MODULI OF FLAT CONFORMAL STRUCTURES OF HYPERBOLIC TYPE

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Abstract. To each flat conformal structure (FCS) of hyperbolic type in the sense of Kulkarni-Pinkall, we associate, for all $\theta \in [(n-1)\pi/2, n\pi/2[$ and for all $r > \tan(\theta/n)$ a unique immersed hypersurface $\Sigma_{r,\theta} = (M, i_{r,\theta})$ in $\mathbb{H}^{n+1}$ of constant θ-special Lagrangian curvature equal to r. We show that these hypersurfaces smoothly approximate the boundary of the canonical hyperbolic end associated to the FCS by Kulkarni and Pinkall and thus obtain results concerning the continuous dependance of the hyperbolic end and of the Kulkarni-Pinkall metric on the flat conformal structure.

1. Introduction

A flat conformal structure (FCS) (or Möbius structure) on an n-dimensional manifold, M, is an atlas of M whose charts lie in S^n and whose transition maps are restrictions of conformal (i.e. Möbius) mappings of S^n. Such structures arise naturally in different domains of mathematics. To every FCS of hyperbolic type may be canonically associated a complete hyperbolic manifold with convex boundary called the hyperbolic end of that structure. The purpose of this paper is to associate to every such FCS defined over a compact manifold families of foliations of neighbourhoods of the finite boundary of its hyperbolic end consisting of smooth, convex hypersurfaces of constant curvature.

The history of FCSs begins with the 2-dimensional case. Here, Thurston shows, for example, that the moduli space of FCSs over a compact surface, M, is homeomorphic to the Cartesian product $\mathcal{T} \times \mathcal{ML}(M)$ of the Teichmüller space of M with the space of measured geodesic laminations over M (see [10] or [20] for details). An important step in Thurston's proof involves the construction of a convex, pleated, equivariant "immersion" $i_T : \tilde{M} \to \mathbb{H}^3$ from the universal cover of M into $\mathbb{H}^3$ which is canonically associated to the FCS. This construction generalises that of the Nielsen Kernel of a quasi-Fuchsian manifold (see [5] for a detailed study of its properties in this case).

In the higher dimensional case, Kapovich [11] provides information on the moduli space of FCSs, but much remains unknown. However, when M is of hyperbolic

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type (see section 2.2), Kulkarni and Pinkall showed in [13] that Thurston's construction may still be carried out. This yields a convex, stratified, equivariant "immersion" $i_{KP} : M \to \mathbb{H}^{n+1}$ canonically associated to the Möbius structure, as well as a canonical $C^{1,1}$ metric over M with a.e. defined sectional curvatures taking values between -1 and 1. We call this metric the Kulkarni-Pinkall metric of the Möbius structure and denote it by g_{KP}.

Heuristically, a hyperbolic end over a manifold M is a complete, hyperbolic manifold with concave, stratified boundary whose interior is homeomorphic to $M \times \mathbb{R}$. A detailed description is provided in Sections 2.1 and 2.3. Strictly speaking, we call the boundary of $\mathcal{E}$ the finite boundary, and we denote it by $\partial_0\mathcal{E}$. This distinguishes it from the ideal boundary, $\partial_\infty\mathcal{E}$, which is the set of equivalence classes of complete half geodesics whose distance from $\partial_0\mathcal{E}$ tends to infinity.

In [13], Kulkarni and Pinkall show that the "immersion" i_{KP} may be interpreted as the finite boundary of a hyperbolic end, $\mathcal{E}$ which is also canonically associated to the FCS and whose ideal boundary $\partial_\infty\mathcal{E}$ is conformally equivalent to M. $\mathcal{E}$ thus provides a cobordism between i_{KP} and M. It is for neighbourhoods of the finite boundaries of these hyperbolic ends that we construct foliations by hypersurfaces of constant curvature. These foliations may thus be considered as families of smoothings of i_{KP}. This construction generalises to higher dimensions the result [15] of Labourie which provides families of parametrisations of the moduli spaces of three dimensional hyperbolic manifolds with geometrically finite ends.

The special Lagrangian curvature, R_θ was first developed by the author in [17]. We recall its construction in section 3.2. Its most important properties are that it is only defined for strictly convex immersed hypersurfaces and that it is regular in a PDE sense, which is summarised in this paper in terms of Theorems 3.6 and 3.7 (proven in [17]) and Theorem 4.4 (proven in [18]).

Of tangential interest, this notion of curvature arises from the natural special Legendrian structure of the unitary bundle of $U\mathbb{H}^3$. Special Legendrian structures are closely related to special Lagrangian structures which are studied under the heading of Calabi-Yau manifolds. Special Lagrangian and Legendrian submanifolds have themselves been of growing interest to mathematicians and physicists since the landmark paper [8] of Harvey and Lawson concerning calibrated geometries. In its classical form, the special Lagrangian operator is a second order, highly non-linear partial differential operator of determinant type closely related to the Monge-Ampère operator, and which is among the archetypical highly non-linear partial differential operators studied in detail in most standard works on nonlinear PDEs ([2] and [3] to name but two).

The main results of this paper are most appropriately described in terms of developing maps (see section 2.2). Let M be a manifold. A Möbius structure over M may be considered as a pair (φ, θ) where $\theta : \pi_1(M) \to \mathrm{Conf}(S^n)$ is a homomorphism and $\varphi : \tilde{M} \to S^n$ is a local homeomorphism from the universal cover of

M into S^n which is equivariant with respect to θ. Two pairs are equivalent if and only if they differ by a conformal mapping of S^n. We furnish the space of Möbius structures with the (quotient of) the topology of local uniform convergence. φ is called the developing map and θ is called the holonomy of the Möbius structure.

We define the Gauss mapping $\overrightarrow{n} : U\mathbb{H}^{n+1} \to \partial_\infty\mathbb{H}^{n+1}$ as follows. For v a unit vector in $U\mathbb{H}^{n+1}$, let $\gamma_v : [0, +\infty[\to \mathbb{H}^{n+1}$ be the half geodesic such that $\partial_t\gamma(0) = v$. We define:

$$\overrightarrow{n}(v) = \gamma_v(+\infty) = \mathrm{Lim}_{t\to+\infty}\gamma_v(+\infty).$$

We recall that $\partial_\infty\mathbb{H}^{n+1}$ may be conformally identified with S^{n+1}. Let $i : M \to \mathbb{H}^{n+1}$ be a convex immersion. Since i is convex, there exists a unique exterior vector field N_i over i in $U\mathbb{H}^{n+1}$. We say that i **projects asymptotically** to the Möbius structure (φ, θ) if and only if i is equivariant with respect to θ, and, up to reparametrisation:

$$\overrightarrow{n} \circ \mathsf{N}_i = \varphi.$$

Theorem 1.1. *Choose $\eta \in [(n-1)\pi/2, n\pi/2[$ and $r > \tan(\eta/n)$. Let M be a compact n dimensional manifold and let (φ, θ) be an FCS of hyperbolic type over M. If $\eta > (n-1)\pi/2$, then there exists a unique, convex, equivariant immersion $i_{r,\eta} : \tilde{M} \to \mathbb{H}^{n+1}$ such that:*

(i) *$i_{r,\eta}$ is a graph over i_{KP};*

(ii) *$i_{r,\eta}$ projects asymptotically to φ;*

(iii) *$R_\eta(i_{r,\eta}) = r$.*

Moreover, the same result holds for $\eta = (n-1)\pi/2$ provided that (φ, θ) is not conformally equivalent to $(S^{n-1} \times \mathbb{R})/\Gamma$, where S^{n-1} is the $(n-1)$-dimensional sphere, and Γ is a properly discontinuous group of conformal actions.

Remark. The proof of this theorem uses the Perron method. The finite boundary forms a barrier, which follows from the Geodesic Boundary Property (see Definition 2.7). In particular, as in the remarks following Definition 2.7, the existence result in fact holds in a much more general class of negatively curved ends of non-constant sectional curvature bounded above by -1 whose finite boundary possesses the Geodesic Boundary Property or even the weak Geodesic Boundary Property.

Since they are graphs over the Kulkarni-Pinkall immersion, these immersed hypersurfaces may be considered as submanifolds of the hyperbolic end of the FCS:

Theorem 1.2. *Let $\mathcal{E}$ be the hyperbolic end of an FCS. Let $\theta \in [(n-1)\pi/2, n\pi/2[$ be an angle. For all $r > \tan(\theta/n)$, let $\Sigma_{r,\theta} = (S, i_{r,\theta})$ be the unique, smooth, convex, immersed hypersurface in $\mathcal{E}$ which is a graph over $\partial_0\mathcal{E}$ and which satisfies $R_\theta(i_{r,\theta}) = r$.*

The family $(\Sigma_{r,\theta})_{r>\tan(\theta/n)}$ foliates a neighbourhood, Ω_θ, of $\partial_0\mathcal{E}$. Morever $(\hat{\Sigma}_{r,\theta})_{r>\tan(\theta/n)}$ converges towards $N\partial_0\mathcal{E}$ in the $C^{0,\alpha}$ sense for all α as r tends to $+\infty$, and, for any compact subset, K, of $\mathcal{E}$, there exists $\theta_0 < n\pi/2$ such that for $\theta > \theta_0$, $K \subseteq \Omega_\theta$.

Remark. The final part of this theorem suggests that by judiciously choosing r as a function of θ, it may be possible to obtain smooth foliations of the entire hyperbolic end.

Remark. Towards completion of this paper, the author was made aware of a recent, complementary result of Mazzeo and Pacard [16]. There, using entirely different techniques, and under different hypotheses on the hyperbolic end, the authors prove the existence of foliations by constant mean curvature hypersurfaces near the ideal boundary, though not near the finite boundary, as is obtained here. It appears reasonable that a happy marriage of these techniques could yield more detailed information concerning the structure of the hyperbolic end and its relation to its ideal boundary.

In the special case where $\mathcal{E}$ is an end of a quasi-Fuchsian manifold, the foliations may be extended up to the ideal boundary, and we obtain:

Theorem 1.3. *Let $\mathcal{E}$ be a hyperbolic end of a quasi-Fuchsian manifold. Let $\theta \in [(n-1)\pi/2, n\pi/2[$ be an angle. For all $r > \tan(\theta/n)$, let $\Sigma_{r,\theta} = (S, i_{r,\theta})$ be the unique, smooth, convex, immersed hypersurface on $\mathcal{E}$ which is a graph over $\partial_0\mathcal{E}$ and which satisfies $R_\theta(i_{r,\theta}) = r$.*

The family $(\Sigma_{r,\theta})_{r>\tan(\theta/n)}$ foliates $\mathcal{E}$. Morever $(\hat{\Sigma}_{r,\theta})_{r>\tan(\theta/n)}$ converges towards $N\partial_0\mathcal{E}$ in the $C^{0,\alpha}$ sense for all α as r tends to $+\infty$, and $(\Sigma_{r,\theta})_{r>\tan(\theta/n)}$ converges to $\partial_\infty\mathcal{E}$ in the Hausdorff sense as r tends to $\tan(\theta/n)$.

Remark. In fact, this result holds for any FCS whose developing map avoids an open subset of $\partial_\infty\mathbb{H}^{n+1}$.

We next consider how these foliations vary with the FCS:

Theorem 1.4. *Let M be a compact manifold. Let $(\theta_x, \varphi_x)_{\|x\|<\epsilon}$ be a continuous family of FCSs of hyperbolic type over M whose holonomy varies smoothly. Let $\theta \in [(n-1)\pi/2, n\pi/2[$ be an angle, and let $r > \tan(\theta/n)$. For all x, let $\Sigma_x = (S, i_x)$ be the unique, smooth, convex, immersed hypersurface in $\mathcal{E}(\theta_x, \varphi_x)$ such that $R_\theta(i_x) = r$. Then, up to reparametrisation, i_x varies smoothly with x.*

Remark. It follows that the space of hypersurfaces of constant special Lagrangian curvature yields smooth moduli for the space of FCSs of hyperbolic type over M which are compatible with the smooth structure obtained from the canonical embedding of this space into $\mathrm{PSO}(n+1,1)^{\pi_1(M)}$, and which also, importantly, encode smooth information about the hyperbolic end and the Kulkarni-Pinkall metric.

As an illustration of these results, we now consider two special cases. The first is when n is equal to 2, and $\theta = \pi/2$. Here the special Lagrangian curvature reduces to the Gaussian curvature and we recover the following, now classical, result of Labourie [15]:

Theorem 1.5 (Labourie (1991)). *Let Σ be a compact surface of hyperbolic type. Let (α, φ) be an FCS over Σ and let $\mathcal{E}$ be the hyperbolic end of (α, φ). There exists a unique, smooth foliation $(\Sigma_k)_{k\in]0,1[}$ of $\mathcal{E}$ such that:*

(i) *for each k, Σ_k is a smooth, immersed surface of constant Gaussian (extrinsic) curvature equal to k;*

(ii) *Σ_k tends to $\partial_0\mathcal{E}$ in the Hausdorff sense as k tends to 0; and*

(iii) *Σ_k tends to $\partial_\infty\mathcal{E}$ in the Hausdorff sense as k tends to 1.*

Remark. The geometric properties particular to this special case allow us to extend the foliations up to the ideal boundary (see also [16] and [19]).

The second special case is when $n = 3$ and $\theta = \pi$. In this case, the special Lagrangian curvature still has a very simple expression:

Theorem 1.6. *Let M be a compact three dimensional manifold. Let (α, φ) be an FCS over M of hyperbolic type. Let $\mathcal{E}$ be the hyperbolic end of (α, φ). There exists a unique, smooth foliation $(\Sigma_r)_{r\in]3,+\infty[}$ of $\mathcal{E}$ such that:*

(i) *for each r, Σ_r is a smooth, immersed hypersurface such that:*

$$H(\Sigma_r)/K(\Sigma_r) = r,$$

where $H(\Sigma_r)$ and $K(\Sigma_r)$ are the mean and Gaussian curvatures of Σ_r respectively; and

(ii) *Σ_r tends to $\partial_0\mathcal{E}$ in the Hausdorff sense as r tends to $+\infty$.*

Towards completion of this paper, the author was made aware of related work by Andersson, Barbot, Béguin and Zeghib [1]. Here the authors study constant mean curvature foliations of Lorentzian, anti de-Sitter and de-Sitter spacetimes. There is a natural duality between hyperbolic ends and de-Sitter spacetimes, and thus a duality between their framework and our own. One interesting consequence is that, in the 4-dimensional case, Theorem 1.6 yields foliations of neighbourhoods of the past ends of four dimensional de-Sitter spacetimes by 3-dimensional space-like hypersurfaces of constant scalar curvature. This may be related to the Yamabe problem of the FCS, which is relevant to [16].

Finally, the proofs of these theorems requires a detailed understanding of the geometric structure of the Kulkarni-Pinkall hyperbolic end of an FCS. We obtain the following characterisation of the Kulkarni-Pinkall end in terms of completeness and local geometric data, which the author is not aware of in the litterature:

Theorem 1.7. *Let $\tilde{N}$ be a hyperbolic end. Suppose that:*

(i) *$\tilde{N}$ possesses the Geodesic Boundary Property; and*

(ii) *$\tilde{N}$ is complete.*

Then $\tilde{N}$ is the Kulkarni-Pinkall hyperbolic end of its quotient Möbius manifold.

Moreover, if N is a compact Möbius manifold, then the family of hyperbolic ends whose quotient Möbius manifold is N is partially ordered, and the Kulkarni-Pinkall hyperbolic end of N is the unique maximal element of this family.

Indeed, as noted in the remark following Theorem 1.4, the foliations constructed here encode smooth information about the hyperbolic end whilst depending smoothly on the conformal structure. We therefore expect them to be of considerable use in the future study of FCSs. Indeed, as examples of possible applications of these results, we state two immediate corollaries. The first concerns continuous dependence of $N\partial_0\mathcal{E}$ which we think of as an equivariant $C^{0,1}$ immersed hypersurface in $U\mathbb{H}^n$:

Theorem 1.8. *Let M be a compact manifold. Let $(\theta_n,\varphi_n)_{n\in\mathbb{N}}, (\theta_0,\varphi_0)$ be FCSs of hyperbolic type over M such that $(\theta_n,\varphi_n)_{n\in\mathbb{N}}$ converges to (θ_0,φ_0), then $(N\partial_0\mathcal{E}(\theta_n,\varphi_n))_{n\in\mathbb{N}}$ converges to $(N\partial_0\mathcal{E}(\theta_0,\varphi_0))$ in the $C^{0,\alpha}$ Cheeger/Gromov sense for all $\alpha\in]0,1[$.*

And the second result concerns the Kulkarni-Pinkall metric. Let D, V and I represent the diameter, volume and injectivity radius respectively of the Kulkarni-Pinkall metric. We obtain the following continuity and compactness result:

Theorem 1.9. *Let M be a compact manifold. Let $(\theta_n,\varphi_n)_{n\in\mathbb{N}}, (\theta_0,\varphi_0)$ be FCSs of hyperbolic type over M such that $(\theta_n,\varphi_n)_{n\in\mathbb{N}}$ converges to (θ_0,φ_0), then the sequence of $C^{0,1}$ Riemannian manifolds $(M,g_{KP}(\varphi_n))_{n\in\mathbb{N}}$ converges to $(M,g_{KP}(\varphi_0))$ in the $C^{0,\alpha}$ Cheeger/Gromov sense for all $\alpha\in]0,1[$.*

In particular, D, V and I define continuous functions over the space of FCSs of hyperbolic type over M. Moreover, the pairs (I,D) and (I,V) define proper functions over the space of FCSs of hyperbolic type.

This paper is structured as follows:

(a) In section 2, we define hyperbolic ends and FCSs, we describe the relationship between the two and prove Theorem 1.7;

(b) In section 3, we define special Lagrangian curvature and prove or recall various analytic properties therof including local rigidity, compactness and the Geometric Maximum Principal;

(c) In section 4, we study immersions of constant special Lagrangian curvature in hyperbolic ends, and prove all the remaining results of this paper; and

(d) In Appendix A, we show how the Kulkarni-Pinkall metric may be used to furnish a simpler proof of a result of Kamishima.

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2. Hyperbolic Ends and Flat Conformal Structures

2.1. **Hyperbolic Ends.** For all m, let $\mathbb{H}^{m+1}$ be $(m+1)$-dimensional hyperbolic space. Let $U\mathbb{H}^{m+1}$ be the unitary bundle over $\mathbb{H}^{m+1}$. Let K be a convex subset of $\mathbb{H}^{m+1}$. We define $\mathcal{N}(K)$, the set of normals over K by:

$$\mathcal{N}(K) = \left\{v_x \in U\mathbb{H}^{m+1} \text{ s.t. } x \in \partial K \text{ and } v_x \text{ is a supporting normal to } K \text{ at } x\right\}.$$

$\mathcal{N}(K)$ is a $C^{0,1}$ submanifold of $U\mathbb{H}^{m+1}$. Let Ω be an open subset of $\mathcal{N}(K)$. We define $\mathcal{E}(\Omega)$, the end over Ω by:

$$\mathcal{E}(\Omega) = \{\mathrm{Exp}(tv_x) \text{ s.t. } t \geq 0, v_x \in \Omega\}.$$

We say that a subset of $\mathbb{H}^{m+1}$ has **concave boundary** if and only if it is the end of some open subset of the set of normals of a convex set. We refer to Ω as the finite boundary of $\mathcal{E}(\Omega)$.

We extend this concept to more general manifolds. Let $(M, \partial M)$ be a smooth manifold with continuous boundary. A **hyperbolic end** over M is an atlas $\mathcal{A}$ such that:

(i) every chart of $\mathcal{A}$ has convex boundary, and
(ii) the transition maps of $\mathcal{A}$ are isometries of $\mathbb{H}^{m+1}$.

We refer to ∂M as the finite boundary of M. In the sequel, we will denote it by $\partial_0 M$ in order to differentiate it from the ideal boundary $\partial_\infty M$ of M.

We can construct hyperbolic ends using continuous maps into $U\mathbb{H}^{m+1}$. Let M be an m-dimensional manifold without boundary. Let $i : M \to U\mathbb{H}^{m+1}$ be a continuous map. We say that i is a **convex immersion** if and only if for every p in M, there exists a neighbourhood Ω of p in M and a convex subset $K \subseteq \mathbb{H}^{m+1}$ such that the restriction of i to Ω is a homeomorphism onto an open subset of $\mathcal{N}(K)$. In this case, we define the mapping $I : M \times [0, \infty[\to \mathbb{H}^{m+1}$ by:

$$I(p, t) = \mathrm{Exp}(ti(p)).$$

We refer to I as the end of i. I is a local homeomorphism from $M\times]0,\infty[$ into $\mathbb{H}^{m+1}$. If g is the hyperbolic metric over $\mathbb{H}^{m+1}$, then I^*g defines a hyperbolic metric over this interior. I^*g degenerates over the boundary, and we identify points that may be joined by curves of zero length. We denote this equivalence by $\sim$ and we define $\mathcal{E}(i)$, which we also call the end of i by:

$$\mathcal{E}(i) = (M\times]0,\infty[)\cup(M/\sim).$$

We shall see presently that every hyperbolic end may be constructed in this manner. Thus, if $\hat{M}$ is an end, and if $i : M \to U\mathbb{H}^{m+1}$ is a convex immersion such that $\hat{M} = \mathcal{E}(i)$, then we say that i is the **boundary immersion** of $\hat{M}$.

2.2. **Flat Conformal Structures.** Let $\mathbb{H}^{n+1}$ be $(n+1)$-dimensional hyperbolic space. We identify $\partial_\infty\mathbb{H}^{n+1}$ with the n-dimensional sphere S^n. $\mathrm{Isom}(\mathbb{H}^{n+1})$ is identified with $\mathrm{PSO}(n+1,1)$. This group acts faithfully on $S^n = \partial_\infty\mathbb{H}^{n+1}$. The image is a subgroup of the group of homeomorphisms of the sphere. We denote this group by Mob (n) and we call elements of Mob (n) **conformal maps**.

Let M be a manifold. A flat conformal structure (FCS) on M is an atlas $\mathcal{A}$ of M in S^n whose transformation maps are restrictions of elements of Mob (n). Trivially, every element of Mob (n) is uniquely determined by its germ at a point. Thus, any chart of $\mathcal{A}$ uniquely extends to a local homeomorphism from $\tilde{M}$, the universal cover of M, into S^n which is equivariant with respect to a given homomorphism. This yields an alternative definition of FCSs which is better adapted to our purposes:

Definition 2.1. *Let M be a manifold. Let $\pi_1(M)$ be its fundamental group and let $\tilde{M}$ be its universal cover. A* **flat conformal structure** *over M is a pair (φ,θ) where:*

(i) *$\theta : \pi_1(M) \to Mob\ (n)$ is a homomorphism, and*

(ii) *$\varphi : \tilde{M} \to S^n$ is a local homeomorphism which is equivariant with respect to θ.*

θ is called the **holonomy** *and φ is called the* **developing map** *of the flat conformal structure.*

We refer to a pair $(M,(\varphi,\theta))$ consisting of a manifold M and a flat conformal structure over M as a **Möbius manifold**. *In the sequel, where no ambiguity arises, we refer to the manifold with its conformal structure merely by M.*

Remark. A canonical differential structure on M is obtained by pulling back the differential structure of S^n through φ.

Möbius manifolds are divided into three types (for more details, see [13]):

(i) manifolds of **elliptic** type, whose universal cover is conformally equivalent to S^n,

(ii) manifolds of **parabolic** type, whose universal cover is conformally equivalent to $\mathbb{R}^n$, and

(iii) manifolds of **hyperbolic** type, consisting of all other cases.

In the sequel, we study FCSs of hyperbolic type over compact manifolds.

Let (φ, θ) be an FCS over M. A **geometric ball** in M is an injective mapping $\alpha : B \to \tilde{M}$ from a Euclidean ball B into $\tilde{M}$ such that $\varphi \circ \alpha$ is the restriction of a conformal mapping. Geometric balls form a partially ordered set with respect to inclusion. In [13], it is shown that when M is of hyperbolic type, every point of $\tilde{M}$ is contained within a maximal geometric ball. Every geometric ball carries a natural complete hyperbolic metric. Indeed, $\partial(\varphi \circ \alpha(B))$ bounds a totally geodesic hyperplane in $\mathbb{H}^{n+1}$ and orthogonal projection defines a homeomorphism from $(\varphi \circ \alpha)(B)$ onto this hyperplane. The hyperbolic metric on B is obtained by pulling back the metric on this hyperplane through the orthogonal projection. We denote this metric by g_B. It is trivially conformal with respect to the conformal structure of M.

We define the **Kulkarni-Pinkall metric** g_{KP} over $\tilde{M}$ by:

$$g_{KP}(p) = \text{Inf } \{g_B(p) \text{ s.t. } B \text{ is a geometric ball and } p \in B\}.$$

This metric is the analogue in the Möbius category of the Kobayashi metric. Trivially, g_{KP} is equivariant and thus quotients to a metric over M. The main result of [13] is:

Theorem 2.2 (Kulkarni, Pinkall). *Let M be a Möbius manifold of hyperbolic type. Then g_{KP} is positive definite and of type $C^{1,1}$.*

Let g_S be a spherical metric over $\partial_\infty \mathbb{H}^{n+1}$. Let $\overline{M}$ be the metric completion of $\tilde{M}$ with respect to $\varphi^* g_S$. Since any two spherical metrics are uniformly equivalent, the topological space $\overline{M}$ is independant of the choice of spherical metric. Trivially φ extends to a continuous map from $\overline{M}$ into $\partial_\infty \mathbb{H}^{n+1}$. We call $\partial \tilde{M} := \overline{M} \setminus \tilde{M}$ the ideal boundary of $\tilde{M}$.

Let (B, α) be a geometric ball. We define $C(B)$ to be the convex hull in B (with respect to the hyperbolic metric) of $\overline{\alpha(B)} \cap \partial_\infty \tilde{M}$. In proposition 4.1 of [13], Kulkarni and Pinkall obtain:

Proposition 2.3 (Kulkarni, Pinkall). *If M is a Möbius manifold of hyperbolic type, then for every point $p \in \tilde{M}$ there exists a unique maximal geometric ball (B, α) such that $p \in \alpha(C(B))$.*

We denote this ball by $B(p)$. Kulkarni and Pinkall show that:

$$g_{KP}(p) = g_{B(p)}(p).$$

In [13], Kulkarni and Pinkall use these maximal geometric balls to associate a canonical hyperbolic end to each FCS. These are the ends that will interest us in the sequel. We refer the reader to [13] for the details of this construction. Let φ be the developing map of the FCS. We denote the canonical hyperbolic end associated to it by Kulkarni and Pinkall by $\mathcal{E}(\varphi)$ and we refer to it as the **Kulkarni-Pinkall hyperbolic end** of φ. Let $U\mathbb{H}^{n+1}$ be the unitary bundle of $\mathbb{H}^{n+1}$, let $\overrightarrow{n} : U\mathbb{H}^{n+1} \to \partial_\infty\mathbb{H}^{n+1}$ be the Gauss map and let $\pi : U\mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ be the canonical projection. Let $\hat{\imath} : \tilde{M} \to U\mathbb{H}^{n+1}$ be the boundary immersion of $\mathcal{E}(\varphi)$ and define $i = \pi \circ \hat{\imath}$. $\mathcal{E}(\varphi)$ has the following useful properties:

(i) $\varphi = \overrightarrow{n} \circ \hat{\imath}$;

(ii) if $p \in \tilde{M}$, if P is the totally geodesic hyperplane in $\mathbb{H}^{n+1}$ normal to $\hat{\imath}(p)$ at $i(p)$, if g is the hyperbolic metric of P and if $\pi_p : \partial_\infty\mathbb{H}^{n+1} \to P$ is the orthogonal projection, then $g_{KP}(p)$ coincides with $(\pi_p \circ \varphi)^* g(p)$; and

(iii) for all $p \in \tilde{M}$, there exists a curve $\gamma :]-\epsilon, \epsilon[\to \tilde{M}$ such that $\gamma(0) = p$ and $i \circ \gamma$ is a geodesic segment in $\mathbb{H}^{n+1}$.

Remark. Condition *(iii)* is a strong statement about the curvature of the finite boundary of $\mathcal{E}(\varphi)$, which can be defined and vanishes in the direction of the geodesic. We shall see presently how this condition alone defines the geometry of the boundary immersion.

2.3. **The Geodesic Boundary Property.** Heuristically, condition *(iii)* of the preceeding section says that the boundary immersion of the Kulkarni-Pinkall hyperbolic end defines a locally ruled hypersurface. To better understand the implications of this property, we study more closely the geometry of hyperbolic ends.

Lemma 2.4. *Let $\tilde{N}$ be a hyperbolic end. $\tilde{N}$ is foliated by complete half-geodesics normal to the finite boundary.*

Remark. In the sequel, we will refer to this foliation as the **vertical foliation**.

Proof. Every subset of $\mathbb{H}^{n+1}$ is foliated in this manner. Since the transition maps preserve the concave boundary, they also preserve the foliation. The result follows. □

This induces an equivalence relation on the hyperbolic end which we denote by $\sim$.

Lemma 2.5. *$\tilde{N}/\sim$ has the structure of a smooth manifold.*

Proof. Let d denote the distance in $\tilde{N}$ from the finite boundary. Choose $r > 0$. We claim that $N_r := d^{-1}(\{r\})$ is a $C^{1,1}$ embedded submanifold of $\tilde{N}$. Indeed, let $\Omega \subseteq \mathbb{H}^{n+1}$ have concave boundary and let d_Ω denote the distance in Ω from the finite boundary. It follows from the properties of convex sets that $d_\Omega^{-1}(\{r\})$ is a $C^{1,1}$ embedded submanifold of Ω. Since these embedded submanifolds are preserved by the transition maps, the assertion follows. Using mollifiers (c.f. [18], for example), we obtain a smooth embedded submanifold N_r' which is close to N_r in the C^1 sense. All such embeddings have the same C^∞ structure, and the result follows. $\square$

We denote $N := \tilde{N}/\sim$.

Lemma 2.6. *If $\tilde{N}$ is simply connected, then there exists a convex immersion, $i : N \to \mathbb{H}^{n+1}$, which is canonical up to composition by isometries of $\mathbb{H}^{n+1}$ such that:*

$$\tilde{N} = \mathcal{E}(i).$$

Remark. In particular, if $\tilde{N}$ is an arbitrary hyperbolic end, then we may define a canonical ideal boundary $\partial_\infty\tilde{N}$ of $\tilde{N}$ as well as a canonical topology of $\tilde{N} \cup \partial_\infty\tilde{N}$.

Proof. Trivially, N is simply connected. Let d be the distance in $\tilde{N}$ from its finite boundary. Choose $r > 0$. By the proof of Lemma 2.5, we may identify N with $d^{-1}(\{r\})$. Choose $p \in N$. Let (α, U, V) be a coordinate chart of $\tilde{N}$ about p. Thus $\alpha : U \to V$, and $V \subseteq \mathbb{H}^{n+1}$ has concave boundary. Define $i_r : N \cap U \to \mathbb{H}^{n+1}$ by:

$$i_r(q) = \alpha|_{N\cap U}.$$

i_r is a $C^{1,1}$ immersion bounding a convex set. For all $q \in N \cap U$, let γ_q be the unit speed geodesic leaving $i_r(q)$ in the direction of the exterior supporting normal of $i_r(N \cap U)$ at q (which is unique). Define $\hat{\imath}(q) : N \cap U \to U\mathbb{H}^{n+1}$ by:

$$\hat{\imath}(q) = \partial_t\gamma_q(-r).$$

Let $K \subseteq \mathbb{H}^{n+1}$ be a convex set such that the finite boundary of V is an open subset, Ω of $\mathcal{N}(K)$. Trivially, $\hat{\imath}$ defines a homeomorphism from $N \cap U$ to Ω. It follows that $\hat{\imath}$ is a convex immersion. Moreover, $\hat{\imath}$ is independant of r, and:

$$V = \mathcal{E}(\hat{\imath}).$$

Since N is simply connected, i_r and $\hat{\imath}$ can be extended to mappings defined over the whole of N which are canonical up to composition by homeomorphisms of $\mathbb{H}^{n+1}$. $\tilde{N} = \mathcal{E}(\hat{\imath})$, and the result follows. $\square$

The convex immersion $\hat{\imath} : N \to \mathbb{H}^{n+1}$ yields an immersion $I : N\times]0,\infty[\to \mathbb{H}^{n+1}$ which is the end of $\hat{\imath}$. I extends continuously to a map from $N\times]0,\infty]$ to $\mathbb{H}^{n+1} \cup \partial_\infty\mathbb{H}^{n+1}$. We define $\varphi : N \to \partial_\infty\mathbb{H}^{n+1}$ by:

$$\varphi(p) = I(p, \infty).$$

Since $\hat{\imath}$ is a convex immersion, φ is a local homeomorphism and φ thus defines an FCS over N. Moreover, φ is smooth with respect to the C^∞ structure of N and thus the underlying C^∞ structure of the FCS induced on N coincides with the preexisting C^∞ structure on N. We refer to (N, φ) as the **quotient Möbius manifold** of the hyperbolic end $\tilde{N}$.

Let $\tilde{N}_1$ and $\tilde{N}_2$ be hyperbolic ends. Let (N_1, φ_1) and (N_2, φ_2) be their respective quotient Möbius manifolds. We define a morphism between $\tilde{N}_1$ and $\tilde{N}_2$ to be a pair $(\Phi, \tilde{\Phi})$ such that:

(i) $\Phi : N_1 \to N_2$ is a locally conformal mapping;

(ii) $\tilde{\Phi} : \tilde{N}_1 \to \tilde{N}_2$ is a local hyperbolic isometry; and

(iii) $\tilde{\Phi}$ extends to a continuous map from $\partial_\infty \tilde{N}_1 = N_1$ to $\partial_\infty \tilde{N}_2 = N_2$ which coincides with Φ.

In the sequel, we denote such a morphism merely by Φ.

We define the relation "$<$" over the family of hyperbolic ends such that, if $\tilde{N}_1$ and $\tilde{N}_2$ are hyperbolic ends, then $\tilde{N}_1 < \tilde{N}_2$ if and only if there exists an injective morphism $\tilde{\Phi} : \tilde{N}_1 \to \tilde{N}_2$. If $\tilde{N}_1 < \tilde{N}_2$, then we say that $\tilde{N}_1$ is contained in $\tilde{N}_2$. We shall see presently that "$<$" defines a partial order over the family of hyperbolic ends whose quotient Möbius manifold is compact.

Definition 2.7 (Geodesic Boundary Property)**.** *Let $\tilde{N}$ be a simply connected hyperbolic end. Let $N = \tilde{N}/\sim$ and let $\hat{\imath} : N \to \mathbb{H}^{n+1}$ be the convex immersion such that $\tilde{N} = \mathcal{E}(i)$. We say that $\tilde{N}$ possesses the* **Geodesic Boundary Property** *if and only if, for every point $p \in N$ there exists:*

(i) *a real number $\epsilon > 0$;*

(ii) *a unit speed geodesic segment $\gamma :]-\epsilon, \epsilon[\to \mathbb{H}^n$; and*

(iii) *a continuous path $\alpha :]-\epsilon, \epsilon[\to N$,*

such that $\alpha(0) = p$ and, for all $t \in]-\epsilon, \epsilon[$:

$$\gamma(t) = (\pi \circ \hat{\imath} \circ \alpha)(t).$$

Remark. Heuristically, $\tilde{N}$ possesses the Geodesic Boundary Property if and only if, at every boundary point, there exists a non-trivial geodesic segment passing through that point which remains in the boundary.

Remark. The Geodesic Boundary Property is a natural property of minimal convex sets in manifolds of constant curvature. Indeed, any such set possesses the Geodesic Boundary Property, since, otherwise, there would be a point at which it would be strictly convex, and therefore not be minimal.

Remark. Importantly, the Geodesic Boundary Property may be substituted by a weaker version, where the geodesic is substituted by a curve whose geodesic curvature vanishes at p. The reader may verify that this Weak Geodesic Boundary Property may be substited for the Geodesic Boundary Property at every stage in the sequel where it is used. As the Geodesic Boundary Property is a natural property of minimal convex sets in manifolds of constant curvature, so the Weak Geodesic Boundary Property is a natural property of minimal convex sets in general manifolds. We thus see how the results of this paper may be extended to a much more general setting than where they are currently presented.

We now obtain a geometric characterisation of the Kulkarni-Pinkall hyperbolic end. Let $\tilde{N}$ be a hyperbolic end. Let d denote the distance in $\tilde{N}$ along the vertical foliation from the finite boundary $\partial_0\tilde{N}$ of $\tilde{N}$. For all $\delta > 0$, let N_δ denote the level hypersurface $d^{-1}(\{\delta\})$. We say that $\tilde{N}$ is **complete** if and only if N_δ is complete for some (and therefore for all) $\delta > 0$.

Lemma 2.8. *Let $\tilde{N}$ be a hyperbolic end. Suppose that:*

(i) *$\tilde{N}$ possesses the Geodesic Boundary Property; and*

(ii) *$\tilde{N}$ is complete.*

Then $\tilde{N}$ is the Kulkarni-Pinkall hyperbolic end of its quotient Möbius manifold.

Proof. Let $p \in \partial_0\tilde{N}$ be a point in the finite boundary of $\tilde{N}$. Let N_p be a supporting normal to $\partial_0\tilde{N}$ at p and let $H_p \subseteq \tilde{N}$ be the supporting totally geodesic hyperspace to $\partial_0\tilde{N}$ at p whose normal at p is N_p. Since $\tilde{N}$ is complete, so is H_p.

Let $K = H_p \cap \partial_0\tilde{N}$ be the intersection of H_p with the finite boundary of $\tilde{N}$. Since the distance to the finite boundary in a hyperbolic end is a convex function, K is a convex subset of H_p. Moreover, K is closed and, by the Geodesic Boundary Property, for every $q \in K$, there exists $\epsilon > 0$ and a unit speed geodesic segment $\gamma :]-\epsilon, \epsilon[\to K$ such that $\gamma(0) = q$. We refer to this as the Local Geodesic Property. Let $\partial_\infty K$ be the intersection of the closure of K with $\partial_\infty H_p$. We claim that K is the convex hull of $\partial_\infty K$.

Let $X \subseteq \mathbb{H}^n$ be a convex subset of hyperbolic space satisfying the Local Geodesic Property. Let $q \in \partial X$ be a boundary point. Let $H_q \subseteq \mathbb{H}^n$ be a supporting totally geodesic hyperplane to X at q. Let $X' \subseteq H_q$ be the intersection of X with H_q. X' is convex, closed and possesses the Local Geodesic Property.

Suppose that for every $q \in \partial X$ and for every supporting hyperplane $H \subseteq \mathbb{H}^n$ to X at q, q lies in the convex hull of $\partial X \cap \partial_\infty H$ in H. Then we claim that X is the convex hull of $\partial_\infty X$. Indeed, ∂X is contained in the convex hull of $\partial_\infty X$. Now consider $q \in X$ and let γ be any geodesic in $\mathbb{H}^n$ passing through q. The endpoints of $\gamma \cap X$ lie either in $\partial_\infty X$ or in ∂X, both of which are subsets of the convex hull of $\partial_\infty X$. $\gamma \cap X$ therefore lies in the convex hull of $\partial_\infty X$ and the assertion now follows.

Suppose that K is not the convex hull of $\partial_\infty K$. Then there exists $q \in \partial K$ and a supporting totally geodesic hyperplane $H_q \subseteq H_p$ to K at q such that q does not lie in the convex hull of $\partial_\infty H_q \cap \partial_\infty K$. Let K_q be the intersection of K with H_q. K_q is convex, closed, and possesses the Local Geodesic Property. Moreover, defining $\partial_\infty K_q$ as before, by definition, K_q is not the convex hull of $\partial_\infty K_q$ in H_q. Proceeding by induction, we obtain a 1-dimensional subset of the real line which is convex, closed, possesses the Local Geodesic Property, but is not contained within the convex hull of its intersection with the ideal boundary of the real line. This is absurd, and the assertion follows.

It follows that p is contained in the convex hull of $K \cap \partial_\infty H_p$. This condition characterises the Kulkarni-Pinkall hyperbolic end, and the result follows. $\square$

In the compact case, moreover, the Kulkarni-Pinkall hyperbolic end is the unique maximal end. First we prove:

Lemma 2.9. *Let $\tilde{N}_1$ and $\tilde{N}_2$ be compact hyperbolic ends. Suppose, moreover that $\tilde{N}_2$ possesses the Geodesic Boundary Property. Let (N_1, φ_1) and (N_2, φ_2) be their respective quotient flat conformal manifolds. If (N_1, φ_1) and (N_2, φ_2) are isomorphic, then $\tilde{N}_1 < \tilde{N}_2$. Moreover, the finite boundary, $\partial_0 \tilde{N}_1$, of $\tilde{N}_1$ is a graph over the finite boundary, $\partial_0 \tilde{N}_2$, of $\tilde{N}_2$.*

Proof. Let $\hat{N}_1$ and $\hat{N}_2$ be the universal covers of $\tilde{N}_1$ and $\tilde{N}_2$ respectivey. Let $\hat{\Phi}_1 : \hat{N}_1 \to \mathbb{H}^{n+1}$ and $\hat{\Phi}_2 : \hat{N}_2 \to \mathbb{H}^{n+1}$ be their respective developing maps. We may assume that $\partial_\infty \hat{N}_1 = \partial_\infty \hat{N}_2$ and that $\hat{\Phi}_1 = \hat{\Phi}_2$ on this set.

The identity on the ideal boundaries extends to an equivariant homeomorphism Ψ from an open subset, U_1, of $\partial_\infty \hat{N}_1$ in $\hat{N}_1$ into an equivariant open subset, U_2, of $\partial_\infty \hat{N}_2$ in $\hat{N}_2$.

Let $d : \hat{N}_1 \to [0, \infty[$ be the distance in $\hat{N}_1$ to $\partial \hat{N}_1$. For all $r > 0$, let $\hat{N}_{1,r}$ be the hypersurface at constant distance r from $\partial \hat{N}_1$:

$$\hat{N}_{1,r} = d^{-1}(\{r\}).$$

For sufficiently large r, $\hat{N}_{1,r}$ is contained in U.

Let V_1 and V_2 be the fields of vertical vectors over $\hat{N}_1$ and $\hat{N}_2$ respectively. Let $(p_n)_{n\in\mathbb{N}} \in U_1$ be a sequence converging to a point $p_0 \in \partial_\infty \hat{N}_1$. Then:

$$(\langle V_1(p_n), \Psi^* V_2(p_n) \rangle)_{n\in\mathbb{N}} \to 1.$$

Thus, by cocompactness, for sufficiently large r, $\Psi(\hat{N}_{1,r})$ is transverse to the vertical foliation of $\hat{N}_2$. Therefore, by cocompactness, the projection from $\Psi(\hat{N}_{1,r})$ onto $\partial_0 \hat{N}_2$ is a covering map, and so $\Psi(\hat{N}_{1,r})$ is a graph over $\partial_0 \hat{N}_2$. Moreover, $\Psi(\hat{N}_{1,r})$ is a strict graph in the sense that it does not intersect $\partial_0 \hat{N}_2$.

By continuously reducing r, U_1 and Ψ may be extended to contain $\hat{N}_{1,r}$ at least as long as $\Psi(\hat{N}_{1,r})$ remains a strict graph over $\partial_0 \hat{N}_2$ (it will always be an immersion). Suppose therefore that there exists $r_0 > 0$ such that $\Psi(\hat{N}_{1,r_0})$ is not a strict graph over $\partial_0 \hat{N}_2$ but $\Psi(\hat{N}_{1,r})$ is for all $r > r_0$.

Suppose that $\Psi(\hat{N}_{1,r_0})$ intersects $\partial_0 \hat{N}_2$ non-trivially. $\Psi(\hat{N}_{1,r_0})$ is an external tangent to $\partial \hat{N}_2$ at this point. However, by Lemma 3.12 the second fundamental form of $\Psi(\hat{N}_{1,r_0})$ is bounded below by $\tanh(r_0)\mathrm{Id}$ in the weak sense. This therefore contradicts the Geodesic Boundary Property of $\hat{N}_2$, and $\Psi(\hat{N}_{1,r_0})$ therefore lies strictly above $\partial_0\hat{N}_2$.

Suppose that $\Psi(\hat{N}_{1,r_0})$ is not a graph over $\partial_0\hat{N}_2$. Then there exists $p \in \hat{N}_{1,r_0}$ such that $\Psi(\hat{N}_{1,r_0})$ is vertical at this point. Let $q \in \partial_0\hat{N}_2$ be the vertical projection of p. Let $\gamma : [0, d(p,q)] \to \hat{N}_2$ be the vertical geodesic segment in $\hat{N}_2$ from q to p. γ lies below the graph of $\Psi(\hat{N}_{1,r})$ for all $r > r_0$. γ is therefore an interior tangent to $\Psi(\hat{N}_{1,r_0})$ at p. However, as in the preceeding paragraph, $\Psi(\hat{N}_{1,r_0})$ is strictly convex at p, and this yields a contradiction.

It follows that $\Psi(\hat{N}_{1,r})$ remains a strict graph over $\partial_0\hat{N}_2$ for all $r > 0$. Letting $r \to 0$, it follows that $U_1 = \hat{N}_{1,r}$ and that $\Psi(\partial_0\hat{N}_1)$ is a graph over $\partial_0\hat{N}_2$. The result now follows by taking quotients. □

Corollary 2.10. *Let $\tilde{N}$ be a compact hyperbolic end. Let (N,φ) be its quotient Möbius manifold. Let $\tilde{N}'$ be the Kulkarni-Pinkall hyperbolic end of (N,φ) then $\tilde{N}$ is contained in $\tilde{N}'$ and $\partial\tilde{N}$ is a graph over $\partial\tilde{N}'$,*

Proof. The Kulkarni-Pinkall hyperbolic end satisfies the geodesic boundary condition. □

Corollary 2.11. *"$<$" defines a partial order over the family of hyperbolic ends whose quotient Möbius manifold is compact.*

Proof. Let $\hat{N}$ be a hyperbolic end. Let $\tilde{N}$ be the universal cover of $\hat{N}$ and let N be the quotient Möbius manifold of $\tilde{N}$. Let $\hat{\imath} : N \to U\mathbb{H}^{n+1}$ be the boundary immersion of $\tilde{N}$. Let $\pi : U\mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ be the canonical projection. Define $i = \pi \circ \hat{\imath}$.

Let g be the hyperbolic metric of $\mathbb{H}^{n+1}$. Since i is $C^{0,1}$, i^*g defines an equivariant L^∞ metric over N. Let dVol_i be the induced equivariant L^∞ volume form. By compactness, integrating dVol_i yields a well defined volume for $\partial_0\hat{N}$, which we denote by $\mathrm{Vol}\ (\hat{N})$.

Now let $\hat{N}_1$ and $\hat{N}_2$ be hyperbolic ends such that $\hat{N}_1 < \hat{N}_2$. By the proof of Lemma 2.9, $\partial_0\hat{N}_1$ may be considered as a graph over $\partial_0\hat{N}_2$. Thus, by convexity:

$$\mathrm{Vol}\ (\hat{N}_2) < \mathrm{Vol}\ (\hat{N}_1).$$

In particular, $\hat{N}_1$ is not contained in $\hat{N}_2$ and so "$<$" is anti-symmetric. Since "$<$" is trivially transitive, we deduce that it is a partial order, and the result follows. □

Corollary 2.12. *Let $\tilde{N}_1$ and $\tilde{N}_2$ be compact hyperbolic ends having the same quotient Möbius manifold. Then there exists a unique hyperbolic end $\tilde{N}_{12}$ such that:*

(i) *$\tilde{N}_1$ and $\tilde{N}_2$ are contained in $\tilde{N}_{12}$; and*

(ii) *if $\tilde{N}_1$ and $\tilde{N}_2$ are contained in $\tilde{N}$, then $\tilde{N}_{12}$ is also contained in $\tilde{N}$.*

Proof. Let $\tilde{N}_{KP}$ be the Kulkarni-Pinkall hyperbolic end of the quotient Möbius manifold. By Corollary 2.10, $\tilde{N}_1$ and $\tilde{N}_2$ are contained in $\tilde{N}_{KP}$ and $\partial_0\tilde{N}_1$ and $\partial_0\tilde{N}_2$ are graphs over $\partial_0\tilde{N}_{KP}$. Let f_1 and f_2 be their respective graph functions. The graph of Min (f_1, f_2) in $\tilde{N}_{KP}$ is convex and yields the desired hyperbolic end. □

This yields uniqueness of the maximal ends in the compact case:

Lemma 2.13. *Let M be a compact Möbius manifold. The Kulkarni-Pinkall hyperbolic end of M is the unique maximal end amongst all ends whose quotient Möbius manifold is M.*

Proof. Let $\tilde{M}_{KP}$ be the Kulkarni-Pinkall hyperbolic end of M. We first show that $\tilde{M}_{KP}$ is maximal. Let $\tilde{M}$ be any other end whose quotient Möbius manifold is M. Suppose that $M_{KP} < M$ and that this inclusion is strict. We thus identify $\tilde{M}_{KP}$ with a subset of $\tilde{M}$.

Let d be the distance in $\tilde{M}$ from $\partial_0\tilde{M}$. Let $p \in \partial_0\tilde{M}_{KP}$ be a point maximising distance from $\partial_0\tilde{M}$. Let N_p be a supporting normal to $\partial_0\tilde{M}_{KP}$ which is parallel to the vertical foliation of $\tilde{M}$. Let U_p be the set of unit vectors, V_p, over p in $T_p\tilde{N}$ such that:

$$\langle V_p, \mathsf{N}(p)\rangle > 0.$$

For all $V_p \in U$, the half geodesic in $\tilde{M}_{KP}$ leaving p in the direction of V_p terminates in a point in $\partial_\infty\tilde{M}_{KP}$. Let B be the image of U_p in $M = \partial_\infty\tilde{M}_{KP}$. By definition of the Kulkarni-Pinkall end, B_p is a maximal ball about the image of N_p in M.

Let $q \in \partial_0\tilde{M}$ be the projection of p. Let N_q be the supporting normal to $\partial_0\tilde{M}$ at q pointing towards p. We define B_q in the same way as B_p. Trivially, B_q contains B_p in its interior, and this contradicts the maximality of B_p. We conclude that $\tilde{M} = \tilde{M}_{KP}$, and maximality follows.

Let $\tilde{M}'$ be another maximal end whose quotient Möbius manifold is M. Since $\tilde{M}_{KP}$ possesses the Geodesic Boundary Property, it follows by Lemma 2.9 that $\tilde{M}' \leq \tilde{M}_{KP}$. By maximality of $\tilde{M}'$, $\tilde{M}' = \tilde{M}_{KP}$, and uniqueness follows. □

The proof of Theorem 1.7 now follows:

Proof of Theorem 1.7. This is the union of Lemma 2.8, Corollary 2.3 and Lemma 2.13. □

3. Special Lagrangian Curvature

3.1. Immersed Submanifolds and the Cheeger/Gromov Topology. Let M be a smooth Riemannian manifold. An **immersed submanifold** is a pair $\Sigma = (S, i)$ where S is a smooth manifold and $i : S \to M$ is a smooth immersion. A **pointed immersed submanifold** in M is a pair (Σ, p) where $\Sigma = (S, i)$ is an immersed submanifold in M and p is a point in S. An **immersed hypersurface** is an immersed submanifold of codimension 1. We give S the unique Riemannian metric i^*g which makes i into an isometry. We say that Σ is **complete** if and only if the Riemannian manifold (S, i^*g) is.

Let UM be the unitary bundle of M (i.e the bundle of unit vectors in TM. In the cooriented case (for example, when I is convex), there exists a unique exterior normal vector field N over i. We denote $\hat{\imath} = \mathsf{N}$ and call it the **Gauss lift** of i. Likewise, we call the manifold $\hat{\Sigma} = (S, \hat{\imath})$ the **Gauss lift** of Σ.

A pointed Riemannian manifold is a pair (M, p) where M is a Riemannian manifold and p is a point in M. Let $(M_n, p_n)_{n\in\mathbb{N}}$ be a sequence of pointed Riemannian manifolds. For all n, we denote by g_n the Riemannian metric over M_n. We say that the sequence $(M_n, p_n)_{n\in\mathbb{N}}$ converges to the pointed manifold (M_0, p_0) in the **Cheeger/Gromov** sense if and only if for all n, there exists a mapping $\varphi_n : (M_0, p_0) \to (M_n, p_n)$, such that, for every compact subset K of M_0, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

(i) the restriction of φ_n to K is a C^∞ diffeomorphism onto its image, and

(ii) if we denote by g_0 the Riemannian metric over M_0, then the sequence of metrics $(\varphi_n^* g_n)_{n\geq N}$ converges to g_0 in the C^∞ topology over K.

We refer to the sequence $(\varphi_n)_{n\in\mathbb{N}}$ as a sequence of **convergence mappings** of the sequence $(M_n, p_n)_{n\in\mathbb{N}}$ with respect to the limit (M_0, p_0). The convergence mappings are trivially not unique.

Let $(\Sigma_n, p_n)_{n\in\mathbb{N}} = (S_n, p_n, i_n)_{n\in\mathbb{N}}$ be a sequence of pointed immersed submanifolds in M. We say that $(\Sigma_n, p_n)_{n\in\mathbb{N}}$ converges to $(\Sigma_0, p_0) = (S_0, p_0, i_0)$ in the **Cheeger/Gromov sense** if and only if the sequence $(S_n, p_n)_{n\in\mathbb{N}}$ of underlying manifolds converges to (S_0, p_0) in the Cheeger/Gromov sense, and, for every sequence $(\varphi_n)_{n\in\mathbb{N}}$ of convergence mappings of $(S_n, p_n)_{n\in\mathbb{N}}$ with respect to this limit, and for every compact subset K of S_0, the sequence of functions $(i_n \circ \varphi_n)_{n\geq N}$ converges to the function $(i_0 \circ \varphi_0)$ in the C^∞ topology over K.

We define $C^{k,\alpha}$ Cheeger/Gromov convergence for manifolds and immersed manifolds in an analogous manner.

3.2. Special Lagrangian Curvature. The special Lagrangian curvature, which only has meaning for strictly convex immersed hypersurfaces, is defined as follows.

Denote by Symm $(\mathbb{R}^n)$ the space of symmetric matrices over $\mathbb{R}^n$. We define $\Phi : \text{Symm}\ (\mathbb{R}^n) \to \mathbb{C}^*$ by:

$$\Phi(A) = \text{Det}\ (I + iA).$$

Since Φ never vanishes and Symm $(\mathbb{R}^n)$ is simply connected, there exists a unique analytic function $\tilde{\Phi} : \text{Symm}\ (\mathbb{R}^n) \to \mathbb{C}$ such that:

$$\tilde{\Phi}(I) = 0, \qquad e^{\tilde{\Phi}(A)} = \Phi(A) \qquad \forall A \in \text{Symm}\ (\mathbb{R}^n).$$

We define the function arctan: Symm $(\mathbb{R}^n) \to (-n\pi/2, n\pi/2)$ by:

$$\arctan(A) = \text{Im}\ (\tilde{\Phi}(A)).$$

This function is trivially invariant under the action of $O(\mathbb{R}^n)$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then:

$$\arctan(A) = \sum_{i=1}^{n} \arctan(\lambda_i).$$

For $r > 0$, we define:

$$\text{SL}_r(A) = \arctan(rA).$$

If A is positive definite, then SL_r is a strictly increasing function of r. Moreover, $SL_0 = 0$ and $SL_\infty = n\pi/2$. Thus, for all $\theta \in]0, n\pi/2[$, there exists a unique $r > 0$ such that:

$$\mathrm{SL_r(A)} = \theta.$$

We define $R_\theta(A) = r$. R_θ is also invariant under the action of $O(n)$ on the space of positive definite, symmetric matrices.

Let M be an oriented Riemannian manifold of dimension $n+1$. Let $\Sigma = (S, i)$ be a strictly convex, immersed hypersurface in M. For $\theta \in]0, n\pi/2[$, we define $R_\theta(\Sigma)$ (the *θ-special Lagrangian curvature* of Σ) by:

$$R_\theta(\Sigma) = R_\theta(A_\Sigma),$$

where A_Σ is the shape operator of Σ.

3.3. **Local Rigidity.** Let N and M be Riemannian manifolds of dimensions n and $(n+1)$ respectively. The special Lagrangian curvature operator sends the space of smooth immersions from N into M into the space of smooth functions over N. These spaces may be viewed as infinite dimensional manifolds (strictly speaking, they are the intersections of infinite nested sequences of Banach manifolds). Let i be a smooth immersion from N into M. Let N be the unit exterior normal vector field of i in M. We identify the space of smooth functions over N with the tangent space at i of the space of smooth immersions from N into M as follows. Let $f : N \to \mathbb{R}$ be a smooth function. We define the family $(\Phi_t)_{t\in\mathbb{R}} : N \to M$ by:

$$\Phi_t(x) = \text{Exp}(tf(x)\mathsf{N}(x)).$$

This defines a path in the space of smooth immersions from N into M such that $\Phi_0 = i$. It thus defines a tangent vector to this space at i. Every tangent vector to this space may be constructed in this manner.

Let A be the shape operator of i. This sends the space of smooth immersions from N into M into the space of sections of the endomorphism bundle of TN. We have the following result:

Lemma 3.1. *Suppose that M is of constant sectional curvature equal to -1, then the derivative of the shape operator at i is given by:*

$$D_i A \cdot f = f\mathrm{Id} - \mathrm{Hess(f)} - \mathrm{fA}^2,$$

where Hess(f) is the Hessian of f with respect to the Levi-Civita covariant derivative of the metric induced over N by the immersion i.

Proof. See the proof of proposition 3.1.1 of [14]. □

We consider the operators $\mathrm{SL}_r = \mathrm{SL}_r(A_\Sigma)$ and $R_\theta = R_\theta(A_\Sigma)$. Using Lemma 3.1, we immediately obtain:

Lemma 3.2. *Suppose that M is of constant sectional curvature equal to -1.*

(i) *The derivative of SL_r at i is given by:*

$$(1/r)D_i SL_r \cdot f = -\mathrm{Tr}((\mathrm{Id} + \mathrm{r}^2\mathrm{A}^2)^{-1}\mathrm{Hess(f)}) + \mathrm{Tr}((\mathrm{Id} - \mathrm{A}^2)(\mathrm{Id} + \mathrm{r}^2\mathrm{A}^2)^{-1})\mathrm{f}.$$

(ii) *Likewise, the derivative of R_θ at i is given by:*

$$\begin{aligned}\mathrm{Tr}(\mathrm{A}(\mathrm{I} + \mathrm{A}^2\mathrm{R}_\theta^2)^{-1})\mathrm{D_iR}_\theta \cdot \mathrm{f} = {}& \mathrm{R}_\theta\mathrm{Tr}((\mathrm{Id} + \mathrm{r}^2\mathrm{A}^2)^{-1}\mathrm{Hess(f)}) \\ & + R_\theta\mathrm{Tr}((\mathrm{Id} - \mathrm{A}^2)(\mathrm{Id} + \mathrm{r}^2\mathrm{A}^2)^{-1})\mathrm{f}.\end{aligned}$$

These operators are trivially elliptic. We wish to establish when they are invertible. We first require the following technical result:

Lemma 3.3. *Let $0 < n < m$ be positive integers. If $t \in]0, \pi/2]$, then:*

$$n\mathrm{sin}^2(\mathrm{t/n}) \geq \mathrm{msin}^2(\mathrm{t/m}),$$

With equality if and only if $n = 1$, $m = 2$ and $t = \pi/2$.

Proof. The function $\sin^2(t/2)$ is strictly convex over the interval $[0, \pi/4]$. Thus, for all $0 < x < y \leq \pi/4$:

$$(1/x)\mathrm{sin}^2(x) < (1/y)\mathrm{sin}^2(y).$$

Thus, for $m > n \geq 2$, we obtain:

$$n\mathrm{sin}^2(t/n) > m\mathrm{sin}^2(t/m).$$

We treat the case $n = 1$ separately. For $t \leq \pi/4$, the result follows as before. We therefore assume that $t > \pi/4$. Since the function $\sin^2(t/2)$ is strictly concave over the interval $[\pi/4, \pi/2]$, it follows that $\sin^2(t) \geq 2t/\pi$, with equality if and only if $t = \pi/2$. However:

$$\sin^2(\pi/4) = 1/2 = (2/\pi)(\pi/4).$$

Since $m \geq 2$, it follows by concavity that:

$$m\sin^2(t/m) \leq \sin^2(t),$$

with equality if and only if $m = 2$ and $t = \pi/2$. The result now follows. □

Using Lagrange multipliers to determine critical points, we obtain:

Lemma 3.4. *If $\theta \geq (n-1)\pi/2$ and $r > \tan(\theta/n)$, then the coefficient of the zeroth order term is non-negative:*

$$Tr((\mathrm{Id} - \mathrm{A}^2)(\mathrm{Id} + \mathrm{r}^2\mathrm{A}^2)^{-1}) \geq 0.$$

Moreover, this quantity reaches its minimum value of 0 if and only if $r = \tan(\theta/n)$ and A is proportional to the identity matrix.

Proof. For all m, we define the functions Φ_m and Θ_m over $\mathbb{R}^m$ by:

$$\Phi_m(x_1, \ldots, x_m) = \sum_{i+1}^{m} \frac{1 - x_i^2}{1 + r^2 x_i^2}, \qquad \Theta_m(x_1, \ldots, x_m) = \sum_{i=1}^{m} \arctan(r x_i).$$

Since the derivative of Θ_m never vanishes, $\Theta_m^{-1}(\theta)$ is a smooth submanifold of $\mathbb{R}^m$. Suppose that Φ_m achieves its minimum value on the interior of $\Theta_m^{-1}(\theta)$. Let $(\tilde{x}_1, \ldots, \tilde{x}_m)$ be a critical point of the restriction of Φ_m to this submanifold. For all i, let $\tilde{\theta}_i \in [0, \pi/2[$ be such that:

$$\tan(\tilde{\theta}_i) = r\tilde{x}_i.$$

Using Lagrange multipliers, we find that there exists $\eta \in [0, \pi/2]$ such that, for all i:

$$\tilde{\theta}_i \in \{\eta, \pi/2 - \eta\}.$$

Let k be the number of values of i such that $\tilde{\theta}_i \geq \pi/4$. Since $\theta \geq (m-1)\pi/2$:

$$k \geq m/2.$$

Choose $\eta \geq \pi/4$. Since $\tilde{\theta}_1 + \cdots + \tilde{\theta}_m = \theta$:

$$\eta = \frac{\theta - (m-k)\pi/2}{2k - m} = \frac{m(\theta/m) - 2(m-k)(\pi/4)}{2k - m}.$$

If $\tilde{\Phi}_m$ is the value acheived by Φ_m at this point, then:

$$\tilde{\Phi}_m = r^{-2}(1 + r^2)(2k - m)\cos^2(\eta) + (m - k)r^{-2}(1 + r^2) - mr^{-2}.$$

However:

$$\pi/4 \leq \theta/m \leq \eta < \pi/2.$$

Thus, since the function $\cos^2$ is convex in the interval $[\pi/4, \pi/2]$:

$$\cos^2(\eta) \geq \frac{m\cos^2(\theta/m) - 2(m-k)\cos^2(\pi/4)}{2k-m},$$

with equality if and only if $k = m$. Thus:

$$\tilde{\Phi}_m \geq mr^{-2}(1+r^2)\cos^2(\theta/m) - mr^{-2},$$

with equality if and only if $\tilde{\theta}_1 = \cdots = \tilde{\theta}_m$. Since $r \geq \tan(\theta/m)$, this is non-negative, and is equal to 0 if and only if $r = \tan(\theta/m)$.

We now show that Φ_m attains its minimum over $\Theta_m^{-1}(\theta)$. We treat first the case $\theta > (m-1)\pi/2$. Suppose the contrary. The functions Φ_m and Θ_m extend to continuous functions over the cube $[0, +\infty]^m$. Let $(\tilde{x}_1, \ldots, \tilde{x}_m)$ be the point in $\Theta_m^{-1}(\theta)$ where Φ_m is minimised, and suppose now that it lies on the boundary of the cube. Since $\theta > (m-1)\pi/2$, $\tilde{x}_i > 0$ for all i. Without loss of generality, there exists $n < m$ such that:

$$x_1, \ldots, x_n < +\infty, \qquad x_{n+1}, \ldots, x_m = +\infty.$$

Let $(\tilde{\theta}_1, \ldots, \tilde{\theta}_m)$ be as before. We define θ' by:

$$\theta' = \tilde{\theta}_1 + \cdots + \tilde{\theta}_n.$$

Since $\tilde{\theta}_{n+1} = \cdots = \tilde{\theta}_m = \pi/2$, it follows that $\theta' = \theta - (m-n)\pi/2$. Moreover:

$$\Phi_m(x_1, \ldots, x_m) = \Phi_n(x_1, \ldots, x_n) - (m-n)r^{-2}.$$

Since $(\tilde{x}_1, \ldots, \tilde{x}_m)$ minimises Φ_m it follows that $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is the minimal valued critical point of Φ_n in $\Theta_n^{-1}(\theta')$. Thus:

$$\Phi_m(x_1, \ldots, x_m) = nr^{-2}(1+r^2)\cos^2(\theta'/n) - mr^{-2}.$$

Let $\eta \in]0, \pi/2[$ be such that:

$$\theta = n\pi/2 - \eta.$$

We have:

$$n\cos^2(\theta'/n) = n\sin^2(\eta/n), \qquad m\cos^2(\theta/m) = m\sin^2(\eta/m).$$

It follows by Lemma 3.3 that:

$$\Phi_m(x_1, \ldots, x_m) > mr^{-2}(1+r^2)\cos^2(\theta/m) - mr^{-2}.$$

It follows that $(\tilde{x}_1, \ldots, \tilde{x}_m)$ cannot be the minimum of Φ_m over $\Theta_m^{-1}(\theta)$, which is absurd. The result now follows in the case $\theta > (m-1)\pi/2$.

It remains to study the case $\theta = (m-1)\pi/2$. This follows as before, with the single exception that it is now possible that $\tilde{x}_1 = 0$, in which case $\tilde{x}_2 = \cdots = \tilde{x}_n = +\infty$. However:

$$\Phi_m(0, +\infty, \ldots, +\infty) = 1 - (m-1)r^{-2}.$$

However, $r \geq \tan((m-1)\pi/2m)$. For $x \in [0,1]$, $\tan(\pi x/4) \leq x$. Thus, since $m \geq 2$:

$$r^{-1} \leq \tan(\pi/2m) = \tan((\pi/4)(2/m)) \leq 2/m.$$

Thus:

$$\Phi_m(0,+\infty,\ldots,+\infty) \geq 1-4(m-1)/m^{-2} = (m-2)^2m^{-2} \geq 0,$$

The result now follows. □

Lemma 3.5. (i) *If $SL_r(i) \geq (n-1)\pi/2$ and $\tan(SL_r(i)/n) \leq r$, then D_iSL_r is invertible.*

(ii) *Likewise, if $\theta \geq (n-1)\pi/2$ and $R_\theta(i) \geq \tan(\theta/n)$, then D_iR_θ is invertible.*

Proof. This follows immediately from the preceeding lemma, the maximum principal and the fact that second order elliptic linear operators on the space of smooth functions over a compact manifold are Fredholm of index 0. □

3.4. **Compactness.** A relatively trivial variant of the reasoning used in [17] yields:

Theorem 3.6. *Let M be a complete Riemannian manifold.*

(i) *Let $(p_n)_{n\in\mathbb{N}}, p_0 \in M$ be such that $(p_n)_{n\in\mathbb{N}}$ converges to p_0;*

(ii) *Let $(\theta_n)_{n\in\mathbb{N}}, \theta_0 \in](n-1)\pi/2, n\pi/2[$ be such that $(\theta_n)_{n\in\mathbb{N}}$ converges to θ_0;*

(iii) *Let $(r_n)_{n\in\mathbb{N}}, r_0 \in C^\infty(M)$ be strictly positive functions such that $(r_n)_{n\in\mathbb{N}}$ converges to r_0 in the C^∞_{loc} sense; and*

(iv) *Let $(\Sigma_n,q_n)_{n\in\mathbb{N}} = (S_n,i_n,q_n)_{n\in\mathbb{N}}$ be pointed, convex immersed hypersurfaces such that, for all n:*

(a) *$i_n(q_n) = p_n$, and*

(b) *Σ_n is complete, convex and $R_{\theta_n}(i_n) = r_n \circ i_n$.*

Then there exists a complete, pointed immersed submanifold $(\Sigma_0,q_0) = (S_0,i_0,q_0)$ in M such that, after extraction of a subsequence, $(\Sigma_n,q_n)_{n\in\mathbb{N}}$ converges to (Σ_0,q_0) in the pointed Cheeger/Gromov sense.

The limit case where $\theta = (n-1)\pi/2$ exhibits more interesting geometric behaviour. We only require it in the constant curvature case:

Theorem 3.7. *Let M be a complete Riemannian manifold.*

(i) *Let $(p_n)_{n\in\mathbb{N}}, p_0 \in M$ be such that $(p_n)_{n\in\mathbb{N}}$ converges to p_0;*

(ii) *Let $(\theta_n)_{n\in\mathbb{N}} \in [(n-1)\pi/2, n\pi/2[$ be such that $(\theta_n)_{n\in\mathbb{N}}$ converges to $(n-1)\pi/2$;*

(iii) *Let $(r_n)_{n\in\mathbb{N}}, r_0 \in]0,\infty[$ be strictly positive real numbers such that $(r_n)_{n\in\mathbb{N}}$ converges to r_0; and*

(iv) *Let $(\Sigma_n,q_n)_{n\in\mathbb{N}} = (S_n,i_n,q_n)_{n\in\mathbb{N}}$ be pointed, convex immersed hypersurfaces such that, for all n:*

(a) *$i_n(q_n) = p_n$, and*

(b) Σ_n *is convex,* $R_{\theta_n}(i_n) = r_n$*, and the Gauss lifting,* $\hat{\Sigma}_n$*, is a complete submanifold of* UM*.*

Then there exists a complete, pointed immersed submanifold $(\hat{\Sigma}_0, q_0) = (S_0, \hat{\imath}_0, q_0)$ *in* UM *such that, after extraction of a subsequence,* $(\hat{\Sigma}_n, q_n)_{n\in\mathbb{N}}$ *converges to* $(\hat{\Sigma}_0, q_0)$ *in the pointed Cheeger/Gromov sense. Moreover:*

(i) *either there exists a convex, immersed hypersurface* Σ_0 *in* M *of constant* $(n-1)\pi/2$*-special Lagrangian curvature equal to* r_0 *such that* $\hat{\Sigma}_0$ *is the Gauss lifting of* Σ_0 *(in other words, if* $\pi : UM \to M$ *is the canonical projection, then* $\pi \circ \hat{\imath}_0$ *is an immersion);*

(ii) *or* $\hat{\Sigma}_0$ *is a covering of a complete sphere bundle over a complete geodesic.*

Remark. Heuristically, if $(\Sigma_n, p_n)_{n\in\mathbb{N}} = (S_n, i_n, p_n)_{n\in\mathbb{N}}$ is a sequence of pointed, immersed submanifolds of constant $(n-1)\pi/2$-special Lagrangian curvature equal to r, then $(\Sigma_n, p_n)_{n\in\mathbb{N}}$ subconverges to (Σ_0, i_0, p_0) where Σ_0 is either another such immersed submanifold or a complete geodesic. This (slightly abusive) language will be use in the sequel.

3.5. **The Geometric Maximum Principal.** Let $\mathcal{E}$ be a hyperbolic end possessing the Geodesic Boundary Property and let $\partial_0\mathcal{E}$ be its finite boundary. For all d, let M_d be the hypersurface in $\mathcal{E}$ at a distance d from $\partial_0\mathcal{E}$. We make the following definition:

Definition 3.8. *Let* M *be a manifold and let* $\Sigma = (S, i)$ *be a* C^0 *convex immersed hypersurface in* M*. Let* A *be a family of positive definite, symmetric, bilinear forms defined on the supporting tangent planes of* Σ*. The second fundamental form of* Σ *at* p *is said to be at least (resp. at most)* A *in the weak sense if and only if, for all* $p \in S$ *and for each supporting tangent space* E_p *of* Σ *at* p*, there exists a smooth, convex, immersed submanifold* $\Sigma' = (S, i')$ *which is an exterior (resp. interior) tangent to* Σ *with tangent space* E_p *at* p *and whose second fundamental form is bounded below (resp. above) by* $A(E_p)$*.*

Likewise, if $p \in S$*, if* $\theta \in]0, n\pi/2[$ *and if* $r > 0$*, then the* θ*-special Lagrangian curvature of* Σ *at* p *is said to be at least (resp. at most)* r *in the weak sense if and only if there exists a smooth, convex, immersed submanifold* $\Sigma' = (S', i')$ *of* θ*-special Lagrangian curvature equal to* r *which is an exterior (resp. interior) tangent to* Σ *at* p*.*

Remark. If the second fundamental form of Σ is bounded above and below, then Σ is necessarily of type $C^{1,1}$.

This definition is well adapted to the Geometric Maximum Principal, whose proof requires the following result concering symmetric matrices:

Lemma 3.9 (Minimax Principal). *Let A be a symmetric matrix of rank n. If $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of A arranged in ascending order, then, for all k:*

$$\lambda_k = \mathrm{Inf}_{\mathrm{Dim(E)=k}}\mathrm{Sup}_{\mathrm{v\in E\setminus\{0\}}}\langle \mathrm{Av}, \mathrm{v}\rangle/\|\mathrm{v}\|^2.$$

Proof. Let $e_1, \ldots, e_n$ be the eigenvectors of A. We define $\hat{E}$ by:

$$\hat{E} = \langle e_1, \ldots, e_k\rangle.$$

Let π be the orthogonal projection onto $\hat{E}$. Let E be a subspace of $\mathbb{R}^n$ of dimension k. For all v in E:

$$\langle A\pi(v), \pi(v)\rangle\|v\|^2 \leq \langle Av, v\rangle\|\pi(v)\|^2.$$

If the restriction of π to E is an isomorphism, then it follows that:

$$\lambda_k = \mathrm{Sup}_{v\in\hat{E}\setminus\{0\}}\langle Av, v\rangle/\|v\|^2 \leq \mathrm{Sup}_{v\in E\setminus\{0\}}\langle Av, v\rangle/\|v\|^2.$$

Otherwise, there exists a non-trivial $v \in E$ such that $\pi(v) = 0$, in which case:

$$\langle Av, v\rangle \geq \lambda_{k+1}\|v\|^2 \geq \lambda_k\|v\|^2.$$

The result now follows. □

Corollary 3.10. *Let A,A' be two symmetric matrices of rank n such that $A' \leq A$. If $\lambda_1 \ldots, \lambda_n$ and $\lambda'_1, \ldots, \lambda'_n$ are the eigenvalues of A and A' respectively arranged in ascending order, then, for all k:*

$$\lambda'_k \leq \lambda_k.$$

We now obtain the Geometric Maximum Principal for hypersurfaces of constant special Lagrangian curvature:

Lemma 3.11 (Geometric Maximum Principal). *Let M be a Riemannian manifold and let $\Sigma = (S, i)$ and $\Sigma' = (S', i')$ be C^0 convex, immersed hypersurfaces in M. For $\theta \in]0, n\pi/2[$, let R_θ and R'_θ be the θ-special Lagrangian curvatures of Σ and Σ' respectively. If $p \in S$ and $p' \in S'$ are such that $q = i(p) = i'(p')$, and Σ' is an interior tangent to Σ at q, then:*

$$R_\theta(p) \geq R'_\theta(p').$$

Proof. If A and A' are the shape operators of Σ and Σ' respectively, then:

$$A'(p') \geq A(p).$$

It follows that:

$$\arctan(R_\theta(p)A'(p')) \geq \arctan(R_\theta(p)A(p)) = \theta = \arctan(R'_\theta(p')A'(p')).$$

The result now follows since the mapping $\rho \mapsto \arctan(\rho A'(p'))$ is strictly increasing. □

Lemma 3.12. *For all $d > 0$, the second fundamental form of M_d is at least* $\tanh(d)$*Id in the weak sense.*

Proof. It suffices to calculate the second fundamental form of a hypersurface equidistant from a supporting totally geodesic hypersurface at some point of $\partial\mathcal{E}$. The result now follows from Lemma 3.1. □

Corollary 3.13. *Let $\theta\in]0,n\pi/2[$ be an angle. For all $d>0$, the θ-special Lagrangian curvature of M_d is at most $\tan(\theta/n)/\tanh(d)$ in the weak sense.*

For $d>0$, define the matrix $A_0(d)$ by:

$$A_0(d)=\begin{pmatrix}\tanh(d) & \\ & \coth\,(d)\mathrm{Id}_{n-1}\end{pmatrix},$$

where Id_{n-1} is the $(n-1)$-dimensional identity matrix.

Lemma 3.14. *For all $d>0$, there exists a (not necessarily continuous) field A of symmetric, bilinear forms over M_d such that:*

(i) *for all $p\in M_d$, $A(p)$ is conjugate to A_0; and*

(ii) *the second fundamental form of M_d is bounded above by A in the weak sense.*

Proof. For all $q\in\partial\mathcal{E}$, there is a geodesic segment passing through p which remains in $\partial\mathcal{E}$. Thus, for all $p\in M_d$, there is a cylinder at a distance d from a geodesic segment which is an interior tangent to M_d at p. By Lemma 3.1, the second fundamental form of this cylinder is conjugate to A_0. The upper bound of the curvature at p thus follows. □

Corollary 3.15. *Let $\theta\in](n-1)\pi/2,n\pi/2[$ be an angle. There exists a function $\kappa:[0,+\infty[\to[0,+\infty[$, which tends to $+\infty$ as d tends to 0, such that the θ-special Lagrangian curvature of M_d is at least $\kappa(d)$ in the weak sense.*

We now obtain upper and lower bounds for the distance between a hypersurface of bounded θ-special Lagrangian curvature and $\partial\mathcal{E}$:

Lemma 3.16. *Let $\mathcal{E}$ be a hyperbolic end. Let $\partial\mathcal{E}$ be the boundary of $\mathcal{E}$. Let $\theta\in](n-1)\pi/2,n\pi/2[$ be an angle. There exists a decreasing function $\delta:[\tan(\theta/n),+\infty[\to]0,+\infty[$ such that if $r\leq R\in]\tan(\theta/n),\infty[$ and if $\Sigma=(S,i)$ is a compact, convex immersed submanifold such that $R_\theta(i)\in[r,R]$, then, for all $p\in S$:*

$$\delta(R)\leq d(i(p),\partial\mathcal{E})\leq\mathrm{arctanh}(\mathrm{r}^{-1}\tan(\theta/\mathrm{n})).$$

Proof. For all $\rho>0$, let M_ρ be the level hypersurface in $\mathcal{E}$ at a distance of R from $\partial\mathcal{E}$. Since Σ is compact, there exists a point $p\in S$ maximising the distance from $\partial\mathcal{E}$. Let d be the distance of $i(p)$ from $\partial\mathcal{E}$. Σ is an interior tangent to M_d at p. The upper bound now follows by Lemma 3.13 and the geometric maximum principle (Lemma 3.11). The lower bound follows in an analogous way, using Lemma 3.15 instead of Lemma 3.13. □

4. Immersions In Hyperbolic Ends

4.1. **Deforming Equivariant Immersions.** The results of the previous section permit us to locally deform equivariant immersions of $\tilde{M}$ in $\mathbb{H}^{n+1}$. Let $\Gamma \subseteq \mathrm{Isom}(\tilde{M})$ be a cocompact subgroup acting properly discontinuously on $\tilde{M}$. Thus $\tilde{M}/\Gamma$ is a compact manifold. Let $\alpha : \Gamma \to \mathrm{Isom}(\mathbb{H}^{n+1})$ be a homomorphism. Let $i : \tilde{M} \to \mathbb{H}^{n+1}$ be an immersion which is equivariant with respect to θ. Thus, for all $\gamma \in \Gamma$:

$$i \circ \gamma = \alpha(\gamma) \circ i.$$

Let $\rho = R_\theta(i)$. Suppose first that i is an embedding. We may therefore extend ρ to a smooth equivariant function over a neighbourhood of $i(\tilde{M})$ in $\mathbb{H}^{n+1}$. We obtain the following local deformation result:

Lemma 4.1. *Let $\theta \geq (n-1)\pi/2$ and suppose that $\rho \geq \tan(\theta/n)$.*

(i) *Let $(\alpha_t)_{t\in]-\epsilon,\epsilon[}$ be a smooth family of homomorphisms such that $\alpha_0 = \alpha$;*
(ii) *let $(\theta_t)_{t\in]-\epsilon,\epsilon[}$ be a smooth family of angles such that $\theta_0 = \theta$; and*
(iii) *let $(\rho_t)_{t\in]-\epsilon,\epsilon[} : \mathbb{H}^{n+1} \to \mathbb{R}$ be a smooth family of smooth functions such that $\rho_0 = \rho$.*

There exists $0 < \delta < \epsilon$ and a unique smooth family of immersions $(i_t)_{t\in]-\delta,\delta[}$ such that $i_0 = i$ and, for all t:

(i) *$R_{\theta_t}(i_t) = \rho_t \circ i_t$, and*
(ii) *i_t is equivariant with respect to α_t.*

Remark. The corresponding result when i is not injective is almost identical. We do not state it in order to avoid notational complexity. In the sequel, we consider embeddings inside smooth manifolds or smooth families of smooth manifolds, and so the distinction is not important.

Proof. For ease of presentation, we only prove the case where both ρ and θ are constant. The general case is proven in a similar manner. The proof is divided into two stages:

(i) We approximate the desired family by constructing a smooth, equivariant family of deformations of i which are not necessarily immersions, and not necessarily of constant θ-special Lagrangian curvature. First we construct a fundamental domain for Γ. Let p be a point in $\tilde{M}$. Let $P \subseteq \tilde{M}$ be the orbit of p under the action of Γ. Thus:

$$P = \Gamma p.$$

We define $\Omega \subseteq \tilde{M}$ to be the set of all points on $\tilde{M}$ which are closer to p than to any other point in the orbit of p:

$$\Omega = \{q \in \mathbb{H}^n \text{ s.t. } d(q,p) < d(q,p') \text{ for all } p' \in P \setminus \{p\}\}.$$

Ω is a polyhedral fundemental domain for Γ.

Using Ω, we now construct the family of deformations. For each t, we construct a (non-continuous) deformation be defining i_t to be equal to i over the interior of Ω and then extending this function to the orbit of Ω (which is almost all of $\tilde{M}$) by equivariance with respect to α_t. These deformations may trivially be smoothed along $\partial\Omega$. The only complication is to ensure that the smoothing is performed in an equivariant manner. The following recipe allows us to achieve exactly this.

For any submanifold $X \in \tilde{M}$ and for all $\epsilon > 0$, let X^ϵ be the set of all points in X which are at a distance (in X) greater than ϵ from the boundary of X. That is:

$$X^\epsilon = \{p \in X \text{ s.t. } d_X(p, \partial X) > \epsilon\}.$$

Choose ϵ_n small. For all $\gamma \in \Gamma$, we define $(\tilde{i}^n_t)_{t\in]\epsilon,\epsilon[}$ over $\gamma\Omega^{\epsilon_n}$ by:

$$\tilde{i}^n_t(p) = \alpha_t(\gamma)i(\gamma^{-1}(p)).$$

This family is trivially equivariant with respect to $(\alpha_t)_{t\in]-\epsilon,\epsilon[}$.

Choose ϵ_{n-1} small. Let F_{n-1} be any $(n-1)$-dimensional face of Ω. We extend $(\tilde{i}^n_t)_{t\in]-\epsilon,\epsilon[}$ smoothly across a neighbourhood of $F^{\epsilon_{n-1}}_{n-1}$. Since every element of Γ is of infinite order, there is no element which fixes any face of Ω (since otherwise it would permute the domains touching that face, and thus be of finite order). It follows that, by choosing ϵ_n and ϵ_{n-1} small enough, we may extend this family further to a smooth equivariant extension over every face in the orbit of F_{n-1}. We then continue extending this family over every face of Ω until all $(n-1)$-dimensional faces are exhausted. By working downwards inductively on the dimension of the faces, we thus obtain a smooth equivariant family $(\tilde{i}_t)_{t\in]-\epsilon,\epsilon[} = (\tilde{i}^0_t)_{t\in]-\epsilon,\epsilon[}$ which extends i.

(ii) We now modify this approximation to obtain the desired family of immersions. Since Ω is relatively compact, there exists $\delta < \epsilon$ such that, for $|t| < \delta$, $\tilde{i}_t$ is an immersion. Moreover, we may suppose that for $\eta > 0$ sufficiently small, we may extend $\tilde{i}_t$ smoothly along normal geodesics to a smooth equivariant immersion from $\tilde{M}\times]-\eta,\eta[$ into $\mathbb{H}^{n+1}$. We thus view $(\tilde{i}_t)_{t\in]-\delta,\delta[}$ as a smooth family of immersions from $\tilde{M}\times]-\eta,\eta[$ into $\mathbb{H}^{n+1}$.

We denote by g the hyperbolic metric over $\mathbb{H}^{n+1}$. We define the family $(g_t)_{t\in]-\delta,\delta[}$ such that, for all t:

$$g_t = \tilde{i}^*_t g.$$

The action of Γ over $\tilde{M}$ trivially extends to an action of Γ over $\tilde{M}\times]-\eta,\eta[$. For all t, g_t is equivariant under this action of Γ. We denote $M = \tilde{M}/\Gamma$ and we obtain a smooth family, which we also call $(g_t)_{t\in]-\delta,\delta[}$, of hyperbolic metrics over $M\times]-\eta,\eta[$.

Let j_0 be the canonical immersion of M into $M\times]-\eta,\eta[$. Trivially, with respect to g_0, $R_\theta(j_0) = \rho$. As in Section 3.3, we view R_θ as a second order, non-linear differential operator sending immersions of M into $M\times]-\eta,\eta[$ into functions over

M. Since infinitesimal variations of immersions may be interpreted as functions over M times the normal vector field of M in $M\times]-\eta,\eta[$, the derivative DR_θ of R_θ may be interpreted as a second order, linear differential operator from $C^\infty(M)$ into $C^\infty(M)$. By Lemma 3.5, the operator DR_θ is invertible. After reducing δ if necessary, the Implicit Function Theorem for non-linear PDEs therefore allows us to extend j_0 to a smooth family $(j_t)_{t\in]-\eta,\eta[}$ of immersions of M into $M\times]-\eta,\eta[$ such that, for all t, the θ-special Lagrangian curvature of j_t with respect to g_t equals ρ. For all t, let $\tilde{j}_t$ be the lift of j_t sending $\tilde{M}$ into $\tilde{M}\times]-\eta,\eta[$. We now define $i_t = \tilde{\imath}_t\circ\tilde{j}_t$. Trivially, $(i_t)_{t\in]-\delta,\delta[}$ is the desired family of immersions, and existence follows.

Let $(i'_t)_{t\in]-\delta,\delta[}$ be another family of immersions having the desired properties. For δ sufficiently small, the image of i'_t is contained in the image of $\tilde{\imath}_t$. For all t, we thus project $\tilde{j}'_t = \tilde{\imath}_t^{-1}\circ i'_t$ to an immersion j'_t of M into $M\times]-\eta,\eta[$. By the uniqueness part of the Implicit Function Theorem for non-linear PDEs, for all sufficiently small t, j'_t coincides with j_t. Uniqueness now follows by a standard open/closed argument. □

4.2. **Uniqueness.** We show that the metric induced by i is uniformly equivalent, up to reparametrisation, with the Kulkarni-Pinkall metric:

Lemma 4.2. *Let $\theta\in](n-1)\pi/2, n\pi/2[$ be an angle, and let $r>\tan(\theta/n)$ be a positive real number. There exists $K=K(r,\theta,n)>0$ which only depends on r, θ and n such that:*

(i) *if M is a compact manifold and (φ,θ) is an FCS of hyperbolic type over M;*

(ii) *if $i: M\to\mathbb{H}^{n+1}$ is a complete, equivariant, convex immersion such that $R_\theta(i)=r$ and $\overrightarrow{n}\circ\hat{\imath}=\varphi$, where $\hat{\imath}$ is the Gauss lifting of i; and*

(iii) *if $\alpha: M\to M$ is a diffeomorphism such that $i\circ\alpha$ is a graph over $\hat{j}$, where $\hat{j}$ is the boundary immersion of $\mathcal{E}(\varphi)$,*

then, if g is the hyperbolic metric on $\mathbb{H}^{n+1}$:

$$K^{-1}g_{KP}\leq(i\circ\alpha)^*g\leq Kg_{KP}.$$

Proof. Let $\mathcal{E}(\varphi)$ be the Kulkarni-Pinkall hyperbolic end of φ. Since, in particular, i is a convex immersion, by Lemma 2.9, i may be viewed as an immersion from M into $\mathcal{E}(\varphi)$ which is a graph over $\hat{j}$. Without loss of generality, we may assume that α is the identity. Thus, for all $p\in M$, $i(p)$ lies above $\hat{j}(p)$. For all $r>0$, let M_r be the hypersurface at distance r from $\partial_0\mathcal{E}(\varphi)$. By Lemma 3.16, there exists $R>\epsilon>0$ such that $i(M)$ lies between M_ϵ and M_R. Let $\pi: U\mathbb{H}^{n+1}\to\mathbb{H}^{n+1}$ be the canonical projection. Define $j=\pi\circ\hat{j}$. For all $p\in M$, let γ_p be the geodesic segment joining $j(p)$ to $i(p)$. Let N_p be the exterior normal to $i(M)$ at p.

We show that there exists $\delta > 0$, which only depends on r, θ and n such that γ_p makes an angle of at most $\pi/2 - \delta$ with N_p. We assume the contrary, and consider the universal covers of M and $\mathcal{E}(p)$. Let $(M_n, p_n)_{n\in\mathbb{N}}$ be a sequence of complete, simply connected, pointed manifolds. For all n, let (θ_n, φ_n) be an FCS of hyperbolic type over M_n and let $i_n : M_n \to \mathbb{H}^{n+1}$ be a complete, equivariant, convex immersion such that $R_\theta(i_n) = r$ and $\varphi_n = \overrightarrow{n} \circ \hat{\imath}_n$. For all n, let γ_n be the geodesic segment joining $j_n(p_n)$ to $i_n(p_n)$. The length of γ_n is greater than ϵ for all n. Suppose that the angle that γ_n makes with N_{p_n} tends to $\pi/2$.

For all n, let B_n be the ball of radius ϵ about $i_n(p_n)$ in $\mathcal{E}(\varphi_n)$. Since (M_n, i_n) is a graph over j_n, there exists convex subset $K_n \subseteq B_n$ such that a portion of (M_n, i_n) coincides with the boundary $\partial K_n \cap B_n$. Moreover, $\gamma_n \subseteq K_n$. For all n, we identify B_n with a ball of radius ϵ in hyperbolic space, which we denote by B_0. Thus, by compactness of the family of convex subsets of hyperbolic space, without loss of generality, there exists a convex subset $K_0 \subseteq B_0$ and a geodesic segment γ_0 to which $(K_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$ converge respectively. By Theorem 3.6, the boundary $\partial K_0 \cap B_0$ is smooth and has constant special Lagrangian curvature, in particular, it is strictly convex. By construction, γ_0 is tangent to $\partial K_0 \cap B_0$ at p_0. However, since $\gamma_0 \subseteq K_0$, it is an interior tangent at this point, which contradicts strict convexity.

It thus follows that γ_p makes an angle of at most $\pi/2 - \delta$ with N_p.

For $p \in M$, let P_p be the supporting totally geodesic hyperspace to $\mathcal{E}(\varphi)$ normal to γ_p at $\pi(p)$. Since $i(M)$ lies below M_R and since its normal makes an angle of at most $\pi/2 - \delta$ with γ_p, there exists K, which only depends on R,ϵ and δ such that the normal projection from $i(M)$ onto P_p is K-bilipschitz at p. The result now follows by the relationship between $\mathcal{E}(\varphi)$ and g_{KP}. $\square$

This yields uniqueness:

Lemma 4.3 (Uniqueness)**.** *Let M be a conformally flat manifold of hyperbolic type. Let $\alpha : \pi_1(M) \to Isom(\mathbb{H}^{n+1})$ be the holonomy and let $\varphi : \tilde{M} \to \partial_\infty\mathbb{H}^{n+1}$ be the developing map.*

Let $\theta \in [(n-1)\pi/2, n\pi/2[$ be an angle, and let $r \geq \tan(\theta/n)$. Let $i, i' : \tilde{M} \to \mathbb{H}^{n+1}$ be complete, equivariant, convex immersions such that $R_\theta(i) = R_\theta(i') = r$ and $\overrightarrow{n} \circ \hat{\imath} = \overrightarrow{n} \circ \hat{\imath}' = \varphi$. Then, up to reparametrisation, $i = i'$.

Moreover $i = i'$ is a graph over the finite boundary of the Kulkarni-Pinkall hyperbolic end of M, and is thus strictly contained within this hyperbolic end.

Proof. By Lemma 2.9, we view i and i' as immersions inside $\mathcal{E}(\varphi)$. We first consider the case where $\theta \neq (n-1)\pi/2$ and extend i and i' to unique foliations $(i_t)_{t\in[r,+\infty[}$ and $(i'_t)_{t\in[r,+\infty[}$ respectively which cover the lower end of $\mathcal{E}(\varphi)$.

Let $I \subseteq [r, +\infty[$ be such that, for all $T \in I$, there exists a foliation $(i^T_t)_{t\in[r,T[}$ of $\mathcal{E}(\varphi)$ such that $i_r = i$ and, for all t, $R_\theta(i_t) = t$. By the local uniqueness part of

Lemma 4.1, these foliations are unique. In other words, for all $r \leq t < T < T'$:

$$i_t^T = t_t^{T'}.$$

By Lemma 4.1, there exists $\delta > 0$ and a smooth family $(i_t)_{t\in[r,r+\delta[}$ such that $i_r = r$, and, for all t, $R_\theta(i_t) = t$. Let N be the normal vector field over i. Let f be the function over M such that $f\mathsf{N}$ is the infinitesimal deformation of $(i_t)_{t\in[r,r+\delta[}$. Then:

$$D_i R_\theta f = 1 \geq 0.$$

It follows by Lemma 3.4 that $f < 0$. Thus, by reducing δ if necessary, $(i_t)_{t\in[r,r+\delta[}$ is a foliation. I is therefore non-empty. Let T be the suprememum of I and suppose that $T < +\infty$. By uniqueness, there exists a foliation $(i_t)_{t\in[r,T[}$ with the given properties.

For all $t \in [r, T[$, by Lemma 2.9, i_t is a graph over $\partial\mathcal{E}(\varphi)$. Since $(i_t)_{t\in[r,T[}$ is a foliation, the corresponding graphs form a monotone family. In fact, the graphs are monotone decreasing. For all t, let Vol_t and Inj_t be the volume and injectivity radius respectively of i_t. By Lemma 4.2, Vol_t is uniformly bounded above and Inj_t is uniformly bounded below for $t \in T$. It follows by Theorem 3.6 that, for every sequence $(t_n)_{n\in\mathbb{N}}$ which converges to T, $(i_{t_n})_{n\in\mathbb{N}}$ subconverges. By monotonicity, all these subsequences converge to the same immersion, and thus $(i_t)_{t\in[r,T]}$ converges as t tends to T. We thus extend $(i_t)_{t\in[r,T[}$ to a foliation $(i_t)_{t\in[r,T]}$ defined over the closed interval.

Applying Lemma 4.1 again, this foliation can be extended to a foliation $(i_T)_{t\in[r,T+\delta[}$. This contradicts the definition of T. We thus obtain the desired foliation.

Let f and f' be the functions of which i and i' are the graphs over $\partial\mathcal{E}(\varphi)$. Suppose that $f' < f$ at some point. For all R, let M_R be the hypersurface of $\mathcal{E}(\varphi)$ at distance R from $\partial\mathcal{E}(\varphi)$. Let $\epsilon > 0$ be such that i and i' lie above M_ϵ. By Lemma 3.16, $(i_t)_{t\in[r,+\infty[}$ converges to $\partial\mathcal{E}(\varphi)$ in the Hausdorff sense as t tends to $+\infty$. In particular, there exists $R_0 > r$ such that i_R lies below M_ϵ and thus does not intersect i'. Let R be the supremum of all $s \in [r, R_0]$ such that i_s intersects i' non-trivially. By compactness i_R is an interior tangent to i' at some point. However, $R_\theta(i_r) = R > R_\theta(i')$, which is a contradiction by the Geometric Maximum Principal (Lemma 3.11).

It follows that $f' \geq f$. By symmetry, $f \geq f'$, and the result now follows for $\theta \neq (n-1)\pi/2$.

Suppose that $\theta = (n-1)\pi/2$. By Lemma 4.1, there exist smooth families (i_η) and (i'_η) for $\eta \in [(n-1)\pi/2, (n-1)\pi/2 + \delta[$ such that $i = i_{(n-1)\pi/2}$, $i' = i'_{(n-1)\pi/2}$ and, for all η:

$$R_\eta(i_\eta) = R_\eta(i'_\eta) = r.$$

By uniqueness for the case where $\theta \neq (n-1)\pi/2$, $i_\eta = i'_\eta$ for all $\eta \neq (n-1)\pi/2$ and the result now follows for $\theta = (n-1)\pi/2$ by taking limits. $\square$

4.3. **Main Results.** Existence follows from Theorem 1.4 of [18]. For the reader's convenience, we include a proof based on the more elementary Theorem 1.2 of the same paper. Throughout the rest of this section, a convex set will be said to be ϵ-convex for some $\epsilon > 0$ if and only if its second fundamental form with respect to every supporting normal is bounded below by $\epsilon\mathrm{Id}$ in the weak sense. We quote Theorem 1.2 of [18]:

Theorem 4.4. *Choose $\theta \in [(n-1)\pi/2, n\pi/2[$. Let $H \subseteq \mathbb{H}^{n+1}$ be a totally geodesic hypersurface. Let $\Omega \subseteq H$ be a bounded open subset. Let $\hat{\Sigma} \subseteq \mathbb{H}^{n+1}$ be a convex hypersurface which is a graph over Ω such that $\partial\hat{\Sigma} = \partial\Omega$ and:*

$$R_\theta(\hat{\Sigma}) \leq R_1,$$

in the weak sense, where $R_1 \geq \tan^{-1}(\theta/n)$. If $\theta > (n-1)\pi/2$, then, for all $r \in [R_1, \infty]$, there exists a unique immersed hypersurface $\Sigma_r \subseteq \mathbb{H}^{n+1}$ such that:

(i) *Σ_r is C^0 and C^∞ in its interior;*

(ii) *$\partial\Sigma_r = \partial\Omega$;*

(iii) *Σ_r is a graph over Ω lying below $\hat{\Sigma}$; and*

(iv) *$R_\theta(\Sigma_r) = r$.*

Moreover, the same result holds for $\theta = (n-1)\pi/2$ provided that, in addition, $\hat{\Sigma}$ is ϵ-convex, for some $\epsilon > 0$.

Remark. The statement of this theorem differs slightly from that appearing in [18] because (for technical reasons) the special Lagrangian curvature as defined in [18] is the reciprocal of the special Lagrangian curvature as defined here.

Following [7] and [18], we use the Perron method to obtain:

Lemma 4.5. *Let $\mathcal{E}$ be a hyperbolic end satisfying the Geodesic Boundary Condition. For all $\theta \in](n-1)\pi/2, n\pi/2[$ and for all $r > \tan(\theta/n)$, there exists a strictly convex immersed hypersurface $\Sigma = (S, i)$ in $\mathcal{E}$ which is a graph over the finite boundary of $\mathcal{E}$ such that $R_\theta(i) = r$.*

Moreover, the same result holds for $\theta = (n-1)\pi/2$ provided that the quotient Möbius manifold of $\mathcal{E}$ is not conformally equivalent to $(S^{n-1} \times \mathbb{R})/\Gamma$, where S^{n-1} is the $(n-1)$-dimensional sphere, and Γ is a properly discontinuous group of conformal actions.

Proof. We first treat the case where the quotient Möbius manifold of $\mathcal{E}$ is compact and $\theta > (n-1)\pi/2$. Let $\partial_0\mathcal{E}$ be the finite boundary of $\mathcal{E}$. For $d > 0$, let Σ^0_d be the level hypersurface at distance d from $\partial_0\mathcal{E}$. By Lemma 3.12, the second fundamental form of Σ^0_d is greater than $\tanh(d)\mathrm{Id}$ in the weak sense. Since $\tanh(d)$ tends to 1 as d tends to $+\infty$, for sufficiently large d, the θ-special Lagrangian curvature of Σ^0_d is at most r in the weak sense. Choose such a d and denote $\Sigma_0 = \Sigma^0_d$.

By definition, Σ_0 is a graph over $\partial_0\mathcal{E}$. Let f_0 be the function whose graph Σ_0 is. Let Σ_1 be a strict graph over $\partial_0\mathcal{E}$ lying below Σ_0 such that $R_\theta(\Sigma_1) \leq r$ in the weak sense. There exists $\epsilon > 0$, which only depends on θ and r such that Σ_1 is ϵ-convex. In particular, by Lemma 3.14 and the Geometric Maximum Principal, there exists $\delta > 0$ such that Σ_1 lies at a distance of at least δ from $\partial_0\mathcal{E}$. Let U_1 be the open set lying between $\partial_0\mathcal{E}$ and Σ_1. Choose $p \in \Sigma_1$. Let N_p be a supporting normal to Σ_1 at p chosen such that, for any other supporting normal N'_p to Σ_1 at p:

$$\langle \mathsf{N}'_p, \mathsf{N}_p \rangle \geq \eta,$$

for some $\eta > 0$. Such an N_p always exists since Σ_1 bounds a convex set with non-trivial interior (c.f. Lemma 4.7 of [18]). Let $\delta_1 > 0$ be smaller than the injectivity radius of $\mathcal{E}$ at p. Let γ be the unit speed geodesic such that:

$$\partial_t\gamma(0) = \mathsf{N}_p.$$

For small t, let $D_{p,t}$ be the totally geodesic disk in $\mathcal{E}$ of radius δ_1 about $\gamma(t)$ whose exterior normal at $\gamma(t)$ is $\partial_t\gamma(t)$. By strict convexity, $D_{p,0}$ only intersects Σ_1 at a single point. There therefore exists $\delta_2 > 0$ such that, for all $t \in]-\delta_2, 0[$, $\Omega_t := U_0 \cap D_{p,t}$ is a convex set and the portion of Σ_1 lying above Ω_t is a graph over Ω_t which we denote by $\Sigma_{1,t}$. Moreover, δ_2 may also be chosen sufficiently small such that it doesn't intersect $\partial_0\mathcal{E}$.

By Theorem 4.4, for all $t \in]-\delta_2, 0[$, there exists a unique graph $\Sigma'_{1,t}$ over Ω_t, lying beneath $\Sigma_{1,t}$ such that:

$$R_\theta(\Sigma'_{1,t}) = r.$$

For all $t \in]-\delta_2, 0[$, let Σ'_t be the hypersurface obtained by replacing the portion $\Sigma_{1,t}$ of Σ_1 with $\Sigma'_{1,t}$. By uniqueness, this is a continuous family. Moreover, for $t_1 > t_2$, Σ'_{t_1} lies above Σ'_{t_2}.

We claim that $R_\theta(\Sigma'_t) \leq r$ in the weak sense. It suffices to verify this property along $\partial\Omega_t = \partial\Sigma'_{1,t}$. However, along $\partial\Omega_t$, this property follows by the convexity of the curvature condition (R_θ is a convex function, c.f. Lemma 2.4 of [18]). The assertion therefore follows.

In particular, Σ'_t is ϵ-convex for all t. We claim that Σ'_t is a graph over $\partial_0\mathcal{E}$. Indeed, since $D_{p,t}$ lies strictly above $\partial_0\mathcal{E}$, so does Σ'_t for all t. Σ'_t therefore only ceases to be a graph if it becomes vertical at some point q_0 for some value t_0 of t. t_0 may be chosen such that Σ_t is a graph over $\partial_0\mathcal{E}$ for all $t \in]t_0, 0[$. Let $\underline{q}_0$ be the projection of q in $\partial_0\mathcal{E}$. Let $\gamma : [0, d(\underline{q}_0, q_0)] \to \mathcal{E}$ be the geodesic segment in $\mathcal{E}$ joining $\underline{q}_0$ to q_0. For all t, let U'_t be the open set lying between $\partial_0\mathcal{E}$ and Σ'_t. For $t > t_0$, since Σ'_t lies above Σ'_{t_0}, γ is contained in U'_t. It follows by continuity that γ is contained in U'_{t_0}, and thus $\partial_t\gamma$ is an interior tangent to Σ_{t_0} at q_0, which contradicts strict convexity. The assertion follows.

We choose any $t \in [-\delta_2, 0]$ and define $\Sigma_2 = \Sigma'_t$. We denote by A this operation for obtaining new immersed hypersurfaces out of old ones.

Let Σ_1 and Σ_2 be two graphs over $\partial_0\mathcal{E}$ and let f_1 and f_2 be the respective functions whose graphs they are. Suppose that:

(i) $f_1, f_2 \leq f_0$; and

(ii) $R_\theta(\Sigma_1), R_\theta(\Sigma_2) \leq r$ in the weak sense.

Define $f_{1,2}$ by:

$$f_{1,2} = \text{Min }(f_1, f_2).$$

Let $\Sigma_{1,2}$ be the graph of $f_{1,2}$. Then $\Sigma_{1,2}$ lies below Σ_0, and, by convexity of the curvature condition (c.f. Lemma 2.4 of [18]):

$$R_\theta(\Sigma_{1,2}) \leq r.$$

We denote this operation by B.

Let $\mathcal{F}$ be the family of immersed hypersurfaces in $\mathcal{E}$ obtained from Σ_0 by a finite number of combinations of the operations A and B. For any $\Sigma \in \mathcal{F}$, let $f(\Sigma)$ be the function of which Σ is the graph, and let $U(\Sigma)$ be the open set contained between $\partial_0\mathcal{E}$ and Σ. Define $V_0 \geq 0$ by:

$$V_0 = \text{Inf }\left\{\text{Vol }(U(\Sigma)) \text{ s.t. } \Sigma \in \mathcal{F}\right\}.$$

There exists a sequence $(\Sigma_n)_{n\in\mathbb{N}} \in \mathcal{F}$ such that:

(i) for all $n \geq m$:

$$f(\Sigma_n) \leq f(\Sigma_m);\text{ and}$$

(ii) $(\text{Vol }(U(\Sigma_n)))_{n\in\mathbb{N}}$ tends to V_0.

Let f_∞ be the function to which $(f(\Sigma_n))_{n\in\mathbb{N}}$ converges pointwise. By Lemma 3.14 and the Geometric Maximum Principal, there exists $d_0 > 0$ such that, for all n:

$$f(\Sigma_n) \geq d_0.$$

It follows that $f_\infty \geq d_0$. Moreover, since the graphs $(f(\Sigma_n))_{n\in\mathbb{N}}$ form the boundaries of a nested sequence of ϵ-convex sets, the graph of f_∞ is also the boundary of an ϵ-convex set, and, by strict convexity as before, the graph of f_∞ is never vertical. It follows that f_∞ is $C^{0,1}$ and that $(f(\Sigma_n))_{n\in\mathbb{N}}$ converges to f_∞ in the $C^{0,\alpha}$ sense for all α.

We claim that f_∞ is smooth. Let Σ_∞ be the graph of f_∞. Choose $p \in \Sigma_\infty$. Let N_p be a supporting normal to Σ_∞ at p chosen such that, for any other supporting normal N'_p to Σ_∞ at p:

$$\langle \mathsf{N}'_p, \mathsf{N}_p\rangle \geq \eta,$$

for some $\eta > 0$. Let $\delta_1 > 0$ be smaller than the injectivity radius of $\mathcal{E}$ at p. Let γ be the unit speed geodesic such that:

$$\partial_t\gamma(0) = \mathsf{N}_p.$$

For small t, let $D_{p,t}$ be the totally geodesic disk in $\mathcal{E}$ of radius δ_1 about $\gamma(t)$ whose exterior normal at $\gamma(t)$ is $\partial_t\gamma(t)$. By strict convexity, $D_{p,0}$ only intersects

Σ_∞ at a single point. There therefore exists $\delta_2>0$ such that, for all $t\in]-\delta_2,0[$, $\Omega_t:=U(\Sigma_\infty)\cap D_{p,t}$ is a convex set and the portion of Σ_∞ lying above Ω_t is a graph over Ω_t. By reducing δ_2 if necessary, there exists $N\in\mathbb{N}$ such that, for all $n\geq N$, and for all $t\in]-\delta_2,0[$, $\Omega_{n,t}:=U(\Sigma_n)\cap D_{p,t}$ is a convex set and the portion of Σ_n lying above $\Omega_{n,t}$ is a graph over $\Omega_{n,t}$. Choose $t\in]-\delta_2,0[$ and for all $n\geq N$, define Σ_n' by replacing the portion of Σ_n lying above $\Omega_{n,t}$ with the smooth graph obtained from Theorem 4.4.

$(f(\Sigma_n'))_{n\in\mathbb{N}}$ is a decreasing sequence and therefore tends towards a $C^{0,1}$ limit, f_∞' in the $C^{0,\alpha}$ sense for all α. For all $n\geq N$, Σ_n' lies below Σ_n. Therefore:

$$f_\infty'\leq f_\infty.$$

We claim that $f_\infty'=f_\infty$. Indeed, suppose that $f_\infty'<f_\infty$, then:

$$\text{Vol}\ (U(f_\infty'))<\text{Vol}\ (U(f_\infty)),$$

which contradicts the minimality of the volume below f_∞. By Theorem 3.6, the portion of $(\Sigma_n')_{n\in\mathbb{N}}$ lying above $\Omega_{n,t}$ converges in the C^∞_{loc} sense to the portion of Σ_∞ lying above $\Omega_{\infty,t}$, which is a non-trivial neighbourhood of p. It follows that Σ_∞ is smooth at p and that $R_\theta(\Sigma_\infty)=r$ near p. Since $p\in\Sigma_\infty$ is arbitrary, the result follows for $\theta>(n-1)\pi/2$ when the quotient Möbius manifold is compact.

Suppose now that $\theta=(n-1)\pi/2$. Let $(\theta_n)_{n\in\mathbb{N}}\in](n-1)\pi/2,n\pi/2[$ be a decreasing sequence converging towards θ. Suppose moreover, that for all n:

$$r>\tan^{-1}(\theta_n/n).$$

For all n, let Σ_n be the immersed hypersurface such that:

$$R_{\theta_n}(\Sigma_n)=r.$$

For all n, let f_n be the function of which Σ_n is the graph and let U_n be the open convex set lying between $\partial_0\mathcal{E}$ and Σ_n. For all $d>0$, let M_d be the level hypersurface at distance d from $\partial_0\mathcal{E}$. By Lemma 3.12, there exists $D>0$ such that, for all n, and for all $d\geq D$, $R_{\theta_n}(M_d)$ is not greater than r. It follows by the Geometric Maximum Principal that, for all n, Σ_n lies below M_D. There therefore exists a convex set U_∞, lying below M_D to which $(U_n)_{n\in\mathbb{N}}$ subconverges in the Haussdorf sense.

Let V be the unit tangent vector field to the vertical foliation of $\mathcal{E}$. For all n, since Σ_n is a graph over $\partial_0\mathcal{E}$, if N_n is the outward unit normal vector to Σ_n, then:

$$\langle V,\mathsf{N}_n\rangle>0.$$

Taking limits, if N_∞ is a supporting normal to U_∞, then:

$$\langle V,\mathsf{N}_n\rangle\geq 0.$$

By Theorem 3.7, the sequence (Σ_n) can only degenerate by converging towards a complete geodesic. If this happens, then the above condition on the supporting normal to U_0 implies one of two possibilities:

(i) either this geodesic is vertical, which is impossible, since Σ_n lies below M_D for all n;

(ii) or this geodesic coincides with $\partial_0\mathcal{E}$, which is excluded by the hypotheses on $\mathcal{E}$.

We thus conclude that Σ_n never degenerates. It follows that the boundary of U_∞ is smooth. Moreover, as before, it is always transverse to V. It follows that $(f_n)_{n\in\mathbb{N}}$ is equicontinuous, and therefore subconverges to a function, f_∞. Since the graph of f_∞ is the boundary of U_∞, f_∞ is smooth and its graph has constant θ-special Lagrangian curvature equal to r. The concludes the proof when the quotient Möbius manifold is compact.

To conclude, we outline the proof in the case when the quotient Möbius manifold is not compact. Let $(U_n)_{n\in\mathbb{N}}$ be an exhaustion of $\partial_0\mathcal{E}$ by relatively compact open sets. For each n, we verify that the Perron method preserves graphs over U_n, and thus, for all n, we obtain a smooth graph over U_n of constant special Lagrangian curvature. Moreover, using the Geometric Maximum Principal, we show that these graphs are uniformly bounded, and thus subconverge to a smooth graph over the whole of $\partial_0\mathcal{E}$ which has the desired properties. The general result now follows. □

Proof of Theorem 1.1. This is the union of Lemmata 4.3 and 4.5. □

Proof of Theorem 1.2. Using Lemma 4.1, these hypersurfaces form a smooth family. Moreover, we can show that the derivative of $i_{r,\theta}$ with respect to r is strictly negative. Thus, if $r' < r$ are close, then $\Sigma_{r,\theta}$ lies strictly below $\Sigma_{r',\theta}$. It follows that this family defines a foliation. By Lemma 3.16, $(\Sigma_{r,\theta})$ converges to $\partial\mathcal{E}$ in the Haussdorf sense as r tends to $+\infty$. Since this concerns the convergence of convex functions, it automatically also implies convergence of the spaces of supporting hyperplanes.

Finally, by Corollary 3.15 and the Geometric Maximum Principle (Lemma 3.11), the distance of $\Sigma_{r,\theta}$ from $\partial_0\mathcal{E}$ is at least R, where:

$$\tanh(R) = \frac{\tan(\theta - (n-1)\pi/2)}{r}.$$

Let $\hat{R}_\theta$ be the maximal value of R which is obtained when $r = \tan(\theta/n)$:

$$\tanh(\hat{R}_\theta) = \frac{\tan(\theta - (n-1)\pi/2)}{\tan(\theta/n)}.$$

This yields a lower bound for the furthest extent of the foliation for each θ. Since $(\theta-(n-1)\pi/2)(\theta/n)$ converges to 1 as θ converges to $n\pi/2$, $\hat{R}_\theta$ converges to ∞ as θ converges to $n\pi/2$ and the result follows. $\square$

Proof of Theorem 1.4. This follows from uniqueness and Lemma 4.1. $\square$

Proof of Theorem 1.8. Let $\tilde{M}$ be the universal cover of M. For all $n\in\mathbb{N}\cup\{0\}$, let $\hat{\imath}_n:\tilde{M}\rightarrow\mathbb{H}^{n+1}$ be the equivariant convex immersion corresponding to $N\partial_0\mathcal{E}(\varphi_n)$. By definition of $C^{0,\alpha}$ Cheeger/Gromov convergence for immersions, it suffices to show that, up to reparametrisation, $(\hat{\imath}_n)_{n\in\mathbb{N}}$ converges to $\hat{\imath}_0$ in the $C^{0,\alpha}$ sense for all α.

Choose $\theta\in](n-1)\pi/2,n\pi/2[$ and $r>\tan^{-1}(\theta/n)$. For all $n\in\mathbb{N}\cup\{0\}$, let $i_{r,n}:\tilde{M}\rightarrow\mathbb{H}^{n+1}$ be the unique equivariant immersion of constant θ-special Lagrangian curvature equal to r which projects asymptotically to φ_n.

For all $n\in\mathbb{N}\cup\{0\}$, we consider $i_{r,n}$ as an immersed submanfiold in $\mathcal{E}(\varphi_n)$ which is a graph over $\partial_0\mathcal{E}(\varphi_n)$. Define i_n such that, for $p\in\tilde{M}$, $i_n(p)$ is the point in $\partial_0\mathcal{E}(\varphi_n)$ lying below $i_{r,n}$. Let $f_n:\tilde{M}\rightarrow[0,\infty[$ be the function of which $i_{r,n}$ is the graph over i_n. By definition, for all $p\in\tilde{M}$:

$$f_n(p)=d(i_n(p),i_{r,n}(p)).$$

By convexity, i_n is a distance decreasing map with respect to the pull back through $i_{n,r}$ of the hyperbolic metric on $\mathbb{H}^{n+1}$. In particular it is 1-Lipschitz and f_n is therefore 2-Lipschitz. Consequently, up to reparametrisation, $(i_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ converge respectively to i_0 and f_0 in the $C^{0,\alpha}$ sense.

By Lemma 3.16, there exists $\epsilon>0$ such that, for all n:

$$f_n\geq\epsilon.$$

Moreover, if Exp denotes the exponential map of $\mathbb{H}^{n+1}$, then, up to reparametrisation, for all $n\in\mathbb{N}\cup\{0\}$ and for all $p\in\tilde{M}$:

$$\hat{\imath}_n(p)=\frac{1}{f_n(p)}\mathrm{Exp}^{-1}_{i_n(p)}(i_{n,r}(p)).$$

Consequently, up to reparametrisation, $(\hat{\imath}_n)_{n\in\mathbb{N}}$ converges to $\hat{\imath}_0$ in the $C^{0,\alpha}$ sense for all α, and the result follows. $\square$

Proof of Theorem 1.9. We continue to use the same notation as in the proof of Theorem 1.8. By definition of $C^{0,\alpha}$ Cheeger/Gromov convergence, to prove the first assertion, it suffices to show that, up to reparametrisation $(g_{KP}(\varphi_n))_{n\in\mathbb{N}}$ converges to $g_{KP}(\varphi_0)$ in the $C^{0,\alpha}$ sense for all α.

For all $n\in\mathbb{N}\cup\{0\}$, let $H_n(p)$ be the hyperspace orthogonal to $\hat{\imath}_n(p)$ in $T_{i_n(p)}\mathbb{H}^{n+1}$, let $g_n(p)$ be the restriction of the hyperbolic metric to $H_n(p)$ and let $\pi_n(p)$ be the orthogonal projection from $T_pi_{r,n}(\tilde{M})$ onto $H_n(p)$. By Theorem 1.8, up to reparametrisation, $(H_n)_{n\in\mathbb{N}}$ converges to H_0 in the $C^{0,\alpha}$ sense for all α, and thus

$(\pi_n)_{n\in\mathbb{N}}$ also converges to π_0 in the $C^{0,\alpha}$ sense for all α. However, for all $n \in \mathbb{N}\cup\{0\}$, up to reparametrisation:

$$g_{KP}(\varphi_n) = \pi_n^* g_n.$$

It follows that $(g_{KP}(\varphi_n))_{n\in\mathbb{N}}$ converges to $g_{KP}(\varphi_0)$ in the $C^{0,\alpha}$ sense for all α, and the first assertion follows. Continuity of D, V and I follows immediately.

Finally, let φ be an FCS over M. Choose $\theta 1](n-1)\pi/2, n\pi/2[$ and $r > \tan^{-1}(\theta/n)$. Let $i_r(\varphi)$ denote the unique equivariant immersion of constant θ-special Lagrangian curvature equal to r which projects asymptotically to φ. By Lemma 4.2, up to reparametrisation $i_r(\varphi)^*g$ and g_{KP} are uniformly equivalent over the space of Flat Conformal Structures over M. The properness of (D, I) and (V, I) now follows from Theorem 3.6 and classical results concerning the compactness of spaces of immersed submanifolds. □

4.4. **Quasi-Fuchsian Manifolds.** Quasi-Fuchsian manifolds provide an interesting special case. For all m, let $\mathbb{H}^m$ be m-dimensional hyperbolic space. Let M be a compact n-dimensional, hyperbolic manifold. We view $\pi_1(M)$ as a subgroup Γ of $\mathrm{Isom}(\mathbb{H}^n)$.

We denote by $\mathrm{Rep}(\mathbb{H}^n, \Gamma)$ the space of pairs (φ, α), where:

(i) $\alpha : \Gamma \to \mathrm{Isom}(\mathbb{H}^{n+1})$ is a properly discontinous representation of Γ in $\mathrm{Isom}(\mathbb{H}^{n+1})$, and

(ii) $\varphi : \partial_\infty\mathbb{H}^n \to \partial_\infty\mathbb{H}^{n+1}$ is an injective, continuous mapping which is equivariant with respect to α.

The set $\mathrm{Rep}(\mathbb{H}^n, \Gamma)$ is a subset of the set of continuous mappings from $\partial_\infty\mathbb{H}^n \cup \Gamma$ into $\partial_\infty\mathbb{H}^{n+1} \cup \mathrm{Isom}(\mathbb{H}^{n+1})$. We furnish this set with the topology of local uniform convergence.

For all n, $\mathbb{H}^n$ embeds totally geodesically into $\mathbb{H}^{n+1}$. This induces a homeomorphism $\alpha_0 : \mathrm{PSO}(n,1) \to \mathrm{PSO}(n+1,1)$ and an injective continuous mapping $\varphi_0 : \partial_\infty\mathbb{H}^n \to \partial_\infty\mathbb{H}^{n+1}$ which is equivariant with respect to α_0. The connected component of $\mathrm{Rep}(\mathbb{H}^n, \Gamma)$ which contains (φ_0, α_0) is called the **quasi-Fuchsian** component. The pair (φ, α) is then said to be **quasi-Fuchsian** if and only if it belongs to the quasi-Fuchsian component.

Let (φ, α) be quasi-Fuchsian. Since $\alpha(\Gamma)$ is properly discontinuous, it defines a quotient manifold $\hat{M}_\alpha = \mathbb{H}^{n+1}/\alpha(\Gamma)$. When $\alpha = \alpha_0$, we call this manifold the **extension** of M. In the sequel, we identify a quasi-Fuchsian pair and its quotient manifold, and we say that a manifold is **quasi-Fuchsian** if and only if it is the quotient manifold of a quasi-Fuchsian pair. In this case it may be isotoped to the extension of a compact, hyperbolic manifold.

Let (φ, α) be quasi-Fuchsian. The image of $\partial_\infty\mathbb{H}^n$ under the action of φ divides $\partial_\infty\mathbb{H}^{n+1}$ into two open, simply connected, connected components. The group

$\alpha(\Gamma)$ acts properly discontinuously on each of these connected components. The quotient of each component is a Möbius manifold homeomorphic to M, and the union of these two quotients forms the ideal boundary of $\hat{M}_\alpha$.

Let K be the convex hull in $\mathbb{H}^{n+1}$ of $\varphi(\partial_\infty\mathbb{H}^n)$. This is the intersection of all closed sets with totally geodesic boundary whose ideal boundary does not intersect $\varphi(\partial_\infty\mathbb{H}^n)$. This set is equivariant under the action of α and thus quotients down to a compact, convex subset of $\hat{M}_\alpha$ which we refer to as the **Nielsen kernel** of $\hat{M}_\alpha$ and which we also denote by K. Trivally $M \setminus K$ consists of two hyperbolic ends arising from FCSs.

Let M be a quasi-Fuchsian manifold, let K be its Nielsen kernel and let D be the diameter of K. Let $\mathcal{E}$ be one of the connected components of $M \setminus K$. Let $\theta \in [(n-1)\pi/2, n\pi/2[$ be an angle. By Theorem 1.1, there exists a family $(\Sigma_r)_{r\in]\tan(\theta/n),\infty[}$ of compact, convex, immersed hypersurfaces in Ω such that, for all r:

(i) $[\Sigma_r]$ is the fundamental class of Ω and

(ii) $R_\theta(\Sigma_r) = r$.

Moreover, this family foliates a neighbourhood of $\partial K \cap \mathcal{E}$. We show that this foliation covers the whole of $\mathcal{E}$:

Lemma 4.6. $(\Sigma_r)_{r\in]\tan(\theta/n),+\infty[}$ *foliates the whole of* $\mathcal{E}$ *and* $\Sigma_r \to \partial_\infty\mathcal{E}$ *in the Hausdorff sense as* $r \to \tan(\theta/n)$.

Proof. Let K_0' be the component of ∂K which does not intersect $\mathcal{E}$ (i.e. K_0' is the boundary component of K lying on the other side of K from Ω). For all $d > 0$, let K_d' be the level hypersurface in $\Omega \cup K$ at a distance of d from K_0'. As in Corollary 3.13, for all $d > 0$, the θ-special Lagrangian curvature of K_d is at most $\tan(\theta/n)/\tanh(d)$ in the weak sense.

For all r, since $\Sigma_r = (S, i_r)$ is compact, there exists a point $p \in S$ such that $d(i_r(p), K_0')$ is minimised. Let d be the distance of $i_r(p)$ from K_0'. Σ is an exterior tangent to K_d at p. By the geometric maximum principal:

$$d(i_r(p), K_0') \geq \mathrm{arctan}h(r^{-1}\tan(\theta/n)) - D.$$

The result now follows. □

The proof of Theorem 1.3 follows immediately:

Proof of Theorem 1.3. This is the union of Theorem 1.1 and Lemma 4.6. □

A - Appendix - On a Result of Kamishima

An earlier revision of this paper relied on a result of Kamishima (Theorem B of [9]) concerning FCSs whose developing maps are not surjective. We discovered that the Kulkarni-Pinkall metric may be used to provide a relatively short proof of this result, which we thus include here.

Let Γ be a subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. The limit set of Γ, $L(\Gamma)$, is the set of all limit points of sequences of the form $(\gamma_n(p))_{n\in\mathbb{N}}$ where $p \in \partial_\infty\mathbb{H}^n$ and $(\gamma_n)_{n\in\mathbb{N}} \in \Gamma$. By definition, this is a closed set. We recall the following important lemma (see, for example [9]):

Lemma A.1 (Chen & Greenberg, [4]). *Let C be a closed subset of $\partial_\infty\mathbb{H}^n$ which contains more than one point and is invariant under Γ, then $L(\Gamma) \subseteq C$.*

This yields the following result of Kamishima:

Theorem A.2 (Kamishima, [9]). *Let M be a closed conformally flat manifold of dimension at least 3. If the developing map is not surjective, then it is a covering map.*

Proof. Let $\tilde{M}$ be the universal cover of M, let $\varphi : \tilde{M} \to \partial_\infty\mathbb{H}^{n+1}$ be its developing map and let $\theta : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^{n+1})$ be its holonomy. We consider the two cases where the complement of $\varphi(\tilde{M})$ contains only one point and where it contains more than one point seperately. Suppose first that $\varphi(\tilde{M})^c$ contains only one point. This point is invariant under the action of $\Gamma := \theta(\pi_1(M))$. Γ is thus conjugate to a subgroup of the symmetry group of Euclidean space. The result then follows by [6]. Suppose now that $\varphi(\tilde{M})^c$ contains more than one point. Since it is closed and invariant under the action of Γ, it follows from Lemma 4.4 that $L(\Gamma) \subseteq \varphi(\tilde{M})$. In other words, $\varphi(\tilde{M}) \subseteq L(\Gamma)^c$. Let g_{KP} be the Kulkarni/Pinkall metric of $L(\Gamma)^c$ (see [13]). since $L(\Gamma)$ contains at least two points, this metric is non-trivial. Moreover, it is complete and invariant under the action of Γ. Thus $\varphi^* g_{KP}$ is invariant under $\pi_1(M)$. Since M is compact, $\varphi^* g_{KP}$ defines a complete metric over $\tilde{M}$. φ is thus a local isometry between complete manifolds, and the result now follows. □

Corollary A.3. *Let M be a closed conformally flat manifold of dimension at least 3. If the developing map φ is not surjective, then $L(\Gamma) = \partial\varphi(\tilde{M})$.*

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