SOME VARIATIONAL PROBLEMS FROM IMAGE PROCESSING

JOHN B. GARNETT, TRIET M. LE, AND LUMINITA A. VESE

Abstract. We consider in this paper a class of variational models introduced for image decomposition into cartoon and texture in [16] (see also [9]), of the form $\inf_u \left\{ |u|_{BV} + \lambda \|K * (f-u)\|_{L^p}^q \right\}$ where K is a real analytic integration kernel. We analyse and characterize the extremals of these functionals and list some of their properties.

1. Introduction and Motivations

A variational model for decomposing a given image-function f into $u+v$ can be given by

$$\inf_{(u,v)\in X_1\times X_2} \Big\{ F_1(u) + \lambda F_2(v) : f = u+v \Big\},$$

where $F_1, F_2 \geq 0$ are functionals and X_1, X_2 are function spaces such that $F_1(u) < \infty$, and $F_2(v) < \infty$, if and only if $(u,v) \in X_1 \times X_2$. The constant $\lambda > 0$ is a tuning (scale) parameter. A good model is given by a choice of X_1 and X_2 so that with the given desired properties of u and v, we have: $F_1(u) \ll F_1(v)$ and $F_2(u) \gg F_2(v)$. The decomposition model is equivalent with:

$$\inf_{u\in X_1} \Big\{ F_1(u) + \lambda F_2(f-u) \Big\}$$

In this work we are interested in the analysis of a class of variational BV models arising in the decomposition of an image function f into cartoon or BV component, and a texture or oscillatory component. This topic has been of much interest in the recent years. We first recall the definition of BV functions.

Definition 1. Let $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ be real. We say $u \in BV$ if

$$\sup\Big\{ \int u \operatorname{div} \varphi dx : \varphi \in C^1_0(\mathbb{R}^d),\ \sup|\varphi(x)| \leq 1 \Big\} = ||u||_{BV} < \infty.$$

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If $u \in BV$ there is an $\mathbb{R}^d$ valued measure $\vec{\mu}$ such that $\frac{\partial u}{\partial x_j} = (\vec{\mu})_j$ as distributions, a positive measure μ, and a Borel function $\vec{\rho}$: $\mathbb{R}^d \to S^{d-1}$ such that

$$Du = \vec{\mu} = \vec{\rho}\mu$$

and

$$||u||_{BV} = \int d\mu.$$

(see Evans-Gariepy [15], for example).

1.1. **History.** Assume $f \in L^2(\mathbb{R}^d)$, f real. We list here several variational BV models that have been proposed for image decomposition models into cartoon and texture.

Rudin-Osher-Fatemi [22] (1992) proposed the minimization

$$\inf_{u \in BV} \Big\{ \|u\|_{BV} + \lambda \int |f-u|^2 dx \Big\}.$$

In this model, we call u a "cartoon" component, and $f-u$ a "noise+texture" component of f, with $f = u + v$. Note that there exists a unique minimizer u by the strict convexity of the functional.

A limitation of this model is illustrated by the following example [20, 12]: let $f = \alpha\chi_D$, $d = 2$, with D a disk centered at the origin and of radius R; if $\lambda R \geq 1/\alpha$, then $u = (\alpha - (\lambda R)^{-1})\chi_D$ and $v = f - u = (\lambda R)^{-1}\chi_D$; if $\lambda R \leq 1/\alpha$, then $u = 0$. Thus, although $f \in BV$ without texture or noise, we do not have $u = f$.

Chan-Esedoglu [11] (2005) considered and analyzed the minimization (see also Alliney [4] for the discrete case)

$$\inf_{u \in BV} \Big\{ \|u\|_{BV} + \lambda \int |f-u| dx \Big\}.$$

The minimizers of this problem exist, but they may not be unique. If $d = 2$, $f = \chi_{B(0,R)}$, then $u = f$ if $R > \frac{2}{\lambda}$ and $u = 0$ if $R < \frac{2}{\lambda}$.

W. Allard [1, 2, 3] (2007) analyzed extremals of

$$\inf_{u \in BV} \Big\{ \|u\|_{BV} + \lambda \int \gamma(u-f) dx \Big\}$$

where $\gamma(0) = 0$, $\gamma \geq 0$, γ locally Lipschitz. Then there exist minimizers u, perhaps not unique, and

$$\partial^*(\{u > t\}) \in C^{1+\alpha}, \quad \alpha \in (0,1)$$

where ∂^* denotes "measure theoretic boundary". Also, Allard gave mean curvature estimates on $\partial^*(\{u > t\})$.

Y. Meyer [20] (2001) in his book *Oscillatory Patterns in Image Processing* analysed further the R-O-F minimization and refined these models proposing

$$\inf_{u \in BV} \Big\{ \|u\|_{BV} + \lambda \|u - f\|_X \Big\}$$

where

$$X = (W^{1,1})^* = \Big\{ \operatorname{div} \vec{g} : \ \vec{g} \in L^\infty \Big\} = G, \quad X = \Big\{ \operatorname{div} \vec{g} : \ \vec{g} \in BMO \Big\} = F,$$

or

$$X = \Big\{ \triangle g : \ g \text{ Zygmund} \Big\} = E.$$

Inspired by the proposals of Y. Meyer, recently a rich literature of models have been proposed and analyzed theoretically and computationally. We list the more relevant ones.

Osher-Vese [25] (2002) proposed

$$\inf_{u,\vec{g}} \Big\{ \|u\|_{BV} + \mu \|f - (u + \operatorname{div} \vec{g})\|_2^2 + \lambda \|\vec{g}\|_p \Big\}, \quad p \to \infty$$

to approximate the (BV, G) Meyer's model and make it computationally amenable. Osher-Solé-Vese [21] proposed the minimization

$$\inf_u \Big\{ \|u\|_{BV} + \lambda \|f - u\|_{H^{-1}} \Big\}$$

and later Lieu and Vese [19] generalized it to

$$\inf_u \Big\{ \|u\|_{BV} + \lambda \|f - u\|_{H^{-s}} \Big\}, \quad s > 0.$$

Similarly, Le-Vese [18] (2005) approximated (BV, F) Meyer's model by

$$\inf_{u,\vec{g}} \Big\{ \|u\|_{BV} + \mu \|f - (u + \operatorname{div} \vec{g})\|_2^2 + \lambda \|\vec{g}\|_{BMO} \Big\}.$$

Aujol et al. [6, 7] addressed the original (BV, G) Meyer's problem and proposed an alternate method to minimize

$$\inf_u \Big\{ \|u\|_{BV} + \lambda ||f - u - v||_2 \Big\},$$

subject to the constraint $||v||_G \leq \mu$.

Garnett-Le-Meyer-Vese [16] (2007) proposed reformulations and generalizations of Meyer's (BV, E) model (see also Aujol-Chambolle [9]), given by

$$\inf_{u,\vec{g}} \Big\{ \|u\|_{BV} + \mu \|f - (u + \triangle \vec{g})\|_2^2 + \lambda \|\vec{g}\|_{\dot{B}^{\alpha}_{p,q}} \Big\}$$

where $1 \leq p, q \leq \infty$, $0 < \alpha < 2$, and exact decompositions from

$$\inf_u \Big\{ \|u\|_{BV} + \lambda \|f - u\|_{\dot{B}^{\alpha-2}_{p,q}} \Big\}.$$

In a subsequent work, Garnett-Jones-Le-Meyer [17] proposed different formulations,

$$\inf_{u,\vec{g}} \Big\{ \|u\|_{BV} + \mu \|f - (u + \triangle \vec{g})\|_2^2 + \lambda \|\vec{g}\|_{B\dot{M}O^\alpha} \Big\},$$

with $B\dot{M}O^\alpha = I_\alpha(BMO)$, $\|v\|_{B\dot{M}O^\alpha} = \|I_\alpha v\|_{BMO}$, and

$$\inf_{u,\vec{g}} \Big\{ \|u\|_{BV} + \mu \|f - (u + \triangle \vec{g})\|_2^2 + \lambda \|\vec{g}\|_{\dot{W}^{\alpha,p}} \Big\},$$

with $\|v\|_{\dot{W}^{\alpha,p}} = \|I_\alpha v\|_p$, $0 < \alpha < 2$.

Generalizing (BV, H^{-s}), $(BV, \dot{B}^{\alpha}_{p,q})$, and the $TV-$Hilbert model [8], an easier cartoon+texture decomposition model can be defined using a smoothing convolution kernel K (previously introduced in [16]):

$$\inf_{u \in BV} \Big\{ \|u\|_{BV} + \lambda \|K * (f-u)\|^q_{L^p} \Big\}. \tag{1}$$

This can be seen as a simplified version of all the previous models.

2. The Variational Problems

In this paper we assume K is a positive, even, bounded and real analytic kernel on $\mathbb{R}^d$ such that $\int K dx = 1$ and such that $L^p \ni u \to K * u$ is injective. For example we may take K to be a Gaussian or a Poisson kernel. We fix $\lambda > 0$, $1 \leq p < \infty$ and $1 \leq q < \infty$. For compactly supported real $f(x) \in L^1$ we consider the extremal problems

$$m_{p,q,\lambda} = \inf\{||u||_{BV} + \mathcal{F}_{p,q,\lambda}(f-u) : u \in BV\} \tag{2}$$

where

$$\mathcal{F}_{p,q,\lambda}(h) = \lambda ||K * h||^q_{L^p}. \tag{3}$$

Since $BV \subset L^{\frac{d}{d-1}}$ and $K \in L^\infty$, a weak-star compactness argument shows that (2) has at least one minimizer u. Our objective is to describe, given f, the set $\mathcal{M}_{p,q,\lambda}(f)$ of minimizers u of (2) .

The papers of Chan-Esedoglu [11] and Allard [1, 2, 3] give very precise results about the minimizers for variations like (2) but without the real analytic kernel K, and this paper is intended to complement those works.

2.1. Convexity.

Since the functional in (2) is convex, the set of minimizers $\mathcal{M}_{p,q,\lambda}(f)$ is a convex subset of BV. If $p > 1$ or if $q > 1$, then the functional (3) is strictly convex and the problem (2) has a unique minimizer because $K * u$ determines u.

Lemma 1. *If $p = q = 1$ and if $u_1 \in \mathcal{M}_{p,q,\lambda}$ and $u_2 \in \mathcal{M}_{p,q,\lambda}$, then*

$$\frac{K * (f-u_1)}{|K * (f-u_1)|} = \frac{K * (f-u_2)}{|K * (f-u_2)|} \text{ almost everywhere,} \tag{4}$$

and

$$\vec{\rho}_k \cdot \frac{d\vec{\mu}_j}{d\mu_k} = \Big|\frac{d\vec{\mu}_j}{d\mu_k}\Big|, \ j \neq k, \tag{5}$$

where for $j = 1, 2$,

$$Du_j = \vec{\mu}_j = \vec{\rho}_j \mu_j$$

with $|\vec{\rho}_j| = 1$ and $\mu_j \geq 0$.

Proof: Since $\frac{u_1+u_2}{2}$ is also a minimizer, we have

$$\Big|\Big|K * \Big(f - \frac{u_1+u_2}{2}\Big)\Big|\Big|_1 = \frac{1}{2}\big(||K * (f-u_1)||_1 + ||K * (f-u_2)||_1\big),$$

which implies (4), and

$$\int \Big|\rho_k + \frac{d\vec{\mu}_j}{\mu_k}\Big| d\mu_k = \int d\mu_k + \int \Big|\frac{d\vec{\mu}_j}{\mu_k}\Big| d\mu_k, \quad j \neq k,$$

which implies (5). □

2.2. Properties of extremals $u \in \mathcal{M}_{p,q,\lambda}(f)$.

Lemma 2. *Let u be a minimizer of (2) and assume $u \neq f$. Let $h \in BV$ be real, write*

$$Dh = \vec{\nu}$$

and

$$\vec{\nu} = \frac{d\vec{\nu}}{d\mu}\mu + \vec{\nu}_s$$

for the Lebesgue decomposition of $\vec{\nu}$ with respect to μ. Then

$$\Big|\int \rho \cdot \frac{d\vec{\nu}}{d\mu} d\mu - \lambda \int h(K * J_{p,q})dx\Big| \leq ||\vec{\nu}_s||, \tag{6}$$

where

$$J_{p,q} = \frac{F|F|^{p-2}}{||F||_p^{p-q}}, \tag{7}$$

$$F = K * (f-u) \tag{8}$$

and $||\vec{\nu}_s||$ denotes the norm of the vector measure $\vec{\nu}_s$. Conversely, if $u \in BV$, $u \neq f$ and (6), (7) and (8) hold, then $u \in \mathcal{M}_{p,q,\lambda}(f)$.

Note that since $u \neq f$ and $K * (f-u)$ is real analytic, $J_{p,q}$ is defined almost everywhere.

Proof: Let $|\epsilon|$ be small. Then since u is extremal,

$$||u+\epsilon h||_{BV} - ||u||_{BV} + \mathcal{F}_{p,q,\lambda}(f-u-\epsilon h) - \mathcal{F}_{p,q,\lambda}(f-u) \geq 0.$$

But

$$\begin{aligned} ||u+\epsilon h||_{BV} - ||u||_{BV} &= |\epsilon|\, ||\nu_s|| + \int \Big(\Big|\rho + \epsilon \frac{d\nu}{d|\mu|}\Big| - 1\Big) d\mu \\ &= |\epsilon|\, ||\nu_s|| + \epsilon \int \rho \cdot \frac{d\nu}{d\mu} d\mu + o(|\epsilon|) \end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_{p,q,\lambda}(f-u-\epsilon h)-\mathcal{F}_{p,q,\lambda}(f-u) &= -q\lambda\epsilon\int (K*h)J_{p,q}dx+o(|\epsilon|)\\ &= -q\lambda\epsilon\int h(K*J_{p,q})dx+o(|\epsilon|)\end{aligned}$$

since K is even. Taking $\pm\epsilon$, we see that (6) holds.

The converse holds because the functional (3) is convex. □

Following Meyer [20], define

$$||v||_* = \inf\Big\{\Big\|\Big(\sum_{j=1}^{d}|u_j|^2\Big)^{\frac{1}{2}}\Big\|_\infty : v=\sum_{j=1}^{d}\frac{\partial u_j}{\partial x_j}\Big\}$$

and note that $||v||_*$ is the norm of the dual of $W^{1,1}\subset BV$, when $W^{1,1}$ is given the norm of BV. By the weak-star density of $W^{1,1}$ in BV,

$$\Big|\int hvdx\Big| \le ||h||_{BV}||v||_* \tag{9}$$

whenever $v\in L^2$. Still following Meyer [20] we have:

Lemma 3. *Let $u\in BV$ and assume $u\neq f$. Then u is a minimizer for the problem (2) if and only if*

$$||K*J_{p,q}||_* = \frac{1}{\lambda} \tag{10}$$

and

$$\int u(K*J_{p,q})dx = \frac{1}{\lambda}||u||_{BV}. \tag{11}$$

Proof: If u is a minimizer, we use Lemma 2. For any $h\in W^{1,1}$, (6) yields

$$||K*J_{p,q}||_* \le \frac{1}{\lambda}.$$

By (9)

$$\Big|\int u(K*J_{p,q})dx\Big| \le ||u||_{BV}||K*J_{p,q}||_*,$$

and by setting $h=u$ in (6), we obtain

$$\lambda\int u(K*J_{p,q})dx = ||u||_{BV}.$$

Therefore (10) and (11) hold.

Conversely, assume $u \in BV$ satisfies (10) and (11) and note that u determines $J_{p,q}$. Still following Meyer [20], we let $h \in BV$ be real. Then for small $\epsilon > 0$, (9), (10) and (11) give

$$\begin{aligned}
&||u + \epsilon h||_{BV} + \lambda ||K * (f - u - \epsilon h)||_1 \\
&\qquad \geq \lambda \int (u + \epsilon h)(K * J_{p,q}) dx + \lambda ||K * (f - u)||_1 \\
&\qquad\quad - \epsilon\lambda \int h(K * J_{p,q}) dx + o(\epsilon) \\
&\qquad = ||u||_{BV} + \epsilon\lambda \int h(K * J_{p,q}) dx - \epsilon\lambda \int h(K * J_{p,q}) dx + o(\epsilon) \\
&\qquad \geq 0.
\end{aligned}$$

Therefore u is a local minimizer for the functional (2), and by convexity that means u is a global minimizer.

2.3. **Radial Functions.** Assume K is radial, $K(x) = K(|x|)$. Also assume f is radial and $f \notin \mathcal{M}_{p,q,\lambda}(f)$. Then averaging over rotations shows that each $u \in \mathcal{M}_{p,q,\lambda}(f)$ is radial, so that

$$Du = \rho(|x|) \frac{\vec{x}}{|x|} \mu$$

where μ is invariant under rotations and where $\rho(|x|) = \pm 1$ a.e. $d\mu$. Let $H \in L^1(\mu)$ be radial and satisfy $\int H d\mu = 0$ and $H = 0$ on $|x| < \epsilon$, and define

$$h(x) = \int_{B(0,|x|)} H(|y|) \frac{1}{|y|^{d-1}} d\mu.$$

Then $h \in BV$ is radial and

$$Dh = \vec{\nu} = H(|x|) \frac{\vec{x}}{|x|} \mu.$$

Consequently $\vec{\nu}_s = 0$ and (6) gives

$$\begin{aligned}
\int \rho H d\mu &= \lambda \int K * J_{p,q}(x) \int_{B(0,|x|)} \frac{H(y)}{|y|^{d-1}} d\mu(y) dx \\
&= \lambda \int \Big(\int_{|x|>|y|} K * J_{p,q}(x) dx \Big) \frac{H(|y|)}{|y|^{d-1}} d\mu(y)
\end{aligned}$$

so that a.e. $d\mu$,

$$\rho(y) = \frac{\lambda}{|y|^{d-1}} \int_{|x|>|y|} K * J_{p,q}(x) dx. \tag{12}$$

But the right side of (12) is real analytic in $|y|$, with a possible pole at $|y| = 0$, and $\rho(|y|) = \pm 1$ almost everywhere μ. Therefore there is a finite set

$$\{r_1 < r_2 < \cdots < r_n\} \tag{13}$$

of radii such that

$$Du = \frac{x}{|x|} \sum_{j=1}^{n} c_j \Lambda_{d-1} | \{|x| = r_j\}$$

for real constants $c_1, \dots, c_n$, where Λ_{d-1} denotes $d-1$ dimensional Hausdorff measure. By Lemma 1, $J_{p,q}$ is uniquely determined by f, and hence the set (13) is also unique. Moreover, it follows from Lemma 1 that for each j, either $c_j \geq 0$ for all $u \in \mathcal{M}_{p,1,\lambda}(f)$ or $c_j \leq 0$ for all $u \in \mathcal{M}_{p,1,\lambda}(f)$. We have proved:

Theorem 1. *If K is radial, if f is radial and if $f \notin \mathcal{M}_{p,q,\lambda}(f)$, then there is a finite set (13) such that all $u \in \mathcal{M}_{p,q,\lambda}(f)$ have the form*

$$\sum_{j=1}^{n} c_j \chi_{B(0,r_j)}. \tag{14}$$

Moreover, there is $X^+ \subset \{1, 2, \dots, n\}$ such that $c_j \geq 0$ if $j \in X^+$ while $c_j \leq 0$ if $j \notin X^+$.

Note that by convexity $\mathcal{M}_{p,q,\lambda}(f)$ consists of a single function unless $p = q = 1$. In Section 2.6 we will say more about the solutions of the form (14).

2.4. **Example.** Unfortunately, Theorem 1 does not hold more generally. The reason is that when u is not radial it is difficult to produce BV functions satisfying $\vec{\nu} \ll \mu$. For simplicity we take $d = 2$ and $p = q = 1$. Let $J = J_{1,1} = \chi_{0<x\leq 1} - \chi_{-1<x\leq 0}$ and $J(x+2, y) = J(x, y)$. Choose $\lambda > 0$ so that $U = \lambda K * J$ satisfies $||U||_* = 1$, and note that $\frac{U}{|U|} = J$. Notice that $u \in C^2$ solves the curvature equation

$$\operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) = U \tag{15}$$

if and only if the level sets $\{u = a\}$ are curves $y = y(x)$ that satisfy the simple ODE $y'' = U(x, 0)(1 + (y')^2)^{3/2}$ on the line. Consequently (15) has infinitely many solutions u and then u and J satisfy (10) and (11). Hence by Lemma 3 u is a minimizer for f provided that

$$J = \frac{K * (f-u)}{|K * (f-u)|} \tag{16}$$

and there are many f that satisfy (16). Note that in this example u can be real analytic except on $U^{-1}(0)$ and not piecewise constant. Similar examples can be made when $(p, q) \neq (1, 1)$.

2.5. **Properties of Minimizers when q =1.** Here we follow the paper of Strang [24].

Lemma 4. *If $q = 1$ and $u \in \mathcal{M}_{p,1,\lambda}(f)$, then $u \in \mathcal{M}_{p,1,\lambda}(u)$.*

Proof: If

$$||h||_{BV} + \lambda ||K * (u-h)||_p < ||u||_{BV},$$

then by the triangle inequality

$$||h||_{BV} + \lambda ||K * (f-h)||_p < ||u||_{BV} + \lambda ||K * (f-u)||_p$$

so that u is not a minimizer for f. □

We write

$$\mathcal{M} = \mathcal{M}_{p,1,\lambda} = \bigcup_f \mathcal{M}_{p,1,\lambda}(f).$$

Lemma 5. *Let $u \in BV$. Then $u \in \mathcal{M}$ if and only if*

$$\Big| \int \rho \cdot \frac{d\vec{\nu}}{d\mu} d\mu \Big| \leq ||(\vec{\nu})_s|| + \lambda ||K * h||_p \tag{17}$$

for all $h \in BV$, where $Dh = \vec{\nu}$.

This follows like the proof of Lemma 2.
Let $a < b$ be such that

$$\mu(\{u = a\} \cup \{u = b\}) = 0. \tag{18}$$

Then $u_{a,b} = \text{Min } \{(u-a)^+, (b-a)\} \in BV$ and $D(u_{a,b}) = \chi_{a<u<b}\, \vec{\rho}\mu$.

Lemma 6. *Assume $q = 1$.*
(a) If $u \in \mathcal{M}$, then $u_{a,b} \in \mathcal{M}$.
(b) More generally, if $u \in \mathcal{M}$ and if $v \in BV$ satisfies $\mu_v \ll \mu_u$ and $\rho_v = \rho_u$ a.e. $d\mu_v$, then $v \in \mathcal{M}$.

Proof: To prove (a) we verify (5). Write $\mu_{a,b} = \chi_{(a,b)}\mu$ so that $D(u_{a,b}) = \vec{\rho}\mu_{a,b}$. Let $h \in BV$ and write $Dh = \vec{\nu}$. Then by (18)

$$\vec{\nu} = \chi_{a<u<b} \frac{d\vec{\nu}}{d\mu}\mu + \Big((\vec{\nu})_s + \chi_{u(x) \notin [a,b]} \frac{d\vec{\nu}}{d\mu}\mu \Big)$$

is the Lebesgue decomposition of $\vec{\nu}$ with respect to $\mu_{a,b}$, and

$$\int \vec{\rho} \cdot \frac{d\vec{\nu}}{d\mu_{a,b}} d\mu_{a,b} = \int \vec{\rho} \cdot \frac{d\vec{\nu}}{d\mu} d\mu - \int_{g(x) \notin [a,b]} \vec{\rho} \cdot \frac{d\vec{\nu}}{d\mu} d\mu.$$

Then (5) for ν and $\mu_{a,b}$ follows from (5) for μ and ν. The proof of (b) is similar. □

For simplicity we assume $u \geq 0$. Write $E_t = \{x : u(x) > t\}$. Then by Evans-Gariepy [15], E_t has finite perimeter for almost every t,

$$||u||_{BV} = \int_0^\infty ||\chi_{E_t}||_{BV} dt, \tag{19}$$

and

$$u(x) = \int_0^\infty \chi_{E_t}(x)dt. \tag{20}$$

Moreover, almost every set E_t has a *measure theoretic boundary* $\partial_* E_t$ such that

$$\Lambda_{d-1}(\partial_* E_t) = ||\chi_{E_t}||_{BV} \tag{21}$$

and a *measure theoretic outer normal* $\vec{n_t}\colon \partial_* E_t \to S^{d-1}$ so that

$$D(\chi_{E_t}) = \vec{n_t}\Lambda_{d-1}\Big|\partial_* E_t. \tag{22}$$

Theorem 2. *Assume $q = 1$.*
(a) If $u \in \mathcal{M}$, then for almost every t, $\chi_{E_t} \in \mathcal{M}$.
(b) If $u \in \mathcal{M}$ and $u \geq 0$, then for all nonnegative $c_1, \ldots, c_n$ and for almost all $t_1 < \cdots < t_n$, $\sum c_j \chi_{E_{t_j}} \in \mathcal{M}$.

Proof: Suppose (a) is false. Then there is $\beta < 1$, and a compact set $A \subset (0, \infty)$ with $|A| > 0$ such that for all $t \in A$ (21) and (22) hold and there exists $h_t \in BV$ such that

$$||\chi_{E_t} - h_t||_{BV} + \lambda||K * h_t||_p \leq \beta||\chi_{E_t}||_{BV}. \tag{23}$$

Choose an interval $I = (a, b)$ such that (18) holds and $|I \cap A| \geq \frac{|I|}{2}$. Define $h_t = 0$ for $t \in I \setminus A$, and take finite sums such that

$$\sum_{j=1}^{N_n} \chi_{E_{t_j^{(n)}}} \Delta t_j^{(n)} \to u_{a,b} \ (n \to \infty), \tag{24}$$

$$\sum_{j=1}^{N_n} ||\chi_{E_{t_j^{(n)}}}||_{BV} \Delta t_j^{(n)} \to ||u_{a,b}|| \ (n \to \infty), \tag{25}$$

and $t_j^{(n)} \in A$ whenever possible. Write $h^{(n)} = \sum_{j=1}^{N_n} h_{t_j^{(n)}} \Delta t_j^{(n)}$. Then by (20) and (23) $\{h^{(n)}\}$ has a weak-star limit $h \in BV$, and by (23), (24) and (25),

$$||u_{a,b} - h||_{BV} + \lambda||K * h||_p \leq \frac{1+\beta}{2}||u_{a,b}||_{BV},$$

contradicting Lemma 6. The proof of (b) is similar. $\square$

We suspect that the converse of Theorem 2 is false, but we have no counterexample.

2.6. **Radial Minimizers.** In this section we assume $q = 1$ and $p = 1$. For convenience we assume the kernel $K = K_t$ is Gaussian, so that K has the form

$$K_t(x) = t^{-d} K(\frac{x}{t}) \tag{26}$$

and

$$K_s * K_t = K_{\sqrt{s^2+t^2}}. \tag{27}$$

Note that (26) and (27) imply that

$$||K_t * f||_1 \text{ decreases in } t \tag{28}$$

and for $f \in L^1$ with compact support

$$\lim_{t\to\infty} ||K_t * f||_1 = |\int f dx|. \tag{29}$$

For fixed λ and t we set

$$R(\lambda, t) = \{r > 0 : \chi_{B(0,r)} \in \mathcal{M}\}.$$

By Theorem 1 and Theorem 2 we have $R(\lambda, t) \neq \emptyset$. For $t = 0$ and $K = I$ our problem (2) becomes the problem

$$\inf\{||u||_{BV} + \lambda||f - u||_{L^1}\}$$

studied by Chan and Esedoglu in [11], and in that case Chan and Esedoglu showed $R(\lambda, 0) = [\frac{2}{\lambda}, \infty)$.

Theorem 3. *There exists* $r_0 = r_0(\lambda, t)$ *such that*

$$R(\lambda, t) = [r_0, \infty). \tag{30}$$

Moreover

$$[0, \infty) \ni t \to r_0(t) \text{ is nondecreasing} \tag{31}$$

and

$$\lim_{t\to\infty} r_0(t) = \infty. \tag{32}$$

Proof: Assume $r \notin R(\lambda, t)$ and $0 < s < r$. Write $\alpha = \frac{r}{s} > 1$ and $f = \chi_{B(0,r)}$. By hypothesis there is $g \in BV$ such that

$$||g||_{BV} + \lambda||K_t * (f - g)||_1 < ||f||_{BV}. \tag{33}$$

We write $\tilde{g}(x) = g(\alpha x)$, $\tilde{f}(x) = f(\alpha x) = \chi_{B(0,s)}(x)$, and change variables carefully in (33) to get

$$\alpha||\tilde{g}||_{BV} + \lambda||\frac{1}{t^d}\int K\big(\frac{x-y}{t}\big)(\tilde{f} - \tilde{g})(\frac{y}{\alpha})dy||_{L^1(x)} < \alpha||\tilde{f}||_{BV}$$

so that

$$\alpha||\tilde{g}||_{BV} + \lambda||\frac{\alpha^d}{t^d}\int K\big(\frac{\alpha x' - \alpha y'}{t}\big)(\tilde{f} - \tilde{g})(y')dy'||_{L^1(\alpha x')} < \alpha||\tilde{f}||_{BV}$$

and

$$\alpha||\tilde{g}||_{BV} + \lambda\alpha^d \int \Big| K_{\frac{t}{\alpha}} * (\tilde{f} - \tilde{g})(x') \Big| dx' < \alpha||\tilde{f}||_{BV}.$$

Since $\alpha > 1$, this and (28) show

$$||\tilde{g}||_{BV} + \lambda||K_t * (\tilde{f} - \tilde{g})||_1 < ||\tilde{f}||_{BV}$$

so that $s \notin R(\lambda, t)$. That proves (30), and (31) now follows easily from (28). To prove (32) take $g = \frac{r^d}{s^d}\chi_B(0,s),\ s > r$ and use (29). □

We note that not all radial minimizers have the form $\chi_{B(0,r)}$. This is seen by considering, for fixed t and λ, the function $\chi_{B(0,r_2)} + \chi_{B(0,r_1)}$ with r_1 and $r_2 - r_1$ large.

2.7. **Characteristic Functions.** Still assuming $q = 1$ we let E be such that $\chi_E \in \mathcal{M}$. Then by Evans-Gariepy [15] $\partial_* E = N \cup \bigcup K_j$, where $D(\chi_E)(N) = \Lambda_{n-1}(N) = 0$, K_j is compact and $K_j \subset S_j$, where S_j is a C^1-hypersurface with continuous unit normal $\vec{n_j}(x), x \in S_j$, and $\vec{n_j}$ is the measure theoretic outer normal of E. After a coordinate change write $S_j = \{x_d = f_j(y)\}, y = (x_1, \ldots, x_{d-1})$ with ∇f_j continuous and $\vec{n_j}(y, f_j(y)) \perp (\nabla f_j, 1)$. Assume $y = 0$ is a point of Lebesgue density of $(f_j, 1)^{-1}(K_j)$, let $V \subset \mathbb{R}^{d-1}$ be a neighborhood of $y = 0$, let $g \in C_0^\infty(V)$ with $g \geq 0$, and consider the variation $u_\epsilon = \chi_{E_\epsilon}$ where $\epsilon > 0$ and

$$E_\epsilon = E \cup \{0 \leq x_d \leq \epsilon u(y), y \in V\}.$$

Then $E \subset E_\epsilon$, and writing $u_0 = \chi_E$, we have

$$||u_\epsilon||_{BV} - ||u_0||_{BV} = \int_V \sqrt{(1 + |\nabla(f_j + \epsilon g)|^2)} - \sqrt{(1 + |\nabla f_j|^2)} dy = o(\epsilon) \tag{34}$$

because by [15]

$$\Lambda_{d-1}((\partial_* E) \cup (E_\epsilon \setminus E)) = o(\epsilon)$$

Λ_{d-1} a.e. on K_j. Also, for a similar reason

$$\lambda||K * (u_\epsilon - u_0)||_p = \lambda|\epsilon| \int_V u dy + o(\epsilon). \tag{35}$$

Together (34) and (34) show

$$\int_V \nabla u \cdot \left(\frac{\nabla f_j}{\sqrt{1 + |\nabla f_j|^2}} \right) dy + \lambda \int_V u dy \geq 0.$$

Repeating this argument with $\epsilon < 0$, we obtain:

Theorem 4. *At Λ_{d-1} almost every $x \in \partial_* E$,*

$$\left| \operatorname{div} \left(\frac{\nabla f_j}{\sqrt{1 + |\nabla f_j|^2}} \right) \right| \leq \lambda. \tag{36}$$

as a distribution on $\mathbb{R}^{d-1}$.

2.8. **Smooth Extremals.** For convenience we assume $d = 2$ and we take $p = q = 1$.

Theorem 5. *Let* $u \in C^2 \cap \mathcal{M}_{1,1,\lambda}(f)$ *and assume* $u \neq f$*. Set* $E_t = \{u > t\}$ *and* $J = \frac{K*(f-u)}{|K*(f-u)|}$*. Then*

(i) $\Lambda_1(\partial_* E_t) = \lambda \iint_{E_t} K * J dxdy$,

(ii) *the level curve* $\{u(z) = c\}$ *has curvature* $\lambda(K * J)(z)$,

and

(iii) *if* $|\nabla u| \neq 0$*, then*

$$\frac{d}{dt}\Lambda_1(\partial_* E_t) = -\int_{\partial E_t} \frac{\lambda(K * J)(z)}{|\nabla u(z)|} ds.$$

Theorem 5 is proved using the variation $u \to u + \epsilon h, h \in C_0^2$. It should be true in greater generality, but we have no proof at this time.

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John B. Garnett
Department of Mathematics
University of California, Los Angeles
405 Hilgard Ave, Los Angeles
CA 90095-1555, USA
E-mail address: jbg@math.ucla.edu

Triet M. Le
Department of Mathematics
Yale Unversity, 10 Hillhouse Ave
New Haven, CT 06511, USA
E-mail address: triet.le@yale.edu

Luminita A. Vese
Department of Mathematics
University of California, Los Angeles
405 Hilgard Ave, Los Angeles
CA 90095-1555, USA
E-mail address: lvese@math.ucla.edu