that, given q a positive real number, we can determine whether or not there is a conjugator g with final slope $g'(\zeta^-) = q$. The proof is the same: we simply start building g from the final box.

Remark 4.17. All the results of subsections 4.1 and 4.2 can be stated and proved by subsituting $\mathrm{PL}_2(J)$ and $\mathrm{PL}_2^<(J)$ for every appearance of $\mathrm{PL}_+(J)$ and $\mathrm{PL}_+^<(J)$. Only a few more remarks must be made in order to prove it. In the Identification Trick we need to observe that α and $z^{-1}(\alpha)$ are dyadic and to take all the extensions in $\mathrm{PL}_2(J)$ through the extension Lemma.

The stair algorithm gives a practical way to find conjugators if they exist and we have chosen a possible initial slope. By modifying the algorithm we can see that, if two elements are in $\mathrm{PL}_2^<(J)$ and they are conjugate through an element with initial slope a power of 2 then the conjugator is an element of $\mathrm{PL}_2(J)$.

Corollary 4.18. *Let $y, z \in \mathrm{PL}_2^<(J)$, $g \in \mathrm{PL}_+(J)$ such that $y^g = z$ and $g'(\eta^+)$ is a power of* 2*. Then $g \in \mathrm{PL}_2(J)$.*

4.3. **The Stair Algorithm for $\mathrm{PL}_2^0(J)$.** Section 3 proves that we can reduce our study to y and z such that $\mathrm{Fix}(y) = \mathrm{Fix}(z)$. It is now important to recall that an intersection point α of the graph of z with the diagonal may not be a dyadic rational (see again figure 1). If this is the case then α cannot be a breakpoint for y, z, g. This means that, for these α's, we have that $y'(\alpha)$, $z'(\alpha)$ and $g'(\alpha)$ are defined, i.e., the left and right derivatives coincide. Recall that a function z belongs to the set $\mathrm{PL}_2^0(J)$ if its graph does not have dyadic intersection points with the diagonal.

Proposition 4.19 (Stair Algorithm for $\mathrm{PL}_2^0(J)$)**.** *Let $y, z \in \mathrm{PL}_2^0(J)$ and suppose that* $\mathrm{Fix}(y) = \mathrm{Fix}(z)$*. Let q be a fixed power of* 2*. We can decide whether or not there is a $g \in \mathrm{PL}_2(J)$ with initial slope $g'(\eta^+) = q$ such that y is conjugate to z through g. If g exists it is unique.*

Proof. This proof will be essentially the same as the previous stair algorithm with a few more remarks. We assume therefore that such a conjugator exists and build it. Let $\mathrm{Fix}(y) = \mathrm{Fix}(z) = \{\eta = \alpha_0 < \alpha_1 < \ldots < \alpha_s < \alpha_{s+1} = \zeta\}$. We restrict our attention to $\mathrm{PL}_2([\alpha_i, \alpha_{i+1}])$ (as defined in Remark 2.1), for each $i = 0, \ldots, s$. If y and z are conjugate on $[\alpha_i, \alpha_{i+1}]$ then we can speak of linearity boxes: let $\Gamma_i := [\alpha_i, \gamma_i] \times [\alpha_i, \gamma_i]$ be the initial linearity box and $\Delta_i := [\delta_i, \alpha_{i+1}] \times [\delta_i, \alpha_{i+1}]$ the final one for $\mathrm{PL}_2([\alpha_i, \alpha_{i+1}])$. Now what is left to do is to repeat the procedure of the stair algorithm for elements in $\mathrm{PL}_2^<(U)$, for some interval U. We build a conjugator g on $[\alpha_0, \alpha_1]$ by means of the stair algorithm. We observe that α_1 is not a breakpoint, hence $g'(\alpha_1^+) = g'(\alpha_1^-)$. Thus we are given an initial slope for g in $[\alpha_1, \alpha_2]$, then we can repeat the same procedure and repeat the stair algorithm on $[\alpha_1, \alpha_2]$. We keep repeating the same procedure until we reach $\alpha_{s+1} = \zeta$. Then we check whether the g we have found conjugates y to z. Finally, we observe that in each square $[\alpha_i, \alpha_{i+1}] \times [\alpha_i, \alpha_{i+1}]$ the determined function is unique, since we can apply Lemma 4.7 on it. ☐

An immediate consequence of the previous result is the following Lemma:

Lemma 4.20. *Suppose $z \in \mathrm{PL}_2^0(J)$ and $g \in \mathrm{PL}_2(J)$ are such that*

(i) *$g'(\eta^+) = 1$,*

(ii) *$(g^{-1}zg)(t) = z(t)$, for all $t \in J$.*

Then $g(t) = t$, for all $t \in J$.

Remark 4.21 (Backwards and Midpoint Stair Algorithm)**.** It is possible to run a backwards version of the stair algorithm also for $\mathrm{PL}_2^0(J)$. Moreover, in this case it also possible to run a midpoint version of it: if we are given a point λ in the interior of J fixed by y and z and q a fixed power of 2, then, by running the stair algorithm at the left and the right of λ we determine whether there is or not a conjugator g such that $g'(\lambda) = q$.

From the previous Lemma and Remark it is immediate to derive:

Corollary 4.22. *Let $y, z \in \mathrm{PL}_2^0(J)$ such that $\mathrm{Fix}(y) = \mathrm{Fix}(z)$ and let $C_{\mathrm{PL}_2(J)}(y,z) = \{g \in \mathrm{PL}_2(J) \,|\, y^g = z\}$ be the set of all conjugators. For any $\tau \in \mathrm{Fix}(y)$ define the map*

$$\begin{array}{rccc} \varphi_{y,z,\tau}: & C_{\mathrm{PL}_2(I)}(y,z) & \longrightarrow & \mathbb{R}_+ \\ & g & \longmapsto & g'(\tau), \end{array}$$

where if τ is an endpoint of J we take only a one-sided derivative. Then

(i) *$\varphi_{y,z,\tau}$ is an injective map.*

(ii) *If $\varphi_{y,z,\tau}$ admits a section, i.e. if there is a map $\mathbb{R}_+ \to C_{\mathrm{PL}_2(I)}(y,z)$, $\mu \to g_\mu$ such that $\varphi_{y,z,\tau}(g_\mu) = \mu$ then g_μ is unique and can be constructed.*

Proposition 4.23. *Let $y, z \in \mathrm{PL}_2^0(J)$ such that $\mathrm{Fix}(y) = \mathrm{Fix}(z)$ and let λ be in the interior of J such that $y(\lambda) \neq \lambda$. Define*

$$\begin{array}{rccc} \psi_{y,z,\lambda}: & C_{\mathrm{PL}_+(J)}(y,z) & \longrightarrow & J \\ & g & \longmapsto & g(\lambda). \end{array}$$

Suppose $y^n(\lambda) \underset{n\to\infty}{\longrightarrow} \tau$. Then

(i) *There is a map $\rho_\lambda : J \to \mathbb{R}_+$ such that the following diagram commutes:*

$$\begin{tikzcd} C_{\mathrm{PL}_+(J)}(y,z) \arrow[r, "\varphi_{y,z,\tau}"] \arrow[dr, "\psi_{y,z,\lambda}"'] & \mathbb{R}_+ \\ & J \arrow[u, "\rho_\lambda"'] \end{tikzcd}$$

(ii) *$\psi_{y,z,\lambda}$ is injective.*

(iii) *If $\psi_{y,z,\lambda}$ admits a section, i.e. if there is a map $J \to C_{\mathrm{PL}_2(I)}(y,z)$, $\mu \to g_\mu$ such that $\psi_{y,z,\lambda}(g_\mu) = \mu$ then g_μ is unique and can be constructed.*

Proof. Let $\mathrm{Fix}(y) = \mathrm{Fix}(z) = \{\eta = \mu_0 < \mu_1 < \ldots < \mu_k < \mu_{k+1} = \zeta\}$ and suppose $\mu_i < \lambda < \mu_{i+1}$ for some i. We define the partial map $\rho_\lambda : J \to \mathbb{R}_+$ as

$$\rho_\lambda(\mu) = \begin{cases} \lim_{n\to\infty} \frac{y^n(\mu)-\tau}{z^n(\lambda)-\tau} & \mu \in [\mu_i, \mu_{i+1}] \\ 1 & \text{otherwise} \end{cases}$$

Since $\mathrm{Fix}(y) = \mathrm{Fix}(z)$, $z^n(\lambda) \underset{n\to\infty}{\longrightarrow} \tau$ and τ is fixed by g. Thus if $\mu = g(\lambda)$, then $y^n(\mu) = g(z^n(\lambda)) \underset{n\to\infty}{\longrightarrow} \tau$. With this definition, the proof follows closely that of Lemma 4.7(ii), Proposition 4.14 and by applying Corollary 4.22 and the previous Remark. $\square$

We conclude this subsection with a technical lemma which we will need later on:

Lemma 4.24. *Let $\tau, \mu \in J$, $h \in \mathrm{PL}_+(J)$. Then:*

(i) *The limit $\varphi_\pm = \lim\limits_{n\to\infty} h^{\pm n}(\tau)$ exists and $h(\varphi_\pm) = \varphi_\pm$,*

(ii) *We can determine whether there is or not an $n \in \mathbb{Z}$, such that $h^n(\tau) = \mu$.*

Proof. If $h(\tau) = \tau$ then it is clear. Otherwise, without loss of generality, we can assume $h(\tau) > \tau$. The two sequences $\{h^{\pm n}(\tau)\}_{n\in\mathbb{N}}$ are strictly monotone, and they have a limit $\lim_{n\to\infty} h^{\pm n}(\tau) = \varphi_{\pm} \in [0, 1]$. Thus, by continuity of h

$$\varphi_{\pm} = \lim_{n\to\infty} h^{n+1}(\tau) = \lim_{n\to\infty} h(h^n(\tau)) = h(\varphi_{\pm}).$$

Thus we have that $\{h^n(\tau)\}_{n\in\mathbb{Z}} \subseteq (\varphi_-, \varphi_+)$ and we have that φ_+ is the closest intersection of h with the diagonal on the right of τ (similarly for φ_-), so we can compute φ_+, φ_- directly, without using the limit. As a first check, we must see if $\mu \in (\varphi_-, \varphi_+)$. Then since the two sequences $\{h^{\pm n}(\tau)\}_{n\in\mathbb{N}}$ are monotone, then after a finite number of steps we find $n_1, n_2 \in \mathbb{Z}$ such that $h^{-n_1}(\tau) < \mu < h^{n_2}(\tau)$ and so this means that either there is an integer $-n_1 \leq n \leq n_2$ with $h^n(\tau) = \mu$ or not, but this is a finite check. $\square$

Remark 4.25. We observe that the results of this Subsection do not depend upon dyadic rationals and can be easily generalized by replacing every occurrence of $\mathrm{PL}_2(I)$ with $\mathrm{PL}_{S,G}(I)$. We will briefly restate them in Section 7.

5. Centralizers and Conjugacy in $\mathrm{PL}_2(I)$

In this Section we show how the techniques developed so far allow us to obtain compute centralizers, roots and solve the conjugacy problem in Thompson's group F. Most of the results of this section were first proved by Guba and Sapir in [9] in 1997 using different methods.

5.1. Centralizers and Roots in $\mathrm{PL}_2(I)$.

Proposition 5.1 (Centralizers)**.** *Suppose $x \in F$, then its centralizer is $C_F(x) \cong F^m \times \mathbb{Z}^n$, for some positive integers m, n such that $0 \leq m \leq n+1$ (see figure 7).*

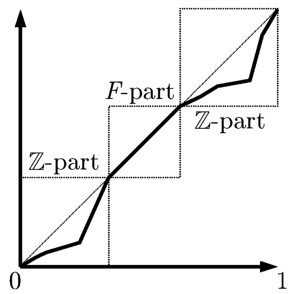

FIGURE 7. The structure of centralizers in F

Proof. Consider the conjugacy problem with $y = z = x$ and let $\partial_2\mathrm{Fix}(x) = \{\eta = \alpha_0 < \alpha_1 < \ldots < \alpha_s < \alpha_{s+1} = \zeta\}$. Since all the points of $\partial_2\mathrm{Fix}(x)$ are fixed by x then $g \in C_{\mathrm{PL}_2(I)}(x)$ must fix the set $\partial_2\mathrm{Fix}(x)$ and thus each of the α_i's. This implies that we can restrict to solve the conjugacy problem in each of the subgroups $\mathrm{PL}_2([\alpha_i, \alpha_{i+1}]) = \mathrm{PL}_2^0([\alpha_i, \alpha_{i+1}])$ and so we can assume that $x \in \mathrm{PL}_2^0(I)$. If $x = id$, then it is immediate $C_{\mathrm{PL}_2(I)}(x) = \mathrm{PL}_2(I)$. Suppose $x \neq id$ on $[0, 1]$, then the map $\varphi_{x,x}$ of Corollary 4.22 is a non-trivial injective group homomorphism. Thus $C_{\mathrm{PL}_2(I)}(x) \cong \log_2(\mathrm{Im}\,\varphi_{x,x}) \leq \mathbb{Z}$, and so $C_{\mathrm{PL}_2(I)}(x)$ is isomorphic to a subgroup of $\mathbb{Z}$. Therefore $C_{\mathrm{PL}_2(I)}(x) \cong \mathbb{Z}$. Let $[\alpha_{i_1}, \alpha_{i_1+1}], \ldots, [\alpha_{i_n}, \alpha_{i_n+1}]$ be the family of intervals such that $x|_{[\alpha_{i_j}, \alpha_{i_j+1}]} \neq id$, then the number of intervals where there restriction of x is trivial cannot be more than $n+1$: x can be trivial only on the intervals $[\eta, \alpha_{i_1}], [\alpha_{i_1+1}, \alpha_{i_2}], \ldots, [\alpha_{i_{n-1}+1}, \alpha_{i_n}], [\alpha_{i_n+1}, \zeta]$. □

This result can be generalized to the groups $\mathrm{PL}_{S,G}(I)$ but it is more difficult since the initial slopes are not necessarily powers of 2 and so we need a different argument to show that the images of the maps $\varphi_{x,x}$ are discrete subgroups.

Proposition 5.2 (Computing Roots)**.** *Let $id \neq x \in \mathrm{PL}_2(I)$, then the function x has only a finite number of roots and every root is constructible, i.e., there is an algorithm to compute it.*

Proof. We suppose that $\partial_2\mathrm{Fix}(z) = \{0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r < \alpha_{r+1} = 1\}$ and we restrict to an interval $[\alpha_i, \alpha_{i+1}]$ (we can repeat the argument for each . We may then assume $\partial_2\mathrm{Fix}(z) = \{0, 1\}$. Let $m = x'(0^+)$ and let $n \in \mathbb{N}$ such that $\sqrt[n]{m}$ is still an integral power of 2 (otherwise it does not make sense to look for a n-th root). We want to determine whether there is, or not, a $g \in \mathrm{PL}_2(I)$ such that $g^{-1}xg = x$ and such that $g'(0^+) = \sqrt[n]{m}$. Suppose that there is such a g, then $g^{-k}xg^k = x$ and $(g^k)'(0^+) = m$. Then, by the uniqueness of the solution of the conjugacy problem with initial slope m (Corollary 4.22), we have that $g^n = x$. Conversely, if we have h such that $h^n = x$, then $h'(0^+) = \sqrt[n]{m}$. But $h^{-1}xh = h^{-1}h^n h = h^n = x$. Thus an element h is a n-th root of x if and only if it is the solution the "differential type" equation with a given initial condition

$$\begin{cases} h^{-1}xh = x \\ h'(0^+) = \sqrt[n]{m}\,. \end{cases}$$

So we can decide whether or not there is a n-th root, by solving the equivalent conjugacy problem with a given initial slope. Moreover, if the n-th root of g exists, it is computable by Proposition 4.19 and unique by Corollary 4.22. Moreover, only finitely many roots are possible because the initial slope of the root divides that of x. □

Corollary 5.3. *Suppose $x \in \mathrm{PL}_2(I)$ is such that $C_{\mathrm{PL}_2(I)}(x) \cong \mathbb{Z}$, then $C_{\mathrm{PL}_2(I)}(x) = \langle \sqrt[k]{x} \rangle$, for some $k \in \mathbb{Z}$.*

Proof. Let $\varphi_{x,x}$ be as in Corollary 4.22, then $\log_2(\operatorname{Im}\varphi_{x,x}) = M\mathbb{Z}$, for some $M \in \mathbb{Z}$. Let $2^n = \varphi_{x,x}(x)$ and let $\widehat{x} = \varphi_{x,x}^{-1}(2^M)$. Thus there is a $k \in \mathbb{Z}$ with $2^n = \varphi_{x,x}(x) = \varphi_{x,x}(\widehat{x}^k) = 2^{kM}$. This implies that $k = n/M$ and that $\widehat{x} = \sqrt[k]{x}$, since $\varphi_{x,x}$ is injective. Thus $C_{\mathrm{PL}_2(I)}(x) = \langle \sqrt[k]{x} \rangle$. □

Proposition 5.4 (Intersection of Centralizers). *Let $x_1, \dots, x_k \in \mathrm{PL}_2(I)$ and define $C := C_{\mathrm{PL}_2(I)}(x_1) \cap \dots \cap C_{\mathrm{PL}_2(I)}(x_k)$. If the interval I is divided by the points in the union $\partial_2\mathrm{Fix}(x_1) \cup \dots \cup \partial_2\mathrm{Fix}(x_k)$ into the intervals J_i then*

$$C = C_{J_1} \cdot C_{J_2} \cdot \ldots \cdot C_{J_r},$$

where $C_{J_i} := \{f \in C \mid f(t) = t, \forall t \notin J_i\} = C \cap \mathrm{PL}_2(J_i)$. Moreover, each C_{J_i} is isomorphic to either $\mathbb{Z}$, or $\mathrm{PL}_2(J_i)$ or the trivial group.

Proof. The set $\partial_2\mathrm{Fix}(x_i)$ is fixed by all elements in $C_{\mathrm{PL}_2(I)}(x_i)$, therefore all elements in C fix the end points of the intervals J_i. The decomposition of C as $C_{J_1} \cdot \ldots \cdot C_{J_r}$ follows from the observation:

Claim: Let J and J' be intervals such that $J' \subseteq J$. Then for any $x \in \mathrm{PL}_2(J)$, such that $\partial_2\mathrm{Fix}(x)$ does not contain any points in the interior of J' we have the restriction of

$$C_{\mathrm{PL}_2(J)}(x) \cap \{g \in \mathrm{PL}_2(J) \mid g(J') = J'\}$$

to the interval J' is either trivial in the case that x does not preserves the interval J' or $C_{\mathrm{PL}_2(J')}(x)$ otherwise.

Proof of the Claim. Let $g \in C_{\mathrm{PL}_2(J)}(x) \cap \{g \in \mathrm{PL}_2(J) \mid g(J') = J'\}$. If $x(J') = J'$ then it is immediate that $g|_{J'} \in C_{\mathrm{PL}_2(J')}(x)$. Suppose now that $x(J') \neq J'$ and $g|_{J'} \neq id$ and say that $J' = [\gamma_1, \gamma_2]$. Thus $x(\gamma_1) \neq \gamma_1$ or $x(\gamma_2) \neq \gamma_2$. Without loss of generality we can assume that $x(\gamma_1) \neq \gamma_1$. Let $[c, d]$ be the largest interval containing γ_1 such that $x(t) \neq t$ for any $t \in \mathbb{Z}[1/2] \cap (c, d)$. The proof of the previous Proposition implies that g coincides with $(\sqrt[M]{x})^k$ for some root of x and some integer k on the interval $[c, d]$. Since $\sqrt[M]{x}(\gamma_1) \neq \gamma_1$, k must be 0 and so g is the identity map on $[c, d]$. The restrictions on J' and J imply that $J' \subseteq [c, d]$, which completes the proof. □

By the previous claim we see that, for each $i = 1, \dots, r$ and $j = 1, \dots, k$, the restriction of the subgroup $C_{\mathrm{PL}_2(I)}(x_j) \cap \{g \in \mathrm{PL}_2(I) \mid g(J_i) = J_i\}$ is either trivial or equal to $C_{\mathrm{PL}_2(J_i)}(x_j)$. Thus $C_{J_i} = id$ or $C_{J_i} = C_{\mathrm{PL}_2(J_i)}(x)$ for some $x \in \mathrm{PL}_2(I)$ which, by the previous Proposition, is isomorphic with $\mathbb{Z}$ or $\mathrm{PL}_2(J_i)$. □

Corollary 5.5. *The intersection of any number $k \geq 2$ centralizers of elements of F is equal to the intersection of 2 centralizers.*

Proof. Let $C = C_{\mathrm{PL}_2(I)}(x_1) \cap \dots \cap C_{\mathrm{PL}_2(I)}(x_k)$ be the intersection of $k \geq 2$ centralizers of elements of F. By the previous Proposition we have $I = J_1 \cup \dots \cup J_r$ and $C = C_{J_1} \cdot \ldots \cdot C_{J_r}$. We want to define $w_1, w_2 \in \mathrm{PL}_2(I)$ such that $C =$

$C_{\mathrm{PL}_2(I)}(w_1) \cap C_{\mathrm{PL}_2(I)}(w_2)$. We define them on each interval $J_i := [\alpha_i, \alpha_{i+1}]$, depending on C_{J_i}. *Case 1:* If $C_{J_i} = id$, then we define w_1, w_2 as any two elements in $\mathrm{PL}_2^<(J_i)$ such that are not one a power of another. *Case 2:* If $C_{J_i} \cong \langle x \rangle$ for some $id \neq x \in \mathrm{PL}_2(J_i)$, then we define $w_1 = w_2 = x$. *Case 3:* If $C_{J_i} = \mathrm{PL}_2(J_i)$, then we define $w_1 = w_2 = id$. □

Question 5.6. Corollary 5.5 determines that any intersection of more than one centralizer of elements in F can be expressed as the intersection $C_F(w_1) \cap C_F(w_2)$ for two suitable elements $w_1, w_2 \in F$. Is it possible to build the two elements w_1, w_2 inside the subgroup $\langle x_1, \ldots, x_k \rangle$? The current proof does not give an answer to this question.

5.2. **The conjugacy problem for** $\mathrm{PL}_2(I)$**.** We are now ready to give an alternative proof of the solvability of the ordinary conjugacy problem for Thompson's group F.

Lemma 5.7. *For any $y, z \in \mathrm{PL}_2^0(I)$ we can decide whether there is (or not) a $g \in \mathrm{PL}_2(I)$ with $y^g = z$.*

Proof. Let $y, z \in \mathrm{PL}_2(I)$, $y \neq z$. We use Theorem 3.1 to make $\mathrm{Fix}(y) = \mathrm{Fix}(z)$, if possible. In order to be conjugate, we must have $y'(0^+) = z'(0^+)$ and $y'(1^-) = z'(1^-)$. Up to taking inverses of y and z, we can assume that $2^u = y'(0^+) = z'(0^+) < 1$. Now observe that $g^{-1}yg = z$ is satisfied if and only if $(y^v g)^{-1} y (y^v g) = z$ is satisfied for every $v \in \mathbb{Z}$. If $2^{\rho(g)}$ is the initial slope of g, then $2^{vu+\rho(g)}$ is the initial slope of $y^v g$. Thus, up to taking powers of y, we can assume that the initial slope of g is between 2^u and 2^{-1}. Now we choose all $q \in U := \{2^u, 2^{u+1}, \ldots, 2^{-1}\}$ as possible initial slopes for g, therefore we apply the stair algorithm for $\mathrm{PL}_2^0(I)$ for all the elements of U and check if we find a solution or not. There is only a finite number of "possible" initial slopes, so the algorithm will terminate. Moreover, by Lemma 2.22 we can derive the uniqueness of each solution, for a given initial slope. □

The previous Lemma provides a way to find all possible conjugators, however it is not an efficient way to do it because we are taking all possible slopes into consideration.

Theorem 5.8. *The group $\mathrm{PL}_2(I)$ has solvable conjugacy problem.*

Proof. We use Theorem 3.1 again and suppose that $\partial_2\mathrm{Fix}(y) = \partial_2\mathrm{Fix}(z) = \{0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r < \alpha_{r+1} = 1\}$. Now we restrict to an interval $[\alpha_i, \alpha_{i+1}]$ and consider $y, z \in \mathrm{PL}_2^0([\alpha_i, \alpha_{i+1}])$. If $\mathrm{Fix}(y)$ contains the a subinterval of $[\alpha_i, \alpha_{i+1}]$, then we must have $y = z = id$ on the whole interval $[\alpha_i, \alpha_{i+1}]$ and so any function $g \in \mathrm{PL}_2([\alpha_i, \alpha_{i+1}])$ will be a conjugator. Otherwise, $\mathrm{Fix}(y)$ does not contain any subinterval of $[\alpha_i, \alpha_{i+1}]$ and so we can apply the previous Lemma on it. If we find a solution on each such interval, then the conjugacy problem is solvable. Otherwise, it is not. □

Remark 5.9. The idea of the solution of the conjugacy problem extends to the groups $\mathrm{PL}_{S,G}(I)$, provided that we have an analogue of Theorem 3.1 to make $\partial\mathrm{Fix}(y) = \partial\mathrm{Fix}(z)$. Then will prove that the set X of possible initial slopes for g is at most countable and that the intersection $X \cap [y'(0), 1]$ is contained in some finite set. By the proof of Lemma 5.7, to solve the conjugacy problem it is enough to test all the candidate conjugators with initial slopes in $X \cap [y'(0), 1]$.

The argument given to solve the conjugacy problem in F also works, in much the same way, to solve the power conjugacy problem. We say that a group G has *solvable power conjugacy problem* if there is an algorithm such that, given any two elements $y, z \in G$, we can determine whether there is, or not, a $g \in G$ and two non-zero integers m, n such that $g^{-1}y^m g = z^n$, that is, there are some powers of y and z that are conjugate.

Theorem 5.10. *The group* $\mathrm{PL}_2(I)$ *has solvable power conjugacy problem.*

Proof. Again, we can use Theorem 3.1, suppose that $\partial_2\mathrm{Fix}(y) = \partial_2\mathrm{Fix}(z)$ and restrict to a smaller interval $J = [\eta, \zeta]$ with dyadic endpoints and such that $y, z \in \mathrm{PL}_2^0(J)$. If $g \in \mathrm{PL}_2(J)$ and m, n exist then we must have that the initial slope of y^m and z^n must coincide. A simple argument on the exponent of these slopes, implies that this can happen if and only if y^m and z^n are both powers of a common minimal power $(y^\alpha)'(\eta) = (z^\beta)'(\eta)$. Hence the problem can be reduced to finding whether there is a $g \in \mathrm{PL}_2(J)$ and an integer k such that $g^{-1}y^{k\alpha}g = z^{k\beta}$. By Lemma 4.10 (that can be naturally generalized to $\mathrm{PL}_2^0(J)$), we have that this is equivalent to finding a $g \in \mathrm{PL}_2(J)$ such that $g^{-1}y^\alpha g = z^\beta$. Hence solving the power conjugacy problem is equivalent to solving the conjugacy problem for y^α and z^β. □

6. The k-Simultaneous Conjugacy Problem in $\mathrm{PL}_2(I)$

We will make a sequence of reductions to solve a particular case. These reductions will use the fact that we are able to solve the ordinary conjugacy problem. First we notice that, since we know how to solve the ordinary conjugacy problem, then solving the $(k+1)$-simultaneous conjugacy problem is equivalent to find a positive answer to the following problem:

Problem 6.1. Is there an algorithm such that given $(x_1, \ldots, x_k, y)$ and $(x_1, \ldots, x_k, z)$ it can decide whether there is a function $g \in C_{\mathrm{PL}_2(I)}(x_1) \cap \ldots \cap C_{\mathrm{PL}_2(I)}(x_k)$ such that $g^{-1}yg = z$?

Since we understand the structure of the intersection of centralizers, we are going to work on solving this last question. Our strategy now is to reduce the problem to the ordinary conjugacy problem and to isolate a very special case that must be dealt with. Along the way, we discuss what requirements we must assume to generalize the proof to the groups $\mathrm{PL}_{S,G}(I)$.

6.1. **General case: any k and any centralizer.** This subsection deals with the general case. We will first extend Theorem 3.1 and then we will use our description for the intersection of many centralizers to solve the general problem. The argument of Proposition 6.3 will show us that we can build possible conjugators by using the stair algorithm and then check if they are in an intersection of centralizers. This will be verifiable, since we have given a description of such intersection in Proposition 5.4.

Lemma 6.2. *Let $x_1, \ldots, x_k, y, z \in \mathrm{PL}_2(J)$. We can determine whether there is, or not, a $g \in C = C_{\mathrm{PL}_2(J)}(x_1) \cap \ldots \cap C_{\mathrm{PL}_2(J)}(x_k)$ such that $g(\mathrm{Fix}(y)) = \mathrm{Fix}(z)$.*

Proof. The proof is essentially the same as that of Corollary 3.7 on each of the intervals between two dyadic fixed points of y and z. The only new tool required is Lemma 4.24 on the intervals where C is isomorphic to $\mathbb{Z}$. We omit the details of this proof. $\square$

Proposition 6.3. *Let $x_1, \ldots, x_k, y, z \in \mathrm{PL}_2(J)$. We can determine whether there is, or not, a $g \in C = C_{\mathrm{PL}_2(J)}(x_1) \cap \ldots \cap C_{\mathrm{PL}_2(J)}(x_k)$ with $g^{-1}yg = z$.*

Proof. Apply Lemma 6.2 to make $\mathrm{Fix}(y) = \mathrm{Fix}(z)$, if possible. Recall that a candidate conjugator must centralize $x_1, \ldots, x_k$ too, so it has to fix $\bigcup_{i=1}^k \partial_2\mathrm{Fix}(x_i)$ and $\partial_2\mathrm{Fix}(y) = \partial_2\mathrm{Fix}(z)$. Let $\bigcup_{i=1}^k \partial_2\mathrm{Fix}(x_i) = \{\lambda_m\}_m$ and $\partial_2\mathrm{Fix}(y) = \{\mu_1 < \ldots < \mu_k\}$ and let J_i denote the interval $[\mu_i, \mu_{i+1}]$, for $i = 1, \ldots, k-1$. We build g on each interval J_i, depending on how y is defined on J_i. We have the following three cases:

Case 1: y is the identity on J_i. In this case we define g to be the identity on J_i.

Case 2: y is not the identity on J_i and there is a point $\lambda_j \in \bigcup_{i=1}^k \partial_2\mathrm{Fix}(x_i)$ which is in the interior of J_i. Since $\mu_i < \lambda_j < \mu_{i+1}$ and λ_j is dyadic, then $\lambda_j \notin \partial_2\mathrm{Fix}(y)$ and in particular λ_j is not fixed by y and z. Since $g(\lambda_j) = \lambda_j$, the proof of Lemma 4.7(ii) implies that $g'(\mu_i^+) = \lim_{n\to\infty} \frac{y^n(\lambda_j)-\mu_i}{z^n(\lambda_j)-\mu_i}$, hence the slope of g on the right of μ_i is uniquely determined. Therefore we can apply Proposition 4.23(iii) to build the unique candidate conjugator g.

Case 3: y is not the identity on J_i and $\bigcup_{i=1}^k \partial_2\mathrm{Fix}(x_i)$ does not contain any point of the interior of J_i. More precisely, each x_r does not fix any point in J_i and so, by the Claim contained in the proof of Proposition 5.4 we have that the restriction group

$$C_{\mathrm{PL}_2(J)}(x_r) \cap \{g \in \mathrm{PL}_2(J) \mid g(J_i) = J_i\}$$

is the trivial group or $\mathrm{PL}_2(J_i)$ or isomorphic to a copy of $\mathbb{Z}$. Since C_{J_i} is the intersection of all the restriction groups for $r = 1, \ldots, k$, then C_{J_i} will also be trivial or $\mathrm{PL}_2(J_i)$ or infinite cyclic. If C_{J_i} is trivial, we choose g to be trivial on J_i. If $C_{J_i} = \mathrm{PL}_2(J_i)$ then the construction reduces to solving the ordinary conjugacy problem in $\mathrm{PL}_2(J_i)$. The case $C_{J_i} \cong \mathbb{Z}$ will be covered in Subsection 6.2.

Finally we have to verify that the element g constructed by the above procedure conjugates y to z and commutes with x. $\square$

The restatement of the k-simultaneous conjugacy problem given in Problem 6.1 and the previous Proposition imply the result of Theorem A in the introduction.

6.2. **A special case:** $k = 1$ **and** $C_{\mathrm{PL}_+(J)}(x) \cong \mathbb{Z}$**.** This subsection is technical and it deals with a variant of the ordinary conjugacy problem. We want to see if we can solve it when we have a restriction on the possible conjugators. Thus, given x, y, z we want to see if $g^{-1}yg = z$ for a $g \in C_{\mathrm{PL}_2(J)}(x) \cong \mathbb{Z}$. In particular, if $\sqrt[M]{x}$ is the "smallest possible" root (in the sense of the proof of centralizers in $\mathrm{PL}_2(J)$), then we need to find if there is a power of $\sqrt[M]{x}$ which conjugates y to z. Since $C_{\mathrm{PL}_2(J)}(x) = C_{\mathrm{PL}_2(J)}(\sqrt[M]{x}) = \langle \sqrt[M]{x} \rangle$ then we can substitute x with $\widehat{x} := \sqrt[M]{x}$. For simplicity, we assume still call $\widehat{x}$ with x. The plan for this subsection will be to reduce to solving an equation of the type

$$f^k = wh^k$$

where f, h, w are given, $w'(\eta^+) = 1$ and we need to find if there is any $k \in \mathbb{Z}$ satisfying the previous equation. The second step will be to prove that there is only a finite number of k's to that may solve the equation and so we try all of them.

We need first to run the usual conjugacy problem on $[\eta, \zeta]$ between y and z to see if they are conjugate. If they are, we continue. Otherwise we stop. Let $C_{\mathrm{PL}_2(J)}(y, z) = \{g \in \mathrm{PL}_2(J) \mid g^{-1}yg(t) = z(t), \text{ for all } t \in J\} = g_0 \cdot C_{\mathrm{PL}_2(I)}(y)$, for some $g_0 \in \mathrm{PL}_2(J)$. Now $C_{\mathrm{PL}_2(J)}(y) \cong \mathbb{Z}^s \times \mathrm{PL}_2(J)^t$. Notice that $s = t = 0$ is impossible.

If $s + t \geq 2$, then there must be some $\tau \in (\eta, \zeta) \cap \mathbb{Z}[\frac{1}{2}]$ fixed point for every element in $C_{\mathrm{PL}_2(I)}(y)$. So if y and z are conjugate through a power of x then there is a k such that $x^k(\tau) = g_0(\tau)$. Notice $x(\tau) \neq \tau$, so we apply Lemma 4.24 with $\mu := g_0(\tau)$ and find, if possible a unique integer $\bar{k}$ such that $x^{\bar{k}}(\tau) = \mu$. Now we take $g := x^{\bar{k}}$ and we check if it is a conjugator or not.

If $s = 0, t = 1$, then this would mean that $C_{\mathrm{PL}_2(J)}(y) \cong \mathrm{PL}_2(J)$ and so that $y = id$ on $[\eta, \zeta]$ and so do not need to check the powers of x, but simply if the function $z = id$ on $[\eta, \zeta]$.

If $s = 1, t = 0$, then $C_{\mathrm{PL}_2(J)}(y) = \langle \widehat{y} \rangle \cong \mathbb{Z}$, for $\widehat{y}$ a generator. Thus, y and z are conjugate through an element of $C_{\mathrm{PL}_2(J)}(x)$, if and only if there exist $k, m \in \mathbb{Z}$ such that $x^m = g_0 \widehat{y}^n$ in $[\eta, \zeta]$.

Lemma 6.4. *Let $x, y, z \in \mathrm{PL}_2(J)$ be such that $C_{\mathrm{PL}_2(J)}(x) = \langle x \rangle$ and $C_{\mathrm{PL}_2(J)}(y) = \langle \widehat{y} \rangle$. Then there exists $X, Y, G_0 \in \mathrm{PL}_2(J)$ such that $G_0'(\eta^+) = 1$ and following two problems are equivalent:*

(i) *Find powers $k, m \in \mathbb{Z}$ such that $x^m = g_0 \widehat{y}^n$*
(ii) *Find a power $k \in \mathbb{Z}$ such that $X^k = G_0 Y^k$.*

Proof. It is clear that (ii) is a special case of (i), thus it is enough to reduce (i) to (ii). Suppose we have $x'(\eta^+) = 2^\alpha, \widehat{y}'(\eta^+) = 2^\beta, g_0'(\eta^+) = 2^\gamma$ for some $\alpha, \beta, \gamma \in \mathbb{Z}$ and that (m, n) satisfy $x^m = g_0\widehat{y}^n$. Then comparing the slopes at η^+, we obtain $2^{\alpha m} = (x^m)'(\eta^+) = (g_0\widehat{y}^n)'(\eta^+) = 2^{\gamma+\beta n}$ and so $\alpha m = \gamma + \beta n$. Thus, if we have a solution we must have that $\gcd(\alpha, \beta)$ divides γ. That is, $\gamma = \alpha m_0 - \beta n_0$, for some $m_0, n_0 \in \mathbb{Z}$, which can be computed and thus $\alpha(m - m_0) = \beta(n - n_0)$. We can change variables and call $\widetilde{m} = m - m_0$ and $\widetilde{n} = n - n_0$. So we have to find $\widetilde{m}, \widetilde{n}$ such that $\alpha\widetilde{m} = \beta\widetilde{n}$ and so that

$$\frac{\alpha}{\gcd(\alpha, \beta)}\widetilde{m} = \frac{\beta}{\gcd(\alpha, \beta)}\widetilde{n}.$$

Thus there must exist a $k \in \mathbb{Z}$ such that

$$\widetilde{m} = \frac{\beta}{\gcd(\alpha, \beta)}k \text{ and } \widetilde{n} = \frac{\alpha}{\gcd(\alpha, \beta)}k.$$

Going backwards, we write

$$m := \frac{\beta}{\gcd(\alpha, \beta)}k + m_0 \text{ and } n := \frac{\alpha}{\gcd(\alpha, \beta)}k + n_0.$$

By substituting these two values in the equation $x^m = g_0y^n$ we get

$$(x^{\frac{\beta}{\gcd(\alpha,\beta)}})^k = x^{-m_0}g_0\widehat{y}^{n_0}(\widehat{y}^{\frac{\alpha}{\gcd(\alpha,\beta)}})^k.$$

We rename $X = x^{\frac{\beta}{\gcd(\alpha,\beta)}}$, $G_0 = x^{-m_0}g_0\widehat{y}^{n_0}$ and $Y = \widehat{y}^{\frac{\alpha}{\gcd(\alpha,\beta)}}$ and so we are left to find a $k \in \mathbb{Z}$, if it exists, such that

$$X^k = G_0Y^k. \tag{2}$$

Thus (i) reduces to solving (ii) for X, Y and G_0 constructed above. Notice that after this reduction $G_0(\eta^+) = 2^0 = 1$. □

In the last case we are examining, both x and y cannot have fixed dyadic points, since their centralizers are cyclic groups. Thus $\mathrm{Fix}(x) \cap (\eta, \zeta)$ and $\mathrm{Fix}(y) \cap (\eta, \zeta)$ must be empty or finite. The same is also true for the new functions X and Y, i.e. $\mathrm{Fix}(X)\cap(\eta, \zeta)$ and $\mathrm{Fix}(Y)\cap(\eta, \zeta)$ must be empty or finite. For sake of simplicity, we will still call X, Y, G_0 with lowercase letters. We will make distinction in the following cases, by checking what are $\mathrm{Fix}(x) \cap (\eta, \zeta)$ and $\mathrm{Fix}(y) \cap (\eta, \zeta)$ and see if they coincide or not.

$\mathrm{Fix}(x) \cap (\eta, \zeta) \neq \mathrm{Fix}(y) \cap (\eta, \zeta)$. There exists a $\tau \in (\eta, \zeta)$ with $y(\tau) = \tau \neq x(\tau)$. Thus, by applying Lemma 4.24, we can determine if there is a k such that $x^k(\tau) = g_0(\tau)$. We act similarly if there is a $\tau \in (\eta, \zeta)$ with $x(\tau) = \tau \neq y(\tau)$.

$\mathrm{Fix}(x) \cap (\eta, \zeta) = \mathrm{Fix}(y) \cap (\eta, \zeta) \neq \emptyset$. Suppose $\mathrm{Fix}(x) = \mathrm{Fix}(y) = \{r_1 < \ldots < r_v\}$. Observe that if the equation has a solution then $g_0(r_i) = r_i$ for all r_i. If these conditions are satisfied, then we can build all the solutions by solving the equation in each interval $[r_i, r_{i+1}]$. This reduces the problem to the next case.

$\mathrm{Fix}(x) \cap (\eta, \zeta) = \mathrm{Fix}(y) \cap (\eta, \zeta) = \emptyset$, that is we have that $x, y \in \mathrm{PL}_2^<(J) \cup \mathrm{PL}_2^>(J)$. We can now assume that both $x, y \in \mathrm{PL}_2^<(J)$. Define

$$K := \{k \in \mathbb{Z} \text{ such that } x^k(t) = g_0(y^k(t)) \text{ for all } t \in J\}.$$

Our goal is to find whether or not $K \neq \emptyset$. The first step will be to prove that the set K is finite, by computing directly its upper and lower bounds. Therefore, we will have that $K \subseteq \mathbb{Z} \cap [l_0, k_0]$, for some integers l_0, k_0, and so we can check all these integers and see if any satisfies $x^k(t) = g_0(y^k(t))$.

Lemma 6.5. *Let $x, y \in \mathrm{PL}_2^<(J)$, $g_0 \in \mathrm{PL}_2(J)$ such that $g_0'(\eta^+) = 1$ and let K be the set defined above. Then K is bounded and there is an algorithm which computes the bounds.*

Proof. Assume $K \neq \emptyset$, then we must have $x'(\eta^+) = y'(\eta^+)$. The first step is to prove that there exists a $k_0 \in \mathbb{Z}$, upper bound for K. Suppose that K has no upper bound. Let $\theta < \zeta$ be a point such that $g_0(t) = t$ and $x(t) = y(t)$ on $[\eta, \theta]$. Let $\psi > \theta$ a number such that $x(\psi) < y(\psi)$ and $x(t) \leq y(t)$ for $t \leq \psi$. Since $y \in \mathrm{PL}_2^<(J)$ then $\lim\limits_{k \to \infty} y^k(\psi) = \eta$, and so we can choose a $k_0 \in K$ be a large enough number such that $y^{k_0}(\psi) < \theta$. Suppose $k \geq k_0$, by definition of θ and $k_0 \in K$ we have

$$x^k(\psi) = g_0(y^k(\psi)) = y^k(\psi).$$

Now recall that $x(\psi) < y(\psi) < \theta + \varepsilon$ and so, since $x \leq y$ on $[\eta, \psi]$

$$\begin{aligned} x^k(\psi) &= x^{k-1}(x(\psi)) < x^{k-1}(y(\psi)) \\ &= x^{k-2}(x(y(\psi)) \leq x^{k-2}(y^2(\theta + \varepsilon)) \leq \ldots \leq x(y^{k-1}(\psi)) \leq y^k(\psi). \end{aligned}$$

By comparing the last two expressions, we get $x^k(\psi) < y^k(\psi) = x^k(\psi)$. Contradiction. Therefore k_0 is an upper bound for K.

We now want to bound the K from below, and so we use a similar technique. If $k \in K$ is negative, then we consider the equation

$$y^{-k} = x^{-k} g_0 = g_0(g_0^{-1} x^{-k} g_0) = g_0(g_0^{-1} x g_0)^{-k} = g_0 \widehat{x}^{-k}$$

where we have set $\widehat{x} := g_0^{-1} x g_0$. Since $\mathrm{Fix}(x) = \emptyset$, then $\mathrm{Fix}(\widehat{x}) = \emptyset$ and $\widehat{x} \in \mathrm{PL}_2^<(I)$. So we have reduced to the situation of the previous claim (with $\widehat{x}$ and y switched in their role) and we obtain that the set of possible $(-k)$'s is bounded above, so that k is bounded below. □

Since K is finite the k's to be checked are finite and we can find its bound in finite time. Now we can check all possible the elements of K and we are done with this case.

In order to generalize this argument to the groups $\mathrm{PL}_{S,G}(I)$ introduced in Section 2 we need to make further assumptions on the ring S and the group G. We will specify all the requirements in Remark 7.4, but we discuss here the one needed to generalize the argument of this subsection:

- We assume that there is an algorithm such that, given $a, b, c \in G$, it is able to determine whether or not there exist $x, z \in \mathbb{Z}$ such that $a^x = b^y c^z$.

Remark 6.6. By taking logarithms in base b, we can rewrite all of the terms of the requirement above, so that it becomes equivalent to the following: given any $\alpha, \beta, \gamma \in \mathbb{R}$, determine whether or not they are linearly dependent over $\mathbb{Q}$ and, if they are, we can find $q_1, q_2 \in \mathbb{Q}$ such that $\gamma = q_1\alpha + q_2\beta$. This rewriting transforms the equation $a^x = bc^z$ into a $\mathbb{Q}$-linearity dependence relation, hence if there is a solution, it is unique.

The requirement on S and G is sufficient to extend the special case of this subsetion to the groups $\mathrm{PL}_{S,G}(I)$: it allows us to generalize the proof of Lemma 6.4, since we cannot take logarithms in base 2 anymore. Moreover, it is straightforward to verify that the rest of this Subsection does not rely on dyadic rationals and hence that it generalizes to any ring S and subgroup G.

7. Generalizing to the groups $\mathrm{PL}_{S,G}(I)$

We now move on to prove the solvability of the simultaneous conjugacy problem of the subgroups of $\mathrm{PL}_{S,G}(I)$ of $\mathrm{PL}_+(I)$ introduced in Section 2 and whose structure generalizes that of Thompson's group F. We remark that Brin and Squier [4] give a criterion for conjugacy in $\mathrm{PL}_+(I)$. We need to introduce some notation for the groups $\mathrm{PL}_{S,G}(I)$.

Definition 7.1. We define an ideal in S given by $\mathcal{I}_{S,G} = \langle (g-1) \mid g \in G \rangle$. We denote with $\pi_{S,G} : S \to S/\mathcal{I}$ the natural quotient map. Unless otherwise stated, we will drop the subscript and write $\mathcal{I}$ and π instead of $\mathcal{I}_{S,G}$ and $\pi_{S,G}$.

The following two results are used to detect when two points of S are in the same $\mathrm{PL}_{S,G}$-orbit.

Lemma 7.2. *Let $J \subseteq [0, 1]$ be a closed interval with at least one of the endpoints η in S and let $g \in \mathrm{PL}_{S,G}(J)$. Then, for every $t \in J \cap S$, we have $\pi(g(t)) = \pi(t)$.*

Proof. We can assume that the η is the left one and we apply induction on the number of breakpoints before t. In case the endpoint in S is the right one, we apply induction on the breakpoints after t. Let $\{\eta_1, \ldots, \eta_r\}$ be the set of all breakpoints of g on the interval $[\eta, t)$. Then $g(t) = c_r(t - \eta_r) + g(\eta_r)$ for some suitable $c_i \in G$. By induction on r we have that $\pi(g(\eta_r)) = \pi(\eta_r)$ and thus

$$\begin{aligned}\pi(g(t)) &= \pi(c_r(t - \eta_r) + g(\eta_r)) = \\ &= \pi(c_r - 1)\pi(t - \eta_r) + \pi(1)\pi(t - \eta_r) + \pi(g(\eta_r)) = \\ &= \pi(t - \eta_r) + \pi(\eta_r) = \pi(t).\end{aligned}$$

□

This result gives us a necessary condition on how homeomorphisms can be built. We want to know what orbits of elements are under the action of $\mathrm{PL}_{S,G}(J)$.

Proposition 7.3. *Let $J \subseteq [0,1]$ be a closed interval with both endpoints in S and let $u, v \in J \cap S$. Then $\pi(u) = \pi(v)$ if and only if there is a $g \in \mathrm{PL}_{S,G}(J)$ such that $g(u) = v$.*

The proof of this proposition can be found in the Appendix (see Proposition A.1).

Remark 7.4 (Computational requirements)**.** We need to add a few requirements to the ring S in order to make a machine able to work with the algorithm. It is reasonable to make the following assumptions to work in the ring S:

- There is solution to the membership problem in S (*i.e.* an algorithm to determine whether an element $s \in \mathbb{R}$ lies in S or not).
- There is a solution to the membership problem in $\mathcal{I}$.
- There is an algorithm that, for every $q \in S$, is able to determine whether two elements in the quotient ring $S/q\mathcal{I}$ are equal or not.
- There is an algorithm such that, given $a, b, c \in G$, it is able to determine whether or not there exist $x, z \in \mathbb{Z}$ such that $a^x = bc^z$.

All these requirements are reasonable to assume in order to make computations inside S and will be checkable in the special cases that we take as examples in Section 8.

Remark 7.5. In general, given two intervals J_1, J_2 with endpoints in S, the groups $\mathrm{PL}_{S,G}(J_1)$ and $\mathrm{PL}_{S,G}(J_2)$ may not be isomorphic. Proposition 7.3 tells us that two elements in S are in the same $\mathrm{PL}_{S,G}$-orbit if their image under the map π is the same. For example in the cases $S = \mathbb{R}, G = \mathbb{R}_+$ and $S = \mathbb{Q}, G = \mathbb{Q}^*$ and $S = \mathbb{Z}\left[\frac{1}{2}\right], G = \langle 2 \rangle$, it is not difficult to see that every two points in S have the same image under π and that any two groups $\mathrm{PL}_{S,G}(J_1)$ and $\mathrm{PL}_{S,G}(J_2)$ are thus isomorphic, for any two intervals J_1, J_2 with endpoints in S. On the other hand, if we consider generalized Thompson's groups (see Section 8), it can be shown that the number of orbits is finite but more than one, so that are only finitely many isomorphism classes for the groups $\mathrm{PL}_{S,G}(J)$, for $S = \mathbb{Z}\big[\frac{1}{n_1}, \dots, \frac{1}{n_k}\big]$ and $G = \langle n_1, \dots, n_k \rangle$ for $n_1, \dots, n_k \in \mathbb{Z}$. We observe that the generalized Thompson's groups which are most often studied are those where we assume that $\mathrm{GCD}(n_1 - 1, \dots, n_k - 1) = 1$, which implies that $S/\mathcal{I}$ is trivial. In general, it seems likely that if two elements $\alpha, \beta \in S$ have different image under π then the groups $\mathrm{PL}_{S,G}([0,\alpha])$ and $\mathrm{PL}_{S,G}([0,\beta])$ are not isomorphic, but it is not easy to prove it.

7.1. **Making** $\mathrm{Fix}(y)$ **and** $\mathrm{Fix}(z)$ **coincide.** We start by generalizing Proposition 7.3 to a finite number of points.

Lemma 7.6. *Let $J = [\eta, \zeta] \subseteq [0,1]$ be a closed interval with endpoints in S and suppose we have $u_1, v_1, \dots, u_k, v_k \in J \cap S$ such that $u_1 < \ldots < u_k$, $v_1 < \ldots < v_k$ and $\pi(u_i) = \pi(v_i)$ for all $i = 1, \dots, k$. Then there exists a $g \in \mathrm{PL}_{S,G}(J)$ such that $g(u_i) = v_i$ for all $i = 1, \dots, k$.*

Proof. We can assume that $J = [\eta, \zeta]$ and that the u_i's are ordered in an increasing sequence $u_1 < \ldots < u_k$ and therefore $v_1 < \ldots < v_k$. By Proposition 7.3, there is an $g_1 \in \mathrm{PL}_{S,G}(J)$ such that $g_1(u_1) = v_1$. Now we notice that $v_1 = g_1(u_1) < g_1(u_2) < \ldots < g_1(u_k)$ and so we restrict to the interval $[v_1, \zeta]$ and, since $\pi(g_1(u_2)) = \pi(u_2) = \pi(v_2)$ we can use again Proposition 7.3 to find an $h_2 \in \mathrm{PL}_{S,G}([v_1, \zeta])$ such that $h_2(g_1(u_2)) = v_2$. Define

$$g_2(t) := \begin{cases} t & t \in [\eta, v_1] \\ h_2(t) & t \in [v_1, \zeta] \end{cases}$$

so that $g_2 g_1(u_1) = v_1, g_2 g_1(u_2) = v_2$ and $g_2 \in \mathrm{PL}_{S,G}(J)$. By iterating this procedure, we build functions $g_i \in \mathrm{PL}_{S,G}(J)$ such that $g_i g_{i-1} \ldots g_1(u_j) = v_j$ for all $j = 1, \ldots, i$ and $i = 1, \ldots, k$. Thus we define $g := g_k g_{k-1} \ldots g_1 \in \mathrm{PL}_{S,G}(J)$ and we get a function such that $g(u_i) = v_i$. $\square$

The previous Lemma yields the following natural generalization of the Extension Lemma 3.3 which we state without proof.

Lemma 7.7 (Extension of Partial Maps)**.** *Let $J \subseteq [0,1]$ be a closed interval with endpoints in S and suppose $I_1, \ldots, I_k \subseteq J$ is a finite family of disjoint closed intervals in increasing order and of the form $I_i = [a_i, b_i]$, for all $i = 1, \ldots, k$ and $a_i, b_i \in S$. Let $J_1, \ldots, J_k \subseteq J$, with $J_i = [c_i, d_i]$, be another family of intervals with the same property and such that $\pi(a_i) = \pi(c_i)$ and $\pi(b_i) = \pi(d_i)$. Suppose that $g_i : I_i \to J_i$ is a piecewise-linear function with a finite number of breakpoints, occurring at points in S and with slopes in G. Then there exists a $\widetilde{g} \in \mathrm{PL}_{S,G}(J)$ such that $\widetilde{g}|_{I_i} = g_i$.*

Let $g \in \mathrm{PL}_{S,G}(J)$ be equal to $g(t) = at + b$ around a point $q \in \mathbb{R}$ fixed by f, for some $a \in G, b \in S$, then $q = b/(1-a)$ and so the intersection points of f with the diagonal lie in Q_S, the field of fractions of S. Now that we have a way to recognize whether we can make two elements of S coincide through an element of $\mathrm{PL}_{S,G}(J)$, we need to see if it is possible to do the same for the field of fractions Q_S.

Proposition 7.8. *Let $J = [\eta, \zeta] \subseteq [0,1]$ be a closed interval with endpoints in S and let $\alpha, \beta \in J \cap Q_S$. There is a $g \in \mathrm{PL}_{S,G}(J)$ with $g(\alpha) = \beta$ if and only if we can find $p, q, r \in S$ such that $\alpha = p/q, \beta = r/q$ and*

$$pG = rG \pmod{q\mathcal{I}}$$

where $q\mathcal{I}$ denotes the product of the ideal generated by q and $\mathcal{I}$.

Proof. Suppose there is a map $g \in \mathrm{PL}_{S,G}(J)$ with $g(\alpha) = \beta$ and let $g(t) = ct + d$ in a small neighborhood J_α of α. We can choose representatives $p, q, r \in S$ such that $\alpha = p/q, \beta = r/q$ and then, since $g \in \mathrm{PL}_{S,G}(J)$, we use Lemma 7.2 to get

$$\pi(t) = \pi(g(t)) = \pi(c-1)\pi(t) + \pi(t) + \pi(d)$$

for all $t \in J_\alpha \cap S$ and therefore $\pi(d) = 0$, which implies $d \in \mathcal{I}$. Conversely, suppose that we can write $\alpha = p/q, \beta = r/q$, for some $p, q, r \in S$ and that $pG = rG$ (mod $q\mathcal{I}$). The second condition implies that there exist $c_1, c_2 \in G, d_2 \in \mathcal{I}$ such that

$$c_1 r = c_2 p + q d_2$$

and so if we set $c = c_2/c_1$ and $d = d_2/c_1$, we get $r = cp + qd$. Let $f(t) = ct + d$ be a line through the point (α, β) and let $[\gamma, \delta] \subseteq J$ be a small interval such that $\gamma, \delta \in S$. Finding the interval $[\gamma, \delta]$ can be accomplished this way: we can assume $G \neq 1$ and pick any $1 \neq c \in G$ such that $0 < c < 1$. Then we choose $m, n \in \mathbb{N}$ such that $\eta + c^m < \alpha < \eta + nc^m < \zeta$ and we set $\gamma := \eta + c^m, \delta := \eta + nc^m$. Since $\pi(d) = 0$ we have that $\pi(f(\gamma)) = \pi(\gamma)$ and $\pi(f(\delta)) = \pi(\delta)$ and so, by the Extension Lemma 7.7 there is an $g \in \mathrm{PL}_{S,G}(J)$ with $g|_{[\gamma,\delta]} = f$. By construction $g(\alpha) = \beta$ as required. □

In a similar fashion, we can get the same result for a finite number of points. This amounts to finding small segments passing through the rational pairs (α_i, β_i) and then applying the Extension Lemma to obtain a homeomorphism of the whole interval J. We thus state without proof the following Lemma.

Lemma 7.9. *Let $J = [\eta, \zeta] \subseteq [0,1]$ be a closed interval with endpoints in S and let $\alpha_i, \beta_i \in J \cap Q_S$ for $i = 1, \dots, k$. There is a $g \in \mathrm{PL}_{S,G}(J)$ with $g(\alpha_i) = \beta_i$ if and only if there exist $g_1, \dots, g_k \in \mathrm{PL}_{S,G}(J)$ such that $g_i(\alpha_i) = \beta_i$.*

By the assumptions made in Remark 7.4, we can detect whether or not two elements in Q_S are equal, thus we obtain the following generalizations of Corollary 3.7 and Lemma 3.1:

Corollary 7.10. *Let $J = [\eta, \zeta] \subseteq [0,1]$ be a closed interval with endpoints in S and let $\alpha_i, \beta_i \in J \cap Q_S$ for $i = 1, \dots, k$. We can determine whether there is or not an $f \in \mathrm{PL}_{S,G}(J)$ such that $g(\alpha_i) = \beta_i$ for every $i = 1, \dots, k$.*

Proposition 7.11. *Given $y, z \in \mathrm{PL}_{S,G}(I)$, we can determine whether there is or not a $g \in \mathrm{PL}_{S,G}(I)$ such that $g(\mathrm{Fix}(y)) = \mathrm{Fix}(g^{-1}yg) = \mathrm{Fix}(z)$. If such a g exists, we can construct it.*

7.2. **Linearity Boxes and Stair Algorithm .** In this Subsection we briefly generalize the results of Subsections 4.1, 4.2 and 4.3.

Lemma 7.12 (Linearity Boxes). *Suppose $y, z, g \in \mathrm{PL}_{S,G}(J)$ and $g^{-1}yg = z$.*

(i) *If there exist two numbers $\alpha > 0$ and $c \geq 1$ such that $y(t) = z(t) = c(t - \eta) + \eta$ for $t \in [\eta, \eta + \alpha]$, then the graph of g is linear inside the square $[\eta, \eta + \alpha] \times [\eta, \eta + \alpha]$*

(ii) *If there exist $\beta, c \in (0,1)$ such that $y(t) = z(t) = c \cdot (t - \zeta) + \zeta$ on $[\beta, \zeta]$, then the graph of g is linear inside the square $[\beta, \zeta] \times [\beta, \zeta]$.*

Proof. These results follow from the proofs of Lemma 4.2 and Remark 4.3. □

We recall that $\mathrm{PL}^0_{S,G}(J)$ denotes the set of functions $f \in \mathrm{PL}_{S,G}(J)$ such that the set $\mathrm{Fix}(f)$ does not contain elements of S other than the endpoints of J.

Proposition 7.13 (Stair Algorithm for $\mathrm{PL}^0_{S,G}(J)$)**.** *Let $J \subseteq [0,1]$ be a closed interval with endpoints in S, let $y, z \in \mathrm{PL}^0_{S,G}(J)$ such that $\mathrm{Fix}(y) = \mathrm{Fix}(z)$ and define $C_{\mathrm{PL}_{S,G}(J)}(y,z) = \{g \in \mathrm{PL}_{S,G}(J) | y^g = z\}$ the set of all conjugators. For any $\tau \in \mathrm{Fix}(y)$ we define the map*

$$\begin{array}{rccc} \varphi_{y,z,\tau}: & C_{\mathrm{PL}_{S,G}(J)}(y,z) & \longrightarrow & \mathbb{R}_+ \\ & g & \longmapsto & g'(\tau), \end{array}$$

where if τ is an endpoint of J we take only a one-sided derivative. Then

(i) *$\varphi_{y,z,\tau}$ is an injective map. In particular, if we define $\varphi_{z,\tau} := \varphi_{z,z,\tau}$, then $\varphi_{z,\tau}$ is a group homomorphism.*

(ii) *If $q \in G$ is a fixed number we can decide whether or not there is a $g \in \mathrm{PL}_{S,G}(J)$ with initial slope $g'(\eta^+) = q$ such that $y^g = z$. If g exists, it is unique. In other words, if there is a g such that $\varphi_{y,z,\tau}(g) = \mu \in G$ then g is unique and can be constructed.*

Proof. Immediate generalization of Corollary 4.22. □

Corollary 7.14. *Let $y, z \in \mathrm{PL}^<_{S,G}(J)$ and $g \in \mathrm{PL}_+(J)$ such that $y^g = z$ and $g'(\eta) \in G$. Then $g \in \mathrm{PL}_{S,G}(J)$.*

7.3. **Centralizers and Roots in** $\mathrm{PL}_{S,G}(J)$ **.** This section proves a generalization of Proposition 5.1. The centralizers $C_{\mathrm{PL}_{S,G}(J)}(z)$ of elements will be direct products of copies of $\mathbb{Z}$'s and of $\mathrm{PL}_{S,G}(U)$'s, for some suitable intervals U. In order to prove this, we will use the Stair Algorithm to build a "section" of the map φ_x. As in the proof of Proposition 5.1, we will reduce the study to functions in $\mathrm{PL}^0_{S,G}(J)$. Consider the conjugacy problem with $y = z$ and let $\partial_S\mathrm{Fix}(z) = \{0 = \alpha_0 < \alpha_1 < \ldots < \alpha_s < \alpha_{s+1} = 1\}$. Since all the points of $\partial_S\mathrm{Fix}(z)$ are fixed by z, then $g \in C_{\mathrm{PL}_{S,G}(I)}(z)$ must fix the set $\partial_S\mathrm{Fix}(z)$ and thus each of the α_i's. This implies that we can restrict to solving the conjugacy problem in each of the subgroups $\mathrm{PL}_{S,G}([\alpha_i, \alpha_{i+1}]) = \mathrm{PL}^0_{S,G}([\alpha_i, \alpha_{i+1}])$. If $z = 1$, it is immediate that $C_{\mathrm{PL}_{S,G}(J)}(x) = \mathrm{PL}_{S,G}(J)$, so now we can focus on $1 \neq z \in \mathrm{PL}^0_{S,G}(J)$. Consider $\mathbb{R}_+$ to be the multiplicative group of positive real numbers. Let $A \subset \mathbb{R}_+$ be the set of all possible initial slopes of functions g such that $g^{-1}zg = z$. The set A is not empty, since $\langle z \rangle \subseteq C_{\mathrm{PL}_{S,G}(J)}(z)$. For a given closed interval J with endpoints in S we define a map

$$\psi : A \to C_{\mathrm{PL}_{S,G}(J)}(z)$$

$$\alpha \mapsto g_\alpha$$

which sends an initial slope α to its associated conjugating function g_α. By the uniqueness of a conjugator with a given initial slope, we notice immediately that $g_\alpha \circ g_\beta = g_{\alpha\cdot\beta}$ and so A is a subgroup of $\mathbb{R}_+$ and ψ is a homomorphism of groups.

Moreover, the uniqueness of the conjugator implies also that ψ is an isomorphism. The main result of this section is the following:

Theorem 7.15. *Let $J \subseteq [0,1]$ be a closed interval with endpoints in S and let $id \neq z \in \mathrm{PL}^0_{S,G}(J)$. Then $C_{\mathrm{PL}_{S,G}(J)}(z)$ is isomorphic with $\mathbb{Z}$.*

We remark that Theorem 7.15 has also been proved by Brin and Squier (Theorem 5.5 in [4]) for the case of $\mathrm{PL}_+(I)$. Altinel and Muranov have also proved it independently (Lemma 4.2 in [1]) using different methods for a family of subgroups of $\mathrm{PL}_+(I)$ which is analogous to the subgroups $\mathrm{PL}_{S,G}(I)$. We mention that the second author of the current paper also has an alternative version of this proof using a conjugacy invariant equivalent to that of Brin and Squier (see Theorem 5.1 in [11]). The tools that we will use in the version of the proof that we are about to give are relevant for Lemma 7.19.

Proof of Theorem 7.15. By the discussion above we have that the group $A = \{g'(\eta^+) \mid g \in C_{\mathrm{PL}_{S,G}(J)}(z)\}$ is isomorphic with $C_{\mathrm{PL}_{S,G}(J)}(z)$. We start by assuming that $z \in \mathrm{PL}^<_{S,G}(J)$ and we want to prove that A is discrete. We assume, by contradiction that A is not discrete.
Step 1: If A is not discrete, then A is dense in $\mathbb{R}_+$.

Proof. This is a standard well known fact. $\square$

Step 2: Let $[\eta,\alpha]^2$ be the first initial linearity box and $[\beta,\tau]^2$ be the first final linearity box, for some $\tau \leq \zeta$ fixed point for z. Without loss of generality, we can assume that the restriction $z|_{[\eta,\tau]} \in \mathrm{PL}^<_+([\eta,\tau])$. Let r be a positive integer big enough so that $z^{-r}(\alpha) > \beta$. Then z^r is linear on $[\beta, z^{-r}(\alpha)]$, say with slope b.

Proof. Since A is dense in $\mathbb{R}_+$, we can pick a $c \in C_{\mathrm{PL}_{S,G}(J)}(z)$ such that $c'(\eta^+) < 1$ is arbitrarily close to 1. Now, observe that $c \in \mathrm{PL}^<_+([\eta,\tau])$ and look at the two hand sides of $cz^r = z^r c$, by restricting this equality to the interval $[\beta, z^{-r}(\alpha)]$. Suppose $\{\mu_1 < \ldots < \mu_k\}$ are the breakpoints of z^r on $[\beta, z^{-r}(\alpha)]$, hence they are also the breakpoints of cz^r on $[\beta, z^{-r}(\alpha)]$, since c is linear on $[\eta,\alpha]$. On the interval $[\beta,\tau]$ we can write $c^{-1}(t) = \lambda(t-1)+1$, where $\lambda = c'(\tau^-)$: if we have chosen $c'(\eta^+) \neq 1$ to be close enough to 1, then $\lambda < 1$ is also arbitrarily close to 1. Since c^{-1} is linear on $[\beta,\tau]$ then, if we choose λ close enough to 1, the set of breakpoints of $z^r c$ on $[\beta, z^{-r}(\alpha)]$ will be $c^{-1}(\{\mu_1,\ldots,\mu_k\}) = \{\lambda(\mu_1-1)+1,\ldots,\lambda(\mu_k-1)+1\}$. As $cz^r = z^r c$ on $[\beta, z^{-r}(\alpha)]$ we must have that $\{\mu_1,\ldots,\mu_k\} = c^{-1}(\{\mu_1,\ldots,\mu_k\})$ and so $\lambda = 1$, which is a contradiction. $\square$

Step 3: Define $a = \frac{d}{dt}z^r(t)\Big|_{t=\eta^+} < 1$ to be the initial slope of z^r. For every positive integer i, the map z^r is linear on $[z^{-ir}(\beta), z^{-(i+1)r}(\alpha)]$ with slope a.

Proof. We assume by induction that the result is true for any integer less than i. Consider now the map $z^{(i+1)r}$ and apply the chain rule on two intervals, recalling that $\frac{d}{dt}z^r(t) = a$ on the intervals $[z^{-jr}(\beta), z^{-(j+1)r}(\alpha)]$ for any $j < i$:

$$\tfrac{d}{dt}z^{(i+1)r}(t) = a^i b \qquad\qquad t \in [\beta, z^{-ir}(\alpha)]$$

$$\tfrac{d}{dt}z^{(i+1)r}(t) = a^{i-1}b\tfrac{d}{dt}z^r(t) \qquad t \in [z^{-ir}(\beta), z^{-(i+1)r}(\alpha)].$$

We apply Step 2 using the positive integer $(i+1)r$, hence we have that $z^{(i+1)r}$ must be linear on $[\beta, z^{-(i+1)r}(\alpha)]$ and we can equate the two derivatives computed above to get $a^i b = a^{i-1}b\frac{d}{dt}z^r(t)$ on the interval $[z^{-ir}(\beta), z^{-(i+1)r}(\alpha)]$. We simplify both sides and get the thesis of the Claim. $\square$

By sending $i \to \infty$ in Claim 2 we see that the slope of z^r around τ^- must be equal to $a < 1$. However, since the restriction $z^r|_{[\eta,\tau]} \in \mathrm{PL}_+^<([\eta,\tau])$, we must have that $\frac{d}{dt}z^r(t)\Big|_{t=\tau^-} > 1$, which is a contradiction. Therefore A is a discrete subgroup of $\mathbb{R}_+$ and so it is isomorphic with $\mathbb{Z}$. $\square$

Theorem 7.16. *Let $J = [\eta, \zeta] \subseteq [0,1]$ be a closed interval with endpoints in S and $z \in \mathrm{PL}_{S,G}(J)$, then:*

(i) *$C_{\mathrm{PL}_{S,G}(I)}(z)$ is isomorphic with a direct product of copies of $\mathbb{Z}$'s and $\mathrm{PL}_2(J_i)$'s for some suitable intervals $J_i \subseteq I$.*
(ii) *For every positive integer n we can decide whether or not $\sqrt[n]{z}$ exists.*

Proof. The proofs of (1) and (2) follow from the proofs of Propositions 5.2 and 5.1 by replacing every occurrence of ∂_2 with ∂_S and by applying the previous corollary to get the centralizers of elements in $\mathrm{PL}^0_{S,G}(J)$. Moreover, to prove (2) we need to observe that, in order to start the procedure, we need to verify whether or not $\sqrt[n]{z'(\eta^+)} \in S$. $\square$

The following is an immediate generalization of Proposition 5.4:

Proposition 7.17 (Intersection of Centralizers)**.** *Let $J = [\eta, \zeta] \subseteq [0,1]$ be a closed interval with endpoints in S, let $z_1, \ldots, z_k \in \mathrm{PL}_{S,G}(J)$ and define the subgroup $C := C_{\mathrm{PL}_{S,G}(I)}(z_1) \cap \ldots \cap C_{\mathrm{PL}_{S,G}(I)}(z_k)$. If the interval J is divided by the points in the union $\partial_S\mathrm{Fix}(z_1) \cup \cdots \cup \partial_S\mathrm{Fix}(z_k)$ into the intervals J_i then*

$$C = C_{J_1} \cdot C_{J_2} \cdot \ldots \cdot C_{J_r},$$

where $C_{J_i} := \{f \in C \mid f(t) = t, \forall t \notin J_i\} = C \cap \mathrm{PL}_{S,G}(J_i)$. Moreover, each C_{J_i} is isomorphic to either $\mathbb{Z}$, or $\mathrm{PL}_{S,G}(J_i)$ or the trivial group.

Corollary 7.18. *Let $J = [\eta, \zeta] \subseteq [0,1]$ be a closed interval with endpoints in S and $y, z \in \mathrm{PL}^0_{S,G}(J)$. Then $C_{\mathrm{PL}_{S,G}(J)}(y,z)$ is either empty or countable.*

Proof. Suppose that the set $C_{\mathrm{PL}_{S,G}(J)}(y,z)$ is not empty, then we have that

$$C_{\mathrm{PL}_{S,G}(J)}(y,z) = g_0 \cdot C_{\mathrm{PL}_{S,G}(J)}(y)$$

for a suitable $g_0 \in \mathrm{PL}_{S,G}(J)$. Thus $\#C_{\mathrm{PL}_{S,G}(J)}(y,z) = \#C_{\mathrm{PL}_{S,G}(J)}(y) = \aleph_0$ which is countable by Theorem 7.15. $\square$

In order to solve the conjugacy problem in $\mathrm{PL}_{S,G}(I)$, we need to check whether or not there are candidate conjugators in a given interval of initial slopes.

Lemma 7.19. *Let $J = [\eta, \zeta]$ be a closed interval with endpoints in S and let $W = [w, 1] \cap G$ for some number $w \in \mathbb{R}$. If $y, z \in \mathrm{PL}^0_{S,G}(J)$, then the set*

$$\{g'(\eta^+) \mid g \in C_{\mathrm{PL}_{S,G}(J)}(y, z)\} \cap W$$

is contained in a finite set V that can be constructed directly.

Proof. We will use the notation of Theorem 7.15. Since the argument of this proof will be based on the Stair Algorithm, which works in $\mathrm{PL}_+(J)$, we can restrict our attention on the interval between η and the first fixed point of z. Hence, we can assume $y, z \in \mathrm{PL}^<_+(J)$ without loss of generality. We choose a positive integer r following the proof of Proposition 4.9: that is, we choose the smallest integer r such that

$$\min\{z^{-r}(\alpha), y^{-r}(\eta + w(\alpha - \eta))\} > \beta$$

using the lowest possible initial number w. Using the explicit conjugator formula for an initial slope $q \in W$ (see Corollary 4.13), we know that the candidate conjugator has the shape $g_q := y^{-r} g_{0,q} z^r$ on the interval $[\eta, z^{-r}(\alpha)]$ for a suitable map $g_{0,q}$ that has initial slope $q \in W$. Our choice of r guarantees that, for any $q \in W$, the map g_q lies inside the final linearity box at the point $z^{-r}(\alpha)$.

Claim: Choose an integer i such that $z^{-ir}(\beta) > z^{-r}(\alpha)$. Then z^r must have a breakpoint $p \in [z^{-ir}(\beta), z^{-(i+1)r}(\alpha)]$.

Proof of the Claim. Let $a = \frac{d}{dt} z^r(t)\Big|_{t=\eta^+} < 1$. If z^r were linear on the interval $[z^{-ir}(\beta), z^{-(i+1)r}(\alpha)]$ then, by Step 3 of Theorem 7.16, we would have that z^r is linear on every interval $[z^{-kir}(\beta), z^{-k(i+1)r}(\alpha)]$ with slope a for every positive integer $k \geq 2$. Arguing as in the conclusion of Theorem 7.16, this would imply that $\frac{d}{dt} z^r(t)\Big|_{t=\zeta^-} = a < 1$ which is a contradiction. ☐

By construction, the map $g_{0,q}$ can be built to be linear on the interval $[\eta, z^{-(i+1)r}(\alpha)]$. We observe that z^r has a breakpoint at p, hence $g_0 z^r$ must have a breakpoint at p. Now, for the map $y^{-r} g_{0,q} z^r$ to be a candidate conjugator, it must be linear around the point p, thus the breakpoints of $g_{0,q} z^r$ on the interval $[z^{-ir}(\beta), z^{-(i+1)r}(\alpha)]$ must be canceled by the set $\{c_1, \ldots, c_v\}$ of all the breakpoints of y^{-r} on $[\eta, \zeta]$, thus the image of p under $g_{0,q} z^r$ must go to a breakpoint of y^{-r}. Since $g_{0,q} z^r(p) = q(z^r(p) - \eta) + \eta \in \{c_1, \ldots, c_v\}$, then there are only finitely many choices for $q \in W$. ☐

Remark 7.20. Since the finite set V of Lemma 7.19 can be computed directly, we can run the stair algorithm on all elements of V as possible initial slopes and thus find all possible conjugators with slopes in $[w, 1] \cap G$.

7.4. **The Conjugacy Problem for** $\mathrm{PL}_{S,G}(I)$**.**

Theorem 7.21. *The conjugacy problem in* $\mathrm{PL}_{S,G}(I)$ *is solvable.*

Proof. The proof follows that of Lemma 5.7 and Theorem 5.8 and we reduce to solve a finite number of conjugacy problems, where we can assume $y, z \in \mathrm{PL}^0_{S,G}(I)$ and initial slopes for candidate conjugators contained in the interval $[y'(0), 1]$. By Lemma 7.19, the set of possible initial slopes inside $[y'(0), 1] \cap G$ is finite and can be directly constructed, so we are done as in the proof of Lemma 5.7. □

The algorithm used to solve the k-simultaneous problem in the case of the group F can be extended in full generality to the groups $\mathrm{PL}_{S,G}(I)$, except for one of its steps which must be dealt with using the last of the requirements of Remark 7.4.

Theorem 7.22. *The k-simultaneous conjugacy problem in* $\mathrm{PL}_{S,G}(I)$ *is solvable.*

Proof. To prove the solvability of the k-simultaneous conjugacy problem we can mimic completely the proof used for Thompson's group F. We need to replace every occurrence of ∂_2 with ∂_S and speak of elements of S instead of dyadic rational numbers. The only part in which we need refine the argument is in the case of Subsection 6.2 in which we reduce to solve the equation

$$x^m = g_0 \widehat{y}^n \tag{3}$$

where $x, y, g_0 \in \mathrm{PL}_{S,G}([\eta, \zeta])$ are given and we are looking for $m, n \in \mathbb{Z}$ satisfying the previous equation. We define $q = g_0'(\eta^+) \in \mathbb{R}_+$ and so $x'(\eta^+) = q^\alpha$, $y'(\eta^+) = q^\beta$, $g_0'(\eta^+) = q$ for some $\alpha, \beta \in \mathbb{R}$. Notice that in Subsection 6.2 we have $\alpha, \beta, \gamma \in \mathbb{Z}$, while here not all of them are integers. We must then have

$$q^\alpha = x'(\eta^+)^m = (g_0 \widehat{y}^n)'(\eta^+) = q^{1+\beta n} \tag{4}$$

and therefore we need to solve the equation

$$\alpha m = 1 + \beta n \tag{5}$$

for some $m, n \in \mathbb{Z}$. We observe that if equation (5) is solvable, then α is rational if and only if β is rational. Thus, if either α or β is a rational number it is immediate to check whether there is a solution to (5). If α and β are both irrational, then equation (5) becomes a $\mathbb{Q}$-linearity dependence relation and, if it is solvable, then the dimension of the vector space generated by α, β and 1 over $\mathbb{Q}$ is exactly 2. By Remark 6.6 and the last of the requirements in Remark 7.4 we are able to detect if this last statement is true or not. In case it is true, then there is a unique solution to (5) and it is given by the coordinates of 1 in the basis α and β, thus it is now trivial to check if there is a integer solution or not. In case there is a solution to equation (5), we do not need to find a bound for $m, n \in \mathbb{Z}$ as for the case of Thompson's group F, because there is at most one solution. The remaining part of the algorithm follows as before. □

8. Interesting Examples

Now that we have developed the general theory, we are going to see a few interesting examples where the simultaneous conjugacy problem is solvable. We will not dwell too much on the details here, sketching only why it is possible to verify the requirements.

Example 8.1. $S = \mathbb{Q}$ and $G = \mathbb{Q}^* = \mathbb{Q} \cap (0, \infty)$.

Since $\mathbb{Q}$ is a field, $S/\mathcal{I} = \{0\}$ so all the requirements of Remark 7.4 are satisfied. To solve the simultaneous conjugacy problem, we need to solve equation (4), which becomes

$$\frac{a_1^m}{b_1^m} = \frac{ca_2^n}{db_2^n}$$

where we can assume that all numerators are coprime with the denominators. By equating prime factors in the equation to be solved, we get a system of equations of the type $\alpha_i m = \gamma_i + \beta_i n$, for $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}$. All of them can be solved in the same fashion as in Lemma 6.4 and we can reduce equation (3) to the equation $X^k = G_0 Y^k$ and solve it as in Subsection 6.2.

Example 8.2. $S = \mathbb{Z}\left[\frac{1}{n_1}, \dots, \frac{1}{n_k}\right]$ and $G = \langle n_1, \dots, n_k \rangle$ for $n_1, \dots, n_k \in \mathbb{Z}$.

We observe that $S = \mathbb{Z}\left[\frac{1}{n_1 \dots n_k}\right]$ and it can be shown that, if $r := n_1 \dots n_k$, then $S/\mathcal{I} \cong \mathbb{Z}/r\mathbb{Z}$ as rings and therefore the requirements of Remark 7.4 are also satisfied. Equation (4) can be treated as in the previous example. For $k = 1$, we recall that the groups $\mathrm{PL}_{S,G}(I)$ are known as *generalized Thompson's groups.*

Example 8.3. $S = \mathbb{Z}\left[\frac{1}{n_1}, \dots, \frac{1}{n_k}, \dots\right]$ with $G = \langle \{n_i\}_{i \in \mathbb{N}} \rangle$ for a sequence $\{n_i\}_{i \in \mathbb{N}} \subseteq \mathbb{Z}$.

This example is easily reducible to the previous one, since if we are given a finite set E of elements in $\mathrm{PL}_{S,G}(I)$ we can consider the set $\{n_{i_1}^{\alpha_{i_1}}, \dots, n_{i_v}^{\alpha_{i_v}}\}$ of all slopes of elements of E. Then $E \subseteq \mathrm{PL}_{S',G'}(I)$ with ring $S' := \mathbb{Z}\left[\frac{1}{n_{i_1}}, \dots, \frac{1}{n_{i_v}}\right]$ and group $G' := \langle n_{i_1}, \dots, n_{i_v} \rangle$.

Example 8.4. S finite algebraic extension over $\mathbb{Q}$ and $G = S^* := S \cap (0, \infty)$.

As with the first example, since S is a finite algebraic extension it is not difficult to verify that all the requirements of Remark 7.4 are satisfied.

Example 8.5. $S = \mathbb{R}$ and $G = \mathbb{R}_+$.

In order to verify the requirements for this case, we need to discuss exactly what we mean by real number and how we implement it in a machine. In most cases, we work with numbers which are expressed as roots of polynomials in some subfields of $\mathbb{R}$ and we are able to give a complete answer and the same is true for all the requirements of Remark 7.4.

Appendix A. Transitivity in $\mathrm{PL}_{S,G}(I)$

This appendix contains the proof of Proposition 7.3:

Proposition A.1. *Let $J \subseteq [0,1]$ be a closed interval with endpoints in S and let $u, v \in J \cap S$. Then $\pi(u) = \pi(v)$ if and only if there is a $g \in \mathrm{PL}_{S,G}(J)$ such that $g(u) = v$.*

Proof. The sufficient condition is implied by Lemma 7.2. Suppose now that $J = [\eta, \zeta]$ and let $L = \zeta - \eta$. We recenter the axis at (η, η), so that interval J is now $[0, L]$. For $\alpha \in G, \beta \in J \cap S$ such that $\alpha\beta < L - \beta$ define (see figure 8)

$$g_{\alpha,\beta}(t) := \begin{cases} \alpha t & t \in [0, \beta] \\ t - (1-\alpha)\beta & t \in [\beta, L - \alpha\beta] \\ \frac{1}{\alpha}(t - L) + L & t \in [L - \alpha\beta, L]. \end{cases}$$

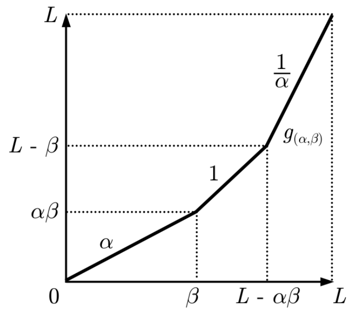

Figure 8. The basic function to get transitivity.

Using the maps $g_{(\alpha,\beta)}$ or $g^{-1}_{(\alpha,\beta)}$ we can send any number $\beta \le t \le L - \alpha\beta$ to $t - (1-\alpha)\beta$ and any number $\alpha\beta \le t \le L - \beta$ to $t + (1-\alpha)\beta$. We define a relation on $J \cap S$ by saying that $t_1 \sim t_2$, if either $t_2 = g_{(\alpha,\beta)}(t_1)$ for some $\alpha \in G, \beta \in J \cap S$ such that $\beta \le t \le L - \alpha\beta$ or $t_2 = g^{-1}_{(\alpha,\beta)}(t_1)$ for some $\alpha \in G, \beta \in J \cap S$ such that $\alpha\beta \le t \le L - \beta$. Then we take the transitive closure of this relation, to get an equivalence relation. Now, since $\pi(u) = \pi(v)$ then we have that $v - u \in \mathcal{I}$ and so

$$v - u = (1-\alpha_1)\beta_1 + \ldots + (1-\alpha_k)\beta_k$$

for some $\alpha_i \in G, \beta_i \in J \cap S$. We want to rewrite $v - u$ as a sum of terms with β_i's small enough so that we can use the defined equivalence relation. We will define a suitable sequence of numbers m_i and $\beta_{i,j}$ with $1 \le j \le m_i$, for each $i = 1, \ldots, k$. Take β_1 and choose a number $\beta_{i,1} \in J \cap S$ small enough such that $g_{(\alpha_i, \beta_{i,1})}$ can be defined. Then choose inductively a number $\beta_{i,j} \in J \cap S$ small enough such that it satisfies all the following three properties

- $g_{(\alpha_i, \beta_{i,j})}$ can be defined
- the number $\beta^0_{i,j+1} := \beta_i - \beta_{i,1} - \ldots - \beta_{i,j} > 0$ is strictly positive

- the number
$$u + (1-\alpha_1)\sum_{s=1}^{m_1}\beta_{1,s} + \ldots + (1-\alpha_i)\sum_{s=1}^{j-1}\beta_{i,s}$$
lies in the interval $[\beta_{i,j}, L-\alpha_i\beta_{i,j}]$.

We stop when we find an index m_i such that the number β^0_{i,m_i} has the property that $g_{(\alpha_i,\beta^0_{i,m_i})}$ can be defined and
$$u + (1-\alpha_1)\sum_{s=1}^{m_1}\beta_{1,s} + \ldots + (1-\alpha_i)\sum_{s=1}^{m_i-1}\beta_{i,s}$$
lies in the interval $[\beta^0_{i,m_i}, L-\alpha_i\beta^0_{i,m_i}]$ and so we define $\beta_{i,m_i} := \beta^0_{i,m_i}$. We iterate this argument for each $i = 1, \ldots, k$ and thus we can rewrite
$$v - u = (1-\alpha_1)\sum_{j=1}^{m_1}\beta_{1,j} + \ldots + (1-\alpha_k)\sum_{j=1}^{m_k}\beta_{k,j}$$
and so
$$\begin{aligned}u \sim & u + (1-\alpha_1)\beta_{1,1} \sim u + (1-\alpha_1)(\beta_{1,1}+\beta_{1,2}) \sim \ldots \sim \\ & u + (1-\alpha_1)\sum_{j=1}^{m_1}\beta_{1,j} + \sim \ldots \sim \\ & u + (1-\alpha_1)\sum_{j=1}^{m_1}\beta_{1,j} + \ldots + (1-\alpha_k)\sum_{j=1}^{m_k}\beta_{k,j} = v\end{aligned}$$
implying that there exists an element $g \in \mathrm{PL}_{S,G}(J)$ such that $g(u) = v$. □

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