# BROWN REPRESENTABILITY DOES NOT COME FOR FREE

CARLES CASACUBERTA AND AMNON NEEMAN

Abstract. We exhibit a triangulated category $\mathcal{T}$ having both products and coproducts and a triangulated subcategory $\mathcal{S} \subset \mathcal{T}$ which is both localizing and colocalizing, and for which neither a Bousfield localization nor a colocalization exists. It follows that neither the category $\mathcal{S}$ nor its dual satisfy Brown representability. Our example involves an abelian category whose derived category does not have small Hom-sets.

## Introduction

In recent years, several authors have proved remarkable generalizations of Brown's representability theorem [1]; see, for example, [3, 6, 7, 8]. It therefore becomes important to have an example of a triangulated category where Brown representability fails. In this short note we produce such a category.

There has also been considerable activity on the subject of localization in homotopy theory, and in particular on Bousfield's old problem of proving the existence of localization of spaces or spectra with respect to cohomology theories. In [2] it was shown that the existence of cohomological localizations follows from a suitable large-cardinal axiom, although Bousfield's problem remains open under the ZFC axioms alone.

In a similar vein, it was asked in [5, p. 35] if every localizing subcategory (i.e., one which is closed under triangles and coproducts) of a stable homotopy category admits a Bousfield localization. Although the answer is not

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known in ZFC either, the counterexample displayed in the present article shows that Bousfield localizations need not exist for localizing subcategories of arbitrary triangulated categories.

More explicitly, we show that there is an abelian category $\mathcal{A}$, due to Freyd, for which the following holds:

(i) The category $\mathcal{A}$ satisfies the [AB5] and [AB4*] conditions (it has exact products and coproducts, and filtered colimits are exact).

(ii) Nevertheless, the derived category $\mathbf{D}(\mathcal{A})$ does not have small Hom-sets. That is, there is a proper class of morphisms between certain objects of $\mathbf{D}(\mathcal{A})$.

(iii) Let $\mathbf{K}(\mathcal{A})$ be the homotopy category of chain complexes in $\mathcal{A}$, and let $\mathbf{A}(\mathcal{A})$ be the full subcategory of acyclic complexes. Then $\mathbf{A}(\mathcal{A})$ is both a localizing and a colocalizing subcategory, but neither a Bousfield localization nor a colocalization exist for $\mathbf{A}(\mathcal{A})$ in $\mathbf{K}(\mathcal{A})$.

(iv) Neither the category $\mathbf{A}(\mathcal{A})$ nor its dual satisfy Brown representability.

## 1. Description and proof

In his 1966 book [4, Chapter 6, Exercise 1, pp. 131–132], Freyd constructed an interesting abelian category. Let us briefly recall the construction.

Let $I$ be the class of all ordinals, and let $R = \mathbb{Z}[I]$ be the polynomial ring freely generated by $I$. The ring $R$ has a proper class of elements, but for what we will do this is no problem. Let $\mathcal{A}$ be the abelian category of all small $R$-modules. Thus an object in $\mathcal{A}$ is a (small) abelian group $M$ together with endomorphisms $\varphi_i \colon M \longrightarrow M$ for every $i \in I$, such that all the $\varphi_i$ commute. The morphisms in $\mathcal{A}$ are the $R$-module homomorphisms. Given two objects $M$ and $N$ in $\mathcal{A}$, there is only a set of morphisms $\mathrm{Hom}_{\mathcal{A}}(M, N)$; it is a subset of the set of abelian group homomorphisms.

Note that the abelian category $\mathcal{A}$ has many good properties. It satisfies the [AB5] and [AB4*] conditions. After all, it is the category of modules over a ring, albeit a very large ring. However, there is no generator or cogenerator, and it will follow from our remarks that there are not enough projectives or injectives.

Let $\mathbb{Z} \in \mathcal{A}$ be the trivial $R$-module. Thus the underlying abelian group is the additive group of integers $\mathbb{Z}$, and all the maps $\varphi_i \colon \mathbb{Z} \longrightarrow \mathbb{Z}$ are zero.

The following observation is due to Freyd [4].

**Lemma 1.1.** *With the notation as above,* $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{Z}, \mathbb{Z})$ *is a proper class.*

*Proof.* For every ordinal $i \in I$ we construct a module $M_i$ such that, as an abelian group, $M_i = \mathbb{Z} \oplus \mathbb{Z}$. The endomorphisms $\varphi_j \colon M_i \longrightarrow M_i$ are given by the following rule:

(i) If $j \neq i$, then $\varphi_j \colon M_i \longrightarrow M_i$ is zero.
(ii) The map $\varphi_i \colon M_i \longrightarrow M_i$ is determined by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is clear that the $M_i$ are pairwise non-isomorphic as $R$-modules, since the element $j \in I$ for which $\varphi_j$ is nonzero on $M_i$ changes as we change $i$. Hence, we have a proper class of non-isomorphic modules $M_i$, each of which fits in an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow M_i \longrightarrow \mathbb{Z} \longrightarrow 0,$$

and we have produced a proper class of elements in $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{Z}, \mathbb{Z})$. ☐

Now consider the category $\mathbf{K}(\mathcal{A})$, the homotopy category of $\mathcal{A}$. The objects are chain complexes of small $R$-modules, and the morphisms are homotopy equivalence classes of chain maps. Each $R$-module is viewed as a chain complex concentrated in degree zero. Let $\mathbf{A}(\mathcal{A}) \subset \mathbf{K}(\mathcal{A})$ be the full subcategory of all acyclic complexes. Both $\mathbf{K}(\mathcal{A})$ and $\mathbf{A}(\mathcal{A})$ are triangulated categories with small Hom-sets.

In what follows, we refer to [8] for the necessary terminology and basic facts. The category $\mathbf{K}(\mathcal{A})$ satisfies the [TR5] and [TR5*] conditions; that is, it has small products and coproducts. The subcategory $\mathbf{A}(\mathcal{A})$ is *localizing* and *colocalizing*, meaning that it is closed under both coproducts and products. (In a triangulated category with coproducts, every triangulated subcategory which is closed under coproducts is automatically thick by [8, Proposition 1.6.8]; that is, it contains all direct summands of its objects.)

The derived category of $\mathcal{A}$ is the Verdier quotient

$$\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})/\mathbf{A}(\mathcal{A}).$$

Since $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(\mathbb{Z}, \Sigma\mathbb{Z}) \cong \mathrm{Ext}^1_{\mathcal{A}}(\mathbb{Z}, \mathbb{Z})$, Lemma 1.1 implies the following.

**Corollary 1.2.** *There is a proper class of morphisms $\mathbb{Z} \longrightarrow \Sigma\mathbb{Z}$ in $\mathbf{D}(\mathcal{A})$.* ☐

We remark that it does not help if we restrict attention to bounded derived categories, since the category $\mathbf{D}^b(\mathcal{A})$ does not have small Hom-sets either.

Recall that a *Bousfield localization* for the pair $\mathbf{A}(\mathcal{A}) \subset \mathbf{K}(\mathcal{A})$ is a right adjoint of the canonical functor $\mathbf{K}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{A})$, and a *Bousfield colocalization* is a left adjoint. As shown in [8, Proposition 9.1.18], a Bousfield localization exists for the pair $\mathbf{A}(\mathcal{A}) \subset \mathbf{K}(\mathcal{A})$ if and only if the inclusion

$$i\colon \mathbf{A}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A})$$

has a right adjoint. Dually, a colocalization exists if and only if $i$ has a left adjoint.

**Corollary 1.3.** *There is neither a Bousfield localization nor a Bousfield colocalization for* $\mathbf{A}(\mathcal{A})$ *in* $\mathbf{K}(\mathcal{A})$*. The inclusion functor* $i\colon \mathbf{A}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A})$ *has neither a right adjoint nor a left adjoint.*

*Proof.* By [8, Theorem 9.1.16], if a Bousfield localization existed for $\mathbf{A}(\mathcal{A}) \subset \mathbf{K}(\mathcal{A})$, then the quotient category $\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})/\mathbf{A}(\mathcal{A})$ would be equivalent to a full subcategory ${}^{\perp}\mathbf{A}(\mathcal{A}) \subset \mathbf{K}(\mathcal{A})$, namely the one whose objects are those $X$ such that

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, X) = 0$$

for all $A \in \mathbf{A}(\mathcal{A})$. For this, the category $\mathbf{D}(\mathcal{A})$ would have to have small Hom-sets. Since this is not the case by Corollary 1.2, a Bousfield localization cannot exist. Dually, there can be no Bousfield colocalization. Therefore, By [8, Proposition 9.1.18], the inclusion of $\mathbf{A}(\mathcal{A})$ into $\mathbf{K}(\mathcal{A})$ has neither a right adjoint nor a left adjoint. □

Let $\mathcal{A}b$ denote the category of abelian groups. A functor from a triangulated category to $\mathcal{A}b$ is called *homological* if it takes triangles to long exact sequences. A triangulated category $\mathcal{T}$ *satisfies Brown representability* if it has small coproducts and every homological functor $H\colon \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{A}b$ that takes products to products is representable; that is, there is an object $A$ in $\mathcal{T}$ such that $H$ is naturally isomorphic to $\mathrm{Hom}_{\mathcal{T}}(-, A)$. (Note that, since products in the dual category $\mathcal{T}^{\mathrm{op}}$ are coproducts in $\mathcal{T}$, our assumption is in fact that $H$ takes coproducts in $\mathcal{T}$ to products in $\mathcal{A}b$.)

**Corollary 1.4.** *Neither the category* $\mathbf{A}(\mathcal{A})$ *nor its dual satisfy Brown representability.*

*Proof.* The category $\mathbf{A}(\mathcal{A})$ has small products and coproducts, and the inclusion

$$i\colon \mathbf{A}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A})$$

respects both. If Brown representability held for $\mathbf{A}(\mathcal{A})$, then the inclusion would have a right adjoint by [8, Proposition 9.1.19]. If Brown representability held for the dual of $\mathbf{A}(\mathcal{A})$, then a left adjoint would have to exist. Corollary 1.3 tells us that we have neither. $\square$

The failure of Brown representability for $\mathbf{A}(\mathcal{A})^{\mathrm{op}}$ can be displayed more explicitly, without referring to results in [8], as follows. (The argument for $\mathbf{A}(\mathcal{A})$ is similar.) The functor $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(\mathbb{Z}, -)$ is a representable functor from $\mathbf{K}(\mathcal{A})$ to $\mathcal{A}b$. The composite

$$\mathbf{A}(\mathcal{A}) \xrightarrow{\ i\ } \mathbf{K}(\mathcal{A}) \xrightarrow{\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(\mathbb{Z},-)} \mathcal{A}b \tag{1}$$

is a homological functor taking products to products, and we assert that it is not representable by any object of $\mathbf{A}(\mathcal{A})$.

Suppose the contrary. If the composite (1) were representable, then there would exist a map $\varphi\colon \mathbb{Z} \longrightarrow A$ where $A \in \mathbf{A}(\mathcal{A})$ and such that all other maps from $\mathbb{Z}$ to acyclic complexes factor uniquely through $\varphi$. Let us complete $\varphi$ to a triangle

$$X \xrightarrow{\ \alpha\ } \mathbb{Z} \xrightarrow{\ \varphi\ } A \longrightarrow \Sigma X$$

in $\mathbf{K}(\mathcal{A})$. Now any morphism $\mathbb{Z} \longrightarrow \Sigma\mathbb{Z}$ in $\mathbf{D}(\mathcal{A})$ can be realized as a pair of maps

$$\mathbb{Z} \xleftarrow{\ \beta\ } Y \longrightarrow \Sigma\mathbb{Z} \tag{2}$$

where $\beta$ is a quasi-isomorphism. This fits into a triangle

$$Y \xrightarrow{\ \beta\ } \mathbb{Z} \xrightarrow{\ \psi\ } B \longrightarrow \Sigma Y$$

with $B \in \mathbf{A}(\mathcal{A})$. Hence, $\psi$ would factor through the universal map $\varphi\colon \mathbb{Z} \longrightarrow A$, and we discover that the above diagram (2) would be equivalent to a diagram

$$\mathbb{Z} \xleftarrow{\ \alpha\ } X \longrightarrow \Sigma\mathbb{Z}\,.$$

Thus each morphism $\mathbb{Z} \longrightarrow \Sigma\mathbb{Z}$ in $\mathbf{D}(\mathcal{A})$ would be represented by some map $X \longrightarrow \Sigma\mathbb{Z}$ in $\mathbf{K}(\mathcal{A})$, where $X$ is fixed. Since there is only a (small) set of such maps, we have contradicted Corollary 1.2.

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Departament d'Àlgebra i Geometria, Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, 08007 Barcelona, Spain

*E-mail address*: `Carles.Casacuberta@ub.edu`

Centre for Mathematics and its Applications, Mathematical Sciences Institute, John Dedman Building, The Australian National University, Canberra, ACT 0200, Australia

*E-mail address*: `Amnon.Neeman@anu.edu.au`