SHARP WEIGHTED BOUNDS FOR FRACTIONAL INTEGRAL OPERATORS

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ABSTRACT. The relationship between the operator norms of fractional integral operators acting on weighted Lebesgue spaces and the constant of the weights is investigated. Sharp bounds are obtained for both the fractional integral operators and the associated fractional maximal functions. As an application improved Sobolev inequalities are obtained. Some of the techniques used include a sharp off-diagonal version of the extrapolation theorem of Rubio de Francia and characterizations of two-weight norm inequalities.

1. Introduction

Recall that a non-negative locally integrable function, or weight, w is said to belong to the A_p class for $1 < p < \infty$ if it satisfies the condition

$$[w]_{A_p} \equiv \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\, dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\, dx\right)^{p-1} < \infty,$$

where p' is the dual exponent of p defined by the equation $1/p + 1/p' = 1$. Muckenhoupt [17] showed that the weights satisfying the A_p condition are exactly the weights for with the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\, dy$$

is bounded on $L^p(w)$. Hunt, Muckenhoupt, and Wheeden [12] extended the weighted theory to the study of the Hilbert transform

$$\mathcal{H}f(x) = p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

They showed that the A_p condition also characterizes the $L^p(w)$ boundedness of this operator. Coifman and Fefferman [3] extended the A_p theory to general Calderón-Zygmund operators. For example, to operators that are bounded, say on $L^2(\mathbb{R}^n)$, and of the form

$$Tf(x) = p.v. \int_{\mathbb{R}^n} f(y)K(x,y)\, dy,$$

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where

$$|\partial^\beta K(x,y)| \le c|x-y|^{-n-|\beta|}.$$

Bounds on the operators norms in terms of the A_p constants of the weights have been investigated as well. Buckley [2] showed that for $1 < p < \infty$, M satisfies

$$\|M\|_{L^p(w)\to L^p(w)} \le c\,[w]_{A_p}^{1/(p-1)} \tag{1.1}$$

and the exponent $1/(p-1)$ is the best possible. A new and rather simple proof of both Muckenhoupt's and Buckley's results was recently given by Lerner [13]. The weak-type bound also observed by Buckley [2] is

$$\|M\|_{L^p(w)\to L^{p,\infty}(w)} \le c\,[w]_{A_p}^{1/p}. \tag{1.2}$$

For singular integrals operators, however, only partial results are known. The interest in sharp weighted norm for singular integral operators is motivated in part by applications in partial differential equations. We refer the reader to Astala, Iwaniec, and Saksman [1]; and Petermichl and Volberg [23] for such applications. Petermichl [21], [22] showed that

$$\|T\|_{L^p(w)\to L^p(w)} \le c\,[w]_{A_p}^{\max\{1,1/(p-1)\}}, \tag{1.3}$$

where T is either the Hilbert or one of the Riesz transforms in $\mathbb{R}^n$,

$$R_j f(x) = c_n\, p.v. \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy.$$

Petermichl's results were obtained for $p = 2$ using Bellman function methods. The general case $p \neq 2$ then follows by the sharp version of the Rubio de Francia extrapolation theorem given by Dragičević, Grafakos, Pereyra, and Petermichl [4]. We recall that the original proof of the extrapolation theorem was given by Rubio de Francia in [24] and it was not constructive. García-Cuerva then gave a constructive proof that can be found in [6, p.434] and which has been used to get the sharp version in [4]. It is important to remark that so far no proof of the L^p version of Petermilch's result is know without invoking extrapolation. These are the best known results so far and whether (1.3) holds for general Calderón-Zygmund operators is not known.

There are also other estimates for Calderón-Zygmund operators involving weights which have received attention over the years. In particular, there is the "Muckenhoupt-Wheeden conjecture"

$$\|Tf\|_{L^{1,\infty}(w)} \le c\,\|f\|_{L^1(Mw)}, \tag{1.4}$$

for arbitrary weight w, and the "linear growth conjecture" for $1 < p < \infty$,

$$\|T\|_{L^p(w)\to L^{p,\infty}(w)} \le c_p [w]_{A_p}. \tag{1.5}$$

Both these conjectures remain very difficult open problems. Some progress has been recently made by Lerner, Ombrosi and Pérez [14], [15].

Motivated by all these estimates, we investigate in this article the sharp weighted bounds for fractional integral operators and the related maximal functions.

For $0 < \alpha < n$, the fractional integral operator or Riesz potential I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

while the related fractional maximal operator M_α is given by

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| \, dy.$$

These operators play an important role in analysis, particularly in the study of differentiability or smoothness properties a functions. See the books by Stein [29] or Grafakos [7] for the basic properties of these operators.

Weighted inequalities for these operators and more general potential operators have been studied in depth. See e.g. the works of Muckenhoupt and Wheeden [18], Sawyer [26], [27], Gabidzashvili and Kokilashvili [5], Sawyer and Wheeden [28], and Pérez [19], [20]. Such estimates naturally appear in problems in partial differential equations and quantum mechanics.

In [18], the authors characterized the weighted strong-type inequality for fractional operators in terms of the so-called $A_{p,q}$ condition. For $1 < p < n/\alpha$ and q defined by $1/q = 1/p - \alpha/n$, they showed that for all $f \geq 0$,

$$\left(\int_{\mathbb{R}^n} (wT_\alpha f)^q \, dx\right)^{1/q} \leq c \left(\int_{\mathbb{R}^n} (wf)^p \, dx\right)^{1/p}, \tag{1.6}$$

where $T_\alpha = I_\alpha$ or M_α, if and only if $w \in A_{p,q}$. That is,

$$[w]_{A_{p,q}} \equiv \sup_Q \left(\frac{1}{|Q|}\int_Q w^q \, dx\right)\left(\frac{1}{|Q|}\int_Q w^{-p'} \, dx\right)^{q/p'} < \infty.$$

The connection between the $A_{p,q}$ constant $[w]_{A_{p,q}}$ and the operator norms of these fractional operators is the main focus of this article. We will obtain the analogous estimates of (1.1), (1.2), (1.3), (1.4), and (1.5) in the fractional integral case.

At a formal level, the case $\alpha = 0$ corresponds to the Calderón-Zygmund case where, as mentioned, some estimates have not been obtained yet. Though for $\alpha > 0$ one deals with positive operators, the corresponding estimates still remain difficult to be proved and we need to use a set of tools different from the ones used in the Calderón-Zygmund situation.

Our main result, Theorem 2.6 below, is the sharp bound

$$\|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} \leq c[w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max\{1,\frac{p'}{q}\}}.$$

This is the analogous estimate of (1.3) for fractional integral operators.

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2. Description of the main results

We start by observing that to obtain sharp bounds for the strong-type inequalities for I_α it is enough to obtain sharp bounds for the weak-type ones. This is due to Sawyer's deep results on the characterization of two-weight norm inequalities for I_α. In fact, he proved in [27] that for two positive locally integrable function v and u, and $1 < p \le q < \infty$,

$$I_\alpha \colon L^p(v) \to L^q(u)$$

if and only if u and the function $\sigma = v^{1-p'}$ satisfy the *(local) testing conditions*

$$[u,\sigma]_{S_{p,q}} \equiv \sup_Q \sigma(Q)^{-1/p} \|\chi_Q I_\alpha(\chi_Q \sigma)\|_{L^q(u)} < \infty$$

and

$$[\sigma,u]_{S_{q',p'}} \equiv \sup_Q u(Q)^{-1/q'} \|\chi_Q I_\alpha(\chi_Q u)\|_{L^{p'}(\sigma)} < \infty.$$

Moreover, his proof shows that actually

$$\|I_\alpha\|_{L^p(v)\to L^q(u)} \approx [u,\sigma]_{S_{p,q}} + [\sigma,u]_{S_{q',p'}}. \tag{2.1}$$

On the other hand in his characterization of the weak-type, two-weight inequalities for I_α, Sawyer [26] also showed that

$$\|I_\alpha\|_{L^p(v)\to L^{q,\infty}(u)} \approx [\sigma,u]_{S_{q',p'}}.$$

Combining (2.1) and (5.3) it follows that

$$\|I_\alpha\|_{L^p(v)\to L^q(u)} \approx \|I_\alpha\|_{L^{q'}(u^{1-q'})\to L^{p',\infty}(v^{1-p'})} + \|I_\alpha\|_{L^p(v)\to L^{q,\infty}(u)}. \tag{2.2}$$

If we now set $u = w^q$ and $v = w^p$, we finally obtain the one-weight estimate

$$\|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} \approx \|I_\alpha\|_{L^{q'}(w^{-q'})\to L^{p',\infty}(w^{-p'})} + \|I_\alpha\|_{L^p(w^p)\to L^{q,\infty}(w^q)}. \tag{2.3}$$

We will obtain sharp bounds for the weak-type norms in the right hand side of (2.3) in two different ways, each of which is of interest on its own. Our first approach is based on an off-diagonal extrapolation theorem by Harboure, Macías, and Segovia [10]. A second one is based in yet another characterization of two-weight norm inequalities for I_α in the case $p < q$, in terms of certain *(global) testing condition* and which is due to Gabidzashvili and Kokilashvili [5].

We present now the extrapolation results. The proof follows the original one, except that we carefully track the dependence of the estimates in terms of the $A_{p,q}$ constants of the weights.

Theorem 2.1. *Suppose that T is an operator defined on an appropriate class of functions, (e.g. C_c^∞, or $\bigcup_p L^p(w^p)$). Suppose further that p_0 and q_0 are exponents with $1 \le p_0 \le q_0 < \infty$, and such that*

$$\|wTf\|_{L^{q_0}(\mathbb{R}^n)} \le c[w]^{\gamma}_{A_{p_0,q_0}} \|wf\|_{L^{p_0}(\mathbb{R}^n)}$$

holds for all $w \in A_{p_0,q_0}$ and some $\gamma > 0$. Then,

$$\|wTf\|_{L^q(\mathbb{R}^n)} \le c[w]_{A_{p,q}}^{\gamma \max\{1, \frac{q_0}{p_0'}\frac{p'}{q'}\}} \|wf\|_{L^p(\mathbb{R}^n)}$$

holds for all p and q satisfying $1 < p \le q < \infty$ and

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0},$$

and all weight $w \in A_{p,q}$.

As a consequence we have the following weak extrapolation theorem using an idea from Grafakos and Martell [9].

Corollary 2.2. *Suppose that for some $1 \le p_0 \le q_0 < \infty$, an operator T satisfies the weak-type (p_0, q_0) inequality*

$$\|Tf\|_{L^{q_0,\infty}(w^{q_0})} \le c[w]^{\gamma}_{A_{p_0,q_0}} \|wf\|_{L^{p_0}(\mathbb{R}^n)}$$

for every $w \in A_{p_0,q_0}$ and some $\gamma > 0$. Then T also satisfies the weak-type (p,q) inequality,

$$\|Tf\|_{L^{q,\infty}(w^q)} \le c[w]_{A_{p,q}}^{\gamma \max\{1, \frac{q_0}{p_0'}\frac{p'}{q}\}} \|wf\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p \le q < \infty$ that satisfy

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$$

and all $w \in A_{p,q}$.

We will use the above corollary to obtain sharp weak bounds in the whole range of exponents for I_α. As already described, this leads to strong-type estimates too. Nevertheless, for a certain range of exponents the strong-type estimates can be obtained in a more direct way without relying on the difficult two-weight results.

It is not obvious a priori what the analogous of (1.3) should be for I_α. A possible guess is

$$\|w\, I_\alpha f\|_{L^q(\mathbb{R}^n)} \le c\,[w]_{A_{p,q}}^{\max\{1, \frac{p'}{q}\}} \|w\, f\|_{L^p(\mathbb{R}^n)}. \tag{2.4}$$

Note that formally, the estimate reduces to (1.3) when $\alpha = 0$ suggesting it could be sharp. While it is possible to obtain such estimate, simple examples indicate it is not the best one. In fact, we will show in this article a direct proof of the following estimate.

Theorem 2.3. *Let $1 < p_0 < n/\alpha$ and q_0 be defined by the equations $1/q_0 = 1/p_0 - \alpha/n$ and $q_0/p_0' = 1 - \alpha/n$, and let $w \in A_{p_0,q_0}$. Then,*

$$\|wI_\alpha f\|_{L^{q_0}(\mathbb{R}^n)} \leq c\,[w]_{A_{p_0,q_0}} \|wf\|_{L^{p_0}(\mathbb{R}^n)}. \tag{2.5}$$

We note that from (2.5), the extrapolation results immediately yields for $1/q = 1/p - \alpha/n$ the estimate

$$\|w\,I_\alpha f\|_{L^q(\mathbb{R}^n)} \leq c\,[w]_{A_{p,q}}^{\max(1,(1-\frac{\alpha}{n})\frac{p'}{q})} \|w\,f\|_{L^p(\mathbb{R}^n)}, \tag{2.6}$$

which again simplifies to (1.3) if we formally put $\alpha = 0$. We have, however, examples that show that the optimal exponent should be

$$\|I_\alpha\| \leq c\,[w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max\{1,\frac{p'}{q}\}}. \tag{2.7}$$

We can combine (2.6) with simple duality arguments to obtain

$$\|wI_\alpha f\|_{L^q(\mathbb{R}^n)} \leq c\,[w]_{A_{p,q}}^{\eta(p'/q)} \|wf\|_{L^p(\mathbb{R}^n)}, \tag{2.8}$$

where $\eta(x) = \min\{\max(1-\alpha/n, x), \max(1, (1-\alpha/n)x)\}$. But this estimate only produces sharp results for p'/q in the range $(0, 1-\alpha/n] \cup [n/(n-\alpha), \infty)$.

To obtain the full range of exponents using the direct approach with the strong extrapolation theorem, it seems that one would need to consider the case $p_0' = q_0$ and show the estimate

$$\|I_\alpha\| \leq c\,[w]_{A_{p_0,q_0}}^{1-\frac{\alpha}{n}}.$$

We do not know if this approach is viable. As we already mentioned, even in the Calderón-Zygmund case the known results for the full range of exponents are obtained via extrapolation from just one estimate.

If instead we use Corollary 2.2, we do obtain sharp estimates in the full range of exponents for the weak-type (p, q) inequality for I_α. We have the following result.

Theorem 2.4. *Suppose that $1 \leq p < n/\alpha$ and that q satisfies $1/q = 1/p - \alpha/n$. Then*

$$\|I_\alpha f\|_{L^{q,\infty}(w^q)} \leq c\,[w]_{A_{p,q}}^{1-\frac{\alpha}{n}} \|w\,f\|_{L^p(\mathbb{R}^n)}. \tag{2.9}$$

Furthermore, the exponent $1 - \frac{\alpha}{n}$ is sharp.

We will also present a second proof of Theorem 2.4 for $p > 1$ without using extrapolation.

Remark 2.5. Once again, the estimate in the above weak-type result should be contrasted with the case $\alpha = 0$ and the linear growth conjecture for a Calderón-Zygmund operator T. Namely,

$$\|T\|_{L^p(w)\to L^{p,\infty}(w)} \le c_p[w]_{A_p}.$$

Such results have remained elusive so far. For the best available result see [15].

The extrapolation proof of Theorem 2.4 will also show that for any weight u the weak-type inequality

$$\|I_\alpha f\|_{L^{(n/\alpha)',\infty}(u)} \le c\,\|f\|_{L^1((Mu)^{1-\frac{\alpha}{n}})}$$

holds. For $\alpha = 0$ the analogous version of this inequality is the Muckenhoupt-Wheeden conjecture

$$\|Tf\|_{L^{1,\infty}(w)} \le c\,\|f\|_{L^1(Mw)},$$

which is an open problem.

As a consequence of the weak-type estimate (2.9) we obtain the sharp bounds indicated by examples.

Theorem 2.6. *Let $1 < p < n/\alpha$ and q be defined by the equation $1/q = 1/p - \alpha/n$, and let $w \in A_{p,q}$. Then,*

$$\|I_\alpha\| \le c\,[w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max\{1,\frac{p'}{q}\}}. \tag{2.10}$$

Furthermore this estimate is sharp.

Another consequence of (2.9) is a Sobolev-type estimate. We obtain this when we use the fact that weak-type inequalities implies strong-type inequalities when a gradient operator is involved. We have the following result based on the ideas of Long and Nie [16]. See also Hajlasz [11].

Theorem 2.7. *Let $p \ge 1$ and let $w \in A_{p,q}$ with q satisfying $1/p - 1/q = 1/n$. Then, for any Lipschitz function f with compact support,*

$$\left(\int_{\mathbb{R}^n} (|f(x)|w(x))^q\,dx\right)^{1/q} \le c\,[w]_{A_{p,q}}^{1/n'}\left(\int_{\mathbb{R}^n} (|\nabla f(x)|w(x))^p\,dx\right)^{1/p}. \tag{2.11}$$

Remark 2.8. We note that this estimate is better than what the strong bound on I_1 in Theorem 2.6 gives. In fact, for f sufficiently smooth and compactly supported, we have the estimate

$$|f(x)| \le cI_1(|\nabla f|)(x).$$

Hence, if we applied Theorem 2.6 we obtain the estimate

$$\|fw\|_{L^q} \le c\,[w]_{A_{p,q}}^{1/n'\max\{1,p'/q\}}\|\nabla fw\|_{L^p}.$$

However, Theorem 2.7 gives a better growth in terms of the weight, simply $[w]_{A_{p,q}}^{1/n'}$. This is a better growth in the range $1 < p < \min(2n', n)$ (i.e. $p'/q > 1$) where the estimate (2.10) only gives $[w]_{A_{p,q}}^{p'/(qn')}$. Note also that (2.11) includes the case $p = 1$, which cannot be obtained using Theorem 2.6.

We also find the sharp constant for M_α in the full range of exponents.

Theorem 2.9. *Suppose $0 \le \alpha < n$, $1 < p < n/\alpha$ and q is defined by the relationship $1/q = 1/p - \alpha/n$. If $w \in A_{p,q}$, then*

$$\|wM_\alpha f\|_{L^q} \le c[w]_{A_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{n})}\|wf\|_{L^p}. \tag{2.12}$$

Furthermore, the exponent $\frac{p'}{q}(1-\frac{\alpha}{n})$ is sharp.

Note one more time that formally replacing $\alpha = 0$ the estimates clearly generalize the result in [2].

Remark 2.10. We also note that there is a weak-type estimate for M_α. For $p \ge 1$ and $1/q = 1/p - \alpha/n$, standard covering methods give

$$\|M_\alpha\|_{L^p(w^p)\to L^{q,\infty}(w^q)} \le c\,[w]_{A_{p,q}}^{1/q}. \tag{2.13}$$

See for instance the book by Garcia-Cuerva and Rubio de Francia [6, pp. 387–393], for the estimate in the case $\alpha = 0$.

Remark 2.11. Continuing with the formal comparison with the case $\alpha = 0$, it would be interesting to know if the analog of (2.3) also holds for Calderón-Zygmund singular integrals. Namely,

$$\|T\|_{L^p(w)\to L^p(w)} \approx \|T^*\|_{L^{p'}(w^{1-p'})\to L^{p',\infty}(w^{1-p'})} + \|T\|_{L^p(w)\to L^{p,\infty}(w)}. \tag{2.14}$$

This estimate, if true, may be beyond reach with the current available techniques.

The rest of the paper is organized as follows. We separate the proofs of the main results in different sections which are essentially independent of each other. In Section 3 we collect some additional definitions and the proof of the version of the extrapolation result Theorem 2.1. We repeat the proof of such result from [10] for the convenience of the reader, but also to show that the constant we need can indeed be tracked through the computations. A faithful reader familiar with the extrapolation result may skip the details, move directly to the following sections of the article, and come back later to Section 3 to verify our claims. Section 4 contains the proof of of Theorem 2.3. We also include in this section a duality argument to conclude the estimate (2.8). The proof of Corollary 2.2 and the two proofs of the weak-type result for I_α, Theorem 2.4, are in Section 5. The proof of Theorem 2.6 as a corollary of Theorem 2.4 is in this section too. The proof of the result for the fractional maximal function, Theorem 2.9, is presented in Section 6. In Section 7 we present the examples for the sharpness in Theorems 2.4, 2.6, and 2.9. Finally, in Section 8 we present the proof of the application to Sobolev-type inequalities.

3. Constants in the off-diagonal extrapolation theorem

For a Lebesgue measurable set E, $|E|$ will denote its Lebesgue measure and $w(E) = \int_E w(x)\,dx$ will denote its weighted measure. We will be working on weighted versions of the classical L^p spaces, $L^p(w)$, and also on the weak-type ones, $L^{p,\infty}(w)$, defined in the usual way with the Lebesgue measure dx replaced by the measure $w\,dx$. Often, however, it will be convenient to viewed the weight not as a measure but as a multiplier. For example $f \in L^p(w^p)$ if

$$\|fw\|_{L^p} = \left(\int_{\mathbb{R}^n} (|f(x)|w(x))^p\,dx\right)^{1/p} < \infty.$$

This is more convenient when dealing with the $A_{p,q}$ condition already defined in the introduction. Recall, that for $1 < p \le q < \infty$, we say $w \in A_{p,q}$ if

$$[w]_{A_{p,q}} \equiv \sup_Q \left(\frac{1}{|Q|}\int_Q w^q\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}\,dx\right)^{q/p'} < \infty. \tag{3.1}$$

Also, for $1 \le q < \infty$ we define the class $A_{1,q}$ to be the weights w that satisfy,

$$\left(\frac{1}{|Q|}\int_Q w^q\,dx\right) \le c \inf_Q w^q. \tag{3.2}$$

Here $[w]_{A_{1,q}}$ will denote the smallest constant c that satisfies (3.2). Notice that $w \in A_{p,q}$ if and only if $w^q \in A_{1+q/p'}$ with

$$[w]_{A_{p,q}} = [w^q]_{A_{1+q/p'}}. \tag{3.3}$$

In particular, $[w]_{A_{p,q}} \ge 1$. We also note for later use that

$$[w^{-1}]_{A_{q',p'}} = [w]_{A_{p,q}}^{p'/q}. \tag{3.4}$$

The term cube will always refer to a cube Q in $\mathbb{R}^n$ with sides parallel to the axis. A multiple rQ of a cube is a cube with the same center of Q and side-length r times as large. By $\mathcal{D}$ we denote the collection of all dyadic cubes in $\mathbb{R}^n$. That is, the collection of all cubes with lower-felt corner $2^{-l}m$ and side-length 2^{-l} with $l \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. As usual, $B(x,r)$ will denote the Euclidean ball in $\mathbb{R}^n$ centered at the point x and with radius r.

To prove Theorem 2.1 we will need the sharp version of the Rubio de Francia algorithm given by García-Cuerva. The proof can be found in the article [4].

Lemma 3.1. *Suppose that $r > r_0$, $v \in A_r$, and g is a non-negative function in $L^{(r/r_0)'}(v)$. Then, there exists a function G such that*

1. $G \ge g$,
2. $\|G\|_{L^{(r/r_0)'}(v)} \le 2\|g\|_{L^{(r/r_0)'}(v)}$,
3. $Gv \in A_{r_0}$ *with* $[Gv]_{A_{r_0}} \le c\,[v]_{A_r}$.

Proof of Theorem 2.1. First suppose $w \in A_{p,q}$ and $1 \le p_0 < p$, which implies $q > q_0$. Then,

$$\left(\int_{\mathbb{R}^n} |Tf|^q w^q\right)^{1/q} = \left(\int_{\mathbb{R}^n} (|Tf|^{q_0})^{q/q_0} w^q\right)^{\frac{q_0}{q}\frac{1}{q_0}}$$
$$= \left(\int_{\mathbb{R}^n} |Tf|^{q_0} g w^q\right)^{\frac{1}{q_0}}$$

for some non-negative $g \in L^{(q/q_0)'}(w^q)$ with $\|g\|_{L^{(q/q_0)'}(w^q)} = 1$. Now, let $r = 1 + q/p'$ and $r_0 = 1 + q_0/p_0'$. Since $p > p_0$ we have $r > r_0$. Furthermore, by the relationship

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0},$$

we have $q/q_0 = r/r_0$. Hence by Lemma 3.1 and using that $w^q \in A_r$, there exists G with $G \ge g$, $\|G\|_{L^{(r/r_0)'}(w^q)} \le 2$, $Gw^q \in A_{r_0}$, and $[Gw^q]_{A_{r_0}} \le c\,[w^q]_{A_r} = c\,[w]_{A_{p,q}}$. Also, since $Gw^q \in A_{r_0}$ then $(Gw^q)^{1/q_0} \in A_{p_0,q_0}$ since,

$$[(Gw^q)^{1/q_0}]_{A_{p_0,q_0}} = \sup_Q \left(\frac{1}{|Q|}\int_Q (G^{1/q_0} w^{q/q_0})^{q_0}\right)\left(\frac{1}{|Q|}\int_Q (G^{1/q_0} w^{q/q_0})^{-p_0'}\right)^{q_0/p_0'}$$
$$= \sup_Q \left(\frac{1}{|Q|}\int_Q Gw^q\right)\left(\frac{1}{|Q|}\int_Q (Gw^q)^{-p_0'/q_0}\right)^{q_0/p_0'}$$
$$= [Gw^q]_{A_{r_0}}.$$

Then, we can proceed with

$$\left(\int_{\mathbb{R}^n} |Tf|^q w^q\right)^{1/q} = \left(\int_{\mathbb{R}^n} |Tf|^{q_0} g w^q\right)^{\frac{1}{q_0}}$$
$$\le \left(\int_{\mathbb{R}^n} |Tf|^{q_0} G w^q\right)^{\frac{1}{q_0}}$$
$$= \left(\int_{\mathbb{R}^n} |Tf|^{q_0} (G^{1/q_0} w^{q/q_0})^{q_0}\right)^{\frac{1}{q_0}}$$
$$\le c\,[G^{1/q_0} w^{q/q_0}]^{\gamma}_{A_{p_0,q_0}} \left(\int_{\mathbb{R}^n} |f|^{p_0} (G^{1/q_0} w^{q/q_0})^{p_0}\right)^{\frac{1}{p_0}}$$
$$= c\,[Gw^q]^{\gamma}_{A_{r_0}} \left(\int_{\mathbb{R}^n} |f|^{p_0} w^{p_0} G^{p_0/q_0} w^{q/(p/p_0)'}\right)^{\frac{1}{p_0}}$$

$$\leq c\,[w]^{\gamma}_{A_{p,q}} \left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p} \left(\int_{\mathbb{R}^n} G^{(r/r_0)'} w^q\right)^{(p-p_0)/pp_0}$$
$$\leq c\,[w]^{\gamma}_{A_{p,q}} \left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p},$$

where we have used the relationship

$$\frac{1}{p}-\frac{1}{q}=\frac{1}{p_0}-\frac{1}{q_0}.$$

For the case $1 < p < p_0$, and hence $q < q_0$, notice that we can write

$$\left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p} = \left(\int_{\mathbb{R}^n} (|f w^{p'}|^{p_0})^{p/p_0} w^{-p'}\right)^{1/p}.$$

Since $p/p_0 < 1$, there exists a function $g \geq 0$ satisfying

$$\int_{\mathbb{R}^n} g^{p/(p-p_0)} w^{-p'} = 1$$

such that

$$\left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p} = \left(\int_{\mathbb{R}^n} |f w^{p'}|^{p_0} g w^{-p'}\right)^{1/p_0},$$

see [8, pp. 335]. Let $h = g^{-p_0'/p_0}$, $r = 1 + p'/q$ and $r_0 = 1 + p_0'/q_0$, so that $r > r_0$. Notice that

$$\frac{1}{p}-\frac{1}{q}=\frac{1}{p_0}-\frac{1}{q_0}$$

implies $r/r_0 = p'/p_0'$, which in turn yields

$$\frac{p_0'}{p_0}\left(\frac{r}{r_0}\right)' = \frac{p}{p_0-p}\,. \tag{3.5}$$

Hence,

$$\int_{\mathbb{R}^n} h^{(r/r_0)'} w^{-p'} = \int_{\mathbb{R}^n} g^{p/(p-p_0)} w^{-p'} = 1\,.$$

Observe that $w^{-p'} \in A_r$, so by Lemma 3.1 we obtain a function H such that $H \geq h$, $\|H\|_{L^{(r/r_0)'}(w^{-p'})} \leq 2$, and $Hw^{-p'} \in A_{r_0}$ with $[Hw^{-p'}]_{A_{r_0}} \leq c\,[w^{-p'}]_{A_r} = c\,[w]^{p'/q}_{A_{p,q}}$. Now, for $Hw^{-p'} \in A_{r_0}$ we claim that $(Hw^{-p'})^{-1/p_0'} \in A_{p_0,q_0}$ with

$[(Hw^{-p'})^{-1/p_0'}]_{A_{p_0,q_0}} = [Hw^{p'}]_{A_{r_0}}^{q_0/p_0'}$. Indeed,

$$\begin{aligned}[(Hw^{-p'})^{-1/p_0'}]_{A_{p_0,q_0}} &= \sup_Q \left(\frac{1}{|Q|}\int_Q (H^{-1/p_0'}w^{p'/p_0'})^{q_0}\right)\left(\frac{1}{|Q|}\int_Q (H^{-1/p_0'}w^{p'/p_0'})^{-p_0'}\right)^{q_0/p_0'} \\ &= \sup_Q \left(\frac{1}{|Q|}\int_Q (Hw^{-p'})^{-q_0/p_0'}\right)\left(\frac{1}{|Q|}\int_Q Hw^{-p'}\right)^{q_0/p_0'} \\ &= [Hw^{-p'}]_{A_{r_0}}^{q_0/p_0'}.\end{aligned}$$

Finally expressing g in terms for h and using (3.5), working backwards we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p} &= \left(\int_{\mathbb{R}^n} |f|^{p_0} h^{-p_0/p_0'} w^{p'(p_0-1)}\right)^{1/p_0} \\ &\geq \left(\int_{\mathbb{R}^n} |f|^{p_0} H^{-p_0/p_0'} w^{p'(p_0-1)}\right)^{1/p_0} \\ &= \frac{[(Hw^{-p'})^{-1/p_0'}]^{\gamma}_{A_{p_0,q_0}}}{[(Hw^{-p'})^{-1/p_0'}]^{\gamma}_{A_{p_0,q_0}}} \left(\int_{\mathbb{R}^n} |f|^{p_0} (H^{-1/p_0'} w^{p'/p_0'})^{p_0}\right)^{1/p_0} \\ &\geq \frac{c}{[(Hw^{-p'})^{-1/p_0'}]^{\gamma}_{A_{p_0,q_0}}} \left(\int_{\mathbb{R}^n} |Tf|^{q_0} (H^{-1/p_0'} w^{p'/p_0'})^{q_0}\right)^{1/q_0} \\ &\geq \frac{c}{[(Hw^{-p'})^{-1/p_0'}]^{\gamma}_{A_{p_0,q_0}}} \left(\int_{\mathbb{R}^n} |Tf|^{q} w^{q}\right)^{1/q} \left(\int_{\mathbb{R}^n} H^{(r/r_0)'} w^{p'}\right)^{q-q_0/qq_0} \\ &\geq \frac{c}{[(Hw^{-p'})^{-1/p_0'}]^{\gamma}_{A_{p_0,q_0}}} \left(\int_{\mathbb{R}^n} |Tf|^{q} w^{q}\right)^{1/q}.\end{aligned}$$

In the second to last inequality we have used Hölder's inequality for exponents less than one, i.e., if $0 < s < 1$ then

$$\|fg\|_{L^1} \geq \|f\|_{L^s}\|g\|_{L^{s'}},$$

where as usual $s' = s/(s-1)$. See [7, pp. 10] for more details. Thus we have shown,

$$\left(\int_{\mathbb{R}^n} |Tf|^q w^q\right)^{1/q} \leq c\,[(Hw^{-p'})^{-1/p_0'}]^{\gamma}_{A_{p_0,q_0}} \left(\int_{\mathbb{R}^n} |f|^p w^p\right)^{1/p}.$$

From here we have

$$\|T\| \leq c\,[(Hw^{-p'})^{-1/p_0'}]^{\gamma}_{A_{p_0,q_0}} = c\,[Hw^{-p'}]_{A_{r_0}}^{\gamma\frac{q_0}{p_0'}} \leq c\,[w^{-p'}]_{A_{1+p'/q}}^{\gamma\frac{q_0}{p_0'}} = c\,[w]_{A_{p,q}}^{\gamma\frac{q_0}{p_0'}\frac{p'}{q}}.$$

This proves the theorem. □

4. Proofs of strong-type results using extrapolation

We will need to use the following weighted versions of M_α. For $0 \le \alpha < n$, let

$$M^c_{\alpha,\nu} f(x) = \sup_{Q_x} \frac{1}{\nu(Q_x)^{1-\alpha/n}} \int_{Q_x} |f(y)|\, d\nu,$$

where the supremum is over all cubes Q_x with center x. A dyadic version of M_α was first introduced by Sawyer in [25]. This maximal function will be an effective tool in obtained the estimates for I_α. The following lemma will be used in the proofs of Theorems 2.3 and 2.9.

Lemma 4.1. *Let $0 \le \alpha < n$ and ν be a positive Borel measure. Then,*

$$\|M^c_{\alpha,\nu} f\|_{L^q(\nu)} \le c\, \|f\|_{L^p(\nu)}$$

for all $1 < p \le q < \infty$ that satisfy $1/p - 1/q = \alpha/n$. Furthermore, the constant c is independent of ν (it depends only on the dimension and p).

The proof of Lemma 4.1 can be obtained by interpolation. In fact, the strong $(n/\alpha, \infty)$ inequality follows directly from Hölder's inequality, while a weak-$(1,(n/\alpha)')$ estimate is a consequence of the Besicovich covering lemma.

Proof of Theorem 2.3. The equation $q_0/p_0' = 1 - \alpha/n$ along with the fact that $1/p_0 - 1/q_0 = \alpha/n$ yields

$$p_0 = \frac{2-\alpha/n}{\alpha/n - (\alpha/n)^2 + 1} \quad \text{and} \quad q_0 = \frac{2-\alpha/n}{1-\alpha/n}.$$

We want to show the linear estimate

$$\|wI_\alpha f\|_{L^{q_0}} \le c\,[w]_{A_{p_0,q_0}} \|wf\|_{L^{p_0}}. \tag{4.1}$$

Notice that (4.1) is equivalent to

$$\|I_\alpha(f\sigma)\|_{L^{q_0}(u)} \le c\,[w]_{A_{p_0,q_0}} \|f\|_{L^{p_0}(\sigma)}, \tag{4.2}$$

where $u = w^{q_0}$ and $\sigma = w^{-p_0'}$. Moreover, by duality, showing (4.2) is equivalent to prove

$$\int_{\mathbb{R}^n} I_\alpha(f\sigma) g u\, dx \le c\,[w]_{A_{p_0,q_0}} \left(\int_{\mathbb{R}^n} f^{p_0}\sigma\, dx\right)^{1/p_0} \left(\int_{\mathbb{R}^n} g^{q_0'} u\, dx\right)^{1/q_0'} \tag{4.3}$$

for all f and g non-negative bounded functions with compact support.

We first discretize the operator I_α as follows. Given a non-negative function f,

$$\begin{aligned} I_\alpha f(x) &= \sum_{k\in\mathbb{Z}} \int_{2^{k-1}<|x-y|\le 2^k} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \\ &\le c \sum_k \sum_{\substack{Q\in\mathcal{D} \\ \ell(Q)=2^k}} \chi_Q(x) \frac{1}{\ell(Q)^{n-\alpha}} \int_{|x-y|\le \ell(Q)} f(y)\, dy \end{aligned}$$

$$\leq c \sum_{Q\in\mathcal{D}} \chi_Q(x) \frac{|Q|^{\alpha/n}}{|Q|} \int_{3Q} f\ dy$$

where the last inequality holds because if $x \in Q$, then $B(x, \ell(Q)) \subseteq 3Q$.

One immediately gets then

$$\int_{\mathbb{R}^n} I_\alpha(f\sigma) gu\ dx \leq c \sum_{\mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_{3Q} f\sigma\, dx \int_Q gu\, dx.$$

The next crucial step is to pass to a more convenient sum where the family of dyadic cubes is replaced by an appropriate subset formed by a family of Calderón-Zygmund dyadic cubes. We combine ideas from the work of Sawyer and Wheeden in [28, pp. 824–829], together with some techniques from [20] (see also [19]).

Fix $a > 2^n$. Since g is bounded with compact support, for each $k \in \mathbb{Z}$, one can construct a collection $\{Q_{k,j}\}_j$ of pairwise disjoint maximal dyadic cubes (maximal with respect to inclusion) with the property that

$$a^k < \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} gu\, dx.$$

By maximality the above also gives

$$\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} gu\, dx \leq 2^n a^k.$$

Although the maximal cubes in the whole family $\{Q_{k,j}\}_{k,j}$ are disjoint in j for each fixed k, they may not be disjoint for different k's. If we define for each k the collection

$$\mathcal{C}^k = \left\{ Q \in \mathcal{D} : a^k < \frac{1}{|Q|} \int_Q gu\, dx \leq a^{k+1} \right\},$$

then each dyadic cube Q belongs to only one $\mathcal{C}^k$ or gu vanishes on it. Moreover, each $Q \in \mathcal{C}^k$ has to be contained in one of the maximal cubes Q_{k,j_0} and verifies for all $Q_{k,j}$

$$\frac{1}{|Q|} \int_Q gu\, dx \leq a^{k+1} \leq \frac{a}{|Q_{k,j}|} \int_{Q_{k,j}} gu\, dx.$$

From these properties and the fact that for any dyadic cube Q_0,

$$\sum_{Q\in\mathcal{D}, Q\subset Q_0} |Q|^{\alpha/n} \int_{3Q} f\sigma\, dx \leq c_\alpha |Q_0|^{\alpha/n} \int_{3Q_0} f\sigma\, dx,$$

one easily deduces as in [28] that

$$\sum_{\mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_{3Q} f\sigma\, dx \int_Q gu\, dx \leq a\, c_\alpha \sum_{k,j} \frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|} \int_{3Q_{k,j}} f\sigma\, dx \int_{Q_{k,j}} gu\, dx.$$

Notice also that,

$$[w]_{A_{p_0,q_0}} = \sup_Q \frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^{1-\alpha/n} < \infty,$$

so we can estimate

$$\begin{aligned}\int_{\mathbb{R}^n} I_\alpha(f\sigma)gu\,dx &\le c\sum_{k,j}\frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|}\int_{3Q_{k,j}} f\sigma\,dx\int_{Q_{k,j}} gu\,dx\\ &= c\sum_{k,j}\frac{1}{\sigma(5Q_{k,j})^{1-\alpha/n}}\int_{3Q_{k,j}} f\sigma\ dx\ \frac{1}{u(3Q_{k,j})}\int_{Q_{k,j}} gu\ dx\\ &\qquad\times\frac{u(3Q_{k,j})}{|Q_{k,j}|}\left(\frac{\sigma(5Q_{k,j})}{|Q_{k,j}|}\right)^{1-\alpha/n}|Q_{k,j}|\\ &\le c\,[w]_{A_{p_0,q_0}}\sum_{k,j}\frac{1}{\sigma(5Q_{k,j})^{1-\alpha/n}}\int_{3Q_{k,j}} f\sigma dx\frac{1}{u(3Q_{k,j})}\int_{Q_{k,j}} gu\ dx|Q_{k,j}|, \qquad (4.4)\end{aligned}$$

where we have set up things to use, in a moment, certain centered maximal functions.

Before we do so, we need one last property about the Calderón-Zygmund cubes $Q_{k,j}$. We need to pass to a disjoint collection of sets $E_{k,j}$ each of which retains a substantial portion of the mass of the corresponding cube $Q_{k,j}$.

Define the sets

$$E_{k,j} = Q_{k,j}\cap\{x\in R^n : a^k < M^d(gu)\le a^{k+1}\},$$

where M^d is the dyadic maximal function. The family $\{E_{k,j}\}_{k,j}$ is pairwise disjoint for all j and k. Moreover, suppose that for some point $x\in Q_{k,j}$ it happens that $M^d(gu)(x) > a^{k+1}$. By the maximality of $Q_{k,j}$, this implies that there exist some dyadic cube Q such that $x\in Q\subset Q_{k,j}$ and so that the average of gu over Q is larger than a^{k+1}. It must also hold then that $M^d(gu\chi_{Q_{k,j}})(x) > a^{k+1}$. But

$$|\{M^d(gu\chi_{Q_{k,j}})(x) > a^{k+1}\}| \le \frac{1}{a^{k+1}}\int_{Q_{k,j}} gu\,dx \le \frac{2^n|Q_{k,j}|}{a}.$$

It follows that

$$|E_{k,j}| \ge (1-\frac{2^n}{a})|Q_{k,j}|.$$

Recalling now that $1 = u^{\frac{n}{n-\alpha}}\sigma = u^{\frac{1}{q_0}\frac{n}{n-\alpha}}\sigma^{\frac{1}{q_0}}$, we can use Hölder's inequality to write

$$|Q_{k,j}| \approx |E_{k,j}| = \int_{E_{k,j}} u^{\frac{1}{q_0}\frac{n}{n-\alpha}}\sigma^{\frac{1}{q_0}} \le u(E_{k,j})^{1/q_0'}\sigma(E_{k,j})^{1/q_0}, \qquad (4.5)$$

since

$$\frac{q_0'}{q_0}\frac{n}{n-\alpha} = 1.$$

With (4.5) we go back to the string of inequalities to estimate $\int I_\alpha(f\sigma)\, gu\, dx$. Using the discrete version of Hölder's inequality, we can estimate in (4.4)

$$\leq c\,[w]_{A_{p_0,q_0}} \left(\sum_{k,j} \left(\frac{1}{\sigma(5Q_{k,j})^{1-\alpha/n}} \int_{3Q_{k,j}} f\sigma \; dx \right)^{q_0} \sigma(E_{k,j}) \right)^{1/q_0}$$

$$\times \left(\sum_{k,j} \left(\frac{1}{u(3Q_{k,j})} \int_{Q_{k,j}} gu \; dx \right)^{q_0'} u(E_{k,j}) \right)^{1/q_0'}$$

$$\leq c\,[w]_{A_{p_0,q_0}} \left(\sum_{k,j} \int_{E_{k,j}} (M^c_{\alpha,\sigma} f)^{q_0} \sigma \; dx \right)^{1/q_0} \left(\sum_{k,j} \int_{E_{k,j}} (M^c_{u} g)^{q_0'} u \; dx \right)^{1/q_0'}$$

$$\leq c\,[w]_{A_{p_0,q_0}} \left(\int_{\mathbb{R}^n} (M^c_{\alpha,\sigma} f)^{q_0} \sigma \; dx \right)^{1/q_0} \left(\int_{\mathbb{R}^n} (M^c_{u} g)^{q_0'} u \; dx \right)^{1/q_0'}$$

$$\leq c\,[w]_{A_{p_0,q_0}} \left(\int_{\mathbb{R}^n} f^{p_0} \sigma \; dx \right)^{1/p_0} \left(\int_{\mathbb{R}^n} g^{q_0'} u \; dx \right)^{1/q_0'}.$$

Here we have denoted by $M^c_u = M^c_{0,u}$, the centered maximal function with respect to the measure u. We have also used in the last step Lemma 4.1, which gives the boundedness of M^c_u and $M^c_{\alpha,\sigma}$ with operator norms independent of the corresponding measure. We obtain then the desired linear estimate

$$\|wI_\alpha f\|_{L^{q_0}} \leq c\,[w]_{A_{p_0,q_0}} \|wf\|_{L^{p_0}}. \tag{4.6}$$ □

From this estimate we can extrapolate (Theorem 2.1) to get,

$$\|wI_\alpha f\|_{L^{q}} \leq c\,[w]_{A_{p,q}}^{\max\{1,(1-\alpha/n)p'/q\}} \|wf\|_{L^{p}} \tag{4.7}$$

for all $1 < p < q < \infty$ with $1/p - 1/q = \alpha/n$.

We can further improve on this using duality. We first observe that, with the pairing

$$\langle g, f\rangle = \int_{\mathbb{R}^n} f(x)g(x)\, dx,$$

we can also isometrically identify the dual of $L^q(w^q)$ with $L^{q'}(w^{-q'})$.

It follows then that for any $1 < p < q < \infty$ with $1/p - 1/q = \alpha/n$,

$$\begin{aligned} \|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} &= \sup_{f\in L^p(w^p),\, g\in L^{q'}(w^{-q'})} \left| \int_{\mathbb{R}^n} I_\alpha f(x) g(x) dx \right| \\ &= \sup_{f\in L^p(w^p),\, g\in L^{q'}(w^{-q'})} \left| \int_{\mathbb{R}^n} f(x) I_\alpha g(x) dx \right| \\ &= \|I_\alpha\|_{L^{q'}(w^{-q'})\to L^{p'}(w^{-p'})}. \end{aligned}$$

It follows that from (4.7) we also get

$$\|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} \le c\,[w^{-1}]_{A_{q',p'}}^{\max\{1,(1-\alpha/n)q/p'\}}, \tag{4.8}$$

which combined with (3.4) gives

$$\|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} \le c\,[w]_{A_{p,q}}^{\max\{p'/q,(1-\alpha/n)\}}, \tag{4.9}$$

again for any $1 < p < q < \infty$ with $1/p - 1/q = \alpha/n$. The combination of (4.7) and (4.9) yields

$$\|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} \le c[w]_{A_{p,q}}^{\min\{\max(1-\frac{\alpha}{n},\frac{p'}{q}),\max(1,(1-\frac{\alpha}{n})\frac{p'}{q})\}}.$$

As we mentioned earlier, this last estimate is only sharp for $p'/q \in (0, 1-\alpha/n] \cup [n/(n-\alpha),\infty)$. We obtain the right estimate in the full range of exponents in the next section. The sharpness will be obtained in Section 7.

5. Proof of the weak-type results and sharp bounds for the full range of exponents

We start with the weak-type version of the extrapolation theorem.

Proof of Corollary 2.2. Note that Theorem 2.1 does not require T to be linear. We can simply apply then the result to the operator $T_\lambda f = \lambda\chi_{\{|Tf|>\lambda\}}$. Fix $\lambda > 0$, then

$$\begin{aligned}\|wT_\lambda f\|_{L^{q_0}} &= \lambda w^{q_0}(\{x : |Tf(x)| > \lambda\})^{1/q_0}\\ &\le \|Tf\|_{L^{q_0,\infty}(w^{q_0})}\\ &\le c[w]_{A_{p_0,q_0}}^{\gamma}\|wf\|_{L^{p_0}},\end{aligned}$$

with constant independent of λ. Hence by Theorem 2.1 if $w \in A_{p,q}$, T_λ maps $L^q(w^q) \to L^p(w^p)$ for all $1/p - 1/q = 1/p_0 - 1/q_0$ and with bound

$$\|wT_\lambda f\|_{L^q} \le c\,[w]_{A_{p,q}}^{\gamma\max\{1,\frac{q_0}{p_0'}\frac{p'}{q}\}}\|fw\|_{L^p}.$$

with c independent of λ. Hence,

$$\|Tf\|_{L^{q,\infty}(w^q)} = \sup_{\lambda>0}\|wT_\lambda f\|_{L^q} \le c\,[w]_{A_{p,q}}^{\gamma\max\{1,\frac{q_0}{p_0'}\frac{p'}{q}\}}\|fw\|_{L^p}. \qquad \square$$

Proof of Theorem 2.4. First Proof (valid for $p \ge 1$). We apply Corollary 2.2 with $p_0 = 1$, $q_0 = n/(n-\alpha) = (n/\alpha)'$, and $u = w^{q_0}$.

Actually, we are going to prove a better estimate, namely

$$\|I_\alpha f\|_{L^{q_0,\infty}(u)} \le c\,\|f\|_{L^1((Mu)^{1/q_0})} \tag{5.1}$$

for any weight u. From this estimate, and since by (3.2) the $A_{1,(n/\alpha)'}$ condition for w is equivalent to

$$M(u) \le [w]_{A_{1,(n/\alpha)'}}u,$$

we can deduce

$$\|I_\alpha f\|_{L^{q_0,\infty}(u)} \le c\,[w]^{1-\alpha/n}_{A_{1,(n/\alpha)'}}\|fw\|_{L^1}.$$

The weak extrapolation Corollary 2.2 with $\gamma = 1-\alpha/n$ gives the right estimate.

In order to prove (5.1), we note that $\|\cdot\|_{L^{q_0,\infty}(u)}$ is equivalent to a norm since $q_0 > 1$. Hence, we may use Minkowski's integral inequality as follows

$$\|I_\alpha f\|_{L^{q_0,\infty}(u)} \le c_q \int_{\mathbb{R}^n} |f(y)| \| |\cdot - y|^{\alpha-n}\|_{L^{q_0,\infty}(u)}\, dy. \tag{5.2}$$

We can finally calculate the inner norm by

$$\begin{aligned} \||\cdot - y|^{\alpha-n}\|_{L^{q_0,\infty}(w^q)} &= \sup_{\lambda>0} \lambda u(\{x : |x-y|^{\alpha-n} > \lambda\})^{1/q_0} \\ &= (\sup_{t>0} \frac{1}{t^n} u(\{x : |x-y| < t\}))^{1/q_0} \\ &= cMu(y)^{1/q_0}. \end{aligned}$$

Once again, the sharpness of the exponent $1-\alpha/n$ will be shown with an example in Section 7.

Second Proof (valid for $p > 1$ only).

We need to recall another characterization of the weak-type inequality for I_α for two weights. This characterization is due to Gabidzashvili and Kokilashvili [5] and establishes that for $1 < p < q < \infty$, the two-weight weak type inequality,

$$\|I_\alpha\|_{L^p(v)\to L^{q,\infty}(u)} < \infty \tag{5.3}$$

hods if and only if

$$\sup_Q \left(\int_Q u(x)\, dx\right)^{1/q} \left(\int_{\mathbb{R}^n} (|Q|^{1/n} + |x_Q - x|)^{(\alpha-n)p'} v(x)^{1-p'}\, dx\right)^{1/p'} < \infty \tag{5.4}$$

where x_Q denotes the center of the cube Q. We will refer to (5.4) as the *global testing condition*, given its global character when compared to the *local testing conditions* of Sawyer. We will use the notation

$$[u,v]_{\mathrm{Glo}(p,q)} = \sup_Q \left(\int_Q u(x)\, dx\right)^{1/q} \left(\int_{\mathbb{R}^n} (|Q|^{1/n} + |x_Q - x|)^{(\alpha-n)p'} v(x)^{1-p'}\, dx\right)^{1/p'}.$$

It follows from the proof in [5] (see also [28]) that

$$\|I_\alpha\|_{L^p(v)\to L^{q,\infty}(u)} \approx [u,v]_{\mathrm{Glo}(p,q)}. \tag{5.5}$$

We now need a reverse doubling property satisfied by w^q when $w \in A_{p,q}$ class (see [28] for precise definitions).

Lemma 5.1. *Let $w \in A_{p,q}$, then for any cube Q we have the estimate*

$$\frac{\int_Q w^q\, dx}{\int_{2Q} w^q\, dx} \le 1 - c[w]^{-1}_{A_{p,q}} \tag{5.6}$$

for an absolute constant c.

Proof. Let $E \subset Q$. Our goal is to show that

$$\left(\frac{|E|}{|Q|}\right)^q [w]_{A_{p,q}}^{-1} \leq \frac{\int_E w^q\, dx}{\int_Q w^q\, dx}. \tag{5.7}$$

Applying this with $E = Q - \frac{1}{2}Q$ will prove the Lemma. We can estimate

$$\begin{aligned}
\frac{|E|}{|Q|} &= \frac{\int_E w \cdot w^{-1}}{|Q|} \\
&\leq \left[\frac{\int_E w^q\, dx}{|Q|}\right]^{1/q} \left[\frac{\int_Q w^{-q'}\, dx}{|Q|}\right]^{1/q'} \\
&\leq \left[\frac{\int_E w^q\, dx}{|Q|}\right]^{1/q} \left[\frac{\int_E w^{-p'}\, dx}{|Q|}\right]^{1/p'} \qquad (q' < p') \\
&= \left[\frac{\int_E w^q\, dx}{\int_Q w^q\, dx}\right]^{1/q} \cdot \left[\frac{\int_Q w^q\, dx}{|Q|}\right]^{1/q} \left[\frac{\int_Q w^{-p'}\, dx}{|Q|}\right]^{1/p'} \\
&\leq \left[\frac{\int_E w^q\, dx}{\int_Q w^q\, dx}\right]^{1/q} [w]_{A_{p,q}}^{1/q}.
\end{aligned}$$

The proof is complete. □

We now claim that in the case $u = w^q$ and $v = w^p$ the constant in the global testing condition and the $A_{p,q}$ constant of w are comparable:

$$[w^q, w^p]_{\mathrm{Glo}(p,q)} \approx [w]_{A_{p,q}}^{(1-\alpha/n)}. \tag{5.8}$$

Proof of (5.8). Observe that $p'(1-\alpha/n) = 1 + p'/q$. One of the inequalities in (5.8) is clear. For the other we estimate

$$\begin{aligned}
&\left(\int_Q w(x)^q\, dx\right)^{1/q} \left(\int_{\mathbb{R}^n} (|Q|^{1/n} + |x_Q - x|)^{(\alpha-n)p'} w(x)^{p(1-p')}\, dx\right)^{1/p'} \\
&\qquad \leq c \left(\int_Q w^q\right)^{1/q} \left[\sum_{j=0}^{\infty} |2^j Q|^{-p'(1-\alpha/n)} \int_{2^j Q} w^{-p'}\right]^{1/p'} \\
&\qquad = c \left[\sum_{j=0}^{\infty} \left(\frac{\int_Q w^q}{\int_{2^j Q} w^q}\right)^{p'/q} \left(\frac{\int_{2^j Q} w^q}{|2^j Q|}\right)^{p'/q} \frac{\int_{2^j Q} w^{p'}}{|2^j Q|}\right]^{1/p'} \\
&\qquad \leq c[w]_{A_{p,q}}^{1/q} \left[\sum_{j=0}^{\infty} \left(\frac{\int_Q w^q}{\int_{2^j Q} w^q}\right)^{p'/q}\right]^{1/p'} \\
&\qquad \leq c[w]_{A_{p,q}}^{1/q} \left[\sum_{j=0}^{\infty} (1 - c[w]_{A_{p,q}}^{-1})^{p' j/q}\right]^{1/p'}
\end{aligned}$$

$$\le c[w]_{A_{p,q}}^{1-\alpha/n}.$$

Note that the next to last line follows from (5.6) and an immediate inductive argument. In the last line, we just use the equality $1/q + 1/p' = 1 - \alpha/n$. ☐

To conclude the second proof of Theorem 2.4 we use (5.5)

$$\|I_\alpha\|_{L^p(w^p)\to L^{q,\infty}(w^q)} \approx [w^p, w^q]_{\mathrm{Glo}(p,q)} \approx [w]_{A_{p,q}}^{1-\alpha/n}. \qquad \square$$

We conclude this section by verifying that (2.3) and (2.9) yield Theorem 2.6. Indeed

$$\begin{aligned}\|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} &\approx \|I_\alpha\|_{L^p(w^p)\to L^{q,\infty}(w^q)} + \|I_\alpha\|_{L^{q'}(w^{-q'})\to L^{p',\infty}(w^{-p'})} \\ &\approx [w]_{A_{p,q}}^{1-\frac{\alpha}{n}} + [w^{-1}]_{A_{q',p'}}^{1-\frac{\alpha}{n}} \approx [w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max\{1,\frac{p'}{q}\}}\end{aligned}$$

since $[w^{-1}]_{A_{q',p'}} = [w]_{A_{p,q}}^{p'/q}$ and since $[w]_{A_{p,q}} \ge 1$.

6. Proof of the sharp bounds for the fractional maximal function

Proof of Theorem 2.9. First notice that $M_\alpha \approx M_\alpha^c$ where M_α^c is the centered version. Let $x \in \mathbb{R}^n$, Q a cube centered at x, $u = w^q$, $\sigma = w^{-p'}$ and $r = 1 + q/p'$. Noticing that $p'/q(1-\alpha/n) = r'/q$, we proceed as in [13] to obtain

$$\begin{aligned}\frac{1}{|Q|^{1-\alpha/n}}\int_Q |f|\,dy &\le 3^{nr'/q}[w]_{A_{p,q}}^{p'/q(1-\alpha/n)}\left(\frac{|Q|}{u(Q)}\right)^{p'/q(1-\alpha/n)}\frac{1}{\sigma(3Q)^{1-\alpha/n}}\int_Q \frac{|f|}{\sigma}\sigma\,dy \\ &\le c\,[w]_{A_{p,q}}^{p'/q(1-\alpha/n)}\left(\frac{1}{u(Q)}\int_Q M_{\alpha,\sigma}^c(f/\sigma)^{q/r'}\,dy\right)^{r'/q}.\end{aligned}$$

Taking the supremum over all cubes centered at x we have the pointwise estimate

$$M_\alpha^c f(x) \le c\,[w]_{A_{p,q}}^{p'/q(1-\alpha/n)} M_u^c\{M_{\alpha,\sigma}^c(f/\sigma)^{q/r'}u^{-1}\}(x)^{r'/q}.$$

Using the fact that $M_u : L^{r'}(u) \to L^{r'}(u)$ with operator norm independent of u combined with Lemma 4.1, we get

$$\begin{aligned}\|w\,M_\alpha f\|_{L^q} &\le c\,\|M_\alpha^c f\|_{L^q(u)} \\ &\le c\,[w]_{A_{p,q}}^{p'/q(1-\alpha/n)}\|M_u^c\{M_{\alpha,\sigma}^c(f/\sigma)^{q/r'}u^{-1}\}\|_{L^{r'}(u)}^{r'/q} \\ &\le c\,[w]_{A_{p,q}}^{p'/q(1-\alpha/n)}\|fw\|_{L^p},\end{aligned}$$

which is the desired estimate. ☐

7. Examples

We will use the power weights considered in [2] to show that Theorems 2.4, 2.6, and 2.9 are sharp.

Suppose again $0 < \alpha < n$ with

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}.$$

Let $w_\delta(x) = |x|^{(n-\delta)/p'}$ so that $w_\delta \in A_{p,q}$, with

$$[w_\delta]_{A_{p,q}} = [w_\delta^q]_{A_{1+q/p'}} \approx \delta^{-q/p'}.$$

Then, if $f_\delta(x) = |x|^{\delta-n}\chi_B$, where B is the unit ball in $\mathbb{R}^n$, we have

$$\|w_\delta f_\delta\|_{L^p} \approx \delta^{-1/p}.$$

For $x \in B$,

$$M_\alpha f_\delta(x) \geq \frac{C}{|x|^{n-\alpha}} \int_{B(0,|x|)} |f_\delta(y)|\, dy \approx \frac{|x|^{\delta-n+\alpha}}{\delta},$$

and so we have

$$\int_{\mathbb{R}^n} w_\delta^q M_\alpha f_\delta(x)^q\, dx \geq \delta^{-q} \int_B |x|^{(\delta-n+\alpha)q} |x|^{(n-\delta)\frac{q}{p'}}\, dx \approx \delta^{-q-1}.$$

It follows that

$$\delta^{-1-1/q} \leq c\, \|w_\delta M f_\delta\|_{L^q} \leq c\, [w_\delta]_{A_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{n})} \|w_\delta f_\delta\|_{L^p} \approx \delta^{-(1-\frac{\alpha}{n})} \delta^{-1/p} = \delta^{-1-1/q}, \tag{7.1}$$

showing Theorem 2.9 is sharp.

Next we show that the same example can be used to show that the exponent in Theorem 2.6 is sharp. Assume first that $p'/q \geq 1$We simply observe that, pointwise,

$$M_\alpha \leq C I_\alpha$$

for some universal constant C. Then using the same w_δ and f_δ as above and the estimate in Theorem 2.6 we arrive at the estimate in equation (7.1) with M_α replaced by I_α, showing sharpness. The case when p'/q immediately follows by the duality arguments described after the proof of Theorem 2.3.

Finally, we show that the exponent $1 - \alpha/n$ in the estimate

$$\|I_\alpha f\|_{L^{q,\infty}(w^q)} \leq c\, [w]_{A_{p,q}}^{1-\alpha/n} \|fw\|_{L^p} \tag{7.2}$$

from Theorem 2.4 is sharp for $p \geq 1$.

By (3.3)

$$\|I_\alpha f\|_{L^{q,\infty}(w^q)} \leq c\, [w^q]_{A_{1+q/p'}}^{1-\alpha/n} \|fw\|_{L^p}, \tag{7.3}$$

and if we let $u = w^q$,

$$\|I_\alpha f\|_{L^{q,\infty}(u)} \leq c\, [u]_{A_{1+q/p'}}^{1-\alpha/n} \|f\|_{L^p(u^{p/q})}. \tag{7.4}$$

Assume now that $u \in A_1$. Then (7.4) yields

$$\|I_\alpha f\|_{L^{q,\infty}(u)} \le c\,[u]_{A_1}^{1-\alpha/n}\|f\|_{L^p(u^{p/q})}. \tag{7.5}$$

Since $\frac{p}{q} = 1 - \frac{p\alpha}{n}$, this is equivalent to

$$\|I_\alpha(u^{\frac{\alpha}{n}} f)\|_{L^{q,\infty}(u)} \le c\,[u]_{A_1}^{1-\alpha/n}\|f\|_{L^p(u)}. \tag{7.6}$$

We now prove that (7.6) is sharp. Let

$$u(x) = |x|^{\delta-n}$$

with $0<\delta<1$. Then standard computations shows that

$$[u]_{A_1} \approx \frac{1}{\delta} \tag{7.7}$$

Consider the function $f = \chi_B$, where B is again the unit ball, we can compute its norm to be

$$\|f\|_{L^p(u)} = u(B)^{1/p} = c\left(\frac{1}{\delta}\right)^{1/p}. \tag{7.8}$$

Let $0 < x_\delta < 1$ be a parameter whose value will be chosen soon. We have

$$\begin{aligned}
\|I_\alpha(u^{\alpha/n} f)\|_{L^{q,\infty}(u)} &\ge \sup_{\lambda>0} \lambda\left(u\{|x|<x_\delta : \int_B \frac{|y|^{(\delta-1)\alpha/n}}{|x-y|^{1-\alpha/n}}dy > \lambda\}\right)^{1/q}\\
&\ge \sup_{\lambda>0} \lambda\left(u\{|x|<x_\delta : \int_{B\setminus B(0,|x|)} \frac{|y|^{(\delta-1)\alpha/n}}{|x-y|^{1-\alpha/n}}dy > \lambda\}\right)^{1/q}\\
&\ge \sup_{\lambda>0} \lambda\left(u\{|x|<x_\delta : \int_{B\setminus B(0,|x|)} \frac{|y|^{(\delta-1)\alpha/n}}{(2|y|)^{1-\alpha/n}}dy > \lambda\}\right)^{1/q}\\
&= \sup_{\lambda>0} \lambda\left(u\{|x|<x_\delta : \frac{c_{\alpha,n}}{\delta}(1-|x|^{\delta\alpha/n}) > \lambda\}\right)^{1/q}\\
&\ge \frac{c_{\alpha,n}}{2\delta}\left(u\{|x|<x_\delta : \frac{c_{\alpha,n}}{\delta}(1-|x|^{\delta\alpha/n}) > \frac{c_{\alpha,n}}{2\delta}\}\right)^{1/q}\\
&= \frac{c_{\alpha,n}}{2\delta} u(B(0,x_\delta))^{1/q}.
\end{aligned}$$

if $x_\delta = (\frac{1}{2})^{n/\alpha\delta}$. It now follows that for $0<\delta<1$,

$$\|I_\alpha(u^{\alpha/n} f)\|_{L^{q,\infty}(u)} \ge \frac{c}{\delta}\left(\frac{x_\delta^\delta}{\delta}\right)^{1/q} = c\frac{1}{\delta}\left(\frac{1}{\delta}\right)^{1/q}. \tag{7.9}$$

Finally, combining (7.7), (7.8), (7.9), and using that $\frac{1}{q} - \frac{1}{p} = -\frac{\alpha}{n}$, we have that (7.5) is sharp.

8. Proof of the Sobolev-type estimate

Proof of Theorem 2.7. Since $|f(x)| \leq cI_1(|\nabla f|)(x)$ we can use Theorem 2.4 to obtain

$$\|f\|_{L^{q,\infty}(w^q)} \leq c[w]_{A_{p,q}}^{1/n'} \|\nabla f w\|_{L^p}. \tag{8.1}$$

From this weak-type estimate we can pass to a strong one with the procedure that follows. We use the so-called truncation method from [16].

Given a non-negative function g and $\lambda > 0$ we define its truncation about λ, $\tau_\lambda g$, to be

$$\tau_\lambda g(x) = \min\{g, 2\lambda\} - \min\{g, \lambda\} = \begin{cases} 0 & g(x) \leq \lambda \\ g(x) - \lambda & \lambda < g(x) \leq 2\lambda \\ \lambda & g(x) > 2\lambda. \end{cases}$$

A well-know fact about Lipschitz functions is that they are preserved by absolute values and truncations. Define $\Omega_k = \{x : 2^k < |f(x)| \leq 2^{k+1}\}$ and let $u = w^q$. Then,

$$\begin{aligned} \left(\int_{\mathbb{R}^n} (|f(x)|w(x))^q \, dx\right)^{1/q} &\leq \left(\sum_k \int_{\{2^{k+1} < |f(x)| \leq 2^{k+2}\}} |f(x)|^q u(x) \, dx\right)^{1/q} \\ &\leq c\left(\sum_k 2^{kq} u(\Omega_{k+1})\right)^{1/q} \\ &\leq c\left(\sum_k 2^{kp} u(\Omega_{k+1})^{p/q}\right)^{1/p}. \end{aligned}$$

Notice that if $x \in \Omega_{k+1}$, then $\tau_{2^k}|f|(x) = 2^k > 2^{k-1}$ and hence

$$\Omega_{k+1} \subseteq \{x : \tau_{2^k}|f|(x) > 2^{k-1}\}.$$

Furthermore, notice that $|\nabla \tau_{2^k}(|f|)| = |\nabla |f||\chi_{\Omega_k} \leq |\nabla f|\chi_{\Omega_k}$, a.e.. Continuing and using the weak-type estimate (8.1) we have

$$\begin{aligned} \|f\|_{L^q(w^q)} &\leq c\left(\sum_k (2^k u(\{x : \tau_{2^k}|f|(x) > 2^{k-1}\})^{1/q})^p\right)^{1/p} \\ &\leq c\,[w]_{A_{p,q}}^{1/n'} \left(\sum_k \int_{\Omega_k} (|\nabla \tau_{2^k}|f|(x)|w(x))^p \, dx\right)^{1/p} \\ &\leq c\,[w]_{A_{p,q}}^{1/n'} \left(\int_{\mathbb{R}^n} (|\nabla f(x)|w(x))^p \, dx\right)^{1/p}, \end{aligned}$$

since $p < q$ and the sets Ω_k are disjoint. This finishes the proof of the theorem. □

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