A CHARACTERIZATION OF TWO WEIGHT NORM INEQUALITIES FOR MAXIMAL SINGULAR INTEGRALS

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Abstract. Let $\sigma$ and $\omega$ be positive Borel measures on $\mathbb{R}^n$ and let $1 < p < \infty$. We characterize boundedness of (dual pairs of) certain maximal singular integrals $T_\natural$ from $L^p(\sigma)$ to $L^p(\omega)$ in terms of two testing conditions. The first applies to a restricted class of functions and is a strong-type testing condition,

$$
\int_Q T_\natural(\chi_E \sigma)(x)^p d\omega(x) \leq C_1 \int_Q d\sigma(x), \quad \text{for all } E \subset Q,
$$

and the second is a weak-type or dual cube testing condition,

$$
\int_Q T_\natural(\chi_Q f \sigma)(x) d\omega(x) \leq C_2 \left( \int_Q |f(x)|^p d\sigma(x) \right)^{\frac{1}{p}} \left( \int_Q d\omega(x) \right)^{\frac{1}{p'}},
$$

for all cubes $Q$ in $\mathbb{R}^n$ and all functions $f \in L^p(\sigma)$. We also characterize the weak-type two weight inequality for $T_\natural$ in terms of the second condition and the $A_p$ condition.

1. Introduction

Two weight inequalities for Maximal Functions and other positive operators have been characterized in [17], [16], [18], with these characterizations being given in terms of obviously necessary conditions, that the operators be uniformly bounded on a restricted class of functions, namely indicators of intervals and cubes. Thus, these characterizations have a form reminiscent of the $T1$ Theorem of David and Journé.

Corresponding results for even the Hilbert transform are not known, and evidently much harder to obtain. We comment in more detail on prior results below, including the innovative work of Nazarov, Treil and Volberg [7], [8], [9], [10].

Our focus is on providing characterizations of the boundedness of certain maximal truncations of a fixed operator of singular integral type. The singular integrals will be of the usual type, for example the Hilbert transform, or a generalized fractional integral, for instance the Cauchy transform in the plane. The characterizations are in terms of certain obviously necessary conditions, in which the class of functions being tested is simplified. For such examples, we prove unconditional

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characterizations of both strong-type and weak-type two weight inequalities for certain maximal truncations of these integrals.

As the precise statements of our characterizations are somewhat complicated, we illustrate them with two important cases here. Let $T$ denote either the Hilbert or Beurling transform, let $T_\sigma$ denote the usual maximal singular integral associated with $T$, and finally let $T_\natural$ denote the new strongly maximal singular integral associated with $T$ that is defined below. Suppose $\sigma$ and $\omega$ are two locally finite positive measures. Then we have the following weak and strong type characterizations which we emphasize hold for all $1 < p < \infty$.

- If $\sigma$ and $\omega$ have no point masses in common, then the operator $T_\flat$ is weak type $(p, p)$ with respect to $(\sigma, \omega)$, i.e.

$$\| T_\flat (f_\sigma) \|_{L^p, \infty(\omega)} \leq C \| f \|_{L^p(\sigma)},$$

for all $f$ bounded with compact support, if and only if the two weight $A_p$ condition

$$\frac{1}{|Q|} \int_Q d\omega \left( \frac{1}{|Q|} \int_Q d\sigma \right)^{p-1} \leq C,$$

holds for all cubes $Q$; and the dual $T_\flat$ cube testing condition

$$\int_Q T_\flat (\chi_Q f_\sigma) d\omega \leq C \left( \int_Q |f|^p d\sigma \right)^{\frac{1}{p}} \left( \int_Q d\omega \right)^{\frac{1}{p}},$$

holds for all cubes $Q$ and $f \in L^p_Q(\sigma)$ (part 4 of Theorem 1.19). The same is true for $T_\natural$. It is easy to see that (1.2) is equivalent to the more familiar dual cube testing condition

$$\int_Q \left| L^* \left( \chi_Q \omega \right) \right|^{p'} d\sigma \leq C \int_Q d\omega,$$

for all cubes $Q$ and linearizations $L$ of the maximal singular integral $T_\flat$ (see (2.14)).

- The operator $T_\natural$ is both strong type $(p, p)$ with respect to $(\sigma, \omega)$ and strong type $(p', p')$ with respect to $(\omega, \sigma)$, i.e.

$$\| T_\natural (f_\sigma) \|_{L^p(\omega)} \leq C \| f \|_{L^p(\sigma)},$$

$$\| T_\natural (h_\omega) \|_{L^{p'}(\sigma)} \leq C \| h \|_{L^{p'}(\omega)},$$

for all $f$ and $h$ bounded with compact support, if and only if the dual $T_\natural$ cube testing conditions

$$\int_Q T_\natural (\chi_Q f_\sigma) d\omega \leq C \left( \int_Q |f|^p d\sigma \right)^{\frac{1}{p}} \left( \int_Q d\omega \right)^{\frac{1}{p}},$$

$$\int_Q T_\natural (\chi_Q h_\omega) d\sigma \leq C \left( \int_Q |h|^{p'} d\omega \right)^{\frac{1}{p'}} \left( \int_Q d\sigma \right)^{\frac{1}{p}},$$
hold for all cubes $Q$ and $f \in L^p_Q(\sigma), h \in L^p_Q(\omega)$; and the forward $T_\sharp$ testing conditions

\begin{align}
\int_Q T_\sharp (\chi_{E\sigma})^p d\omega &\leq C \int_Q d\sigma, \\
\int_Q T_\sharp (\chi_{E\omega})^{p'} d\sigma &\leq C \int_Q d\omega,
\end{align}

(1.4)

hold for all cubes $Q$ and all compact subsets $E$ of $Q$ (part 4 of Theorem 1.24). Note that in (1.4) we are required to test over all compact subsets $E$ of $Q$ on the left side, but retain the upper bound over the (larger) cube $Q$ on the right side.

As these results indicate, the imposition of the weight $\sigma$ on both sides of (1.1) is a standard part of weighted theory, in general necessary for the testing conditions to be sufficient. Compare to the characterization of the two weight maximal function inequalities in Theorem 1.6 below.

Problem 1.5. In (1.4), our testing condition is more complicated than one would like, in that one must test over all compact $E \subset Q$ in (1.4). There is a corresponding feature of (1.2), seen after one unwinds the definition of the linearization $L^*$.

We do not know if these testing conditions can be further simplified. The form of these testing conditions is dictated by our use of what we call the ‘maximum principle’, see Lemma 2.9.

We now recall the two weight inequalities for the Maximal Function as they are central to the new results of this paper. Define the Maximal Function

\[ \mathcal{M} \nu(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |\nu|, \quad x \in \mathbb{R}, \]

where the supremum is taken over all cubes $Q$ (by which we mean cubes with sides parallel to the coordinate axes) containing $x$.

Theorem on Maximal Function Inequalities 1.6. Suppose that $\sigma$ and $\omega$ are positive locally finite Borel measures on $\mathbb{R}^n$, and that $1 < p < \infty$. The maximal operator $\mathcal{M}$ satisfies the two weight norm inequality (16)

\[ \| \mathcal{M}(f \sigma) \|_{L^p(\omega)} \leq C \| f \|_{L^p(\sigma)}, \quad f \in L^p(\sigma), \]

if and only if for all cubes $Q \subset \mathbb{R}^n$,

\[ \int_Q \mathcal{M} (\chi_Q \sigma) (x)^p d\omega(x) \leq C_1 \int_Q d\sigma(x). \]

The maximal operator $\mathcal{M}$ satisfies the weak-type two weight norm inequality (5)

\[ \| \mathcal{M}(f \sigma) \|_{L^{p,\infty}(\omega)} \equiv \sup_{\lambda > 0} \lambda \left\{ \mathcal{M}(f \sigma) > \lambda \right\}^{\frac{1}{p}} \leq C \| f \|_{L^p(\sigma)}, \quad f \in L^p(\sigma), \]
if and only if the two weight $A_p$ condition holds for all cubes $Q \subset \mathbb{R}^n$:

$$\left[ \frac{1}{|Q|} \int_Q d\omega \right]^\frac{1}{p} \left[ \frac{1}{|Q|} \int_Q d\sigma \right]^\frac{1}{p'} \leq C_2. \tag{1.10}$$

The necessary and sufficient condition (1.8) for the strong type inequality (1.7) states that one need only test the strong type inequality for functions of the form $\chi_Q \sigma$. Not only that, but the full $L^p(\omega)$ norm of $M(\chi_Q \sigma)$ need not be evaluated. There is a corresponding weak-type interpretation of the $A_p$ condition (1.10).

Finally, the proofs given in [16] and [5] for absolutely continuous weights carry over without difficulty for the locally finite measures considered here.

1.1. Two Weight Inequalities for Singular Integrals. Let us set notation for our Theorems. Consider a kernel function $K(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following size and smoothness conditions,

$$|K(x, y)| \leq C|x - y|^{-n}, \quad (1.11)$$

$$|K(x, y) - K(x', y)| \leq C\delta \left( \frac{|x - x'|}{|x - y|} \right) |x - y|^{-n}, \quad \frac{|x - x'|}{|x - y|} \leq \frac{1}{2},$$

where $\delta$ is a Dini modulus of continuity, i.e. a nondecreasing function on $[0, 1]$ with $\delta(0) = 0$ and $\int_0^1 \delta(s) \frac{ds}{s} < \infty$.

Next we describe the truncations we consider. Let $\zeta, \eta$ be fixed smooth functions on the real line satisfying

$$\zeta(t) = 0 \text{ for } t \leq \frac{1}{2} \text{ and } \zeta(t) = 1 \text{ for } t \geq 1,$$

$$\eta(t) = 0 \text{ for } t \geq 2 \text{ and } \eta(t) = 1 \text{ for } t \leq 1,$$

$\zeta$ is nondecreasing and $\eta$ is nonincreasing.

Given $0 < \varepsilon < R < \infty$, set $\zeta_\varepsilon(t) = \zeta(\frac{t}{\varepsilon})$ and $\eta_R(t) = \eta(\frac{t}{R})$ and define the smoothly truncated operator $T_{\varepsilon, R}$ on $L^1_{\text{loc}}(\mathbb{R}^n)$ by the absolutely convergent integrals

$$T_{\varepsilon, R}f(x) = \int K(x, y)\zeta_\varepsilon(|x - y|)\eta_R(|x - y|) f(y) dy, \quad f \in L^1_{\text{loc}}(\mathbb{R}^n).$$

Define the maximal singular integral operator $T_\circ$ on $L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$T_\circ f(x) = \sup_{0 < \varepsilon < R < \infty} |T_{\varepsilon, R}f(x)|, \quad x \in \mathbb{R}^n.$$

We also define a corresponding new notion of strongly maximal singular integral operator $T^{\circ}_\circ$ as follows. In dimension $n = 1$ we set

$$T^{\circ}_\circ f(x) = \sup_{0 < \varepsilon, \frac{1}{4} \leq \frac{R}{\varepsilon} \leq 4} |T_{\varepsilon, R}f(x)|, \quad x \in \mathbb{R},$$
where \( \varepsilon = (\varepsilon_1, \varepsilon_2) \) and

\[
T_{\varepsilon,R} f(x) = \int K(x, y) \left\{ \zeta_{\varepsilon_1}(x - y) + \zeta_{\varepsilon_2}(y - x) \right\} \eta_R(|x - y|) f(y) dy.
\]

Thus the local singularity has been removed by a noncentered smooth cutoff - \( \varepsilon_1 \) to the left of \( x \) and \( \varepsilon_2 \) to the right of \( x \), but with controlled eccentricity \( \varepsilon_1 \varepsilon_2 \). There is a similar definition of \( T_{\varepsilon} f \) in higher dimensions involving in place of \( \zeta_{\varepsilon}(|x - y|) \), a product of smooth cutoffs,

\[
\zeta_{\varepsilon}(x - y) \equiv 1 - \prod_{k=1}^{n} \left[ 1 - \left\{ \zeta_{\varepsilon_2k-1}(x_k - y_k) + \zeta_{\varepsilon_2k}(y_k - x_k) \right\} \right],
\]

satisfying \( \frac{1}{4} \leq \frac{\varepsilon_2k-1}{\varepsilon_2k} \leq 4 \) for \( 1 \leq k \leq n \). The advantage of this larger operator \( T_{\varepsilon} \) is that in many cases boundedness of \( T_{\varepsilon} \) (or collections thereof) implies boundedness of the maximal operator \( \mathcal{M} \). Our method of proving boundedness of \( T_{\varepsilon} \) requires boundedness of the maximal operator \( \mathcal{M} \) anyway, and as a result we can in some cases give necessary and sufficient conditions for strong boundedness of \( T_{\varepsilon} \). As for weak-type boundedness, we can in many more cases give necessary and sufficient conditions for weak boundedness of the usual truncations \( T_{\delta} \).

**Definition 1.12.** We say that \( T \) is a **standard singular integral operator with kernel** \( K \) if \( T \) is a bounded linear operator on \( L^q(\mathbb{R}^n) \) for some fixed \( 1 < q < \infty \), that is

\[
\|Tf\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^q(\mathbb{R}^n)}, \quad f \in L^q(\mathbb{R}^n),
\]

if \( K(x, y) \) is defined on \( \mathbb{R}^n \times \mathbb{R}^n \) and satisfies both (1.11) and the Hörmander condition,

\[
\int_{B(y,2\varepsilon)^c} |K(x, y) - K(x, y')| dx \leq C, \quad y' \in B(y, \varepsilon), \varepsilon > 0,
\]

and finally if \( T \) and \( K \) are related by

\[
Tf(x) = \int K(x, y) f(y) dy, \quad \text{a.e.-} x \notin \text{supp } f,
\]

whenever \( f \in L^q(\mathbb{R}^n) \) has compact support in \( \mathbb{R}^n \). We call a kernel \( K(x, y) \) **standard** if it satisfies (1.11) and (1.14).

Some of our results will apply to singular integral operators that are not standard. However, for standard singular operators, we have this classical result. (See the appendix on truncation of singular integrals on page 30 of [20] for the case \( R = \infty \); the case \( R < \infty \) is similar.)

**Theorem 1.16.** Suppose that \( T \) is a standard singular integral operator. Then the map \( f \to T_{\delta} f \) is of weak-type \((1,1)\), and bounded on \( L^p(\mathbb{R}) \) for \( 1 < p < \infty \).
There exist sequences $\varepsilon_j \to 0$ and $R_j \to \infty$ such that for $f \in L^p(\mathbb{R})$ with $1 \leq p < \infty$,

$$\lim_{j \to \infty} T_{\varepsilon_j, R_j} f(x) \equiv T_{0, \infty} f(x)$$

exists for a.e. $x \in \mathbb{R}$. Moreover, there is a bounded measurable function $a(x)$ (depending on the sequences) satisfying

$$T f(x) = T_{0, \infty} f(x) + a(x) f(x), \quad x \in \mathbb{R}^n.$$ 

We state a conjecture, so that the overarching goals of this subject are clear.

**Conjecture 1.17.** Suppose that $\sigma$ and $\omega$ are positive Borel measures on $\mathbb{R}^n$, $1 < p < \infty$, and $T$ is a standard singular integral operator on $\mathbb{R}^n$. Then the following two statements are equivalent:

$$\int |T(f \sigma)|^p \omega \leq C \int |f|^p \sigma, \quad f \in C_0^\infty,$$

$$\begin{cases}
\left[ \frac{1}{|Q|} \int_Q d\omega \right]^{\frac{1}{p}} \left[ \frac{1}{|Q|} \int_Q d\sigma \right]^{\frac{1}{p}} \leq C,
\int_Q |T \chi_Q \sigma|^p \leq C' \int_Q \sigma, \quad \text{for all cubes } Q.
\end{cases}$$

**Remark 1.18.** The first of the three testing conditions above is the two-weight $A_p$ condition. We would expect that this condition can be strengthened to a ‘Poisson two-weight $A_p$ condition’. See [9, 23].

The most important instances of this Conjecture occur when $T$ is one of a few canonical singular integral operators, such as the Hilbert transform, the Beurling Transform, or the Riesz Transforms. This question occurs in different instances, such as the Sarason Conjecture concerning the composition of Hankel operators, or the semi-commutator of Toeplitz operators (see [3], [24]), Mathematical Physics [12], as well as perturbation theory of some self-adjoint operators. See references in [23].

To date, this has only been verified for positive operators, such as Poisson integrals, and fractional integral operators [17], [16] and [18]. The two weight Helson-Szego Theorem has been proved by Cotlar and Sadosky [1] and [2], thus the $L^2$ case of the Hilbert transform is completely solved, though not in a manner that can be described as real-variable.

Nazarov, Treil and Volberg [7], [9] have characterized those weights for which the class of Haar multipliers is bounded when $p = 2$. They also have a result for an important special class of singular integral operators, the ‘well-localized’ operators of [8]. Citing the specific result here would carry us too far afield, but this class includes the important Haar shift examples, such as the one found by S. Petermichl [13], and generalized in [14]. Consequently, characterizations are given in [23] and [9] for the Hilbert transform and Riesz transforms in weighted
L^2 spaces under various additional hypotheses. In particular they obtain an analogue of the case p = 2 of the strong type theorem below. Our results can be reformulated in the context there, which theme we do not pursue further here.

We now characterize the weak-type two weight norm inequality for both maximal singular integrals and strongly maximal singular integrals.

**Theorem on Maximal Singular Integral Weak-Type Inequalities 1.19.** Suppose that σ and ω are positive locally finite Borel measures on \( \mathbb{R}^n \), 1 < p < \( \infty \), and let \( T_♭ \) and \( T^♭ \) be the maximal singular integral operators as above with kernel \( K(x, y) \) satisfying (1.11).

1. Suppose that the maximal operator \( M \) satisfies (1.9). Then \( T^♭ \) satisfies the weak-type two weight norm inequality

\[
\| T^♭(f \sigma) \|_{L^{p, \infty} (\omega)} \leq C \| f \|_{L^p (\sigma)}, \quad f \in L^p (\sigma),
\]

if and only if

\[
\int_Q T^♭(\chi_Q f \sigma) (x) d\omega(x) \leq C_2 \left( \int_Q |f(x)|^p d\sigma(x) \right)^{\frac{1}{p}} \left( \int_Q d\omega(x) \right)^{\frac{1}{p'}},
\]

for all cubes \( Q \subset \mathbb{R}^n \) and all functions \( f \in L^p (\sigma) \).

2. The same characterization as above holds for \( T_♭ \) in place of \( T^♭ \) everywhere.

3. Suppose that \( \sigma \) and \( \omega \) are absolutely continuous with respect to Lebesgue measure, that the maximal operator \( M \) satisfies (1.9), and that \( T \) is a standard singular integral operator with kernel \( K \) as above. If (1.20) holds for \( T^♭ \) or \( T_♭ \), then it also holds for \( T \):

\[
\| T(f \sigma) \|_{L^{p, \infty} (\omega)} \leq C \| f \|_{L^p (\sigma)}, \quad f \in L^p (\sigma), f \sigma \in L^\infty, \supp f \sigma \text{ compact}.
\]

4. Suppose \( c > 0 \) and that \( \{K_j\}_{j=1}^J \) is a collection of standard kernels such that for each unit vector \( u \) there is \( j \) satisfying

\[
|K_j(x, x + tu)| \geq ct^{-n}, \quad t \in \mathbb{R}.
\]

Suppose also that \( \sigma \) and \( \omega \) have no common point masses, i.e. \( \sigma (\{x\}) = \omega (\{x\}) = 0 \) for all \( x \in \mathbb{R}^n \). Then

\[
\| (T_j)_b (f \sigma) \|_{L^{p, \infty} (\omega)} \leq C \| f \|_{L^p (\sigma)}, \quad f \in L^p (\sigma), \quad 1 \leq j \leq J,
\]

if and only if the two weight \( A_p \) condition (1.10) holds and

\[
\int_Q (T_j)_b (\chi_Q f \sigma) (x) d\omega(x) \leq C_2 \left( \int_Q |f(x)|^p d\sigma(x) \right)^{\frac{1}{p}} \left( \int_Q d\omega(x) \right)^{\frac{1}{p'}},
\]

\( f \in L^p (\sigma) \), cubes \( Q \subset \mathbb{R}^n \), 1 \leq j \leq J.

While in (1)—(3), we assume that the Maximal Function inequality holds, in point (4), we obtain an uncondotional characterization of the weak-type inequality for a large class of families of (centered) maximal singular integral operators \( T_j \). This class includes the individual maximal Hilbert transform in one dimension,
the individual maximal Beurling transform in two dimensions, and the families of maximal Riesz transforms in higher dimensions, see Lemma 2.19.

Note that in (1) above, there is only size and smoothness assumptions placed on the kernel, so that it could for instance be a degenerate fractional integral operator, and therefore unbounded on \( L^2(dx) \). But, the characterization still has content in this case, if \( \omega \) and \( \sigma \) are not of full dimension.

In (3), we deduce a two weight inequality for standard singular integrals \( T \) without truncations when the measures are absolutely continuous. The proof of this is easy. From (1.20) and the pointwise inequality \( T_{0,\infty}f \sigma(x) \leq T_{f}f \sigma(x) \), we obtain that for any limiting operator \( T_{0,\infty} \), the map \( f \to T_{0,\infty}f \sigma \) is bounded from \( L^{p}(\sigma) \) to \( L^{p,\infty}(\omega) \). By (1.19) \( f \to Mf \sigma \) is bounded, hence \( f \to f \sigma \) is bounded, and so Theorem 1.16 shows that \( f \to Tf \sigma = T_{0,\infty}f \sigma + af \sigma \) is also bounded, provided we initially restrict attention to functions \( f \) for which \( f \sigma \) is bounded with compact support.

The characterizing condition (1.21) is a weak-type condition, with the restriction that one only needs to test the weak-type condition for functions supported on a given cube, and test the weak-type norm over that given cube. It also has an interpretation as a dual inequality \( \int_{Q} |L^{\ast}(\chi_{Q}\omega)|^{p'} d\sigma \leq C_{2} \int_{Q} d\omega \), which we return to below, see (2.14) and (2.15).

We now characterize the two weight norm inequality for a strongly maximal singular integral \( T_{z} \).

**Theorem on Maximal Singular Integral Strong-Type Inequalities 1.24.** Suppose that \( \sigma \) and \( \omega \) are positive locally finite Borel measures on \( \mathbb{R}^{n} \), \( 1 < p < \infty \), and let \( T_{z} \) and \( T_{z}^{\ast} \) be the maximal singular integral operators as above with kernel \( K(x,y) \) satisfying (1.11).

(1) Suppose that the maximal operator \( M \) satisfies (1.7) and also the ‘dual’ inequality

\[
\| M(g\omega) \|_{L^{p'}(\sigma)} \leq C \| g \|_{L^{p'}(\omega)}, \quad g \in L^{p'}(\omega).
\]

Then \( T_{z}^{\ast} \) satisfies the two weight norm inequality

\[
\int_{\mathbb{R}^{n}} T_{z}^{\ast}(f \sigma)(x)^{p} d\omega(x) \leq C \int_{\mathbb{R}^{n}} |f(x)|^{p} d\sigma(x),
\]

for all \( f \in L^{p}(\sigma) \) that are bounded with compact support in \( \mathbb{R}^{n} \), if and only if both the dual cube testing condition (1.21) and the condition below hold:

\[
\int_{Q} T_{z}^{\ast}(\chi_{Q}g\sigma)(x)^{p} d\omega(x) \leq C_{1} \int_{Q} d\sigma(x),
\]

for all cubes \( Q \subset \mathbb{R}^{n} \) and all functions \( |g| \leq 1 \).
The same characterization as above holds for $T_\flat$ in place of $T_\natural$ everywhere. In fact

$$|T_\flat f \sigma(x) - T_\natural f \sigma(x)| \leq CM(f \sigma)(x).$$

Suppose that $\sigma$ and $\omega$ are absolutely continuous with respect to Lebesgue measure, that the maximal operator $M$ satisfies (1.7), and that $T$ is a standard singular integral operator. If (1.26) holds for $T_\natural$ or $T_\flat$, then it also holds for $T$:

$$\int_{\mathbb{R}^n} |T(f \sigma)(x)|^p \, d\omega(x) \leq C \int_{\mathbb{R}^n} |f(x)|^p \, d\sigma(x),$$

$$f \in L^p(\sigma), f \sigma \in L^\infty, \text{supp}(f) \text{ compact.}$$

Suppose that $\{K_j\}_{j=1}^n$ is a collection of standard kernels satisfying for some $c > 0$,

$$\pm \text{Re} \, K_j(x, y) \geq \frac{c}{|x - y|^n}, \quad \text{for } \pm (y_j - x_j) \geq \frac{1}{4} |x - y|;$$

where $x = (x_j)_{1 \leq j \leq n}$. Then (1.26) holds for $(T_j)_\natural$ and $(T_j)_\flat$ for all $1 \leq j \leq n$, if and only if both (1.27) and (1.21) hold for $(T_j)_\natural$ and $(T_j)_\flat$ for all $1 \leq j \leq n$.

Note that the second condition (1.27) is a stronger condition than we would like: it is the $L^p$ inequality, applied to functions bounded by 1 and supported on a cube $Q$, but with the $L^p(\sigma)$ norm of $1_Q$ on the right side. It is easy to see that the bounded function $g$ in (1.27) can be replaced by $\chi_E$ for every compact subset $E$ of $Q$. Indeed if $L$ ranges over all linearizations of $T_\natural$, then with $g_{h,Q} = \frac{L^*(\chi_Q h \omega)}{|L^*(\chi_Q h \omega)|}$ we have

$$\sup_{|g| \leq 1} \int_{Q} T_j(\chi_Q g \sigma)^p \, \omega = \sup_{|g| \leq 1} \sup_{L \|h\|_{L^p(\omega)} \leq 1} \left| \int_{Q} L(\chi_Q g \sigma) \, h \omega \right|$$

$$= \sup_{L \|h\|_{L^p(\omega)} \leq 1} \sup_{|g| \leq 1} \left| \int_{Q} L^* (\chi_Q h \omega) \, g \sigma \right|$$

$$= \sup_{L \|h\|_{L^p(\omega)} \leq 1} \left| \int_{Q} L^* (\chi_Q h \omega) \, g_{h,Q} \sigma \right|$$

$$\leq \sup_{L \|h\|_{L^p(\omega)} \leq 1} \left| \int_{Q} T_j(\chi_Q g_{h,Q} \sigma)^p \, \omega \right|.$$
And in (4), we note that the truncations in the way that we formulate them, dominate the Maximal Function, so that our assumption on $M$ in (1)–(3) is not unreasonable. The main result of [9] assumes $p = 2$ and that $T$ is the Hilbert transform, and makes similar kinds of assumptions. In fact it is essentially the same as our result in the case $p = 2$, but only for $T$ and not $T_\delta$ or $T_\natural$. Finally, we observe that by our definition of the truncation $T_\natural$, we obtain in point (4) an unconditional characterization of the strong-type inequality for appropriate families of standard singular integrals and their adjoints, including the Hilbert and Riesz transforms, see Lemma 2.22.

We do not know if the bounded function $g$ in condition (1.27) can be replaced by the constant function 1.

1.2. Two Weight Inequalities for Generalized Fractional Integrals. Part 1 of Theorem 1.24 and part 1 of Theorem 1.19 extend to generalized fractional integrals, including the Cauchy integral in the plane. The setup is essentially the same as above but with a fractional variant of the size and smoothness conditions (1.11) on the kernel, and a fractional maximal function replacing the standard maximal function. Here are the details.

Let $0 \leq \alpha < n$. Consider a kernel function $K^\alpha(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the fractional size and smoothness conditions,

$$(1.29) \quad |K^\alpha(x, y)| \leq C|x - y|^{\alpha-n},$$

$$|K^\alpha(x, y) - K^\alpha(x', y)| \leq C\delta \left(\frac{|x - x'|}{|x - y|}\right) |x - y|^{\alpha-n}, \quad \frac{|x - x'|}{|x - y|} \leq \frac{1}{2},$$

where $\delta$ is a Dini modulus of continuity.

**Example 1.30.** The Cauchy integral $C^1$ in the complex plane arises when $K(x, y) = \frac{1}{x - y}$, $x, y \in \mathbb{C}$. The fractional size and smoothness condition (1.29) holds with $n = 2$ and $\alpha = 1$ in this case.

Define maximal fractional integrals $T_\delta^\alpha$ and $T_\natural^\alpha$ as above, but with $K^\alpha$ in place of $K$, and define the fractional maximal function by

$$(1.31) \quad \mathcal{M}^\alpha \nu(x) = \sup_{x \in Q} |Q|^{-\frac{n}{n-1}} \int_Q |\nu|, \quad x \in \mathbb{R}.$$}

We have the following solution from [16] to the two weight norm inequality for the fractional maximal operator $\mathcal{M}^\alpha$.

**Theorem on Fractional Maximal Function Inequalities 1.32.** Let $0 \leq \alpha < n$. Suppose that $\sigma$ and $\omega$ are positive locally finite Borel measures on $\mathbb{R}^n$, and $1 < p < \infty$. The fractional maximal operator $\mathcal{M}^\alpha$ satisfies the two weight norm inequality

$$(1.33) \quad \|\mathcal{M}^\alpha(f \sigma)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\sigma)}, \quad f \in L^p(\sigma),$$
if and only if for all cubes $Q \subset \mathbb{R}^n$,
\[
\int_Q \mathcal{M}^\alpha (\chi_Q \sigma) (x)^p d\omega(x) \leq C_1 \int_Q d\sigma(x) < \infty.
\]

The fractional maximal operator $\mathcal{M}^\alpha$ satisfies the weak-type two weight norm inequality
(1.34)
\[
\|[\mathcal{M}^\alpha (f \sigma)]\|_{L^{p,\infty}(\omega)} \equiv \sup_{\lambda > 0} \lambda \left| \{ \mathcal{M}^\alpha (f \sigma) > \lambda \} \right|^\frac{1}{p} \leq C \|[f]\|_{L^p(\sigma)}, \quad f \in L^p(\sigma),
\]
if and only if the two weight $A_\alpha^p$ condition holds for all cubes $Q \subset \mathbb{R}^n$:
\[
|Q|^\frac{n}{2p} \left( \frac{1}{|Q|} \int_Q d\omega \right)^{\frac{1}{p}} \left( \frac{1}{|Q|} \int_Q d\sigma \right)^{\frac{1}{p}} \leq C_2.
\]

The following theorem is proved in exactly the same way as part 1 of Theorems 1.24 and 1.19 above, but with $\mathcal{M}^\alpha$ in place of $\mathcal{M}$.

**Theorem on Fractional Maximal Singular Integral Inequalities 1.35.**

Let $0 \leq \alpha < n$. Suppose that $\sigma$ and $\omega$ are positive locally finite Borel measures on $\mathbb{R}^n$, $1 < p < \infty$, and let $T^\alpha_\flat$ and $T^\alpha_\#$ be maximal fractional integral operators as above with kernel $K^\alpha(x, y)$ satisfying (1.29).

1. Suppose that the fractional maximal operator $\mathcal{M}^\alpha$ satisfies (1.33) and the corresponding ‘dual’ inequality
\[
\|[\mathcal{M}^\alpha (g \omega)]\|_{L^{p'}(\sigma)} \leq C \|[g]\|_{L^{p'}(\omega)}, \quad g \in L^{p'}(\omega).
\]

Then $T^\alpha_\#$ satisfies the two weight norm inequality
(1.36)
\[
\int_{\mathbb{R}^n} T^\alpha_\# (f \sigma)(x)^p d\omega(x) \leq C \int_{\mathbb{R}^n} |f(x)|^p d\sigma(x),
\]
for all $f \in L^p(\sigma)$ that are bounded with compact support in $\mathbb{R}^n$, if and only if both
\[
\int_Q T^\alpha_\# (\chi_Q g \sigma)(x)^p d\omega(x) \leq C_1 \int_Q d\sigma(x), \quad \text{for all } |g| \leq 1,
\]
and
\[
\int_Q T^\alpha_\# (\chi_Q f \sigma)(x) d\omega(x) \leq C_2 \left( \int_Q |f(x)|^p d\sigma(x) \right)^{\frac{1}{p}} \left( \int_Q d\omega(x) \right)^{\frac{1}{p}},
\]
for all cubes $Q \subset \mathbb{R}^n$ and all $f \in L^p(\sigma)$. The same holds with $T^\alpha_\flat$ in place of $T^\alpha_\#$ in (1.36), (1.37) and (1.38).
Suppose that the fractional maximal operator $M^\alpha_s$ satisfies (1.34). Then $T^\alpha_s$ satisfies the weak-type two weight norm inequality

$$\|T^\alpha_s(f\sigma)\|_{L^{p,\infty}(\omega)} \leq C \|f\|_{L^p(\sigma)}, \quad f \in L^p(\sigma) \text{ bounded with compact support,}$$

if and only if (1.38) holds. The same holds with $T^\alpha_s$ in place of $T^\alpha_s$ in (1.39) and (1.38).

In particular, the Cauchy integral $C_1$ in the plane is bounded from one weighted space to another provided the fractional maximal function $M_1$ is bounded and the two testing conditions (1.37) and (1.38) hold. Thus we see that the problem of deciding whether the testing conditions hold is the main issue here, and the cancellation conditions inherent in the Cauchy kernel play a decisive role. For more general fractional integrals, the appropriate notion of cancellation remains mysterious, and so we do not have a corresponding definition of a standard generalized fractional integral operator. Finally, we note that there are analogues of part 4 of both Theorem 1.24 and Theorem 1.19 which the interested reader can easily supply.

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2. Overview of the Proofs, General Principles

If $Q$ is a cube then $\ell(Q)$ is its side length, $|Q|$ is its Lebesgue measure and for a positive Borel measure $\nu$, $|Q|_\nu = \int_Q d\nu$ is its $\nu$-measure.

2.1. Calderón-Zygmund Decompositions. Our starting place is the argument in [18] used to prove a two weight norm inequality for fractional integral operators on Euclidean space. Of course the fractional integral is a positive operator, with a monotone kernel, which properties we do not have in the current setting.

A central tool arises from the observation that for any positive Borel measure $\mu$, one has the boundedness of a maximal function associated with $\mu$. Define the dyadic $\mu$-maximal operator $M^\mu$ by

$$M^\mu f(x) = \sup_{Q \in \mathcal{D}} \frac{1}{|Q|_\mu} \int_Q |f| \mu,$$

with the supremum taken over all dyadic cubes $Q \in \mathcal{D}$ containing $x$. It is immediate to check that $M^\mu$ satisfies the weak-type $(1,1)$ inequality, and the $L^\infty(\mu)$ bound is obvious. Hence we have

$$\int (M^\mu f)^p \mu \leq C \int f^p \mu, \quad f \geq 0 \text{ on } \mathbb{R}^n.$$
This observation places certain Calderón-Zygmund decompositions at our disposal. Exploitation of this brings in the testing condition (1.27) involving the bounded function \( g \) on a cube \( Q \), and indeed, \( g \) turns out to be the “good” function in a Calderón-Zygmund decomposition of \( f \) on \( Q \). The associated ‘bad’ function requires the dual testing condition (1.21) as well.

2.2. Edge effects of dyadic grids. Our operators are not dyadic operators, nor—in contrast to the fractional integral operators—can they be easily obtained from dyadic operators. This leads to the necessity of considering for instance triples of dyadic cubes, which are not dyadic.

Also, dyadic grids distinguish points by for instance making some points on the boundary of many cubes. As our measures are arbitrary, they could conspire to assign extra mass to some of these points. To address this point, Nazarov-Treil-Volberg [9, 10, 11] use a random shift of the grid.

A random approach would likely work for us as well, though the argument would be different from those in the cited papers above. Instead, we will use M. Christ’s non-random technique of shifted dyadic grid from [6]. Define a shifted dyadic grid to be the collection of cubes

\[
D^\alpha = \left\{ 2^j (k + [0,1]^n + (-1)^j \alpha) : j \in \mathbb{Z}, k \in \mathbb{Z}^n \right\}, \quad \alpha \in \left\{ 0, \frac{1}{3}, \frac{2}{3} \right\}^n.
\]

The basic properties of these collections are these: In the first place, each \( D^\alpha \) is a grid, namely for \( Q, Q' \in D^\alpha \) we have \( Q \cap Q' \in \{ \emptyset, Q, Q' \} \) and \( Q \) is a union of \( 2^n \) elements of \( D^\alpha \) of equal volume. In the second place (and this is the novel property for us), for any cube \( Q \subset \mathbb{R}^n \), there is a choice of some \( \alpha \) and some \( Q' \in D_\alpha \) so that \( Q \subset \frac{9}{10} Q' \) and \( |Q'| \leq C |Q| \).

We define the analogs of the dyadic maximal operator in (2.1), not to be confused with \( M^\alpha \) in (1.31), namely

\[
M^\alpha f(x) = \sup_{Q \in D^\alpha} \frac{1}{|Q|} \int_Q |f| \mu.
\]

These operators clearly satisfy (2.2). Shifted dyadic grids will return in § 4.4.

2.3. A Maximum Principle. A second central tool is a ‘maximum principle’ (or good \( \lambda \) inequality) which will permit one to localize large values of a singular integral, provided the Maximal Function is bounded. It is convenient for us to describe this in conjunction with another fundamental tool of this paper, a family of Whitney decompositions.

We begin with the Whitney decompositions. Fix a finite measure \( \nu \) with compact support on \( \mathbb{R}^n \) and for \( k \in \mathbb{Z} \) let

\[
\Omega_k = \left\{ x \in \mathbb{R}^n : T_k \nu(x) > 2^k \right\}.
\]

Note that \( \Omega_k \neq \mathbb{R}^n \) has compact closure for such \( \nu \). Fix an integer \( N \geq 3 \). We can choose \( R_W \geq 3 \) sufficiently large, depending only on the dimension and \( N \), such
that there is a collection of cubes \( \{ Q^k_j \} \) which satisfy the following properties:

\[
\begin{align*}
\Omega_k &= \bigcup_j Q^k_j \quad \text{(disjoint cover)} \\
R^k_W Q^k_j &\subset \Omega_k \quad \text{(Whitney condition)} \\
\sum_j \chi_{\Omega^k_j} &\leq C \chi_{\Omega_k} \quad \text{(finite overlap)} \\
\# \{ Q^k_j : Q^k_j \cap NQ^k_j \neq \emptyset \} &\leq C \quad \text{(crowd control)} \\
Q^k_j \not\subset Q^\ell_k \quad \text{(nested property)}
\end{align*}
\]

Indeed, one should choose the \( \{ Q^k_j \} \) satisfying the Whitney condition, and then show that the other properties hold. The different combinatorial properties above are fundamental to the proof. And alternate Whitney decompositions are constructed in §4.7.1 below.

Remark 2.7. Our use of the Whitney decomposition and the maximum principle are derived from the two weight fractional integral argument of Sawyer, see Sec 2 of [18]. In particular, the properties above are as in [18], aside from the the crowd control property above, which as \( N = 3 \) in op. cit.

Remark 2.8. In our notation for the Whitney cubes, the superscript indicates a ‘height’ and the subscript an arbitrary enumeration of the cubes. We will use super- and sub-scripts below in this manner consistently throughout the paper. It is important to note that a fixed cube \( Q \) can arise in many Whitney decompositions: There are integers \( K_-(Q) \leq K_+(Q) \) with \( Q = Q^k_{j(k)} \) for some choice of \( j(k) \) for all \( K_-(Q) \leq k \leq K_+(Q) \). (The last point follows from the nested property.) There is no a priori upper bound on \( K_+(Q) - K_-(Q) \).

Lemma 2.9. [Maximum Principle] Let \( \nu \) be a finite (signed) measure with compact support. For any cube \( Q^k_j \) as above we have the pointwise inequality

\[
\sup_{x \in Q^k_j} T^k_\nu (\chi_{3Q^k_j}, \nu) (x) \leq 2^k + C \mathbb{P} (Q^k_j, \nu) \leq 2^k + CM (Q^k_j, \nu),
\]

where \( \mathbb{P} (Q, \nu) \) and \( M (Q, \nu) \) are defined by

\[
\mathbb{P} (Q, \nu) \equiv \frac{1}{|Q|} \int_Q d|\nu| + \sum_{\ell=0}^{\infty} \delta \left( \frac{2^{-\ell}}{2^{\ell+1}Q} \right) \int_{2^{\ell+1}Q \setminus 2^\ell Q} d|\nu|,
\]

\[
M (Q, \nu) \equiv \sup_{Q' \supset Q} \frac{1}{|Q'|} \int_{Q'} d|\nu|. 
\]

The bound in terms of \( \mathbb{P} (Q, \nu) \) should be regarded as one in terms of a modified Poisson integral. It is both slightly sharper than that of \( M (Q, \nu) \), and a linear expression in \( |\nu| \), which fact will be used in the proof of the strong type estimates.
Proof. To see this, take \( x \in Q_j^k \) and note that for each \( \eta > 0 \) there is \( \varepsilon \) with \( \ell(Q_j^k) < \max_{1 \leq j \leq n} \varepsilon_j < R < \infty \) and \( \theta \in [0, 2\pi) \) such that

\[
T_\varepsilon \left( \chi_{(3Q_j^k)^c} \nu \right)(x) \leq (1 + \eta) \left| \int_{(3Q_j^k)^c} K(x, y) \zeta_\varepsilon(x - y) \eta_R(x - y) d\nu(y) \right|
= (1 + \eta) e^{i\theta} T_{\varepsilon, R} \left( \chi_{(3Q_j^k)^c} \nu \right)(x).
\]

For convenience we take \( \eta = 0 \) in the sequel. By the Whitney condition in (2.6), there is a point \( z \in 3R_w Q_j^k \cap \Omega_k^c \) and it now follows that (remember that \( \ell(Q_j^k) < \max_{1 \leq j \leq n} \varepsilon_j \)),

\[
\left| T_{\varepsilon, R} \left( \chi_{(3Q_j^k)^c} \nu \right)(x) - T_{\varepsilon, R} \nu(z) \right| \\
\leq C \frac{1}{|6R_w Q_j^k|} \int_{6R_w Q_j^k} d|\nu| + \left| T_{\varepsilon, R} \left( \chi_{(6R_w Q_j^k)^c} \nu \right)(x) - T_{\varepsilon, R} \left( \chi_{(6R_w Q_j^k)^c} \nu \right)(z) \right| \\
= C \frac{1}{|6R_w Q_j^k|} \int_{6R_w Q_j^k} d|\nu| \\
+ \int_{(6R_w Q_j^k)^c} |K(x, y) \zeta_\varepsilon(x - y) \eta_R(x - y) - K(z, y) \zeta_\varepsilon(z - y) \eta_R(z - y)| d|\nu|(y) \\
\leq C \frac{1}{|6R_w Q_j^k|} \int_{6R_w Q_j^k} d|\nu| + C \int_{(6R_w Q_j^k)^c} \delta \left( \frac{|x - z|}{|x - y|} \right) \frac{1}{|x - y|} d|\nu|(y) \\
\leq CP \left( Q_j^k, \nu \right).
\]

Thus

\[
T_\varepsilon \left( \chi_{(3Q_j^k)^c} \nu \right)(x) \leq |T_\varepsilon \nu(z)| + CP \left( Q_j^k, \nu \right) \leq 2^k + CP \left( Q_j^k, \nu \right),
\]

which yields (2.10) since \( P(Q, \nu) \leq CM(Q, \nu) \).

2.4. Linearizations. We now make comments on the linearizations of our maximal singular integral operators. We would like, at different points, to treat \( T_\varepsilon \) as a linear operator, which of course it is not. Nevertheless \( T_\varepsilon \) is a pointwise supremum of the linear truncation operators \( T_{\varepsilon, R} \), and as such, the supremum can be linearized with measurable selection of the parameters \( \varepsilon \) and \( R \), as was just done in the previous proof. We make this a definition.

Definition 2.12. We say that \( L \) is a linearization of \( T_\varepsilon \) if there are measurable functions \( \varepsilon(x) \in (0, \infty)^n \) and \( R(x) \in (0, \infty) \) with \( \frac{1}{4} \leq \varepsilon_j \leq 4 \), \( \max_{1 \leq i \leq n} \varepsilon_i < R(x) < \infty \) and \( \theta(x) \in [0, 2\pi) \) such that

\[
Lf(x) = e^{i\theta(x)} T_{\varepsilon(x), R(x)} f(x), \quad x \in \mathbb{R}^n.
\]
For fixed \( f \) and \( \delta > 0 \), we can always choose a linearization \( L \) so that \( T_\delta f(x) \leq (1 + \delta) L f(x) \) for all \( x \). In a typical application of this Lemma, one takes \( \delta \) to be one.

Note that condition (1.27) is obtained from inequality (1.26) by testing over \( f = \chi_Q g \) with \( |g| \leq 1 \), and then restricting integration on the left to \( Q \). By passing to linearizations \( L \), we can ‘dualize’ (1.21) to the testing conditions

\[
(2.14) \quad \int_Q |L^*(\chi_Q \omega)(x)|^{p'} \, d\sigma(x) \leq C_2 \int_Q d\omega(x),
\]

or equivalently (note that in (1.27) the presence of \( g \) makes a difference, but not here),

\[
(2.15) \quad \int_Q |L^*(\chi_Q g \omega)(x)|^{p'} \, d\sigma(x) \leq C_2 \int_Q d\omega(x), \quad |g| \leq 1,
\]

with the requirement that these inequalities hold uniformly in all linearizations \( L \) of \( T_\delta \).

While the smooth truncation operators \( T_{\varepsilon,R} \) are essentially self-adjoint, the dual of a linearization \( L \) is generally complicated. Nevertheless, the dual \( L^* \) does satisfy one important property which plays a crucial role in the proof of Theorem 1.24, the \( L^p \)-norm inequalities.

**Lemma 2.16.** \( L^* \mu \) is \( \delta \)-Hölder continuous (where \( \delta \) is the Dini modulus of continuity of the kernel \( K \)) with constant \( CP(\mu) \) on any cube \( Q \) satisfying \( \int_{3Q} d|\mu| = 0 \), i.e.

\[
(2.17) \quad |L^* \mu(y) - L^* \mu(y')| \leq CP(\mu) \delta \left( \frac{|y - y'|}{\ell(Q)} \right), \quad y, y' \in Q.
\]

Here, recall the definition (2.11) and that \( P(\mu) \leq CM(Q, \mu) \).

**Proof.** Suppose \( L \) is as in (2.13). Then for any finite measure \( \nu \),

\[
L \nu(x) = e^{i \theta(x)} \int \zeta_{\varepsilon(x)}(x - y) \eta_{R(x)}(x - y) K(x,y) d\nu(y).
\]

Fubini’s theorem shows that the dual operator \( L^* \) is given on a finite measure \( \mu \) by

\[
(2.18) \quad L^* \mu(y) = \int \zeta_{\varepsilon(x)}(x - y) \eta_{R(x)}(x - y) K(x,y) e^{i \theta(x)} d\mu(x).
\]

For \( y, y' \in Q \) and \( |\mu|(3Q) = 0 \), we thus have

\[
L^* \mu(y) - L^* \mu(y') = \int \{ (\zeta_{\varepsilon(x)} \eta_{R(x)})(x - y') - (\zeta_{\varepsilon(x)} \eta_{R(x)})(x - y) \} K(x,y) e^{i \theta(x)} d\mu(x)
\]

\[
+ \int (\zeta_{\varepsilon(x)} \eta_{R(x)})(x - y') \{ K(x,y) - K(x,y') \} e^{i \theta(x)} d\mu(x),
\]

where \( \zeta_{\varepsilon(x)} \eta_{R(x)}(x - y) \) is as in (2.13). Then for any finite measure \( \nu \),

\[
L \nu(x) = e^{i \theta(x)} \int \zeta_{\varepsilon(x)}(x - y) \eta_{R(x)}(x - y) K(x,y) d\nu(y).
\]

Fubini’s theorem shows that the dual operator \( L^* \) is given on a finite measure \( \mu \) by

\[
(2.18) \quad L^* \mu(y) = \int \zeta_{\varepsilon(x)}(x - y) \eta_{R(x)}(x - y) K(x,y) e^{i \theta(x)} d\mu(x).
\]

For \( y, y' \in Q \) and \( |\mu|(3Q) = 0 \), we thus have

\[
L^* \mu(y) - L^* \mu(y') = \int \{ (\zeta_{\varepsilon(x)} \eta_{R(x)})(x - y') - (\zeta_{\varepsilon(x)} \eta_{R(x)})(x - y) \} K(x,y) e^{i \theta(x)} d\mu(x)
\]

\[
+ \int (\zeta_{\varepsilon(x)} \eta_{R(x)})(x - y') \{ K(x,y) - K(x,y') \} e^{i \theta(x)} d\mu(x),
\]

where \( \zeta_{\varepsilon(x)} \eta_{R(x)}(x - y) \) is as in (2.13). Then for any finite measure \( \nu \),

\[
L \nu(x) = e^{i \theta(x)} \int \zeta_{\varepsilon(x)}(x - y) \eta_{R(x)}(x - y) K(x,y) d\nu(y).
\]
from which (2.17) follows easily if we split the two integrals in $x$ over dyadic
annuli centered at the center of $Q$. ■

2.5. Control of Maximal Functions. Next we record the facts that $T$ and $T^\#$
control $\mathcal{M}$ for many (collections of) standard singular integrals $T_j$, including the
Hilbert transform, the Beurling transform and the collection of Riesz transforms
in higher dimensions.

**Lemma 2.19.** Suppose that $\sigma$ and $\omega$ have no point masses in common, and that
$\{K_j\}_{j=1}^J$ is a collection of standard kernels satisfying (1.11) and (1.23). If the
corresponding operators $T_j$ given by (1.15) satisfy
\[
\|\chi_E T_j(f \sigma)\|_{L^p(\omega)} \leq C \|f\|_{L^p(\sigma)}, \quad E = \mathbb{R}^n \setminus \text{supp } f,
\]
for $1 \leq j \leq J$, then the two weight $A_p$ condition (1.10) holds, and hence also the
weak-type two weight inequality (1.9).

**Proof.** Part of the ‘one weight’ argument on page 211 of Stein [21] yields the
asymmetric two weight $A_p$ condition
\[
|Q|_\omega |Q'|_{\sigma}^{p-1} \leq C |Q|^p,
\]
where $Q$ and $Q'$ are cubes of equal side length $r$ and distance approximately $C_0 r$
apart for some fixed large positive constant $C_0$ (for this argument we choose the
unit vector $u$ in (1.23) to point in the direction from the center of $Q$ to the center
of $Q'$, and then with $j$ as in (1.23), $C_0$ is chosen large enough by (1.11) that (1.23)
holds for all unit vectors $u$ pointing from a point in $Q$ to a point in $Q'$). In the
one weight case treated in [21] it is easy to obtain from this (even for a
single direction $u$) the usual (symmetric) $A_p$ condition (1.10). Here we will instead use
our assumption that $\sigma$ and $\omega$ have no point masses in common for this purpose.

So fix an open cube $Q$ in $\mathbb{R}^n$ and let $\{Q_\alpha\}_\alpha$ be a Whitney decomposition (2.6)
of the open set $(Q \times Q) \setminus \mathcal{D}$ relative to $\mathcal{D}$ where $\mathcal{D}$ is the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$.
Note that if $Q_\alpha = Q_\alpha \times Q'_\alpha$, then (2.21) can be written
\[
A_p(\omega; Q_\alpha) \leq C |Q_\alpha|^{\frac{p}{2}}.
\]
where $A_p(\omega; Q_\alpha) = |Q_\alpha|_\omega |Q'_\alpha|_{\sigma}^{p-1}$, $A_2(\omega; Q_\alpha) = |Q_\alpha|_{\omega \times \sigma}$, where $\omega \times \sigma$ denotes
product measure on $\mathbb{R}^n \times \mathbb{R}^n$. We choose $R_{W'}$ sufficiently large in (2.6),
depending on $C_0$, such that (2.21) holds for all the Whitney cubes $Q_\alpha$. For $1 < p \leq \infty$ we
easily compute that $\sum_\alpha |Q_\alpha|^{\frac{p}{2}} \leq C |Q \times Q'|^{\frac{p}{2}} = C |Q|^p$.

Suppose now that $1 < p \leq 2$. We claim that if $R = Q \times Q'$ is a rectangle in
$\mathbb{R}^n \times \mathbb{R}^n$ (i.e. $Q, Q'$ are cubes in $\mathbb{R}^n$), and if $R = \bigcup_\alpha R_\alpha$ is a finite disjoint union
of rectangles $R_\alpha$, then with the obvious extension of $A_p(\omega; \sigma; R)$ to rectangles,
\[
A_p(\omega; \sigma; R) \leq \sum_\alpha A_p(\omega; \sigma; R_\alpha).
\]
This is easy to see using $0 < p - 1 \leq 1$ if the disjoint union consists of just two rectangles, and the general case then follows by induction (the case $p = 2$ is just countable additivity of product measure).

Since $\omega$ and $\sigma$ have no point masses in common, a limiting argument using the above subadditivity of $A_p$ shows that

$$A_p(\omega; Q \times Q) \leq \sum_\alpha A_p(\omega; Q_\alpha) \leq C \sum_\alpha |Q_\alpha|^\frac{p}{2} \leq C |Q|^p,$$

which is (1.10). The case $2 \leq p < \infty$ is proved in the same way using that (2.20) can be written

$$A'_p(\sigma; \omega; Q_\alpha) \leq C'|Q_\alpha|^\frac{p'}{2}.$$

**Lemma 2.22.** If $\{T_j\}_{j=1}^n$ satisfies (1.28), then

$$M_\nu(x) \leq C \sum_{j=1}^n (T_j)_2 \nu(x), \quad x \in \mathbb{R}^n, \quad \nu \geq 0 \quad \text{a finite measure with compact support}.$$

**Proof.** We prove the case $n = 1$, the general case being similar. Then with $T = T_1$ and $r > 0$ we have

$$\text{Re} \left( T_{r,4r,100r} \nu(x) - T_{r,4r,100r} \nu(x) \right)$$

$$= \int \left\{ \zeta_{4r}(y-x) - \zeta_{4r}(y-x) \right\} \text{Re} K(x,y) d\nu(y) \geq \frac{c}{r} \int_{[x+\frac{2r}{4},x+2r]} d\nu(y).$$

Thus

$$T_2 \nu(x) \geq \max \left\{ |T_{r,4r,100r} \nu(x)|, |T_{r,4r,100r} \nu(x)| \right\} \geq \frac{c}{r} \int_{[x+\frac{2r}{4},x+2r]} d\nu(y),$$

and similarly

$$T_2 \nu(x) \geq \frac{c}{r} \int_{[x-2r,x+\frac{2r}{4}]} d\nu(y).$$

It follows that

$$M_\nu(x) \leq \sup_{r>0} \frac{1}{4r} \int_{[x-2r,x+2r]} d\nu(y)$$

$$= \sup_{r>0} \sum_{k=0}^\infty 2^{-k} \frac{1}{2^{2-k}r} \int_{[x-2^{-k}r,x-2^{-1-k}r] \cup [x+2^{-1-k}r,x+2^{-k}r]} d\nu(y)$$

$$\leq C T_2 \nu(x).$$

Finally, we will use the following covering lemma of Besicovitch type for multiples of dyadic cubes (the case of triples of dyadic cubes arises in (4.65) below).
Lemma 2.23. Let $M$ be an odd positive integer, and suppose that $\Phi$ is a collection of cubes $P$ with bounded diameters and having the form $P = MQ$ where $Q$ is dyadic (a product of clopen dyadic intervals). If $\Phi^*$ is the collection of maximal cubes in $\Phi$, i.e. $P^* \in \Phi^*$ provided there is no strictly larger $P$ in $\Phi$ that contains $P^*$, then the cubes in $\Phi^*$ have finite overlap at most $M^n$.

Proof. Let $Q_0 = [0,1)^n$ and assign labels $1, 2, 3, \ldots, M^n$ to the dyadic subcubes of side length one of $MQ_0$. We say that the subcube labeled $k$ is of type $k$, and we extend this definition by translation and dilation to the subcubes of $MQ$ having side length that of $Q$. Now we simply observe that if $\{P_i^*\}_i$ is a set of cubes in $\Phi^*$ containing the point $x$, then for a given $k$, there is at most one $P_i^*$ that contains $x$ in its subcube of type $k$. The reason is that if $P_j^*$ is another such cube and $\ell(P_j^*) \leq \ell(P_i^*)$, we must have $P_j^* \subset P_i^*$ (draw a picture in the plane for example).

2.6. Preliminary Precaution. Given a positive locally finite Borel measure $\mu$ on $\mathbb{R}^n$, there exists a rotation such that all boundaries of rotated dyadic cubes have $\mu$-measure zero (see [4] where they actually prove a stronger assertion when $\mu$ has no point masses, but our conclusion is obvious for a sum of point mass measures). We will assume that such a rotation has been made so that all boundaries of rotated dyadic cubes have $(\omega + \sigma)$-measure zero, where $\omega$ and $\sigma$ are the positive Borel measures appearing in the theorems above. While this assumption is not essential for the proof, it relieves the reader of having to consider the possibility that boundaries of dyadic cubes have positive measure at each step of the argument below.

Recall also (see e.g. Theorem 2.18 in [15]) that any positive locally finite Borel measure on $\mathbb{R}^n$ is both inner and outer regular.

3. The proof of Theorem 1.19: Weak-type Inequalities

We begin with the necessity of condition (1.21):

$$
\int_Q T_\omega (\chi_Q f \sigma) \omega = \int_0^\infty \min \left\{ |Q|_{\omega}, \{ T_\omega (\chi_Q f \sigma) > \lambda \} \right\} d\lambda 
\leq \left\{ \int_0^A + \int_A^\infty \right\} \min \left\{ |Q|_{\omega}, C\lambda^{-p} \int |f|^p d\sigma \right\} d\lambda 
\leq A |Q|_{\omega} + C A^{1-p} \int |f|^p d\sigma 
= (C + 1) |Q|_{\omega}^{1-p} \left( \int |f|^p d\sigma \right)^{\frac{1}{p}} 
$$

if we choose $A = \left( \frac{\int |f|^p d\sigma}{|Q|_{\omega}} \right)^{\frac{1}{p}}$. 

if we choose $A = \left( \frac{\int |f|^p d\sigma}{|Q|_{\omega}} \right)^{\frac{1}{p}}$. 

Now we turn to proving (1.20), assuming both (1.21) and (1.9), and moreover that \( f \) is bounded with compact support. We will prove the quantitative estimate
\[
\|T^{\#} f_{\sigma}\|_{L^p,\infty(\omega)} \leq C \left\{ \mathfrak{A} + \mathfrak{T} \right\} \|f\|_{L^p(\sigma)} \, \omega
\]
(3.1)

\[
\mathfrak{A} = \sup_Q \sup_{\|f\|_{L^p(\sigma)} = 1} \sup_{\lambda > 0} \lambda \|\mathcal{M}(f \sigma) > \lambda\|_{L^p(\sigma)}^{\frac{1}{p}}
\]
(3.2)

\[
\mathfrak{T} = \sup_Q \frac{1}{\|f\|_{L^p(\sigma)}^{1-p'}} \int_Q T^{\#}_T \left( \chi_Q f \sigma \right)(x) \, d\omega(x)
\]
(3.3)

We should emphasize that the term (3.2) is comparable to the two weight \( A_p \) condition (1.10).

Standard considerations ([17]. Section 2) show that it suffices to prove the following good-\( \lambda \) inequality: There is a positive constant \( C \) so that for \( \beta > 0 \) sufficiently small, and provided
\[
\sup_{0 < \lambda < \Lambda} \left\{ x \in \mathbb{R}^n : T^T f \sigma(x) > 2\lambda \text{ and } \mathcal{M} f \sigma(x) \leq \beta \lambda \right\}_{\omega} < \infty, \quad \Lambda < \infty,
\]
we have this inequality:
\[
\left\{ x \in \mathbb{R}^n : T^T f \sigma(x) > 2\lambda \text{ and } \mathcal{M} f \sigma(x) \leq \beta \lambda \right\}_{\omega} \leq C\beta \mathfrak{T} \left\{ x \in \mathbb{R}^n : T^T f \sigma(x) > \lambda \right\}_{\omega} + C\beta^{-p} \lambda^{-p} \int |f|^p \, d\sigma.
\]
(3.4)

Our presumption (3.4) holds due to the \( A_p \) condition (1.10) and the fact that
\[
\left\{ x \in \mathbb{R}^n : T^T f \sigma(x) > \lambda \right\} \subset B \left( 0, c\lambda^{-\frac{1}{p'}} \right), \quad \lambda > 0 \text{ small},
\]
Hence it is enough to prove (3.5).

To prove (3.5) we choose \( \lambda = 2^k \), and apply the decomposition in (2.6). In this argument, we can take \( k \) to be fixed, so that we suppress its appearance as a superscript in this section. (When we come to \( L^p \) estimates, we will not have this luxury.)

Define
\[
E_j = \left\{ x \in Q_j : T^T f \sigma(x) > 2\lambda \text{ and } \mathcal{M} f \sigma(x) \leq \beta \lambda \right\}.
\]
Then for \( x \in E_j \), we can apply Lemma (2.9) to deduce
\[
T^T_\lambda \left( \chi_{(3Q_j)^c} f \sigma \right)(x) \leq (1 + C\beta) \lambda.
\]
(3.6)

If we take \( \beta > 0 \) so small that \( 1 + C\beta \leq \frac{3}{2} \), then (3.6) implies that for \( x \in E_j \)
\[
2\lambda < T^T_\lambda f \sigma(x) \leq T^T_\lambda \chi_{3Q_j} f \sigma(x) + T^T_\lambda \chi_{(3Q_j)^c} f \sigma(x)
\]
\[
\leq T^T_\lambda \chi_{3Q_j} f \sigma(x) + \frac{3}{2} \lambda.
\]

Integrating this inequality with respect to \( \omega \) over \( E_j \) we obtain
\[
\lambda |E_j|_{\omega} \leq 2 \int_{E_j} \left( T^T_\lambda \chi_{3Q_j} f \sigma \right) \, \omega.
\]
(3.7)
The disjoint cover condition in (2.6) shows that the sets $E_j$ are disjoint, and this suggests we should sum their $\omega$-measures. We split this sum into two parts, according to the size of $|E_j|/|3Q_j|_\omega$. The left-hand side of (3.5) satisfies
\[
\sum_j |E_j|_\omega \leq \beta \sum_{j: |E_j|_\omega \leq \beta |3Q_j|_\omega} |3Q_j|_\omega
\]
\[
+ \beta^{-p} \sum_{j: |E_j|_\omega > \beta |3Q_j|_\omega} |E_j|_\omega \left( \frac{2}{\lambda} \frac{1}{|3Q_j|_\omega} \int_{E_j} \left( T_2 \chi_{3Q_j} f \sigma \right) \omega \right)^p
\]
\[
= I + II.
\]
Now
\[
I \leq \beta \sum_j |3Q_j^k|_\omega \leq C \beta |\Omega|_\omega,
\]
by the finite overlap condition in (2.6). From (1.21) with $Q = 3Q_j$ we have
\[
II \leq \left( \frac{2}{\beta \lambda} \right)^p \sum_j |E_j|_\omega \left( \frac{1}{|3Q_j|_\omega} \int_{E_j} \left( T_2 \chi_{3Q_j} f \sigma \right) \omega \right)^p
\]
\[
\leq C \left( \frac{2}{\beta \lambda} \right)^p \sum_j |E_j|_\omega \left( \frac{1}{|3Q_j|_\omega} \int_{3Q_j} |f|^p d\sigma \right)^{p-1}
\]
\[
\leq C \left( \frac{2}{\beta \lambda} \right)^p \sum_j \chi_{3Q_j^k} \left( \int_\sigma |f|^p d\sigma \right)
\]
\[
\leq C \left( \frac{2}{\beta \lambda} \right)^p \sum_j \left( \int_\sigma |f|^p d\sigma \right),
\]
by the finite overlap condition in (2.6) again. This completes the proof of the good-$\lambda$ inequality (3.5).

The proof of assertion 2 regarding $T_\flat$ is similar. Assertion 3 was discussed earlier and assertion 4 follows readily from assertion 2 and Lemma 2.19.

4. The proof of Theorem 1.24: Strong-type Inequalities

Since conditions (1.27) and (1.21) are obviously necessary for (1.26), we turn to proving the weighted inequality (1.26) for the strongly maximal singular integral $T_\sharp$.

4.1. The Quantitative Estimate. In particular, we will prove
\[
\|T_\sharp f \sigma\|_{L^p(\omega)} \leq C \left\{ \mathcal{M} + \mathcal{M}_* + \mathfrak{T} + \mathfrak{T}_* \right\} \|f\|_{L^p(\sigma)} ,
\]
(4.1)
\[
\mathcal{M} = \sup_{\|f\|_{L^p(\sigma)} = 1} \|\mathcal{M}(f \sigma)\|_{L^p(\omega)} ,
\]
(4.2)
\[ M_* = \sup_{\|g\|_{L^p(\omega)} = 1} \|g\|_{\overline{L^p}(\omega)}, \]

(4.3)

\[ \mathcal{T} = \sup_Q \sup_{\|f\|_{L^\infty} \leq 1} |Q|^{-1/p} \|\chi_Q T_f \chi_Q f\|_{L^p(\omega)}, \]

(4.4)

\[ \mathcal{T}_* = \sup_{\|f\|_{L^p(\sigma)} = 1} \sup_Q |Q|^{-1/p'} \int_Q T_f \chi_Q f (x) d\omega(x). \]

(4.5)

The norm estimates on the maximal function (4.2) and (4.3) are equivalent to the testing conditions in (1.8) and its dual formulation. The term \( \mathcal{T}_* \) also appeared in (3.3).

4.2. The Initial Construction. We suppose that both (1.27) and (1.21) hold, and that \( f \) is bounded with compact support on \( \mathbb{R}^n \). Moreover, in the case (1.28) holds, we see that (1.27) implies (1.8) by Lemma 2.22 and so by Theorem 1.6 we may also assume that the maximal operator \( \mathcal{M} \) satisfies the two weight norm inequality (1.7). It now follows that \( \int (T_f \sigma)^p \omega < \infty \) for \( f \) bounded with compact support. Indeed, \( T_f \sigma \leq C \mathcal{M} f \sigma \) far away from the support of \( f \), while \( T_f \sigma \) is controlled by the testing condition (1.27) near the support of \( f \).

Let \( \{Q^k_j\} \) be the cubes as in (2.5) and (2.6), with the measure \( \nu \) that appears in there being \( \nu = f \sigma \). We will use Lemma 2.29 with this choice of \( \nu \) as well. Now define an ‘exceptional set’ associated to \( Q^k_j \) to be

\[ E^k_j = Q^k_j \cap (\Omega_{k+1} \setminus \Omega_{k+2}). \]

See Figure 4.1. One might anticipate the definition of the exceptional set to be more simply \( Q^k_j \cap \Omega_{k+1} \). We are guided to this choice by the work on fractional integrals [18]. And indeed, the choice of exceptional set above enters in a decisive way in the analysis of the bad function at the end of the proof.

![Figure 4.1. The set \( E^k_j(Q) \).](image-url)
We estimate the left side of (1.26) in terms of this family of dyadic cubes \( \{Q^k_j\}_{k,j} \) by
\[
\int (T_\sharp f \sigma)^p \omega(dx) \leq \sum_{k \in \mathbb{Z}} (2^{k+2})^p |\Omega_{k+1} \setminus \Omega_{k+2}|_\omega \\
\leq \sum_{k,j} (2^{k+2})^p |E^k_j|_\omega.
\]

Choose a linearization \( L \) of \( T_\sharp \) as in (2.13) so that (recall \( R(x) \) is the upper limit of truncation)
\[
R(x) \leq \frac{1}{2} \ell(Q^k_j), \quad x \in E^k_j,
\]
and
\[
T_\sharp \left( \chi_{3Q^k_j} f \sigma \right)(x) \leq 2L \left( \chi_{3Q^k_j} f \sigma \right)(x) + C \frac{1}{|3Q^k_j|} \int_{3Q^k_j} |f| \sigma, \quad x \in E^k_j.
\]

For \( x \in E^k_j \), the maximum principle (2.10) yields
\[
T_\sharp \chi_{3Q^k_j} f \sigma(x) \geq T_\sharp f \sigma(x) - T_\sharp \chi_{3Q^k_j} f \sigma(x) \\
> 2^{k+1} - 2^k - CP \left( Q^k_j, f \sigma \right) \\
= 2^k - CP \left( Q^k_j, f \sigma \right).
\]

From (4.7) we conclude that
\[
L \chi_{3Q^k_j} f \sigma(x) \geq 2^{k-1} - CP \left( Q^k_j, f \sigma \right).
\]
Thus either \( 2^k \leq 4 \inf_{E^k_j} L \chi_{3Q^k_j} f \sigma \) or \( 2^k \leq 4CP \left( Q^k_j, f \sigma \right) \leq 4CM \left( Q^k_j, f \sigma \right) \). So we obtain either
\[
|E^k_j|_\omega \leq C2^{-k} \int_{E^k_j} (L \chi_{3Q^k_j} f \sigma) \omega(dx),
\]
or
\[
|E^k_j|_\omega \leq C2^{-pk} |E^k_j|_\omega M \left( Q^k_j, f \sigma \right)^p \leq C2^{-pk} \int_{E^k_j} (M f \sigma)^p \omega(dx).
\]

Now consider the following decomposition of the set of indices \( (k, j) \):
\[
E = \left\{ (k, j) : |E^k_j|_\omega \leq \beta |NQ^k_j|_\omega \right\},
\]
\[
F = \left\{ (k, j) : (4.9) \text{ holds} \right\},
\]
\[
G = \left\{ (k, j) : |E^k_j|_\omega > \beta |NQ^k_j|_\omega \text{ and } (4.8) \text{ holds} \right\},
\]
where \( 0 < \beta < 1 \) will be chosen sufficiently small at the end of the argument. (It will be of the order of \( c^p \) for a small constant \( c \).) By the ‘crowd control’ condition
of (2.6), we have
\[ \sum_j \chi_{NQ_k^j} \leq C, \quad k \in \mathbb{Z}. \]
We then have the corresponding decomposition:

\[
\int (T_0 f \sigma)^p \omega \leq \left\{ \sum_{(k,j) \in E} + \sum_{(k,j) \in F} + \sum_{(k,j) \in G} \right\} (2^{k+2})^p |E_j^k| \omega
\]
\[ \leq \beta \sum_{(k,j) \in E} (2^{k+2})^p |NQ_j^k| \omega + C \sum_{(k,j) \in F} \int_{E_j^k} (Mf \sigma)^p \omega
\]
\[ + C \sum_{(k,j) \in G} |E_j^k| \omega \left( \frac{1}{\beta |NQ_j^k|} \int_{E_j^k} (L \chi_{3Q_j^k} f \sigma) \omega \right)^p \]
\[ = J(1) + J(2) + J(3) \]
\[ \leq C_0 \left\{ \beta \int (T_0 f \sigma)^p \omega + \beta^{-p} \int |f|^p \sigma \right\}, \]

where \( C_0 \leq C \{ \mathfrak{M} + \mathfrak{M}_* + \mathfrak{F} + \mathfrak{F}_* \}^p \). The last line is the claim that we take up in the remainder of the proof. Once it is proved, note that if we take \( 0 < C_0 \beta < \frac{1}{2} \) and use the fact that \( \int (T_0 f \sigma)^p \omega < \infty \) for \( f \) bounded with compact support, we have proved assertion (1) of Theorem 1.24, and in particular (4.1).

The proof of the strong type inequality requires a complicated series of decompositions of the dominating sums, which are illustrated for the reader’s convenience as a schematic tree in Figure 4.2.

4.3. Two Easy Estimates. Note that the first term \( J(1) \) in (4.12) satisfies
\[ J(1) = \beta \sum_{(k,j) \in E} (2^{k+2})^p |NQ_j^k| \omega \leq C \beta \int (T_0 f \sigma)^p \omega, \]
by the finite overlap condition (4.11). The second term \( J(2) \) is dominated by
\[ C \sum_{(k,j) \in F} \int_{E_j^k} (Mf \sigma)^p \omega \leq C \mathfrak{M} \|f\|_{L^p(\sigma)}^p, \]
by our assumption (1.7). It is useful to note that this is the only time in the proof that we use the maximal function inequality (1.7) - from now on we use the dual maximal function inequality (1.25).

Remark 4.14. In the arguments below we can use Theorem 2 of [18] to replace the dual maximal function assumption \( \mathfrak{M}_* < \infty \) with two assumptions, namely a ‘Poisson two weight Ap condition’ and the analogue of the dual pivotal condition of Nazarov, Treil and Volberg [9]. The Poisson two weight Ap condition is in fact
necessary for the two weight inequality, but the necessity of the dual pivotal conditions for singular integral weighted inequalities is still an open question. On the other hand, the assumption $\mathfrak{M} < \infty$ cannot be weakened here, reflecting that our method requires the maximum principle in Lemma 2.9.

It is the third term $J(3)$ that is the most involved, see Figure 4.2. The remainder of the proof is taken up with the proof of
26 M.T. LACEY, E.T. SAWYER, AND I. URIARTE-TUERO

(4.15) \[ \sum_{(k,j) \in G} R^k_j \left| \int_{E^k_j} (L \chi_{3Q^k_j} f \sigma) \omega \right|^p \leq C \{ M^p + T^p + T^p \} \| f \|_{L^p(\sigma)} \],

where

(4.16) \[ R^k_j = \frac{|E^k_j|}{|N Q^k_j|} \].

Once this is done, the proof of (4.12) is complete, and the proof of assertion (1) is finished.

4.4. The Calderón-Zygmund Decompositions. To carry out this proof, we implement Calderón-Zygmund Decompositions relative to the measure \( \sigma \). These Decompositions will be done at all heights simultaneously. We will use the shifted dyadic grids, see (2.3). For \( \alpha \in \{0, \frac{1}{3}, \frac{2}{3}\} \), let

(4.17) \[ M^\alpha_{s} f(x) = \sup_{x \in Q \in D^\alpha} \frac{1}{|Q|} \int_{Q} |f| d\sigma, \]

\( \Gamma^\alpha_{t} = \{ x \in \mathbb{R} : M^\alpha_{s} f(x) > 2^t \} = \bigcup_s G^{\alpha, t}_s, \)

where \( \{G^{\alpha, t}_s\}_{t,s} \) are the maximal \( D^\alpha \) cubes in \( \Gamma^\alpha_t \). This implies that we have the nested property: If \( G^{\alpha, t}_s \subseteq G^{\alpha, t'}_{s'} \) then \( t > t' \). Moreover, if \( t > t' \) there is some \( s' \) with \( G^{\alpha, t}_s \subseteq G^{\alpha, t'}_{s'} \). These are the cubes used to make a Calderón-Zygmund Decomposition at height \( 2^t \) for the grid \( D^\alpha \) with respect to the measure \( \sigma \).

Of course we have from the maximal inequality in (2.2)

(4.18) \[ \sum_{t,s} 2^{pt} |G^{\alpha, t}_s| \sigma \leq C \| f \|_{L^p(\sigma)}^p. \]

The point of these next several definitions is associate to each dyadic cube \( Q \), a good shifted dyadic grid, and an appropriate height, at which we will build our Calderón-Zygmund Decomposition.

**Principal Labels:** We identify a distinguished subset of the labelling set of the cubes \( \{G^{\alpha, t}_s\} \), which we refer to as the `principal labels’, of the pairs \((t, s)\) parameterizing the cubes \( \{G^{\alpha, t}_s\} \). Define a set of indices \((t, s)\) by

(4.19) \[ L^\alpha = \{ (t, s) : \text{there is no cube } G^{\alpha, t+1}_{s'} \text{ equal to } G^{\alpha, t}_s \}. \]

In other words, if there is a maximal chain of equal cubes \( G^{\alpha, t_0}_s = G^{\alpha, t_0+1}_s = \cdots = G^{\alpha, t_0+N}_s \) we discard all of these indices but \((s_N, t_0 + N)\), the one for which

(4.20) \[ 2^{t_0+N} \leq \frac{1}{|G^{\alpha, t_0+N}_s|} \int_{G^{\alpha, t_0+N}_s} |f| d\sigma \leq 2^{t_0+N+1}. \]
We have this variant of (4.18).

\[
\sum_{(t,s) \in \mathbb{L}^{\alpha}} \left( \frac{1}{|G_{s,t}^{\alpha,t}|_\sigma} \int_{G_{s,t}^{\alpha,t}} |f| \, \sigma \right)^p |G_{s,t}^{\alpha,t}|_\sigma \leq C \|f\|_{L^p(\sigma)}^p.
\]

For \((t,s) \in \mathbb{L}^{\alpha}\), and any \(s'\) we have \(G_{s,t}^{\alpha,t} \cap G_{s',t+1}^{\alpha,t+1} \in \{\emptyset, G_{s',t+1}^{\alpha,t+1}\}\). We will refer to a cube \(G_{s,t}^{\alpha,t}\) with principal label \((t,s)\) as \textit{a \textcolor{red}{principal} cube} (thus every cube \(G_{s,t}^{\alpha,t}\) is a principal cube when properly labelled).

**Select a shifted grid:** Let \(\tilde{\alpha} : \mathcal{D} \rightarrow \{0, \frac{1}{3}, \frac{2}{3}\}^{\alpha}\) be a map so that for \(Q \in \mathcal{D}\), there is a \(\tilde{Q} \in \mathcal{D}^{\tilde{\alpha}(Q)}\) so that \(3\tilde{Q} \subset \mathcal{D}^{\tilde{\alpha}(Q)}\) and \(|\tilde{Q}| \leq C|Q|\). Here, \(C\) is an appropriate constant depending only on dimension. Thus, \(\tilde{\alpha}(Q)\) picks a ‘good’ shifted dyadic grid for \(Q\). Note that

\[
\tilde{Q} \subset MQ,
\]

for some positive dimensional constant \(M\). The cubes \(\tilde{Q}_j^k\) will play a critical role below. See Figure 4.4

**Select a principal cube:** Define \(\mathcal{A}(Q)\) to be the smallest cube from the collection \(\{G_{s,t}^{\tilde{\alpha}(Q),t} : (t,s) \in \mathbb{L}^{\alpha}\}\) that contains \(3Q\); \(\mathcal{A}(Q)\) is uniquely determined by \(Q\) and the choice of function \(\tilde{\alpha}\). Define

\[
\mathbb{H}_s^{\alpha,t} = \{(k,j) : \mathcal{A}(Q_j^k)^{\alpha,t} = G_s^{\alpha,t}\}, \quad (s,t) \in \mathbb{L}^{\alpha}.
\]

This is an important definition for us. The combinatorial structure this places on the corresponding cubes is essential for this proof to work. Note that \(3Q_j^k \subset \tilde{Q}_j^k \in \mathcal{A}(Q_j^k)\).

**Parents:** For any of the shifted dyadic grids \(\mathcal{D}^{\alpha}\), a \(Q \in \mathcal{D}^{\alpha}\) has a unique parent denoted as \(P(Q)\), the smallest member of \(\mathcal{D}^{\alpha}\) that strictly contains \(Q\). We suppress the dependence upon \(\alpha\) here.

**Select maximal parents:** Let

\[
\mathbb{K}_{s,t}^{\alpha,t} = \{r \mid P(G_r^{\alpha,t+1}) \text{ is maximal in } \{P(G_r^{\alpha,t+1}) \mid G_r^{\alpha,t+1} \subset G_s^{\alpha,t}\}\}.
\]

Note that the labels \((t + 1,r)\) with \(r \in \mathbb{K}_{s,t}^{\alpha,t}\) are not necessarily principal labels, although the actual cubes \(G_r^{\alpha,t+1}\) are principal when properly labelled.

**The good and bad functions:** Let \(A_{P(G_r^{\alpha,t+1})} = \frac{1}{|P(G_r^{\alpha,t+1})|_\sigma} \int_{P(G_r^{\alpha,t+1})} f \, \sigma\) be the \(\sigma\)-average of \(f\) on \(P(G_r^{\alpha,t+1})\). Define functions \(g_{s,t}^{\alpha,t}\) and \(h_{s,t}^{\alpha,t}\) satisfying \(f = g_{s,t}^{\alpha,t} + h_{s,t}^{\alpha,t}\) on \(G_{s,t}^{\alpha,t}\) by

\[
g_{s,t}^{\alpha,t}(x) = \begin{cases} A_{P(G_r^{\alpha,t+1})} & x \in P(G_r^{\alpha,t+1}) \text{ with } r \in \mathbb{K}_{s,t}^{\alpha,t} \\ f(x) & x \in G_{s,t}^{\alpha,t} \setminus \bigcup \{P(G_r^{\alpha,t+1}) : r \in \mathbb{K}_{s,t}^{\alpha,t}\}, \end{cases}
\]
that (any of the standard proofs can be adapted to the dyadic setting for $r \in \mathbb{K}_s^{\alpha,t}$. Note that $g_s^{\alpha,t}$ is supported on $g_s^{\alpha,t}$, and has $L^\infty$ norm at most $2^{t+1}$, and that the function $h_s^{\alpha,t}$ is supported on the cubes $P(G_r^{\alpha,t+1})$, and has integral zero with respect to $\sigma$-measure on each such cube.

\begin{equation}
(4.26) \quad h_s^{\alpha,t}(x) = \begin{cases} 
    f(x) - A_{P(G_r^{\alpha,t+1})} & x \in P(G_r^{\alpha,t+1}) \text{ with } r \in \mathbb{K}_s^{\alpha,t} \\
    0 & x \in G_s^{\alpha,t} \setminus \bigcup \{P(G_r^{\alpha,t+1}) : r \in \mathbb{K}_s^{\alpha,t}\}.
\end{cases}
\end{equation}

We extend both $g_s^{\alpha,t}$ and $h_s^{\alpha,t}$ to all of $\mathbb{R}^n$ by defining them to vanish outside $G_s^{\alpha,t}$. Now the average $A_{P(G_r^{\alpha,t+1})}$ is at most $2^{t+1}$ by maximality of the cubes in (4.17). Thus Lebesgue’s differentiation theorem shows that (any of the standard proofs can be adapted to the dyadic setting for positive locally finite Borel measures on $\mathbb{R}^n$)

\begin{equation}
(4.27) \quad |g_s^{\alpha,t}(x)| \leq 2^{t+1} < \frac{2}{|G_s^{\alpha,t}|} \int_{G_s^{\alpha,t}} |f| \, \sigma, \quad \sigma\text{-a.e. } x \in G_s^{\alpha,t}, \ (t, s) \in \mathbb{L}^\alpha.
\end{equation}

That is, $g_s^{\alpha,t}$ is the ‘good’ function and $h_s^{\alpha,t}$ is the ‘bad’ function. See Figure 4.3

We can now refine the final sum on the left side of (4.15) according to the decomposition of $\mathcal{M}_\beta f$. We carry out these steps. In the first step, we fix an $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, and for the remainder of the proof, we only consider $Q_j^s$ for which $\alpha(Q_j^s) = \alpha$. Namely, we will modify the important definition of $G$ in (4.10) to

\begin{equation}
(4.28) \quad G^\alpha = \left\{ (k, j) : \alpha(Q_j^k) = \alpha, \ |E_j^k|_\omega > \beta |NQ_j^k|_\omega \text{ and (4.8) holds} \right\}.
\end{equation}

In the second step, we partition the indices $(k, j)$ into the sets $\mathbb{H}^{\alpha,t}_s$ in (4.23) for $(t, s) \in \mathbb{L}^\alpha$. In the third step, for $(k, j) \in \mathbb{H}^{\alpha,t}_s$, we split $f$ into the corresponding
good and bad parts. This yields the decomposition

\begin{equation}
\sum_{(k,j) \in \mathbb{G}^\alpha} R_j^k \left| \int_{E_j^k} \left( L_{X_{3Q_j}^k} f \right) \sigma \omega(dx) \right|^p \leq C (I + II),
\end{equation}

(4.29)

\begin{equation}
I = \sum_{(t,s) \in \mathbb{L}^\alpha} I_s^t, \quad II = \sum_{(t,s) \in \mathbb{L}^\alpha} II_s^t
\end{equation}

(4.30)

\begin{equation}
I_s^t = \sum_{(k,j) \in \mathbb{I}_{s,t}^\wedge} R_j^k \left| \int_{E_j^k} \left( L_{X_{3Q_j}^k} g_{s,t}^{\alpha,t} \right) \sigma \omega(dx) \right|^p
\end{equation}

(4.31)

\begin{equation}
II_s^t = \sum_{(k,j) \in \mathbb{I}_{s,t}^\wedge} R_j^k \left| \int_{E_j^k} \left( L_{X_{3Q_j}^k} h_{s,t}^{\alpha,t} \right) \sigma \omega(dx) \right|^p
\end{equation}

(4.32)

\begin{equation}
\mathbb{I}_{s,t}^\alpha = \mathbb{G}^\alpha \cap \mathbb{H}_{s,t}^\alpha
\end{equation}

(4.33)

Recall the definition of $R_j^k$ in (4.16). In the definitions of $I$, $I_s^t$ and $II$, $II_s^t$ we will suppress the dependence on $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, the same will be done for the subsequent decompositions of the (difficult) term $II$, although we usually retain the superscript $\alpha$ in the quantities arising in the estimates. In particular, we emphasize that the combinatorial properties of the cubes associated with $I_s^t$ are essential to completing this proof.

Term $I$ requires only condition (1.27) and the maximal theorem (2.2), while term $II$ will require both conditions (1.27) and (1.21), along with the dual maximal function inequality (1.25) and the maximal theorem (2.2). It will be convenient at this point to map out the various decompositions of the terms and the conditions used to control them.

4.5. The Analysis of the good function. We claim that

\begin{equation}
I \leq C \mathcal{M}^p \|f\|_{L^p(\sigma)}^p.
\end{equation}

(4.34)

\textbf{Proof.} We use boundedness of the ‘good’ function $g_{s,t}^{\alpha,t}$, as defined in (4.25), the testing condition (1.27) for $T$, see also (4.4), and finally the universal maximal function bound (2.2) with $\mu = \omega$. Here are the details. For $x \in E_j^k$, (4.7) implies that $L_{X_{3Q_j}^k} g_{s,t}^{\alpha,t}(x) = Lg_{s,t}^{\alpha,t}(x)$ and so

\begin{equation}
I = \sum_{(t,s) \in \mathbb{L}^\alpha} I_s^t
\end{equation}

\begin{equation}
= C \sum_{(t,s) \in \mathbb{L}^\alpha} \sum_{(k,j) \in \mathbb{G}^\alpha \cap \mathbb{H}_{s,t}^\alpha} R_j^k \left| \int_{E_j^k} \left( Lg_{s,t}^{\alpha,t} \right) \sigma \omega \right|^p
\end{equation}

\begin{equation}
\leq C \sum_{(t,s) \in \mathbb{L}^\alpha} \int \mathcal{M}_{\omega}^p \left( X_{\mathbb{G}^\alpha \cap \mathbb{H}_{s,t}^\alpha} \right) \sigma \omega
\end{equation}
\[
\leq C \sum_{(t,s) \in \mathbb{L}^a} \int_{G_{s}^{\alpha,t}} |L G_{s}^{\alpha,t} \sigma|^p \omega \\
\leq C \sum_{(t,s) \in \mathbb{L}^a} 2^{pt} \int_{G_{s}^{\alpha,t}} \left( T_2 g_{s}^{\alpha,t} \right)^p \omega \\
\leq C \mathcal{F} p \sum_{(t,s) \in \mathbb{L}^a} 2^{pt} |G_{s}^{\alpha,t}| \sigma ,
\]

where we have used (4.27) and (1.27) with \( g = \frac{g_{s}^{\alpha,t}}{2^{pt}} \) in the final inequality. This last sum is controlled by (4.18), and completes the proof of (4.34). \( \Box \)

4.6. The Analysis of the Bad Function: Part 1. It remains to estimate term \( II \), as in (4.32), but this is in fact the harder term. Recall the definition of \( K_{s}^{\alpha,t} \) in (4.24). We now write

\[
h_{s}^{\alpha,t} = \sum_{r \in K_{s}^{\alpha,t}} \left( f - A_{P(G_{s}^{\alpha,t}+1)} \right) \chi_{P(G_{s}^{\alpha,t}+1)} \equiv \sum_{r \in K_{s}^{\alpha,t}} b_r,
\]

where the ‘bad’ functions \( b_r \) are supported in the cube \( P(G_{s}^{\alpha,t}+1) \) and have \( \sigma \)-mean zero, \( \int_{P(G_{s}^{\alpha,t}+1)} b_r \sigma = 0 \). To take advantage of this, we will pass to the dual \( L^* \) below.

But first we must address the fact that the triples of the \( D^\alpha \) cubes \( P(G_{r}^{\alpha,t}+1) \) do not form a grid. For \((t, s) \in \mathbb{L}^a \) and \((k, j) \in \mathbb{I}^a \) let \( \tilde{Q}_j^k \) be the largest \( D^\alpha \) cube containing \( \hat{Q}_j^k \) and satisfying

\[
R_{W}^* \tilde{Q}_j^k \subset \Omega_k.
\]

where \( R_{W} = \frac{R_{W}}{M} \) and \( M \) is defined in (1.22). Note that such a cube \( \tilde{Q}_j^k \) exists since \( \hat{Q}_j^k \subset M Q_j^k \) by (1.22) and \( R_{W} Q_j^k \subset \Omega_k \) by (2.6). Moreover, we can arrange to have

\[
3Q_j^k \subset NQ_j^k,
\]

where \( N \) is as in Remark 2.7 by choosing \( R_{W} \) sufficiently large in (2.6). (Recall that the cubes \( \tilde{Q}_j^k \) are chosen at (1.22) above.) See Figure 4.4.

Now momentarily fix \((k, j) \in \mathbb{I}^a \) and set

\[
K_{s}^{\alpha,t}(k, j) = \left\{ r \in K_{s}^{\alpha,t} : P(G_{r}^{\alpha,t}+1) \subset \tilde{Q}_j^k \right\}.
\]

Let

\[
C_{s}^{\alpha,t}(k, j) = \left\{ 3P(G_{r}^{\alpha,t}+1) : r \in K_{s}^{\alpha,t}(k, j) \right\},
\]

be the collection of triples of the \( D^\alpha \) cubes \( P(G_{r}^{\alpha,t}+1) \) with \( r \in K_{s}^{\alpha,t}(k, j) \). We select the maximal triples \( \left\{ 3P(G_{r}^{\alpha,t+1}) \right\}_r \equiv \{ T_\ell \}_\ell \) from the collection \( C_{s}^{\alpha,t}(k, j) \), and assign to each \( r \in K_{s}^{\alpha,t}(k, j) \) the maximal triple \( T_\ell = T_\ell(r) \) containing
$$3P (G_r^{\alpha,t+1})$$ with least \(\ell\). By Lemma 2.23 applied to \(D^\alpha\) the maximal triples \(\{T_\ell\}_\ell\) have finite overlap \(3^n\), and this will prove crucial in (4.65) below. Note that \(T_{t(r)}\) depends on \((k, j)\) as well.

We will pass to the dual of the linearization.

\[
\int_{E_j^k} (L h_{s,t}^{a,t}) \omega(dx) = \sum_{r \in \mathbb{K}_s^{a,t}} \int_{E_j^k} (L b_r) \omega \\
= \sum_{r \in \mathbb{K}_s^{a,t}} \int_{P(G_r^{a,t+1}) \cap 3Q_j^k} \left( L^* \chi_{E_j^k} \omega(dx) \right) b_r \sigma
\]

Note that (4.7) implies \(L^* \nu\) is supported in \(3Q_j^k\) if \(\nu\) is supported in \(E_j^k\), explaining the range of integration above. Continuing, we have

\[
\sum_{r \in \mathbb{K}_s^{a,t}} \int_{P(G_r^{a,t+1}) \cap 3Q_j^k} \left( L^* \chi_{E_j^k \cap T_{t(r)}} \omega \right) b_r \sigma \\
+ C \sum_{r \in \mathbb{K}_s^{a,t}} \mathbf{P} \left( P(G_{r}^{a,t+1}), \chi_{E_j^k \setminus 3P(G_r^{a,t+1})} \omega \right) \int_{P(G_r^{a,t+1})} |f| \sigma \\
+ \left| \sum_{r \in \mathbb{K}_s^{a,t} \setminus \mathbb{K}_s^{a,t}} \int_{P(G_r^{a,t+1}) \cap 3Q_j^k} \left( L^* \chi_{E_j^k \cap T_{t(r)}} \omega \right) b_r \sigma \right|
\]

To see the above inequality, note that for \(r \in \mathbb{K}_s^{a,t}\) we are splitting the set \(E_j^k\) into \(E_j^k \cap T_{t(r)}\) and \(E_j^k \setminus T_{t(r)}\). On the latter set, the hypotheses of Lemma 2.10 are in force, namely the set \(E_j^k \setminus T_{t(r)}\) does not intersect \(3P(G_r^{a,t+1})\), whence we have an estimate on the \(\delta\)-Hölder modulus of continuity of \(L^* \chi_{E_j^k \setminus T_{t(r)}} \omega\). Combine this
with the fact that $b_r$ has $\sigma$-mean zero on $P(G^{a,t+1}_r)$ to derive the estimate below, in which $y^{t+1}_r$ is the center of the cube $G^{a,t+1}_r$.

\[
\left| \int_{P(G^{a,t+1}_r)} \left( L^* \chi^{E_k}_j \setminus T_{t(r)} \right) y \right| \leq \left| \int_{P(G^{a,t+1}_r)} \left( L^* \chi^{E_k}_j \setminus T_{t(r)} \right) (y - L^* \chi^{E_k}_j \setminus T_{t(r)}(y^{t+1}_r)) \right| (b_r \sigma) \]

(4.44)

\[
\leq \int_{P(G^{a,t+1}_r) \cap 3Q^k_j} CP \left( P(G^{a,t+1}_r), \chi^{E_k}_j \setminus T_{t(r)} \right) \delta \left( \frac{|y - y^{t+1}_r|}{\ell(P(G^{a,t+1}_r))} \right) |b_r(y)| d\sigma(y)
\]

\[
\leq CP \left( P(G^{a,t+1}_r), \chi^{E_k}_j \cap 3P(G^{a,t+1}_r) \right) \int_{P(G^{a,t+1}_r)} |f| d\sigma.
\]

We have after application of (4.44),

(4.45) \[ II^t_s = \sum_{(k,j)\in E^{a,t}_s} R^k_j \left[ \int_{E^{a,t}_j} (Lh^s_{\alpha,t}) \omega \right]^p \]

(4.46) \[ \leq II^t_s(1) + II^t_s(2) + II^t_s(3), \]

(4.47) \[ II^t_s(1) = \sum_{(k,j)\in E^{a,t}_s} R^k_j \left[ \sum_{r\in K^{a,t} \setminus E^{a,t}_j} \int_{P(G^{a,t+1}_r)} \left( L^* \chi^{E_k}_j \setminus T_{t(r)} \right) y \right| \leq \left| \int_{P(G^{a,t+1}_r)} \left( L^* \chi^{E_k}_j \setminus T_{t(r)} \right) (y - L^* \chi^{E_k}_j \setminus T_{t(r)}(y^{t+1}_r)) \right| (b_r \sigma) \]

(4.48) \[ \leq \int_{P(G^{a,t+1}_r) \cap 3Q^k_j} CP \left( P(G^{a,t+1}_r), \chi^{E_k}_j \setminus T_{t(r)} \right) \delta \left( \frac{|y - y^{t+1}_r|}{\ell(P(G^{a,t+1}_r))} \right) |b_r(y)| d\sigma(y)
\]

(4.49) \[ \leq CP \left( P(G^{a,t+1}_r), \chi^{E_k}_j \cap 3P(G^{a,t+1}_r) \right) \int_{P(G^{a,t+1}_r)} |f| d\sigma.
\]

Note that we may further restrict the integrations in (4.47) and (4.48) to $P(G^{a,t+1}_r) \cap 3Q^k_j$ since $L^* \chi^{E_k}_j \setminus T_{t(r)} \omega$ is supported in $3Q^k_j$.

4.6.1. Analysis of $II(2)$. We claim that

(4.50) \[ \sum_{(t,s)\in E^{a,t}_s} II^t_s(2) \leq C \mathfrak{M}_s \int |f|^p \sigma. \]

Recall the definition of $\mathfrak{M}_s$ in (4.3).
Proof. We begin by defining a linear operator by

\[ P^k_j(\mu) \equiv \sum_{r \in K^{a,t}_s} P \left( P(\varphi_{a,t}^{\alpha+1}), \chi_{E^k_j \mu} \right) \chi_{P(\varphi_{a,t}^{\alpha+1})}. \]  

In this notation, we have for \((k, j) \in \mathbb{N}_s \times (See (4.23) and (4.32))

\[ \sum_{r \in K^{a,t}_s} P \left( P(\varphi_{a,t}^{\alpha+1}), \chi_{E^k_j \omega} \right) \int_{P(\varphi_{a,t}^{\alpha+1})} |f| \sigma \]

\[ = \sum_{r \in K^{a,t}_s} P \left( P(\varphi_{a,t}^{\alpha+1}), \chi_{E^k_j \omega} \right) \int_{P(\varphi_{a,t}^{\alpha+1})} |f| \sigma \]

\[ \times \left\{ \frac{1}{P(\varphi_{a,t}^{\alpha+1})} \int_{P(\varphi_{a,t}^{\alpha+1})} |f| \sigma \right\} \]

\[ \leq 2^{t+1} \int_{E^k_j} (\varphi_{a,t}^{\alpha+1})^k(\chi_{E^k_j \sigma}) \omega. \]

By assumption, the maximal function \( M(\omega) \) maps \( L^{p'}(\omega) \) to \( L^{p'}(\sigma) \), and we now note a particular consequence of this. In the definition (4.51) we were careful to insert \( \chi_{E^k_j} \) on the right hand side. These sets are pairwise disjoint, whence we have the inequality below for measures \( \mu \).

\[ \sum_{(k, j) \in \mathbb{N}_s \times (4.52) \总结}{P^k_j(\mu)(x)} \]

\[ \leq \sum_{(k, j) \in \mathbb{N}_s \times (4.52) \sum}{\infty \sum{\infty}} \frac{\delta(2^{-\ell})}{|2^\ell P(\varphi_{a,t}^{\alpha+1})|} \left( \int_{2^\ell P(\varphi_{a,t}^{\alpha+1})} \chi_{E^k_j \mu} \right) \chi_{P(\varphi_{a,t}^{\alpha+1})}(x) \]

\[ \leq \sum_{\ell=0}^{\infty} \sum_{r \in K^{a,t}_s} \frac{\delta(2^{-\ell})}{|2^\ell P(\varphi_{a,t}^{\alpha+1})|} \left( \int_{2^\ell P(\varphi_{a,t}^{\alpha+1})} \mu \right) \chi_{P(\varphi_{a,t}^{\alpha+1})}(x) \]

\[ \leq C \chi_{G_a^{\alpha,t}} M(\chi_{G_a^{\alpha,t} \mu})(x). \]

Thus the inequality

\[ \| \chi_{G_a^{\alpha,t}} \|_{L^{p'}(\omega)} \leq C M_\omega \| \chi_{G_a^{\alpha,t}} g \|_{L^{p'}(\omega)} \]

follows immediately. By duality we then have

\[ \| \chi_{G_a^{\alpha,t}} \|_{L^{p'}(\omega)} \leq C M_\omega \| \chi_{G_a^{\alpha,t}} f \|_{L^p(\sigma)}. \]

Note that it was the linearity that we wanted in (4.51), so that we could appeal to the dual maximal function assumption.
We thus obtain
\[
II_s^t(2) \leq 2p(t+1) \sum_{(k,j) \in \ell^s_{t,s}} R^k_j \left[ \int_{\Omega^t_j} (P^k_j)^* (\chi_{G_{s,t}^t}) d\omega \right]^p.
\]
Summing in \((t, s)\) and using \((P^k_j)^* \leq \sum_{(i,j) \in \ell^s_{t,s}} (P^i_j)^*\) for \((k, j) \in \ell^s_{t,s}\) we obtain (4.54)
\[
\sum_{(t,s) \in L^\alpha} II_s^t(2) \leq C \sum_{(t,s) \in L^\alpha} 2^{pt} \sum_{(k,j) \in \ell^s_{t,s}} R^k_j \left[ \int_{\Omega^t_j} (P^k_j)^* (\chi_{G_{s,t}^t}) d\omega \right]^p
\]
\[
= C \sum_{(t,s) \in L^\alpha} 2^{pt} \sum_{(k,j) \in \ell^s_{t,s}} |E^k_j|_\omega \left[ \frac{1}{|N^k_j|_\omega} \int_{\Omega^t_j} (P^k_j)^* (\chi_{G_{s,t}^t}) \omega \right]^p
\]
\[
\leq C \sum_{(t,s) \in L^\alpha} 2^{pt} \int_{G^t_{s,t}} \left[ \sum_{(i,j) \in \ell^s_{t,s}} (P^i_j)^* (\chi_{G_{s,t}^t}) \right]^p \omega
\]
\[
\leq C \mathbb{M}^p \sum_{(t,s) \in L^\alpha} 2^{pt} |G^s_{s,t}|_\sigma^p,
\]
which is bounded by \(C \mathbb{M}^p \int |f|^p \sigma\). In the last line we are applying (4.53).

4.6.2. Decomposition of \(II(3)\). We now dominate the term \(II_s^t(3)\) in (4.49) by the sum of the two terms
\[
II_s^t(3)' \equiv \sum_{(k,j) \in \ell^s_{t,s}} R^k_j \left| \sum_{r \in K_{s,t}^{\alpha,t} \setminus K_{s,t}^{\alpha,t}(k,j)} \int_{P(G_{r,t}^{\alpha,t+1}) \setminus \Omega_{k+2}} \left( L^* \chi_{E_j^f \cap T_{(r,j)}} \omega \right) b_r \sigma(dy) \right|^p
\]
\[
II_s^t(3)'' \equiv \sum_{(k,j) \in \ell^s_{t,s}} R^k_j \left| \sum_{r \in K_{s,t}^{\alpha,t} \setminus K_{s,t}^{\alpha,t}(k,j)} \int_{P(G_{r,t}^{\alpha,t+1}) \setminus \Omega_{k+2}} \left( L^* \chi_{E_j^f \cap T_{(r,j)}} \omega \right) b_r \sigma(dy) \right|^p
\]
The term \(II_s^t(3)''\) is the same as term \(IV'_{s,t}^l\) in (4.63) except that the sum in \(r\) is over \(K_{s,t}^{\alpha,t} \setminus K_{s,t}^{\alpha,t}(k,j)\) in term \(II_s^t(3)''\) and over \(K_{s,t}^{\alpha,t}(k,j)\) in term \(IV'_{s,t}^l\). These two terms will be handled together below.

Remark 4.59. The key difference between these two terms is the range of integration: \(P(G_{r,t}^{\alpha,t+1}) \setminus \Omega_{k+2}\) for \(II_s^t(3)'\) and \(P(G_{r,t}^{\alpha,t+1}) \cap \Omega_{k+2}\) for \(II_s^t(3)''\). Just as for the
fractional integral case, it is the latter case that is harder, requiring combinatorial facts, which we come to at the end of the argument. An additional fact that we return to in different forms, is that the set \( P(G^{a,b+1}_r) \cap \Omega_{k+2} \) can be further decomposed by the Whitney decompositions \( \{Q^k_{j+2}\}_j \).

Recalling the definition in (4.3), we next claim that

\[
(4.60) \quad \sum_{(t,s) \in L^\alpha} H^t_s(3)' \leq C \mathcal{X}_s^p \int |f|^p \sigma.
\]

**Proof.** The term \( H^t_s(3)' \) in (4.40) is handled by observing that if \( r \in \mathbb{K}_{s,j}^a \setminus \mathbb{K}_{s,j}^a(k,j) \) then \( P(G^{a,b+1}_r) \nsubseteq \tilde{Q}_j^k \) yet \( P(G^{a,b+1}_r) \cap 2Q^k_j \neq \phi \) as the support of \( L^* \chi_{E_j^k} \cap T_r(y) \) is contained in \( 2Q^k_j \) by (4.17). The key point here is that there are only a bounded number (depending on \( n \) and \( N \)) of such cubes \( P(G^{a,b+1}_r) \). We have using the form (2.15) of (1.21) with \( g = \chi_{E_j^k} \) and \( Q = 3Q^k_j \) the following estimate for each such cube \( P(G^{a,b+1}_r) \):

\[
(4.61) \quad \left| \int_{P(G^{a,b+1}_r) \setminus \Omega_{k+2}} \left( L^* \chi_{E_j^k} \right) b \sigma(dy) \right|^p \leq \left[ \int_{3Q^k_j} \left| L^* \chi_{E_j^k} \right|' \sigma \right]^{p-1} \times \int_{E_j^k} |h_{s,t}^a|^p \sigma \leq C \mathcal{X}_s^p \left| 3Q^k_j \sigma \right|^{p-1} \int_{E_j^k} |h_{s,t}^a|^p \sigma,
\]

where \( \tilde{E}_j^k = 3Q^k_j \setminus \Omega_{k+2} \) (note that \( \tilde{E}_j^k \) is much larger than \( E_j^k \)). Thus we have

\[
H^t_s(3)' \leq C \mathcal{X}_s^p \sum_{(k,j) \in \mathbb{K}_{s,j}^a} \left| \frac{E_j^k}{\nu} \right| \left| 3Q^k_j \sigma \right|^{p-1} \int_{E_j^k} |h_{s,t}^a|^p \sigma
\]

\[
\leq C \mathcal{X}_s^p \sum_{(k,j) \in \mathbb{K}_{s,j}^a} \int_{\tilde{E}_j^k} |h_{s,t}^a|^p \sigma \leq C \mathcal{X}_s^p \sum_{(k,j) \in \mathbb{G} \cap \mathbb{H}_{s,j}^a} \int_{E_j^k} (|f|^p + |\mathcal{M}_s f|^p) \sigma,
\]

since \( |h_{s,t}^a| \leq |f| + |\mathcal{M}_s f| \). Using

\[
(4.62) \quad \sum_{(t,s) \in L^\alpha} \sum_{(k,j) \in \mathbb{G} \cap \mathbb{H}_{s,j}^a} \chi_{\tilde{E}_j^k} = \sum_{\text{all } k,j} \chi_{\tilde{E}_j^k} \leq 2,
\]

we thus obtain that

\[
\sum_{(t,s) \in L^\alpha} H^t_s(3)' \leq C \mathcal{X}_s^p \int |f|^p \sigma. \]

\[\blacksquare\]
Next we note that the term $II_1^t(1)$ in (4.47) is dominated by $II_3^t(1) \leq III_3^t + IV_3^t$, where

$$III_3^t = \sum_{(k,j) \in \mathbb{K}^t_{3,t}} \sum_{r \in \mathbb{K}^t_{3,t}(k,j)} \left| \sum_{\ell \in \mathbb{K}^t_{3,t}(k,j): \ell = \ell(r)} \int_{P(G_{\ell}^{\alpha,t+1}) \cap \Omega_{k+2}^t} \left( L^* \chi_{E_\ell^t \cap T_{\ell}(r)} \omega \right) b_\sigma(dy) \right|^p$$

(4.63)

$$IV_3^t = \sum_{(k,j) \in \mathbb{K}^t_{3,t}} \sum_{r \in \mathbb{K}^t_{3,t}(k,j)} \left| \sum_{\ell \in \mathbb{K}^t_{3,t}(k,j)} \int_{P(G_{\ell}^{\alpha,t+1}) \cap \Omega_{k+2}^t} \left( L^* \chi_{E_\ell^t \cap T_{\ell}(r)} \omega \right) b_\sigma(dy) \right|^p$$

The term $III_3^t$ includes that part of $b_t$ supported on $P(G_{\ell}^{\alpha,t+1}) \setminus \Omega_{k+2}$, and the term $IV_3^t$ includes that part of $b_t$ supported on $P(G_{\ell}^{\alpha,t+1}) \cap \Omega_{k+2}$, which is the more delicate case.

Recall the definition of $\mathfrak{T}_s$ in (4.5). We claim

$$\sum_{(t,s) \in \mathcal{L}^t} III_3^t \leq C\mathfrak{T}_s \int |f|^p \sigma(dy).$$

Proof. Recall that $r \in \mathbb{K}^{\alpha,t}(k,j)$ in the sum defining $III_3^t$, the definition above (4.39). We will use the definition of $R^t_3$ in (1.16), and the fact that

$$\sum_{\ell} |X_{T_\ell}| \leq 3^n$$

provided $N \geq 9$. We will apply the form (2.15) of (1.21) with $g = \chi_{E_\ell^t \cap T_{\ell}^t}$ and $Q$ equal to $T_{\ell}^t \cap \widetilde{Q}_j^k$, also see (4.34), but we must first address the fact that $T_{\ell}^t \cap \widetilde{Q}_j^k$ may not be a cube. Recall that $\widetilde{Q}_j^k$ is the cube in the shifted grid $\mathcal{D}^\alpha$ that is selected by $Q_j^k$ as in the definition ‘Select a shifted grid’ above, and that $\widetilde{Q}_j^k$ is defined by (4.36) and satisfies $3\widetilde{Q}_j^k \subset NQ_j^k$ - see (4.37). Now $T_{\ell}^t$ is a triple of a cube in the grid $\mathcal{D}^\alpha$ and $Q_j^k$ is a cube in $\mathcal{D}^\alpha$. Thus if $T_{\ell}^t \cap \widetilde{Q}_j^k$ is not a cube, then we must have $T_{\ell}^t \subset 3\widetilde{Q}_j^k$, and in this case we apply (2.15) to the cube $3\widetilde{Q}_j^k$ with $g = \chi_{E_\ell^t \cap T_{\ell}^t}$.

$$III_3^t \leq \sum_{(k,j) \in \mathbb{K}^t_{3,t}} R^t_3 \left[ \sum_{\ell} \sum_{r \in \mathbb{K}^t_{3,t}(k,j): \ell = \ell(r)} \int_{P(G_{\ell}^{\alpha,t+1}) \cap \Omega_{k+2}^t} \left| L^* \chi_{E_\ell^t \cap T_{\ell}(r)} \omega (dx) \right|^{p'} \sigma \right]^{p-1} \int_{E_j^t} |h_{\alpha,t}^s|^{p} \sigma$$

$$\leq \sum_{(k,j) \in \mathbb{K}^t_{3,t}} R^t_3 \left[ \sum_{\ell} \int_{T_{\ell} \cap \widetilde{Q}_j^k} \left| L^* \chi_{E_\ell^t \cap T_{\ell}(r)} \omega (dx) \right|^{p'} \sigma \right]^{p-1} \int_{E_j^t} |h_{\alpha,t}^s|^{p} \sigma$$

$$\leq \mathfrak{T}_s \sum_{(k,j) \in \mathbb{K}^t_{3,t}} R^t_3 \left[ \sum_{\ell} \left| T_{\ell} \cap 3\widetilde{Q}_j^k \right| \omega \right]^{p-1} \int_{E_j^t} |h_{\alpha,t}^s|^{p} \sigma$$
\[ \leq \mathcal{X}^p \sum_{(k,j) \in \mathcal{T}^s_t} \left| E_k^\alpha \right| NQ_k^\beta \int_{E_k^\alpha} \left| h_s \right|^p \sigma \]

\[ \leq C \mathcal{X}^p \sum_{(k,j) \in \mathcal{T}^s_t} \int_{E_k^\alpha} \left| h_s \right|^p \sigma \leq C \mathcal{X}^p \sum_{(k,j) \in \mathcal{G}^s \cap \mathcal{H}^t_s} \int_{E_k^\alpha} \left( |f|^p + |M^s f|^p \right) \sigma , \]

and then (4.64) follows using (4.62) again.

4.7. The Analysis of the Bad Function: Part 2. This is the most intricate and final case. We consider each of the terms \( \mathcal{N}^s_t \) and \( \mathcal{M}^s_t(3)^r \), which differ only in that the sum in \( r \) is over \( \mathcal{K}^s_t(k,j) \) in term \( \mathcal{N}^s_t \) and over \( \mathcal{K}^s_t \setminus \mathcal{K}^s_t(k,j) \) in term \( \mathcal{M}^s_t(3)^r \). We recall these two terms here.

\[ \mathcal{N}^s_t = \sum_{(k,j) \in \mathcal{T}^s_t} R_k^\alpha \sum_{r \in \mathcal{K}^s_t(k,j)} \int_{P(G^s_t \cap \Omega_{k+2})} \left( L^* \chi_{E_k^s \cap \Omega_{k+2}} \right) b_r \sigma(dy) \]

\[ \mathcal{M}^s_t(3)^r = \sum_{(k,j) \in \mathcal{T}^s_t} R_k^\alpha \sum_{r \in \mathcal{K}^s_t \setminus \mathcal{K}^s_t(k,j)} \int_{P(G^s_t \cap \Omega_{k+2})} \left( L^* \chi_{E_k^s \cap \Omega_{k+2}} \right) b_r \sigma(dy) \]

For these terms we will prove

\[ \sum_{(l,s) \in \mathcal{L}^s} \mathcal{N}^s_t \leq C \{ \mathcal{X}^p + \mathcal{X}^p + \mathcal{M}^p_s \} \int |f|^p \sigma , \]

\[ \sum_{(l,s) \in \mathcal{L}^s} \mathcal{M}^s_t(3)^r \leq C \{ \mathcal{X}^p + \mathcal{X}^p + \mathcal{M}^p_s \} \int |f|^p \sigma . \]

where \( \mathcal{X}^p \), \( \mathcal{X}^p \), and \( \mathcal{M}^p_s \) are defined in (4.4), (4.5) and (4.3) respectively. The estimates (4.34), (4.50), (4.60), (4.64), (4.65) prove (4.12), and so complete the proof of assertion 1 of the strong type characterization in Theorem 1.24. Assertions 2 and 3 of Theorem 1.24 follow as in the weak-type Theorem 1.19. Finally, to prove assertion 4 we note that Lemma 2.22 and condition (1.27) imply (1.8), which by Theorem 1.6 yields (1.7).

The argument below handles both terms \( \mathcal{M}^s_t(3)^r \) and \( \mathcal{N}^s_t \) equally well. We will explicitly address the latter term, and discuss the variants needed for the former term in remarks below.

4.7.1. Whitney decompositions with shifted grids. We now use the shifted grid \( \mathcal{D}^\alpha \) in place of the dyadic grid \( \mathcal{D} \) to form a Whitney decomposition of \( \Omega_{k+2} \) in the spirit of (2.6). More precisely, recalling the definition of \( \widetilde{Q}^k_{i,j} \) in (4.36). Note that we could have for \( i < j \)

\[ \widetilde{Q}^k_{i+2} = \widetilde{Q}^k_{j+2} . \]
Let us define
\[(4.69) \quad S^{α,k+2} = \{ i : \text{for all } j, \text{if } (4.68) \text{ holds, then } i \leq j \} .\]

It is immediate that
\[
\left\{ Q_i^{k+2} \right\}_{i : \tilde{\alpha}(Q_i^{k+2}) = α}
\]
is a pairwise disjoint collection of cubes in $\mathcal{D}^α$. These cubes satisfy a modified Whitney-type condition, namely
\[
3Q_i^{k+2} \subset \hat{Q}_i^{k+2} \subset \tilde{Q}_i^{k+2} \subset NQ_i^{k+2} \subset Ω_{k+2},
\]
and hence with $R_W' = \frac{R_W}{M}$, we have
\[
R_W' \hat{Q}_i^{k+2} \subset Ω_{k+2},
\]
\[
3R_W' \hat{Q}_i^{k+2} \cap Ω_{k+2}^c \neq ∅,
\]
\[
\tilde{Q}_j^k \not\subset \tilde{Q}_i^k, \quad j \in S^{α,k+2}, \quad i \in S^{α,ℓ+2} \quad \text{implies} \quad k > ℓ.
\]

We now complete this collection to a pairwise disjoint Whitney covering of $Ω_{k+2}$ by cubes $B_i^{k+2}$ in $\mathcal{D}^α$ satisfying
\[
R_W' B_i^{k+2} \subset Ω_{k+2},
\]
\[
3R_W' B_i^{k+2} \cap Ω_{k+2}^c \neq ∅,
\]
and the following analogue of the nested property in (2.6):
\[(4.70) \quad B_j^k \not\subset B_i^k \text{ implies } k > ℓ.\]

Indeed, we simply use the decomposition in (2.6) with $\mathcal{D}$ replaced by $\mathcal{D}^α$ and $R_W$ replaced by $R_W'$. In particular we have decomposed
\[
Ω_{k+2} = \bigcup_i B_i^{k+2}
\]
into a Whitney decomposition of pairwise disjoint cubes $B_i^{k+2}$ in $\mathcal{D}^α$ that include among them all of the cubes $\hat{Q}_i^{k+2}$ with $i \in S^{α,k+2}$, namely:
\[(4.71) \quad \hat{Q}_i^{k+2} = B_i^{k+2} \text{ for some } ℓ \text{ if } α = \tilde{α}(Q_i^{k+2}) \text{ and } (4.36) \text{ holds}.\]

Note that the set of indices $i$ arising in the decomposition of $Ω_{k+2}$ into cubes $B_i^{k+2}$ is not the same as the set of indices $i$ arising in the decomposition of $Ω_{k+2}$ into cubes $Q_i^{k+2}$, but this should not cause confusion.

Define
\[
S_ℓ = \bigcup \{ P(G^α,t+1) : 3P(G^α,t+1) \subset T_i \},
\]
This is a union of pairwise disjoint cubes. Note that $S_\ell \subset T_\ell$ and now split the term $IV_s^t$ into two pieces as follows:

$$
IV_s^t = \sum_{(k,j) \in I_s^t} R_j \left| \sum_\ell \sum_{i \in T_\ell \cap B_i^{k+2}} \int_{S_\ell \cap Q_i^{k+2}} \left( L^* \chi_{E_i^{k+2} \cap T_\ell} \right) b_\sigma \right|^p
$$

$$
= IV_s^t (1) + IV_s^t (2),
$$

where

$$
I V_s^t = \{ i : A_i^{k+2} > 2^{t+2} \} \quad \text{and} \quad J_s^t = \{ i : A_i^{k+2} \leq 2^{t+2} \},
$$

and where

$$
A_i^{k+2} = \frac{1}{|B_i^{k+2}|} \int_{B_i^{k+2}} |f| d\sigma
$$

denotes the $\sigma$-average of $|f|$ on the cube $B_i^{k+2}$. Thus $IV (1)$ corresponds to the case where the averages are ‘big’ and $IV (2)$ where the averages are ‘small’. The analysis of $IV_s^t (1)$ in (4.72) is the hard case, taken up later.

4.7.2. Replace bad functions by averages. The first task in the analysis of these terms will be to replace part of the ‘bad functions’ $b_\tau$ by their averages over $B_i^{k+2}$, or more exactly the averages $A_i^{k+2}$. We again appeal to the Hölder continuity of $L^* \chi_{E_i^{k+2} \cap T_\ell} \omega$. By construction, $3B_i^{k+2}$ does not meet $E_i^{k+2}$, so that Lemma 2.16 applies. We have for some constant $c_i^{k+2}$ satisfying $|c_i^{k+2}| \leq 2$

$$
\left| \int_{S_\ell \cap B_i^{k+2}} \left( L^* \chi_{E_i^{k+2} \cap T_\ell} \right) b_\sigma - \left\{ c_i^{k+2} \int_{B_i^{k+2}} \left( L^* \chi_{E_i^{k+2} \cap T_\ell} \right) \sigma \right\} A_i^{k+2} \right| 
\leq CP \left( B_i^{k+2}, \chi_{E_i^{k+2} \cap T_\ell} \omega \right) \int_{B_i^{k+2}} |f| \sigma.
$$

Indeed, if $z_i^{k+2}$ is the center of the cube $B_i^{k+2}$, we have

$$
\int_{S_\ell \cap B_i^{k+2}} \left( L^* \chi_{E_i^{k+2} \cap T_\ell} \right) b_\sigma
$$

$$
= L^* \chi_{E_i^{k+2} \cap T_\ell} \omega (z_i^{k+2}) \int_{B_i^{k+2}} \chi_{S_\ell} b_\sigma + O \left\{ P \left( B_i^{k+2}, \chi_{E_i^{k+2} \cap T_\ell} \omega \right) \int_{B_i^{k+2}} |b_\tau| \sigma \right\}
$$

$$
= \left\{ \int_{B_i^{k+2}} \left( L^* \chi_{E_i^{k+2} \cap T_\ell} \right) \sigma \right\} \frac{1}{|B_i^{k+2}|} \int_{S_\ell \cap B_i^{k+2}} b_\sigma
$$
+ O \left\{ \mathbb{P} \left( B_{i}^{k+2}, \chi_{E_{j}^{k+2} \cap T_{\ell}} \right) \int_{B_{i}^{k+2}} \left| b_{r} \right| \sigma \right\}.

Now, the functions $b_{r}$ are given in (4.35), and by construction, we note that

$$\frac{1}{B_{i}^{k+2} \sigma} \left| \int_{S_{i} \cap B_{i}^{k+2}} b_{r} \sigma \right| \leq \frac{2}{B_{i}^{k+2} \sigma} \int_{B_{i}^{k+2}} \left| f \right| \sigma = 2A_{i}^{k+2},$$

so that with

$$c_{i}^{k+2} = \frac{1}{A_{i}^{k+2}} \frac{1}{B_{i}^{k+2} \sigma} \int_{S_{i} \cap B_{i}^{k+2}} b_{r} \sigma,$$

we have $|c_{i}^{k+2}| \leq 2$ and

$$\int_{S_{i} \cap B_{i}^{k+2}} \left( L^{*} \chi_{E_{j}^{k+2} \cap T_{\ell}} \right) b_{r} \sigma = \left\{ c_{i}^{k+2} \int_{B_{i}^{k+2}} \left( L^{*} \chi_{E_{j}^{k+2} \cap T_{\ell}} \right) \sigma \right\} A_{i}^{k+2}$$

$$+ O \left\{ \mathbb{P} \left( B_{i}^{k+2}, \chi_{E_{j}^{k+2} \cap T_{\ell}} \right) \int_{B_{i}^{k+2}} \left| b_{r} \right| \sigma \right\}.$$

We apply (4.75) to be able to write

$$Iv_{s}^{t} (2) = \sum_{(k,j) \in \mathcal{I}_{v, t}} R_{j}^{k} \left| \sum_{i \in \mathcal{I}_{j}^{l} : B_{i}^{k+2} \subset T_{\ell}} \int_{S_{i} \cap B_{i}^{k+2}} \left( L^{*} \chi_{E_{j}^{k+2} \cap T_{\ell}} \right) b_{r} \sigma \right|^{p}$$

$$\leq \sum_{(k,j) \in \mathcal{I}_{v, t}} R_{j}^{k} \left| \sum_{i \in \mathcal{I}_{j}^{l} : B_{i}^{k+2} \subset T_{\ell}} \left[ \int_{B_{i}^{k+2}} \left( L^{*} \chi_{E_{j}^{k+2} \cap T_{\ell}} \right) \sigma \right] c_{i}^{k+2} A_{i}^{k+2} \right|^{p}$$

$$+ \sum_{(k,j) \in \mathcal{I}_{v, t}} R_{j}^{k} \left| \sum_{i \in \mathcal{I}_{j}^{l} : B_{i}^{k+2} \subset T_{\ell} \cap \partial Q_{j}^{k+2}} \mathbb{P} \left( Q_{i}^{k+2}, \chi_{E_{j}^{k+2} \cap T_{\ell}} \right) \int_{Q_{i}^{k+2}} \left| f \right| \sigma \right|^{p}$$

$$= V_{s}^{t} (1) + V_{s}^{t} (2).$$

We claim that

$$\sum_{(t,s) \in L_{s}^{t}} V_{s}^{t} (1) \leq C \mathcal{F} \| f \|_{L^{p}(\sigma)}^{p}.$$

**Proof.** We estimate $V_{s}^{t} (1)$ by

$$V_{s}^{t} (1) = \sum_{(k,j) \in \mathcal{I}_{v, t}} R_{j}^{k} \left| \sum_{i \in \mathcal{I}_{j}^{l} : B_{i}^{k+2} \subset T_{\ell}} L \left( \sum_{i \in \mathcal{I}_{j}^{l} : B_{i}^{k+2} \subset T_{\ell}} c_{i}^{k+2} A_{i}^{k+2} \right) \right|^{p}$$
\[
\leq 2^{p(t+3)} \sum_{(k,j) \in \mathbb{N}^2} R_j^k \left( \sum_{\ell} \int_{E_j^k \cap T\ell} T_\ell \left( \sum_{i \in J_{i}^k : B_i^{k+2} \subset C T\ell} c_i^{k+2} \frac{A_{i}^{k+2}}{2t^{k+2}} \chi_{B_i^{k+2}} \sigma \right) \right)^p \omega \\
\leq 2^{p(t+3)} \sum_{(k,j) \in \mathbb{N}^2} R_j^k \left( \int_{3Q_j^k} T_\ell \left( \chi_{G_{\alpha,t}^{k+1} h_{t,s,\ell} \sigma} \right) \right)^p \omega \\
\leq 2^{p(t+3)} \int M_\omega \left( T_\ell \left( \chi_{G_{\alpha,t}^{k+1} h_{t,s,\ell} \sigma} \right) \right)^p \omega \\
\leq 2^{p(t+3)} \int T_\ell \left( \chi_{G_{\alpha,t}^{k+1} h_{t,s,\ell} \sigma} \right)^p \omega \leq 2^{p(t+3)} |G_{\alpha,t}^{k+1} \sigma| \omega
\]

by the testing condition (1.27) on \( T\ell \), see (4.4). Here the function

\[
h_{t,s,\ell} = \sum_{i \in J_{i}^k : B_i^{k+2} \subset C T\ell} c_i^{k+2} \frac{A_{i}^{k+2}}{2t^{k+2}} \chi_{B_i^{k+2}}
\]

has modulus bounded by 1.

We see that (4.18) then implies (4.77). \( \square \)

4.7.3. The bound for \( V(2) \).

We next claim that

\[
\sum_{(t,s) \in \mathbb{N}^2} V_s^l (2) \leq C M_{s}^p \|f\|_{L^p(\sigma)}^p.
\]

Here, \( M_{s} \) is defined in (4.3).

**Proof.** The estimate for term \( V_s^l (2) \) is similar to that of \( II_s^l (2) \) above, see (4.50). We define

\[
P_j^k (\mu) \equiv \sum_{\ell} \sum_{i \in J_{i}^k : B_i^{k+2} \subset C T\ell \cap 3Q_j^k} P \left( B_i^{k+2}, \chi_{E_j^k \cap T\ell} \sigma \right) \chi_{B_i^{k+2}}.
\]

We observe that this operator satisfies \( \sum_{k,j} P_j^k (\mu) \leq CM(\mu) \) due to the fact that the sets \( E_j^k \) are pairwise disjoint and the cubes \( T\ell \) have bounded overlaps. See the discussion following (4.39).

With this notation, the summands in the definition of \( V_s^l (2) \), as given in (4.76), are

\[
\sum_{\ell} \sum_{i \in J_{i}^k : B_i^{k+2} \subset C T\ell \cap 3Q_j^k} P \left( B_i^{k+2}, \chi_{E_j^k \cap T\ell} \sigma \right) \left\{ \int_{B_i^{k+2}} \frac{1}{|B_i^{k+2}|} \int_{B_i^{k+2}} |f| \sigma \right\}
\leq 2^{l+2} \int P_j^k (\omega) \sigma \quad \text{(since } i \in J_{i}^l) \\quad (4.79)
\]

\[
\leq 2^{l+2} \int_{Q_j^k} (P_j^k)^*(\sigma) \omega.
\]
We can restrict the integration to $Q_j^k$ due to the definition of $P_j^k$. Our assertion is that we have the inequality

\begin{equation}
\left\| \chi_{G_{a,t}^k} \sum_{k,j} (P_j^k)^* (\chi_{G_{a,t}^k} | f) \right\|_{L^p(\omega)} \leq C \mathfrak{M}_s \| \chi_{G_{a,t}^k} f \|_{L^p(\omega)}.
\end{equation}

Indeed, this follows from the same reasoning as the proof of (4.53).

We then have from (4.76) and (4.79)

\begin{align*}
\sum_{(t,s) \in L^\alpha} V_s^t (2) & \leq C \left( \sum_{(t,s) \in L^\alpha} 2^{pt} \sum_{(k,j) \in I_{s,t}^a} R_j^k \left| \int_{Q_j^k} (P_j^k)^* (\sigma) \omega \right|^p \right)^{1/p} \\
& \leq C \left( \sum_{(t,s) \in L^\alpha} 2^{pt} \sum_{(k,j) \in I_{s,t}^a} |E_{j,k}|_\omega |NQ_j^k|_\omega^{-1} \int_{Q_j^k} (P_j^k)^* (\sigma) \omega \right)^{1/p} \\
& \leq C \left( \sum_{(t,s) \in L^\alpha} 2^{pt} \left[ \mathcal{M}_\omega \left( \chi_{G_{a,t}^k} \sum_{(k,j) \in I_{s,t}^a} (P_j^k)^* (\chi_{G_{a,t}^k} \sigma) \right) \right] \right)^{1/p} \\
& \leq C \left( \sum_{(t,s) \in L^\alpha} 2^{pt} \left| \chi_{G_{a,t}^k} \sum_{(k,j) \in I_{s,t}^a} (P_j^k)^* (\chi_{G_{a,t}^k} \sigma) \right| \right)^{1/p} \\
& \leq C \mathfrak{M}_s \left( \sum_{(t,s) \in L^\alpha} 2^{pt} |G_{a,t}^k|_\sigma \right) \leq C \mathfrak{M}_s \left( \int |f| \sigma \right).
\end{align*}

In last line we are using (4.80) and (4.18).

4.7.4. The bound for IV and VI(2). To estimate the first term $IV_s^t (1)$ in (4.72), we again apply (4.75) to be able to write

\begin{align}
IV_s^t (1) & \leq C \left( \sum_{(k,j) \in I_{s,t}^a} R_j^k \left[ \sum_{\ell \in I_j^k} \sum_{T_i^k \subseteq T_i^j} \left[ \int_{B_i^{k+2}} |L^* \chi_{E_j^k \cap \tau T_i^j} \sigma| \right] A_{i}^{k+2} \right] \right)^{1/p} \\
& \quad + C \left( \sum_{(k,j) \in I_{s,t}^a} R_j^k \left[ \sum_{\ell \in I_j^k} \sum_{T_i^k \subseteq T_i^j} \mathbf{P} \left( B_i^{k+2} \cap \chi_{E_j^k \cap \tau T_i^j} \sigma \right) \int_{B_i^{k+2}} f \sigma \right] \right)^{1/p} \\
& = VI_s^t (1) + VI_s^t (2). 
\end{align}

We comment that we are able to replace the averages of the bad function by $2A_{i}^{k+2}$ since $i \in I_s^k$ in this case, see [4.73], and note that membership in $I_s^k$ implies that the average of $|b_i|$ is dominated by the average of $|f|$ over the cube $B_i^{k+2}$. 
We claim that
\[ \sum_{(t,s) \in L^α} VI^I_{s}(2) \leq C \mathcal{M}^P \| f \|_{L^p(σ)}^p \]

Proof. The term \( VI^I_{s}(2) \) can be handled the same way as the term \( II^I_{s}(2) \), see (4.50), and the term \( V^I_{s}(2) \), see (4.78). We omit the details.

4.7.5. The bound for \( VI^I_{s}(1) \). Our final estimate in the proof of (4.66) is for the term \( VI^I_{s}(1) \). We claim that
\[ \sum_{(t,s) \in L^α} VI^I_{s}(1) \leq C T^P \| f \|_{L^p(σ)}^p. \]

Proof. This proof will require combinatorial facts related to the principal cubes, and the definition of the collection \( G \) in (4.28). Also essential is the implementation of the shifted dyadic grids. We detail the arguments below.

Remark 4.85. A difference in the arguments for \( IV^I_{s} \) and \( II^I_{s}(3)'' \) in (4.66) and (4.67) arises. Recall that for the term \( IV^I_{s} \) every cube \( B_{k}^{i+2} \) that arises is contained in the associated cube \( \tilde{Q}^{k}_{j} \) since \( r \in \mathbb{K}^{a,t}(k,j) \)—see (4.38) and the definition of \( IV^I_{s} \). We claim that the same is true for the term \( II^I_{s}(3)'' \), i.e. every cube \( B_{k}^{i+2} \) that arises is contained in the associated cube \( \tilde{Q}^{k}_{j} \). To see this, note that \( Q^{k}_{j} \) decomposes as a pairwise disjoint union of cubes \( B_{k}^{i+2} \) and thus we have
\[
\int_{P(G^{a,t+1}_{r}) \cap \Omega_{k+2}} \left( L^{*} \chi_{E_{j}^{k} \cap 3P(G^{a,t+1}_{r})} \omega \right) b_{r} \sigma = \sum_{i:B_{k}^{i+2} \cap \tilde{Q}^{k}_{j} \neq \emptyset} \int_{B_{k}^{i+2}} \left( L^{*} \chi_{E_{j}^{k} \cap 3P(G^{a,t+1}_{r})} \omega \right) b_{r} \sigma,
\]
since the support of \( L^{*} \chi_{E_{j}^{k} \cap 3P(G^{a,t+1}_{r})} \omega \) is contained in \( 2Q^{k}_{j} \subset \tilde{Q}^{k}_{j} \subset \tilde{Q}^{k}_{j} \) by (4.7).

Since both \( B_{k}^{i+2} \) and \( \tilde{Q}^{k}_{j} \) lie in the grid \( D^α \), one of these cubes is contained in the other. Now \( B_{k}^{i+2} \) cannot strictly contain \( \tilde{Q}^{k}_{j} \) since \( \tilde{Q}^{k}_{j} = B_{k}^{i} \) for some \( \ell \) and the cubes \( \{ B_{k}^{i} \}_{k,j} \) satisfy the nested property (4.70). It follows that we must have
\[ B_{k}^{i+2} \subset \tilde{Q}^{k}_{j}. \]

We now continue giving the proof the estimate for \( IV^I_{s} \) in (4.66), noting that the remaining arguments apply also to (4.67) because of (4.86).

We first estimate the sum in \( i \) inside term \( VI^I_{s}(1) \). Recall that the sum in \( i \) is over those \( i \) such that \( B_{k}^{i+2} \subset T_{\ell} \) for some \( \ell \) where \( \{ T_{\ell} \}_{\ell} \) is the set of maximal cubes in the collection \( \{ 3P(G^{a,t+1}_{r}) : r \in \mathbb{K}^{a,t}(k,j) \} \). See the discussion at (4.39), and (4.65). It is also important to note that since \( L^{*} \chi_{E_{j}^{k} \cap T_{\ell}} \omega \) is supported in
are the Whitney decomposition of $\Omega^k$ is also restricted to those $i$ such that $B_{i+2}^{k+2} \subset \hat{Q}^j$, while the sum in $i$ deriving from term $II^j_i (3)^\prime$ is as well by (4.86). We have

$$
\left| \sum_i \left[ \int_{B_{i+2}^{k+2}} \left| L^* \chi_{E^j_i \cap T \omega} \right| \right] A_i^{k+2} \right|^p \\
\leq \sum_i \left| B_{i+2}^{k+2} \right|_\sigma \left( A_i^{k+2} \right)^p \left[ \sum \left| B_{i+2}^{k+2} \right|_\sigma \left[ \int_{B_{i+2}^{k+2}} \left| L^* \chi_{E^j_i \cap T \omega} \right| \sigma \right] \right]^{p-1} \\
\leq \sum_i \left| B_{i+2}^{k+2} \right|_\sigma \left( A_i^{k+2} \right)^p \left[ \sum \left| T \right|_\omega \right]^{p-1} \\
\leq C \sum_i \left| B_{i+2}^{k+2} \right|_\sigma \left( A_i^{k+2} \right)^p \left[ NQ_j^k \right]_\omega^{p-1},
$$

where the second to last inequality uses the form (2.15) of (1.21) with $g = \chi_{E^j_i \cap T \omega}$ and $Q = T \ell$. With this we obtain,

$$(4.87) \quad \text{VI}^l_i(1) \leq C \beta^{-1} \sum_{(k,j) \in \Gamma_k^l} \sum_{i \in I^k_j} R_k^l \left| B_{i+2}^{k+2} \right|_\sigma \left( A_i^{k+2} \right)^p \left[ NQ_j^k \right]_\omega^{p-1},$$

Here we have also used the fact that a given cube $Q$ occurs as $B_{i+2}^{k+2}$ at most a bounded number of times in the sum for $\text{VI}^l_i(1)$. This property, listed below, is first of some properties that we should formalize at this point.

4.7.6. The combinatorial arguments.

**Bounded Occurrence of Cubes:** A given cube $Q$ can occur only a finite number of times as $B_{i+2}^{k+2}$ in (4.87). Specifically, let $(k_1, i_1), \ldots, (k_M, i_M) \in \mathcal{G}$, as defined in (4.10), be such that $Q = B_{i+2}^{k+2}$ for $1 \leq \sigma \leq M$. It follows that $M < C \beta^{-1}$, where $\beta$ is the small constant chosen in the definition of $\mathcal{G}$. The constant $C$ depends only on dimension.

*Proof.* The Whitney structure, see (2.6), is decisive here. First we show that a given $Q$ can occur only a finite number of times as $Q_{i+2}^{k+2}$. The distinction between this claim and the property we are proving is that the cubes $\{B_{i+2}^{k+2}\}$ are the Whitney decomposition of $\Omega_{k+2}$ constructed in § 4.7.1.

Suppose that $(k_1, i_1), \ldots, (k_M, i_M) \in \mathcal{G}$ are such that $Q = Q_{i+2}^{k+2}$ for $1 \leq \sigma \leq M$. Let $Q_{i+2}^{k+2}$ be such that $Q \subset 3Q_{i+2}^{k+2}$, with the indices $(k_{\sigma}, i_{\sigma})$ being distinct. Observe that the finite overlap property in (2.6) then gives us the observation
that a single integer $k$ can occur only a bounded number of times among the $k_1, \ldots, k_M$.

After a relabelling, we can assume that all the $k_\sigma$ for $1 \leq \sigma \leq M'$ are distinct, listed in increasing order, and the number of $k_\sigma$ satisfies $CM' > M$. The nested property of $(2.6)$ assures us that $Q$ is an element of the Whitney decomposition of $\Omega_k$ for all $k_1 \leq k \leq k_M$.

**Remark 4.88.** Note that the $k_\sigma$ are not necessarily consecutive since we require that $(k_\sigma, j_\sigma) \in \mathbb{G}^\alpha$. Nevertheless, the cube $Q$ does occur among the $Q^k_{j_\sigma}$ for any $k$ that lies between $k_\sigma$ and $k_{\sigma+1}$. These latter occurrences of $Q$ may be unbounded, but we are only concerned with bounding those for which $(k_\sigma, j_\sigma) \in \mathbb{G}^\alpha$, and it is these occurrences that our argument is treating.

Thus for $2 \leq \sigma \leq M'$, both $Q$ and $Q^k_{j_\sigma}$ are members of the Whitney decomposition of the open set $\Omega_{k_\sigma}$. By the Whitney condition, we have $R_W Q^k_{j_\sigma} \subset \Omega_k$ but $3R_W Q \not\subset \Omega_{k_\sigma}$, whence $3R_W Q \not\subset R_W Q^k_{j_\sigma}$. Since we are free to take $R_W \geq 4$, this last conclusion shows that the number of possible locations for the cubes $Q^k_{j_\sigma}$ is bounded by a constant depending only on dimension.

Apply the pigeonhole principle to the locations of the $Q^k_{j_\sigma}$. After a relabelling, we can argue under the assumption that all $Q^k_{j_\sigma}$ equal the same cube $Q'$ for all choices of $1 \leq \sigma \leq M''$ where $CM'' > M$. There is another condition that the indices $(k_\sigma, j_\sigma)$ must satisfy: They are members of $\mathbb{G}$, as given in $(4.10)$. In particular we have $|E^k_{j_\sigma}| \omega \geq \beta |NQ'| \omega$ where $N$ is as in Remark 2.7. The $k_\sigma$ are distinct, and the sets $E^k_{j_\sigma} \subset Q'$ are pairwise disjoint, hence $M'' \beta |NQ'| \omega \leq |Q'| \omega$ implies $M'' \leq \beta^{-1}$. Our proof of the bounded occurrence of $Q$ as one of the $Q^k_{j_\sigma}$ is complete.

Since $Q^k_{i+2} \subset 3Q^k_{i+2} \subset \tilde{Q}^k_{i+2} \subset \widetilde{Q}^k_{i+2} = B^k_{i+2} \subset NQ^k_{i+2}$, the above argument shows that there are only a bounded number of times that a given cube $Q$ can arise as $B^k_{i+2}$ with $i \in \mathcal{S}^{\alpha, k+2}$, the latter collection defined in $(2.6)$. Finally, to deal with those $B^k_{i+2}$ that do not arise as some $\tilde{Q}^k_{i+2}$, i.e. $i \notin \mathcal{S}^{\alpha, k+2}$, we simply apply the entire Whitney argument above using that the $B^k_{i+2}$ are defined as in $(2.6)$ but with $\mathcal{D}^\alpha$ in place of $\mathcal{D}$ and $R'_{W}$ in place of $R_W$.

**Definition 4.89.** We say that a cube $B^k_{i+2}$ satisfying the defining condition in $VT_i'(1)$, namely

there is $(k, j) \in \mathbb{I}^{\alpha, t} = \mathbb{G}^\alpha \cap \mathbb{H}^{\alpha, t}_s$ such that

$B^k_{i+2} \subset \tilde{Q}^k_{j}$ and

$B^k_{i+2} \subset \text{some } P \left( G_{r, t+1}^{\alpha, t} \right) \subset G_s^{\alpha, t}$ satisfying $A_s^{k+2} > 2^{t+2}$,

is a final type cube for the pair $(t, s) \in \mathbb{L}^\alpha$ generated from $Q^k_{j}$. 


We can now complete the bound for \( \sum_{(t,s) \in \mathbb{L}^a} V_l^a(t) \). The collection of cubes
\[
\mathcal{F} \equiv \{ B_i^{k+2} : B_i^{k+2} \text{ is a final type cube generated from some } Q_j^k \text{ with } (k, j) \in \mathbb{I}_s^{a,t} \text{ for some pair } (t, s) \in \mathbb{L}^a \}
\]
satisfies the following three properties:

**Property 1:** \( \mathcal{F} \) is a nested grid in the sense that given any two distinct cubes in \( \mathcal{F} \), either one is strictly contained in the other, or they are disjoint (ignoring boundaries).

**Property 2:** If \( B_i^{k+2} \) and \( B_{i'}^{k'+2} \) are two distinct cubes in \( \mathcal{F} \) with \( Q_{i'}^{k'+2} \subsetneq Q_i^{k+2} \), and \( k, k' \) have the same parity, then
\[
A_i^{k+2} > 2A_{i'}^{k'+2}.
\]

**Property 3:** A given cube \( B_i^{k+2} \) can occur at most a bounded number of times in the grid \( \mathcal{F} \).

**Proof of Properties 1, 2 and 3.** Property 1 is obvious from the properties of the dyadic shifted grid \( \mathcal{D}^a \). Property 3 follows from the ‘Bounded Occurrence of Cubes’ noted above. So we turn to Property 2. It is this Property that prompted the use of the shifted dyadic grids.

Indeed, since \( B_{i'}^{k'+2} \subsetneq B_i^{k+2} \), it follows from the nested property \( (4.70) \) that \( k' > k \). By Definition \( (4.89) \) there are cubes \( Q_{i'}^{k'} \) and \( Q_j^k \) satisfying
\[
B_{i'}^{k'+2} \subset \widetilde{Q}_{i'}^{k'} \quad \text{and} \quad B_i^{k+2} \subset \widetilde{Q}_j^k,
\]
and also cubes \( G_s^{a,t'} \subset G_s^{a,t} \) such that \( (k', j') \in \mathbb{I}_s^{a,t'} \) and \( (k, j) \in \mathbb{I}_s^{a,t} \) with \( (t', s') = (t, s) \), so that in particular,
\[
\widetilde{Q}_{i'}^{k'} \subset G_{s'}^{a,t'} \quad \text{and} \quad \widetilde{Q}_j^k \subset G_s^{a,t}.
\]

Now \( k' \geq k + 2 \) and in the extreme case where \( k' = k + 2 \), it follows from \( (4.71) \) that the \( \mathcal{D}^a \)-cube \( \widetilde{Q}_{i'}^{k'} \) is one of the cubes \( B_i^{k+2} \), so in fact it must be \( B_i^{k+2} \) since \( B_{i'}^{k'+2} \subset B_i^{k+2} \). Thus we have
\[
B_{i'}^{k'+2} \subset \widetilde{Q}_{i'}^{k'} = B_i^{k+2}.
\]

In the general case \( k' \geq k + 2 \) we have instead
\[
B_{i'}^{k'+2} \subset \widetilde{Q}_{i'}^{k'} \subset B_i^{k+2}.
\]

Now \( A_i^{k+2} > 2^{t+2} \) by Definition \( (4.89) \) and so there is \( t_0 \geq t + 2 \) determined by the condition
\[
(4.90) \quad 2^{t_0} < A_i^{k+2} \leq 2^{t_0+1},
\]
and also \( s_0 \) such that
\[
B_i^{k+2} \subset G_{s_0}^{a,t_0} \subset G_s^{a,t}.
\]
Combining inclusions we have
\[ \tilde{Q}^{\alpha}_{ij} \subset B_{k}^{i+2} \subset C_{s_{0}}^{\alpha}, \]
and since \((k', j') \in \mathbb{E}^{\alpha}_{\beta} \), we obtain \(G_{s_{0}}^{\alpha} \subset C_{s_{0}}^{\alpha}. \) Since \((t', s') \in \mathbb{L}^\alpha \) is a principal label, we have the key property that
\[ (4.91) \quad t' \geq t_{0}. \]
Thus using \((4.91)\) and \((4.90)\) we obtain
\[ A_{i}^{k+2} > 2^{t'+2} \geq 2^{t_{0}+2} \geq 2A_{i}^{k+2}, \]
which is Property 2. \(\blacksquare\)

**Proof of \((4.84)\).** Now for \(Q = B_{k}^{i+2} \in \mathcal{F}\) set
\[ A(Q) = \frac{1}{|Q|_{\sigma}} \int_{Q} |f|_{\sigma} = A_{k}^{i+2} = \frac{1}{|B_{k}^{i+2}|_{\sigma}} \int_{B_{k}^{i+2}} |f|_{\sigma}. \]

With the above three properties we can continue from \((4.87)\) to estimate as follows. Use the trivial inequality \(R_{k}^{j} |NQ_{j}|_{\omega}^{p-1} \leq 1. \) In the display below by \(\sum_{i}^{*}\) we mean the sum over \(i\) such that \(B_{k}^{i+2}\) is contained in some \(P(G_{r,t}^{\alpha}, s_{0}) \subset \tilde{Q}_{j}^{\alpha}\), and also in some \(\tilde{Q}_{j}^{k}\) with \((k, j) \in \mathbb{E}^{\alpha}_{\beta}\), and satisfying \(A_{i}^{k+2} > 2^{t'+2}\).

\[
\sum_{(t,s) \in \mathbb{L}^\alpha} V_{k}^{i}(1) \leq C \Sigma_{k}^{p} \sum_{(t,s) \in \mathbb{L}^\alpha} \sum_{(k,j) \in \mathbb{E}^{\alpha}_{\beta}} \sum_{i}^{*} |B_{k}^{i+2}|_{\sigma} (A_{i}^{k+2})^{p}
\]
\[= C \Sigma_{k}^{p} \sum_{Q \in \mathcal{F}} |Q|_{\sigma} A(Q)^{p} = \sum_{Q \in \mathcal{F}} |Q|_{\sigma} \left[ \frac{1}{|Q|_{\sigma}} \int_{Q} |f|_{\sigma} \right]^{p}
\]
\[= C \Sigma_{k}^{p} \int_{\mathbb{R}^{n}} \sum_{Q \in \mathcal{F}} \chi_{Q} (x) \left[ \frac{1}{|Q|_{\sigma}} \int_{Q} |f|_{\sigma} \right]^{p} d\sigma(x)
\]
\[\leq C \Sigma_{k}^{p} \int_{\mathbb{R}^{n}} \sup_{x \in Q, Q \in \mathcal{F}} \left[ \frac{1}{|Q|_{\sigma}} \int_{Q} |f|_{\sigma} \right]^{p} d\sigma(x)
\]
\[\leq C \Sigma_{k}^{p} \int_{\mathbb{R}^{n}} M_{\alpha}^{p} f(x)^{p} \sigma(dx) \leq C \Sigma_{k}^{p} \int_{\mathbb{R}^{n}} |f(x)|^{p} d\sigma(x),
\]
where the second to last line follows since for fixed \(x \in \mathbb{R}^{n}\), the sum
\[\sum_{Q \in \mathcal{F}} \chi_{Q} (x) \left[ \frac{1}{|Q|_{\sigma}} \int_{Q} |f|_{\sigma} \right]^{p}\]
is supergeometric by properties 1, 2 and 3 above, i.e. for any two distinct cubes \(Q\) and \(Q'\) in \(\mathcal{F}\) each containing \(x\), the ratio of the corresponding values is bounded.
away from 1, more precisely,

$$\left( \frac{1}{|Q|} \int_Q |f| \right)^p \sigma \left( \frac{1}{|Q'|} \int_{Q'} |f| \right)^p
\notin [2^{-p}, 2]$$

This completes the proof of (4.84). \(\square\)

REFERENCES


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