HSU-ROBBINS AND SPITZER'S THEOREMS FOR THE VARIATIONS OF FRACTIONAL BROWNIAN MOTION

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Abstract. Using recent results on the behavior of multiple Wiener-Itô integrals based on Stein's method, we prove Hsu-Robbins and Spitzer's theorems for sequences of correlated random variables related to the increments of the fractional Brownian motion.

1. Introduction

A famous result by Hsu and Robbins [6] says that if $X_1, X_2, \ldots$ is a sequence of independent identically distributed random variables with zero mean and finite variance and $S_n := X_1 + \ldots + X_n$, then

$$\sum_{n\geq 1} P\left(|S_n| > \varepsilon n\right) < \infty$$

for every $\varepsilon > 0$. Later, Erdös ([3], [4]) showed that the converse implication also holds, namely if the above series is finite for every $\varepsilon > 0$ and $X_1, X_2, \ldots$ are independent and identically distributed, then $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$. Since then, many authors extends this result in several directions.

Later, Spitzer's showed in [12] that

$$\sum_{n\geq 1} \frac{1}{n} P\left(|S_n| > \varepsilon n\right) < \infty$$

for every $\varepsilon > 0$ if and only if $\mathbf{E}X_1 = 0$ and $\mathbf{E}|X_1| < \infty$. Also, Spitzer's theorem has been the object of various generalizations and variants. One of problems related to the Hsu-Robbins' and Spitzer's theorems is to find the precise asymptotics as $\varepsilon \to 0$ of the quantities $\sum_{n\geq 1} P\left(|S_n| > \varepsilon n\right)$ and $\sum_{n\geq 1} \frac{1}{n} P\left(|S_n| > \varepsilon n\right)$. Heyde [5] showed that

$$\lim_{\varepsilon\to 0} \varepsilon^2 \sum_{n\geq 1} P\left(|S_n| > \varepsilon n\right) = \mathbf{E}X_1^2 \tag{1}$$

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whenever $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$. In the case when X is attracted to a stable distribution of exponent $\alpha > 1$, Spataru proved that

$$\lim_{\varepsilon\to 0} \frac{1}{-\log \varepsilon} \sum_{n\geq 1} \frac{1}{n} P\left(|S_n| > \varepsilon n\right) = \frac{\alpha}{\alpha - 1}. \tag{2}$$

The purpose of this paper is to prove Hsu-Robbins and Spitzer's theorems for sequences of correlated random variables, related to the increments of fractional Brownian motion. Recall that the fractional Brownian motion $(B_t^H)_{t\geq 0}$ is a centered Gaussian process with covariance function $R^H(t,s) = \mathbf{E}(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. It can be also defined as the unique self-similar Gaussian process with stationary increments. Concretely, in this paper we will studies the behavior of the tail probabilities of the sequence

$$V_n = \sum_{k=0}^{n-1} H_q(B_{k+1} - B_k) \tag{3}$$

where B is a fractional Brownian motion with Hurst parameter $H \in (0,1)$ (in the sequel we will omit the superscript H for B) and H_q is the Hermite polynomial of degree $q \geq 1$ given by $H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}(e^{-\frac{x^2}{2}})$. The sequence V_n behaves as follows (see [8]):

Theorem 1. *Let $q \geq 2$ an integer and let $(B_t)_{t\geq 0}$ a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Then, with some explicit positive constants $c_{1,q,H}, c_{2,q,H}$ depending only on q and H we have*

i. If $0 < H < 1 - \frac{1}{2q}$ then

$$\frac{V_n}{c_{1,q,H}\sqrt{n}} \xrightarrow[n\to\infty]{Law} N(0,1) \tag{4}$$

ii. If $1 - \frac{1}{2q} < H < 1$ then

$$\frac{V_n}{c_{2,q,H} n^{1-q(1-H)}} \xrightarrow[n\to\infty]{Law} Z \tag{5}$$

where Z is a Hermite random variable. See Section 2 for the definition.

In the case $H = 1 - \frac{1}{2q}$ the limit is still Gaussian but the normalization is different. However we will not treat this case in the present work. We will explain later (in Remark 3) why.

We note that the techniques generally used in the literature to prove the Hsu-Robbins and Spitzer's results are strongly related to the independence of the random variables $X_1, X_2, \ldots$. In our case the variables are correlated. Indeed, for any $k, l \geq 1$ we have $\mathbf{E}\left(H_q(B_{k+1} - B_k) H_q(B_{l+1} - B_l)\right) = \frac{1}{(q!)^2}\rho_H(k-l)$ where the correlation function is $\rho_H(k) = \frac{1}{2}\left((k+1)^{2H} + (k-1)^{2H} - 2k^{2H}\right)$ which is not equal to zero unless $H = \frac{1}{2}$ (which is the case of the standard Brownian motion). We use new techniques based on the estimates for the multiple Wiener-Itô integrals obtained in [2], [9] via Stein's method and Malliavin calculus.

The paper is organized as follows. Section 2 contains some preliminaries on calculus on Wiener chaos. In Section 3 we prove the Spitzer's theorem for variations of the fractional Brownian motion while Section 4 is devoted to the the Hsu-Robbins theorem for this sequence.

Throughout the paper we will denote by c a generic strictly positive constant which may vary from line to line (and even on the same line).

2. Preliminaries

Let $(W_t)_{t\in[0,1]}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, \mathbf{P})$. If $f \in L^2([0,1]^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener-Itô integral of f with respect to W. The basic reference is [10]. We will work with processes indexed by the time interval $[0,1]$ because by the scaling property of the fractional Brownian motion we can write the sequence V_n in terms of the increments of the fractional Brownian motion between points of the interval $[0,1]$.

Let $f \in \mathcal{S}_m$ be an elementary functions with m variables that can be written as

$$f = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} 1_{A_{i_1}\times\ldots\times A_{i_m}}$$

where the coefficients satisfy $c_{i_1,\ldots i_m} = 0$ if two indices i_k and i_l are equal and the sets $A_i \in \mathcal{B}([0,1])$ are disjoints. For a such step function f we define

$$I_m(f) = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} W(A_{i_1})\ldots W(A_{i_m})$$

where we put $W([a,b]) = W_b - W_a$. It can be seen that the application I_n constructed above from $\mathcal{S}_m$ to $L^2(\Omega)$ is an isometry on $\mathcal{S}_m$, i.e.

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = n!\langle f, g\rangle_{L^2([0,1]^n)} \text{ if } m = n \tag{6}$$

and

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = 0 \text{ if } m \neq n.$$

Since the set $\mathcal{S}_n$ is dense in $L^2([0,1]^n)$ for every $n \geq 1$ the mapping I_n can be extended to an isometry from $L^2([0,1]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension.

We will need the following bound for the tail probabilities of multiple Wiener-Itô integrals (see [7], Theorem 4.1)

$$P\left(|I_n(f)| > u\right) \leq c \exp\left(\left(\frac{-cu}{\sigma}\right)^{\frac{2}{n}}\right) \tag{7}$$

for all $u > 0$, $n \geq 1$, with $\sigma = \|f\|_{L^2([0,1]^2)}$.

The Hermite random variable of order q that appears as limit in Theorem 1, point ii. is defined as (see [8])

$$Z = d(q,H) I_q(L) \tag{8}$$

where the kernel $L \in L^2([0,1]^q)$ is given by

$$L(y_1,\dots,y_n) = 1_{[0,1]^q} \int_{y_1\vee\ldots\vee y_q}^1 \partial_1 K^H(u,y_1)\ldots\partial_1 K^H(u,y_q)du.$$

The constant $d(q,H)$ is a positive normalizing constant that guarantees that $\mathbf{E}Z^2 = 1$ and K^H is the standard kernel of the fractional Brownian motion (see [10], Section 5). We will not need the explicit expression of this kernel. Note that the case $q=1$ corresponds to the fractional Brownian motion and the case $q=2$ corresponds to the Rosenblatt process.

3. Spitzer's theorem

In order to transfer the problem on the unit interval, we note that the scaling property of the fractional Brownian motion implies that the sequence (3) has the same distribution as

$$V_n = \sum_{n\geq 1} H_q\left(n^H\left(B_{\frac{k+1}{n}} - B_{\frac{k}{n}}\right)\right) \tag{9}$$

where B is a fractional Brownian motion with time interval $[0,1]$. We set

$$Z_n^{(1)} = \frac{V_n}{c_{1,q,H}\sqrt{n}},\quad Z_n^{(2)} = \frac{V_n}{c_{2,q,H}n^{1-q(1-H)}} \tag{10}$$

with the constants $c_{1,q,H}, c_{2,q,H}$ from Theorem 1. We note that in this case it follows from [8] that the sequence $Z_n^{(2)}$ converges in L^2 to the Hermite random variable introduced in Section 2.

Let us denote, for every $\varepsilon > 0$,

$$f_1(\varepsilon) = \sum_{n\geq} \frac{1}{n} P\left(V_n > \varepsilon n\right) = \sum_{n\geq 1} \frac{1}{n} P\left(Z_n^{(1)} > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right) \tag{11}$$

and

$$f_2(\varepsilon) = \sum_{n\geq} \frac{1}{n} P\left(V_n > \varepsilon n^{2-2q(1-H)}\right) = \sum_{n\geq 1} \frac{1}{n} P\left(Z_n^{(2)} > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right). \tag{12}$$

Remark 1. *It is natural to consider the tail probability of order $n^{2-2q(1-H)}$ in (12) because the L^2 norm of the sequence V_n is in this case of order $n^{1-q(1-H)}$.*

We are interested to study the behavior of $f_i(\varepsilon)$ $(i=1,2)$ as $\varepsilon \to 0$. For a given random variable X, we set $\Phi_X(z) = 1 - P(X<z) + P(X<-x)$.

The first lemma gives the asymptotics of the functions $f_i(\epsilon)$ as $\varepsilon \to 0$ when $Z_n^{(i)}$ are replaced by their limits.

Lemma 1. *Consider $c > 0$.*

i. Let $Z^{(1)}$ be a standard normal random variable given by (8). Then as

$$\frac{1}{-\log c\varepsilon} \sum_{n\geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) \underset{\varepsilon\to 0}{\longrightarrow} 2.$$

ii. Let $Z^{(2)}$ be a Hermite random variable or order q. Then, for any integer $q \geq 1$

$$\frac{1}{-\log c\varepsilon} \sum_{n\geq 1} \frac{1}{n} \Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{1-q(1-H)}.$$

Proof: The case when $Z^{(1)}$ follows the standard normal law is hidden in [11]. We will give the ideas of the proof. We can write, for any m (see [11])

$$\sum_{n=1}^{m} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) = \int_1^m \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx + \frac{1}{2m}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{m})$$
$$- \frac{1}{2}\Phi_{Z^{(1)}}(c\varepsilon) - \int_1^m P_1(x) d\left[\frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})\right]$$

with $P_1(x) = [x] - x + \frac{1}{2}$. Letting $m \to \infty$

$$\sum_{n\geq 1} \frac{1}{n} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{n}) = \int_1^\infty \frac{1}{x} \Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx - \frac{1}{2}\Phi_{Z^{(1)}}(c\varepsilon)$$
$$- \int_1^\infty P_1(x) d\left[\frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})\right].$$

Clearly as $\varepsilon \to 0$, $\frac{1}{\log\varepsilon}\Phi_{Z^{(1)}}(c\varepsilon) \to 0$ because $\Phi_{Z^{(1)}}$ is a bounded function and concerning the last term it is also trivial to observe that

$$\frac{1}{-\log c\varepsilon} \int_1^\infty P_1(x) d\left[\frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})\right]$$
$$= \frac{1}{-\log c\varepsilon}\left(-\int_1^\infty P_1(x)\left(\frac{1}{x^2}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx + c\varepsilon\frac{1}{2}x^{-\frac{1}{2}}\frac{1}{x}\Phi'_{Z^{(1)}}(\varepsilon\sqrt{x})\right)dx\right) \underset{\varepsilon\to 0}{\longrightarrow} 0$$

since $\Phi_{Z^{(1)}}$ and $\Phi'_{Z^{(1)}}$ are bounded. Therefore the asymptotics of the function $f_1(\varepsilon)$ as $\varepsilon \to 0$ will be given by $\int_1^\infty \frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx$. By making the change of variables $c\varepsilon\sqrt{x} = y$, we get

$$\lim_{\varepsilon\to 0} \frac{1}{-\log c\varepsilon} \int_1^\infty \frac{1}{x}\Phi_{Z^{(1)}}(c\varepsilon\sqrt{x})dx$$
$$= \lim_{\varepsilon\to 0} \frac{1}{-\log c\varepsilon} 2\int_{c\varepsilon}^\infty \frac{1}{y}\Phi_{Z^{(1)}}(y)dy = \lim_{\varepsilon\to 0} 2\Phi_{Z^{(1)}}(c\varepsilon) = 2.$$

Let us consider now the case of the Hermite random variable. We will have as above

$$\lim_{\varepsilon\to 0}\frac{1}{-\log c\varepsilon}\sum_{n\geq 1}\frac{1}{n}\Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)})$$

$$=\lim_{\varepsilon\to 0}\frac{1}{-\log c\varepsilon}\left(\int_1^\infty \frac{1}{x}\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx - \int_1^\infty P_1(x)d\left[\frac{1}{x}\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})\right]\right).$$

By making the change of variables $c\varepsilon x^{1-q(1-H)}=y$ we will obtain

$$\lim_{\varepsilon\to 0}\frac{1}{-\log c\varepsilon}\int_1^\infty \frac{1}{x}\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx$$

$$=\lim_{\varepsilon\to 0}\frac{1}{-\log c\varepsilon}\frac{1}{1-q(1-H)}\int_{c\varepsilon}^\infty \frac{1}{y}\Phi_{Z^{(2)}}(y)dy$$

$$=\lim_{\varepsilon\to 0}\frac{1}{1-q(1-H)}\Phi_{Z^{(2)}}(c\varepsilon)=\frac{1}{1-q(1-H)}$$

where we used the fact that $\Phi_{Z^{(2)}}(y)\leq y^{-2}\mathbf{E}|Z^{(2)}|^2$ and so $\lim_{y\to\infty}\log y\Phi_{Z^{(2)}}(y)=0$.

It remains to show that $\frac{1}{-\log c\varepsilon}\int_1^\infty P_1(x)d\left[\frac{1}{x}\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})\right]$ converges to zero as ε tends to 0 (note that actually it follows from a result by [1] that a Hermite random variable has a density, but we don't need it explicitly, we only use the fact that $\Phi_{Z^{(2)}}$ is differentiable almost everywhere). This is equal to

$$\lim_{\varepsilon}\frac{1}{-\log c\varepsilon}\left[-\int_1^\infty dxP_1(x)x^{-2}\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)}) + \int_1^\infty dxP_1(x)c\varepsilon(1-q(1-H))x^{-q(1-H)-1}\Phi'_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})\right]$$

$$=\lim_{\varepsilon}\frac{1}{-\log c\varepsilon}\int_1^\infty P_1(x)c\varepsilon(1-q(1-H))x^{-q(1-H)-1}\Phi'_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx$$

$$=c\frac{\varepsilon}{-\log\varepsilon}(c\varepsilon)^{\frac{q(1-H)}{1-q(1-H)}}\int_{c\varepsilon}^\infty P_1\left(\left(\frac{y}{c\varepsilon}\right)^{\frac{1}{1-q(1-H)}}\right)\Phi'_{Z^{(2)}}(y)y^{-\frac{1}{1-q(1-H)}}dy$$

$$\leq c\frac{1}{-\log\varepsilon}\int_{c\varepsilon}^\infty P_1\left(\left(\frac{1}{c\varepsilon}\right)^{\frac{1}{1-q(1-H)}}\right)\Phi'_{Z^{(2)}}(y)dy$$

which clearly goes to zero since P_1 is bounded and $\int_0^\infty \Phi'_{Z^{(2)}}(y)dy=1$.

Remark 2. *Lemma 1 is valid for any $H\in(0,1)$ and for any random variable $Z^{(1)},Z^{(2)}$ with finite variance.*

The next result estimates the limit of the difference between the functions $f_i(\varepsilon)$ given by (11), (12) and the sequence in Lemma 1.

Proposition 1. *Let $q \geq 2$ and $c > 0$.*

i. If $H < 1 - \frac{1}{2q}$, let $Z_n^{(1)}$ be given by (10) and let $Z^{(1)}$ be standard normal random variable. Then it holds

$$\frac{1}{-\log c\varepsilon}\left[\sum_{n\geq 1}\frac{1}{n}P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - \sum_{n\geq 1}\frac{1}{n}P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \xrightarrow[\varepsilon\to 0]{} 0.$$

ii. Let $Z^{(2)}$ be a Hermite random variable of order $q \geq 2$ and $H > 1 - \frac{1}{2q}$. Then

$$\frac{1}{-\log c\varepsilon}\left[\sum_{n\geq 1}\frac{1}{n}P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right.$$
$$\left. - \sum_{n\geq 1}\frac{1}{n}P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right)\right] \xrightarrow[\varepsilon\to 0]{} 0.$$

Proof: We can consider separately the two cases when H is lesser or bigger than $1 - \frac{1}{2q}$. *Case $H < 1 - \frac{1}{2q}$.* We can write

$$\sum_{n\geq 1}\frac{1}{n}P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - \sum_{n\geq 1}\frac{1}{n}P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)$$
$$= \sum_{n\geq 1}\frac{1}{n}\left[P\left(Z_n^{(1)} > c\varepsilon\sqrt{n}\right) - P\left(Z^{(1)} > c\varepsilon\sqrt{n}\right)\right]$$
$$+ \sum_{n\geq 1}\left[\frac{1}{n}P\left(Z_n^{(1)} < -c\varepsilon\sqrt{n}\right) - P\left(Z^{(1)} < -c\varepsilon\sqrt{n}\right)\right]$$
$$\leq 2\sum_{n\geq 1}\frac{1}{n}\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(1)} > x\right) - P\left(Z^{(1)} > x\right)\right|.$$

It follows from [9], Theorem 4.1 that

$$\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(1)} > x\right) - P\left(Z^{(1)} > x\right)\right| \leq c\begin{cases}\frac{1}{\sqrt{n}}, & H \in (0, \frac{1}{2}] \\ n^{H-1}, & H \in [\frac{1}{2}, \frac{2q-3}{2q-2}) \\ n^{qH-q+\frac{1}{2}}, & H \in [\frac{2q-3}{2q-2}, 1 - \frac{1}{2q})\end{cases} \tag{13}$$

and this implies that
(14)

$$\sum_{n\geq 1}\frac{1}{n}\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(i)}>x\right)-P\left(Z^{(i)}>x\right)\right|\leq c\begin{cases}\sum_{n\geq 1}\frac{1}{n\sqrt{n}}, & H\in(0,\frac{1}{2}]\\ \sum_{n\geq 1}n^{H-2}, & H\in[\frac{1}{2},\frac{2q-3}{2q-2})\\ \sum_{n\geq 1}n^{qH-q-\frac{1}{2}}, & H\in[\frac{2q-3}{2q-2},1-\frac{1}{2q}).\end{cases}$$

and the last sums are finite (for the last one we use $H<1-\frac{1}{2q}$). The conclusion follows.

Case $H>1-\frac{1}{2q}$. In this case, by a result in Proposition 3.1 of [2]

$$\sup_{x\in\mathbb{R}}\left|P\left(Z_n^{(i)}>x\right)-P\left(Z^{(i)}>x\right)\right|\leq c\left(\mathbf{E}\left|Z_n^{(2)}-Z^{(2)}\right|^2\right)^{\frac{1}{2q}}\leq cn^{1-\frac{1}{2q}-H} \tag{15}$$

and as a consequence

$$\sum_{n\geq 1}\frac{1}{n}P\left(|Z_n^{(2)}|>c\varepsilon n^{1-q(1-H)}\right)-\sum_{n\geq 1}\frac{1}{n}P\left(|Z^{(2)}|>c\varepsilon n^{1-q(1-H)}\right)\leq c\sum_{n\geq 1}n^{-\frac{1}{2q}-H}$$

and the above series is convergent because $H>1-\frac{1}{2q}$. ■

We state now the Spitzer's theorem for the variations of the fractional Brownian motion.

Theorem 2. *Let f_1, f_2 be given by (11), (12) and the constants $c_{1,q,H}, c_{2,q,H}$ be those from Theorem 1.*

i. If $0<H<1-\frac{1}{2q}$ *then*

$$\lim_{\varepsilon\to 0}\frac{1}{\log(c_{1,H,q}^{-1}\varepsilon)}f_1(\varepsilon)=2.$$

ii. If $1>H>1-\frac{1}{2q}$ *then*

$$\lim_{\varepsilon\to 0}\frac{1}{\log(c_{2,H,q}^{-1}\varepsilon)}f_2(\varepsilon)=\frac{1}{1-q(1-H)}.$$

Proof: It is a consequence of Lemma 1 and Proposition 1. ■

Remark 3. *Concerning the case $H=1-\frac{1}{2q}$, note that the correct normalization of V_n (9) is $\frac{1}{\sqrt{n}}$. In this case result for the behavior of $\sum_{n\geq 1}P\left|V_n\right|>\varepsilon\sqrt{\log n}n\big)$ can be obtained by following the above proof. To be coherent, the natural version of the Spitzer's theorem would be to normalize the quantity $\sum_{n\geq 1}P\left|V_n\right|>\varepsilon n\log n)$ and this demands a new proof. To keep our approach unitary, we decide to avoid this case.*

4. Hsu-Robbins theorem for the variations of fractional Brownian motion

In this section we prove a version of the Hsu-Robbins theorem for the variations of the fractional Brownian motion. Concretely, we denote here by, for every $\varepsilon > 0$

$$g_1(\varepsilon) = \sum_{n\geq 1} P\left(|V_n| > \varepsilon n\right) \tag{16}$$

if $H < 1 - \frac{1}{2q}$ and by

$$g_2(\varepsilon) = \sum_{n\geq 1} P\left(|V_n| > \varepsilon n^{2-2q(1-H)}\right) \tag{17}$$

if $H > 1 - \frac{1}{2q}$. and we estimate the behavior of the functions $g_i(\varepsilon)$ as $\varepsilon \to 0$. Note that we can write

$$g_1(\varepsilon) = \sum_{n\geq 1} P\left(|Z_n^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right), \quad g_2(\varepsilon) = \sum_{n\geq 1} P\left(|Z_n^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right)$$

with $Z_n^{(1)}, Z_n^{(2)}$ given by (10).

We decompose it as: for $H < 1 - \frac{1}{2q}$

$$g_1(\varepsilon) = \sum_{n\geq 1} P\left(|Z^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right) + \sum_{n\geq 1}\left[P\left(|Z_n^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c_{1,q,H}^{-1}\varepsilon\sqrt{n}\right)\right].$$

and for $H > 1 - \frac{1}{2q}$

$$g_2(\varepsilon) = \sum_{n\geq 1} P\left(|Z^{(2)}| > \varepsilon c_{2,q,H}^{-1} n^{1-q(1-H)}\right) + \sum_{n\geq 1}\left[P\left(|Z_n^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right) - P\left(|Z^{(2)}| > c_{2,q,H}^{-1}\varepsilon n^{1-q(1-H)}\right)\right].$$

We start again by consider the situation when $Z_n^{(i)}$ are replaced by their limits.

Lemma 2. *i. Let $Z^{(1)}$ be a standard normal random variable. Then*

$$\lim_{\varepsilon\to 0}(c\varepsilon)^2 \sum_{n\geq 1} P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right) = 1.$$

ii. Let $Z^{(2)}$ be a Hermite random variable with $H > 1 - \frac{1}{2q}$. Then

$$\lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}} \sum_{n\geq 1} P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)}\right) = \mathbf{E}|Z^{(2)}|^{\frac{1}{1-q(1-H)}}.$$

Proof: The part i. is a consequence of the result of Heyde [5]. Indeed take $X_i \sim N(0,1)$ in (1). Concerning part ii. we can write

$$\begin{aligned}\lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}}\sum_{n\geq 1}\Phi_{Z^{(2)}}(c\varepsilon n^{1-q(1-H)}) \\ &= \lim_{\varepsilon\to 0}(c\varepsilon)^{\frac{1}{1-q(1-H)}}\left[\int_1^\infty \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx \right.\\ &\quad \left. -\int_1^\infty P_1(x)d\left[\Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})\right]\right]\\ &:= \lim_{\varepsilon\to 0}(A(\varepsilon)+B(\varepsilon))\end{aligned}$$

with $P_1(x)=[x]-x+\frac{1}{2}$. Moreover

$$\begin{aligned}A(\varepsilon) &= (c\varepsilon)^{\frac{1}{1-q(1-H)}}\int_1^\infty \Phi_{Z^{(2)}}(c\varepsilon x^{1-q(1-H)})dx\\ &= \frac{1}{1-q(1-H)}\int_{c\varepsilon}^\infty \Phi_{Z^{(2)}}(y)y^{\frac{1}{1-q(1-H)}-1}dy.\end{aligned}$$

Since $\Phi_{Z^{(2)}}(y)\leq y^{-2}$ we have $\Phi_{Z^{(2)}}(y)y^{\frac{1}{1-q(1-H)}}\to_{y\to\infty} 0$ and therefore

$$A(\varepsilon) = -\Phi_{Z^{(2)}}(c\varepsilon)(c\varepsilon)^{\frac{1}{1-q(1-H)}} - \int_{c\varepsilon}^\infty \Phi'_{Z^{(2)}}(y)y^{\frac{1}{1-q(1-H)}}dy$$

where the first terms goes to zero and the second to $\mathbf{E}\left|Z^{(2)}\right|^{\frac{1}{1-q(1-H)}}$. The proof that the term $B(\varepsilon)$ converges to zero is similar to the proof of Lemma 2, point ii. ■

Remark 4. *Lemma 2 also holds for any random variables that has a moments of order $\frac{1}{1-q(1-H)}$. The Hermite random variable has moments of all orders since it is the value at time 1 of a selfsimilar process with stationary increments.*

Proposition 2. *i. Let $H<1-\frac{1}{2q}$ and let $Z_n^{(1)}$ be given by (10). Let also $Z^{(1)}$ be a standard normal random variable. Then*

$$(c\varepsilon)^2\sum_{n\geq 1}\left[P\left(|Z_n^{(1)}|>c\varepsilon\sqrt{n}\right)-P\left(|Z^{(1)}|>c\varepsilon\sqrt{n}\right)\right]\underset{\varepsilon\to 0}{\longrightarrow} 0.$$

ii. Let $H>1-\frac{1}{2q}$ and let $Z_n^{(2)}$ be given by (10). Let $Z^{(2)}$ be a Hermite random variable. Then

$$(c\varepsilon)^{\frac{1}{1-q(1-H)}}\sum_{n\geq 1}\left[P\left(|Z_n^{(2)}|>c\varepsilon n^{1-q(1-H)}\right)-P\left(|Z^{(2)}|>c\varepsilon n^{1-q(1-H)}\right)\right]\underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Remark 5. *Note that the bounds (13), (15) does not help here because the series that appear after their use are not convergent.*

Proof: [Proof of Proposition 2] *Case* $H < 1 - \frac{1}{2q}$. We have, for some $\beta > 0$ to be chosen later,

$$\begin{aligned}
&\varepsilon^2 \sum_{n\geq 1} \left[P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \\
&\qquad = \varepsilon^2 \sum_{n=1}^{[\varepsilon^{-\beta}]} \left[P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \\
&\qquad\quad + \varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} \left[P\left(|Z_n^{(1)}| > c\varepsilon\sqrt{n}\right) - P\left(|Z^{(1)}| > c\varepsilon\sqrt{n}\right)\right] \\
&\qquad := I_1(\varepsilon) + J_1(\varepsilon).
\end{aligned}$$

Consider first the situation when $H \in (0, \frac{1}{2}]$. Let us choose a real number β such that $2 < \beta < 4$. By using (13),

$$I_1(\varepsilon) \leq c\varepsilon^2 \sum_{n=1}^{[\varepsilon^{-\beta}]} n^{-\frac{1}{2}} \leq c\varepsilon^2 \varepsilon^{-\frac{\beta}{2}} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

since $\beta < 4$. Next, by using the bound for the tail probabilities of multiple integrals and since $\mathbf{E}\left|Z_n^{(1)}\right|^2$ converges to 1 as $n \to \infty$

$$\begin{aligned}
J_1(\varepsilon) &= \varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} P\left(Z_n^{(1)} > c\varepsilon\sqrt{n}\right) \\
&\leq c\varepsilon^{-2} \sum_{n>[\varepsilon^{-\beta}]} \exp\left(\frac{-c\varepsilon\sqrt{n}}{\left(\mathbf{E}\left|Z_n^{(1)}\right|^2\right)^{\frac{1}{2}}}\right)^{\frac{2}{q}} \\
&\leq \varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} \exp\left(\left(-cn^{-\frac{1}{\beta}}\sqrt{n}\right)^{\frac{2}{q}}\right)
\end{aligned}$$

and since converges to zero for $\beta > 2$. The same argument shows that $\varepsilon^2 \sum_{n>[\varepsilon^{-\beta}]} P\left(Z^{(1)} > c\varepsilon\sqrt{n}\right)$ converges to zero.

The case when $H \in (\frac{1}{2}, \frac{2q-3}{2q-2})$ can be obtained by taking $2 < \beta < \frac{2}{H}$ (it is possible since $H < 1$) while in the case $H \in (\frac{2q-3}{2q-2}, 1 - \frac{1}{2q})$ we have to choose $2 < \beta < \frac{2}{qH - q + \frac{3}{2}}$ (which is possible because $H < 1 - \frac{1}{2q}!$).

Case $H > 1 - \frac{1}{2q}$. We have, with some suitable $\beta > 0$

$$\varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n\geq 1} \left[P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) \right]$$

$$= \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n=1}^{[\varepsilon^{-\beta}]} \left[P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) \right]$$
$$+ \varepsilon^{\frac{1}{1-q(1-H)}} \sum_{n\geq [\varepsilon^{-\beta}]} \left[P\left(|Z_n^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) - P\left(|Z^{(2)}| > c\varepsilon n^{1-q(1-H)} \right) \right]$$
$$:= I_2(\varepsilon) + J_2(\varepsilon).$$

Choose $\frac{1}{1-q(1-H)} < \beta < \frac{1}{(1-q(1-H))(2-H-\frac{1}{2q})}$ (again, this is always possible when $H > 1 - \frac{1}{2q}!$). Then

$$I_2(\varepsilon) \leq c e^{\frac{1}{1-q(1-H)}} \varepsilon^{(-\beta)(2-H-\frac{1}{2q})} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

and by (7)

$$J_2(\varepsilon) \leq c \sum_{n>[\varepsilon^{-\beta}]} \exp\left(\left(\frac{-c\varepsilon n^{1-q(1-H)}}{\left(\mathbf{E}\left| Z_n^{(2)} \right|^2 \right)^{\frac{1}{2}}} \right)^{\frac{2}{q}} \right)$$
$$\leq c \sum_{n>[\varepsilon^{-\beta}]} \exp\left(c n^{-\frac{1}{\beta}} n^{1-q(1-H)} \right)^{\frac{2}{q}} \underset{\varepsilon\to 0}{\longrightarrow} 0. \qquad \blacksquare$$

We state the main result of this section which is a consequence of Lemma 2 and Proposition 2.

Theorem 3. *Let $q \geq 2$ and let $c_{1,q,H}, c_{2,q,H}$ be the constants from Theorem 1. Let $Z^{(1)}$ be a standard normal random variable, $Z^{(2)}$ a Hermite random variable of order $q \geq 2$ and let g_1, g_2 be given by (16) and (17). Then*

i. If $0 < H < 1 - \frac{1}{2q}$, we have $(c_{1,q,H}^{-1}\varepsilon)^2 g_1(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 1 = \mathbf{E}Z^{(1)}$.

ii. If $1 - \frac{1}{2q} < H < 1$ we have $(c_{2,q,H}^{-1}\varepsilon)^{\frac{1}{1-q(1-H)}} g_2(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \mathbf{E}|Z^{(2)}|^{\frac{1}{1-q(1-H)}}$.

Remark 6. *In the case $H = \frac{1}{2}$ we retrieve the result (1) of* [5]. *The case $q = 1$ is trivial, because in this case, since $V_n = B_n$ and $\mathbf{E}V_n^2 = n^{2H}$, we obtain the following (by applying Lemma 1 and 2 with $q = 1$)*

$$\frac{1}{\log \varepsilon} \sum_{n\geq 1} \frac{1}{n} P\left(|V_n| > \varepsilon n^{2H} \right) \underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{H}$$

and

$$\varepsilon^2 \sum_{n\geq 1} P\left(|V_n| > \varepsilon n^{2H}\right) \underset{\varepsilon\to 0}{\longrightarrow} \mathbf{E}\left|Z^{(1)}\right|^{\frac{1}{H}}.$$

References

[1] L.M. Albin (1998): A note on Rosenblatt distributions. *Statist. Probab. Lett.,* **40(1)**, 83–91.

[2] J.-C. Breton and I. Nourdin (2008): Error bounds on the non-normal approximation of Hermite power variations of fractional Brownian motion. *Electronic Communications in Probability,* **13**, 482-493.

[3] P. Erdös (1949): On a theorem of Hsu and Robbins. *Ann. Math. Statistics,* **20**, 286–291.

[4] P. Erdös (1950): Remark on my paper "On a theorem of Hsu and Robbins". *Ann. Math. statistics,* **21**, 138.

[5] C.C. Heyde (1975): A supplement to the strong law of large numbers. *Journal of Applied Probability,* **12**, 173–175.

[6] P. Hsu and H. Robbins (1947): Complete convergence and the law of large numbers. *Proc. Nat. Acad. Sci. U.S.A.,* **33**, 25–31.

[7] P. Major (2005): Tail behavior of multiple integrals and U-statitics. *Probability Surveys,* **2**, 448–505.

[8] I. Nourdin, D. Nualart and C.A. Tudor (2008): Central and non-central limit theorems for weighted power variations of fractional Brownian motion. *Preprint.*

[9] I. Nourdin and G. Peccati (2007): Stein's method on Wiener chaos. To appear in *Probability Theory and Related Fields.*

[10] D. Nualart (2006): *Malliavin Calculus and Related Topics. Second Edition.* Springer.

[11] A. Spataru (1999): Precise Asymptotics in Spitzer's law of large numbers. *Journal of Theoretical Probability,* **12(3)**, 811–819.

[12] F. Spitzer (1956): A combinatorial lemma and its applications to probability theory. *Trans. Amer. Math. Soc.,* **82**, 323–339.

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