PARALLEL SPINORS ON PSEUDO-RIEMANNIAN SPIN^q MANIFOLDS

RAFAEL HERRERA*

Abstract. We study simply-connected irreducible non-locally symmetric pseudo-Riemannian Spin^q manifolds admitting parallel quaternionic spinors.

1. Introduction

In [8], Wang classified the irreducible simply-connected Riemannian Spin manifolds admitting parallel spinors. In particular, such manifolds must be Ricci-flat with holonomy $SU(m)$, $Sp(m)$, G_2 or $Spin(7)$. In [6], Moronoianu classified the simply-connected Riemannian Spin^c manifolds admitting parallel spinors showing that such manifolds must be the product of a Ricci-flat Spin manifold and non-Ricci-flat Kähler manifold endowed with its canonical (or anticanonical) Spin^c structure. In [3], we generalized these results by showing that a Riemannian Spin^q manifold admitting a parallel spinor must be the product of a Ricci-flat Spin manifold and a non-Ricci-flat Kähler manifold endowed with its canonical Spin^q structure.

In [1], Baum and Kath characterized all the simply-connected irreducible non-locally symmetric pseudo-Riemannian Spin manifolds admitting parallel spinors. More precisely, they showed that the holonomy group must be one of the following: $SU(r,s)$, $Sp(r,s)$, G_2, $G'_{2(2)}$, $G_2^{\mathbb{C}}$, $Spin(7)$, $Spin(4,3)$, $Spin(7,\mathbb{C})$. In [4], Ikemakhen generalized this result to simply connected irreducible non-locally symmetric pseudo-Riemannian Spin^c manifolds admitting parallel spinors, showing that the holonomy group must be one in the list of Baum and Kath, or $U(r,s)$. In this note, we study the pseudo-Riemannian Spin^q case.

Theorem 1. *Let M be a connected, simply-connected, non-locally symmetric, irreducible pseudo-Riemannian Spin^q manifold of dimension $r+s$ and index r. Then M admits a parallel spinor if and only if it is either a Spin manifold admitting a parallel spinor or a Kähler non-Ricci-flat manifold.*

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In Section 2 we give some preliminaries on the group $Spin^q(r,s)$ and $Spin^q$-structures on pseudo-Riemannian manifolds. In Section 3 we prove the main Theorem.

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2. Preliminaries on Spin^q structures

2.1. **The group** $Spin^q(r,s)$**.** Let $\langle\, ,\rangle_{r,s}$ denote the usual scalar product of signature (r,s) on $\mathbb{R}^{r+s}$. Let $Cl_{r,s}$ denote the Clifford algebra of $\mathbb{R}^{r,s} := (\mathbb{R}^{r+s}, \langle\, ,\rangle_{r,s})$ and $\mathbb{C}l_{r,s}$ its complexification. Let the dot "$\cdot$" denote Clifford multiplication of $\mathbb{C}l_{r,s}$. The Clifford algebra $\mathbb{C}l_{r,s}$ contains the group

$$Spin(r,s) := \{X_1 \cdot \ldots \cdot X_{2k};\ \ \langle X_i, X_i\rangle_{r,s} = \pm 1;\ \ k \geq 0\}.$$

Let $Sp(1)$ denote the group of unit quaternions, which is isomorphic to $SU(2)$. Let us define the group

$$Spin^q(r,s) = Spin(r,s) \times_{\mathbb{Z}_2} Sp(1).$$

The following sequences are exact (see [7]):

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(r,s) \longrightarrow SO(r,s) \longrightarrow 1,$$

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^q(r,s) \longrightarrow SO(r,s) \times SO(3) \longrightarrow 1.$$

2.2. **Spin^q structures.**

Definition 1. *Let M be an oriented pseudo-Riemannian manifold with a fixed metric and let $P_{SO(r,s)}(M)$ denote its (positively oriented) $SO(r,s)$-frame bundle.*

M is called $Spin^q$ if it admits a $Spin^q$ structure consisting of a $SO(3)$-principal bundle $P_{SO(3)}(M)$, a principal $Spin^q(r,s)$ bundle $P_{Spin^q(r,s)}(M)$ and a $Spin^q$ equivariant projection

$$P_{Spin^Q(r,s)}(M) \longrightarrow P_{SO(r,s)}(M) \times P_{SO(3)}(M)$$

Remark 1. M carries a $Spin^q$-structure if and only if the second Stiefel-Whitney class of M, $w_2(M)$, equals the second Stiefel-Whitney class of $P_{SO(3)}(M)$

$$w_2(M) = w_2(P_{SO(3)}(M)).$$

Recall that on a Spin^c manifold, the auxiliary $U(1)$-bundle of a Spin^c structure has an associated complex line bundle L. Let $\Delta_{r,s}(M)$ denote the locally defined spinor bundle of M. The Spin^c structure has an associated globally defined vector bundle $\Delta^c_{r,s} = \Delta_{r,s}(M) \otimes L^{1/2}$, whose sections are also called spinors. Similarly, a Spin^q structure has a an associated globally defined quaternionic spinor bundle $\Delta^q_{r,s} = \Delta_{r,s}(M) \otimes \Delta(E)$ where $\Delta(E)$ denotes the locally defined spinor bundle of the rank 3 oriented Riemannian vector bundle E associated to the auxiliary bundle $P_{SO(3)}$ of the Spin^q structure.

Remark 2. In general, we have the following situation.

(1) A Spin manifold admits trivial Spin^c and Spin^q structures
(2) A Spin^c manifold canonically admits a Spin^q structure. If M is not spin, the Spin^c bundle is $\Delta^c_{r,s} = \Delta_{r,s}(M) \otimes L^{1/2}$. Therefore, the direct sum bundle $(\Delta_{r,s} \otimes L^{1/2}) \oplus (\Delta_{r,s} \otimes L^{-1/2})$ defines a Spin^q structure whose $SO(3)$ bundle is the underlying real vector bundle of $S^2(L^{1/2} \oplus L^{-1/2}) = L + \mathbb{C} + L^{-1}$. We shall call this structure the *canonical* Spin^q structure of a Spin^c manifold.
(3) A Spin^c manifold is not necessarily Spin.
(4) A Spin^q manifold may be neither Spin nor Spin^c.

Example. Any irreducible pseudo-Riemannian Kähler manifold is canonically a Spin^c manifold. The holonomy group H of (M, g) is $U(r', s')$, where $(r, s) = (2r', 2s')$ is the signature of (M, g). The canonical (resp. anti-canonical) complex line bundle provides the complex line bundle needed to define a Spin^c structure on M, and therefore a canonical Spin^q structure on M.

2.3. **Connections on pseudo-Riemannian Spin^q manifolds.** Let M be a connected oriented pseudo-Riemannian manifold admitting a Spin^q structure $P_{Spin^q(r,s)}(M)$. The Levi-Civita connection ω on M together with a chosen fixed connection θ on $P_{SO(3)}$ define a connection on $P_{Spin^q(r,s)}(M)$. The Levi-Civita connection induces the covariant derivative

$$\nabla : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM),$$

where

$$\nabla v_i = \sum_{j=1}^{n} \omega_{ji} \otimes v_j,$$

$\{v_1, \ldots, v_n\}$ denotes a local orthonormal frame of TM, and the collection of 1-forms $\omega_{ji} = \langle \nabla v_j, v_i \rangle$. The covariant derivative ∇ is compatible with the pseudo-Riemannian metric, and if $R = \nabla \circ \nabla$, for $X, Y \in TM$, $R_{X,Y} v_i = \sum_{j=1}^{r} v_j \Omega_{ji}(X, Y)$, where $\Omega_{ji} = d\omega_{ji} + \sum_{k=1}^{n} \omega_{jk} \wedge \omega_{ki}$. Similarly, let $\{e_1, e_2, e_3\}$ be a local orthonormal frame for the rank 3 oriented Riemannian auxiliary vector bundle E associated to $P_{SO(3)}(M)$, so that the covariant derivative induced by the connection θ is $\nabla^E \colon \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$, $\nabla^E e_i = \sum_{j=1}^{3} \theta_{ji} \otimes e_j$, for a collection of 1-forms θ_{ji}. ∇^E is also compatible with the corresponding metric. Let $R^E = \nabla^E \circ \nabla^E$, $X, Y \in TM$, so that $R^E_{X,Y} e_i = \sum_{j=1}^{3} e_j \Theta_{ji}(X, Y)$, where $\Theta_{ji} = d\theta_{ji} + \sum_{k=1}^{3} \theta_{jk} \wedge \theta_{ki}$.

Let $\Delta_{r,s}(M)$ and $\Delta(E)$ denote the locally defined spinor bundles. The quaternionic spinor bundle $\Delta^q_{r,s}(M) = \Delta_{r,s}(M) \otimes \Delta(E)$ is globally defined and inherits the following covariant derivative. Let $\psi \in \Gamma(\Delta^q_{r,s}(M))$ and ∇^q defined by

$$\nabla^q\psi = d\psi + \frac{1}{2}\sum_{i<j}\omega_{ji}\, v_i\cdot v_j\cdot\psi + \frac{1}{2}\sum_{k<l}\theta_{lk}\, e_l\cdot e_k\cdot\psi, \tag{1}$$

which is compatible with the induced metric. Since $\Delta(E_x)\cong\mathbb{H}$

$$\nabla^q\psi = d\psi + \frac{1}{2}\sum_{i<j}\omega_{ji}\, v_i\cdot v_j\cdot\psi + \frac{1}{2}(i\theta_{23}+j\theta_{31}+k\theta_{12})\cdot\psi$$

and

$$\nabla^q(\nabla^q\psi) = \frac{1}{2}\sum_{i<j}\Omega_{ij}\, v_i\cdot v_j\cdot\psi + \frac{1}{2}(i\Theta_{23}+j\Theta_{31}+k\Theta_{12})\cdot\psi. \tag{2}$$

Now, if $\{\varphi^i\}$ is a frame dual to $\{v_i\}$, we can rewrite (2) as

$$\nabla^q(\nabla^q\psi) = \frac{1}{4}\sum_{i<j}\left(\sum_{k,l}R_{ijkl}\varphi^k\wedge\varphi^l\right) v_i\cdot v_j\cdot\psi + \frac{1}{2}(i\Theta_{23}+j\Theta_{31}+k\Theta_{12})\cdot\psi.$$

Remark. A Spin^q structure on a simply-connected manifold M whose $P_{SO(3)}$ bundle is trivial with a flat connection is canonically identified with a Spin structure and the covariant derivative ∇^q is the same as ∇ on spinor bundles.

Remark. Since $\Delta(E_x)$ can be identified with the quaternions, we also have multiplication by the quaternions on the right which commutes with ∇^q.

3. Parallel spinors

Let M be a simply-connected pseudo-Riemannian Spin^q manifold. A quaternionic spinor $\psi\in\Gamma(\Delta^q)$ is *parallel* if

$$\nabla^q_X\psi = 0$$

for every vector field X.

Lemma 3.1. *Let X be a vector field and ψ a parallel spinor. Then*

$$\mathrm{Ric}(X)\cdot\psi = (X\lrcorner\Theta)\cdot\psi, \tag{3}$$

where Ric *denotes the Ricci tensor as a type $(1,1)$ tensor, and $\Theta = i\Theta_{23}+j\Theta_{31}+k\Theta_{12}$.*

The proof is analogous to that in [2, pages 64-65]. □

Proof of Theorem 1. Consider the sub-bundle V of TM whose fiber at a point $x\in M$ is

$$V_x = \{X\in T_xM | X\cdot\psi = 0\}.$$

First notice that V is parallel since

$$0 = \nabla^q_Z(X\cdot\psi) = \nabla_z X\cdot\psi + X\cdot\nabla^q_z\psi = \nabla_z X\cdot\psi.$$

Secondly, $X\cdot\psi=0$ implies

$$X\cdot X\cdot\psi=-|X|^2\,\psi=0.$$

This means that $|X|=0$ in an open dense subset of M since ψ is non-trivial. Therefore, $|X|=0$ over all of M. Thus the bundle V is isotropic. By the holonomy principle $V=0$.

Step 1. Define the distribution $\mathcal{D}\subset TM$ with fiber

$$\mathcal{D}_x=\{X\in T_xM\,|\,\exists\, Y_1,Y_2,Y_3\in T_xM,\ X\cdot\psi=i\,Y_1\cdot\psi+j\,Y_2\cdot\psi+k\,Y_3\cdot\psi\}.$$

The distribution $\mathcal{D}$ is parallel. First notice

$$\begin{aligned}\nabla^q(i\psi)&=id\psi+i\frac{1}{2}\sum_{i<j}\omega_{ji}v_i\cdot v_j\cdot\psi+\frac{1}{2}(i\theta_{23}+j\theta_{31}+k\theta_{12})i\psi\\&=id\psi+i\frac{1}{2}\sum_{i<j}\omega_{ji}v_i\cdot v_j\cdot\psi+i\frac{1}{2}(i\theta_{23}-j\theta_{31}-k\theta_{12})\psi\\&=i\nabla^q\psi+(-k\theta_{31}+j\theta_{12})\psi\\&=(-k\theta_{31}+j\theta_{12})\psi,\end{aligned}$$

and similarly

$$\begin{aligned}\nabla^q(j\psi)&=(k\theta_{23}-i\theta_{12})\psi,\\\nabla^q(j\psi)&=(-j\theta_{23}+i\theta_{31})\psi.\end{aligned}$$

Let $X\in\Gamma(\mathcal{D})$ and Z be a vector field. Thus

$$\begin{aligned}\nabla_ZX\cdot\psi&=\nabla_ZX\cdot\psi+X\cdot\nabla^q_Z\psi\\&=\nabla^q_Z(X\cdot\psi)\\&=\nabla^q_Z(Y_1\cdot i\psi+Y_2\cdot j\psi+Y_3\cdot k\psi)\\&=(i\nabla_ZY_1\cdot\psi+Y_1\cdot\nabla^q_Z(i\psi))+(j\nabla_ZY_2\cdot\psi+Y_2\cdot\nabla^q_Z(j\psi))\\&\quad+(k\nabla_ZY_3\cdot\psi+Y_3\cdot\nabla^q_Z(k\psi))\\&=\quad i(\nabla_ZY_1-\theta_{12}(Z)Y_2+\theta_{31}(Z)Y_3)\cdot\psi\\&\quad+j(\nabla_ZY_2+\theta_{12}(Z)Y_1-\theta_{23}(Z)Y_3)\cdot\psi\\&\quad+k(\nabla_ZY_3-\theta_{31}(Z)Y_1+\theta_{23}(Z)Y_2)\cdot\psi,\end{aligned}$$

so $\nabla_ZX\in\Gamma(\mathcal{D})$. Since M is irreducible, either $\mathcal{D}=TM$ or $\mathcal{D}=0$. If $\mathcal{D}=0$, by Lemma 3.1,

$$\mathrm{span}\{\mathrm{Ric}(X)|X\text{ vector field}\}\subset\mathcal{D}=\{0\},$$

so that Θ vanishes identically, the connection of $P_{SO(3)}(M)$ is flat and M is Spin as in [1].

If $\mathcal{D}=TM$, we proceed as follows.

Step 2. Let us assume there exists a quaternion $q_0 = ai + bj + ck$ with $a^2 + b^2 + c^2 = 1$, $q_0^2 = -1$, such that the distribution $\mathcal{E}$ with fiber at $x \in M$

$$\mathcal{E}_x = \{X \in T_xM \,|\, \exists Y \in T_xM,\ X \cdot \psi = Y \cdot \psi \cdot q_0\},$$

is non-trivial. The bundle $\mathcal{E}$ is parallel. Namely, let $X \in \Gamma(\mathcal{E})$ and Z be a vector field

$$\begin{aligned}(\nabla_Z X) \cdot \psi = (\nabla_Z X) \cdot \psi + X \cdot (\nabla^q_Z \psi) &= \nabla^q_Z (X \cdot \psi) \\ &= \nabla^q_z (Y \cdot \psi \cdot q_o) \\ &= ((\nabla_Z Y) \cdot \psi + Y \cdot (\nabla^q_Z \psi)) \cdot q_0 \\ &= (\nabla_Z Y) \cdot \psi \cdot q_0,\end{aligned}$$

so that $\nabla_Y X \in \Gamma(\mathcal{E})$. Since M is irreducible, either $\mathcal{E} = TM$ or $\mathcal{E} = 0$.

If $\mathcal{E} = TM$, we can define a parallel complex structure on M as follows. For any vector field X, define the almost complex structure J_0 by the equation

$$X \cdot \psi = J_0(X) \cdot \psi \cdot q_0, \tag{4}$$

since by this definition, $J_0(J_0(X)) = -X$. To see that it is orthogonal, multiply (4) by X on the left

$$X \cdot X \cdot \psi = X \cdot J_0(X) \cdot \psi \cdot q_0,$$

$$-|X|^2\,\psi = X \cdot J_0(X) \cdot \psi \cdot q_0. \tag{5}$$

Multiply (4) by $J_0(X)$ on the left

$$\begin{aligned}J_0(X) \cdot X \cdot \psi &= J_0(X) \cdot J_0(X) \cdot \psi \cdot q_0, \\ J_0(X) \cdot X \cdot \psi &= -|J_0(X)|^2\,\psi \cdot q_0.\end{aligned}$$

Multiply the last equation by $-q_0$ on the right

$$-|J_0(X)|^2\,\psi = -J_0(X) \cdot X \cdot \psi \cdot q_0 = (X \cdot J_0(X) + 2\big\langle X, J_0(X)\big\rangle) \cdot \psi \cdot q_0. \tag{6}$$

Subtract (6) from (5) to get

$$\psi((-|X|^2 + |J_0(X)|^2) + 2\big\langle X, J_0(X)\big\rangle q_0^{-1}) = 0,$$

which is essentially multiplication by a complex number. Since ψ in non-trivial

$$(-|X|^2 + |J_0(X)|^2) + 2\big\langle X, J_0(X)\big\rangle q_0^{-1} = 0$$

and

$$|X| = |J_0(X)| \quad \text{and} \quad \big\langle X, J_0(X)\big\rangle = 0.$$

Now, taking the covariant derivative of (4) with respect to a vector field Z

$$\begin{aligned}\nabla^q_Z(X\cdot\psi) &= (\nabla_Z X)\cdot\psi + X\cdot(\nabla^q_Z\psi) = (\nabla_Y X)\cdot\psi\\ &= \nabla^q_Z(J_0(X)\cdot\psi\cdot q_0)\\ &= (\nabla_Z(J_0(X))\cdot\psi + J_0(X)\cdot(\nabla^q_Z\psi))\cdot q_0\\ &= \nabla_Z(J_0(X))\cdot\psi\cdot q_0\end{aligned}$$

gives

$$(\nabla_Z X)\cdot\psi = \nabla_Z(J_0(X))\cdot\psi\cdot q_0.$$

Substitute X with $\nabla_Z(X)$ in (4)

$$(\nabla_Z X)\cdot\psi = J_0(\nabla_Z X)\cdot\psi\cdot q_0.$$

Subtracting the last two equations gives

$$(\nabla_Z(J_0(X)) - J_0(\nabla_Z X))\cdot\psi\cdot q_0 = 0.$$

As before, this says that $(\nabla_Z(J_0(X)) - J_0(\nabla_Z X)) \in \Gamma(V) = 0$, thus

$$(\nabla J_0)(X,Z) = \nabla_Z(J_0(X)) - J_0(\nabla_Z X) = 0.$$

Since X and Z are arbitrary, $\nabla J_0 = 0$, which means M is Kähler.

If $\mathcal{E} = 0$, we proceed as follows.

Step 3. By Step 2, the following intersections are trivial

$$TM\cdot\psi\cap TM\cdot\psi\cdot i = TM\cdot\psi\cap TM\cdot\psi\cdot j = TM\cdot\psi\cap TM\cdot\psi\cdot k = \{0\},$$

which imply

$$TM\cdot\psi\cdot i\cap TM\cdot\psi\cdot j = TM\cdot\psi\cdot j\cap TM\cdot\psi\cdot k = TM\cdot\psi\cdot i\cap TM\cdot\psi\cdot k = \{0\}.$$

Thus, the bundle $TM\cdot\psi\cdot i\oplus TM\cdot\psi\cdot j\oplus TM\cdot\psi\cdot k$ is a direct sum. Consider the distribution $\mathcal{F}\subset TM$ with fiber at $x\in M$

$$\mathcal{F}_x = \{X\in T_xM \mid \exists Y_1,Y_2,Y_3\in T_xM,\ X\cdot\psi = Y_1\cdot\psi\cdot i + Y_2\cdot\psi\cdot j + Y_3\cdot\psi\cdot k\},$$

i.e. for any vector field $X\in\Gamma(\mathcal{F})$, $X\cdot\psi$ can be uniquely written as

$$X\cdot\psi = Y_1\cdot\psi\cdot i + Y_2\cdot\psi\cdot j + Y_3\cdot\psi\cdot k.$$

The distribution $\mathcal{F}$ is parallel. Let $X\in\Gamma(\mathcal{F})$ and Z be a vector field then

$$\begin{aligned}\nabla_Z X\cdot\psi &= \nabla_Z X\cdot\psi + X\cdot\nabla^q_Z\psi = \nabla^q_Z(X\cdot\psi)\\ &= (\nabla_Z Y_1\cdot\psi + Y_1\cdot\nabla^q_Z\psi)\cdot i + (\nabla_Z Y_2\cdot\psi + Y_2\cdot\nabla^q_Z\psi)\cdot j\\ &\quad + (\nabla_Z Y_3\cdot\psi + Y_3\cdot\nabla^q_Z\psi)\cdot k\\ &= (\nabla_Z Y_1\cdot\psi)\cdot i + (\nabla_Z Y_2\cdot\psi)\cdot j + (\nabla_Z Y_3\cdot\psi)\cdot k,\end{aligned}$$

so $\nabla_Z X\in\Gamma(\mathcal{F})$. Since M is irreducible, either $\mathcal{F} = TM$ or $\mathcal{F} = 0$. If the former, set $I(X) = Y_1,\quad J(X) = Y_2,\quad K(X) = Y_3$, so that

$$X\cdot\psi = I(X)\cdot\psi\cdot i + J(X)\cdot\psi\cdot j + K(X)\cdot\psi\cdot k, \tag{7}$$

which multiplied by i, j and k gives the following equations

$$I(X)\cdot\psi = (-X)\cdot\psi\cdot i + (-K(X))\cdot\psi\cdot j + J(X)\cdot\psi\cdot k,$$
$$J(X)\cdot\psi = K(X)\cdot\psi\cdot i + (-X)\cdot\psi\cdot j + (-I(X))\cdot\psi\cdot k,$$
$$K(X)\cdot\psi = (-J(X))\cdot\psi\cdot i + I(X)\cdot\psi\cdot j + (-X)\cdot\psi\cdot k.$$

Therefore I, J, K are three almost complex structures satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K, \ldots .$$

In order to show they are orthogonal almost complex structures, multiply (7) by X on the left

$$-|X|^2\psi = X\cdot I(X)\cdot\psi\cdot i + X\cdot J(X)\cdot\psi\cdot j + X\cdot K(X)\cdot\psi\cdot k. \tag{8}$$

Now multiply (7) on the left by $I(X)$ and on the right by $-i$

$$-|I(X)|^2\psi = -I(X)\cdot X\cdot\psi\cdot i + I(X)\cdot K(X)\cdot\psi\cdot j - I(X)\cdot J(X)\cdot\psi\cdot k. \tag{9}$$

Subtract (9) from (8)

$$\psi((-|X|^2 + |I(X)|^2) + 2\langle X, I(X)\rangle i) =$$
$$= (X\cdot J(X) - I(X)\cdot K(X))\cdot\psi\cdot j + (X\cdot K(X) + I(X)\cdot J(X))\cdot\psi\cdot k; \tag{10}$$

substitute X with $I(X)$ in (10)

$$\psi((-|I(X)|^2 + |X|^2) - 2\langle I(X), X\rangle i) =$$
$$= (X\cdot J(X) - I(X)\cdot K(X))\cdot\psi\cdot j + (X\cdot K(X) + I(X)\cdot J(X))\cdot\psi\cdot k. \tag{11}$$

Finally subtract (11) from (10) to get

$$\psi(2(-|X|^2 + |I(X)|^2) + 4\langle X, I(X)\rangle i) = 0,$$

which implies

$$|X|^2 = |I(X)|^2$$
$$\langle X, I(X)\rangle = 0.$$

Similarly for all the other orthongonality relations between X, $I(X)$, $J(X)$ and $K(X)$.

Taking the covariant derivative of (7) with respect to a vector field Z yields

$$\nabla_Z X\cdot\psi = \nabla_Z(I(X))\cdot\psi\cdot i + \nabla_Z(J(X))\cdot\psi\cdot j + \nabla_Z(K(X))\cdot\psi\cdot k, \tag{12}$$

where Z is a vector field. Now substituting X with $\nabla_Z X$ in (7)

$$\nabla_Z X\cdot\psi = I(\nabla_Z X)\cdot\psi\cdot i + J(\nabla_Z X)\cdot\psi\cdot j + K(\nabla_Z X)\cdot\psi\cdot k. \tag{13}$$

By subtracting (13) from (12) we get

$$0 = (\nabla_Z(I(X)) - I(\nabla_Z X))\cdot\psi\cdot i$$
$$+ (\nabla_Z(J(X)) - J(\nabla_Z X))\cdot\psi\cdot j + (\nabla_Z(K(X)) - K(\nabla_Z X))\cdot\psi\cdot k.$$

Given that such a linear combination is unique

$$(\nabla_Z(I(X)) - I(\nabla_Z X)) \cdot \psi = 0,$$
$$(\nabla_Z(J(X)) - J(\nabla_Z X)) \cdot \psi = 0,$$
$$(\nabla_Z(K(X)) - K(\nabla_Z X)) \cdot \psi = 0,$$

so that the vectors fields $\nabla_Z(I(X)) - I(\nabla_Z X)$, $\nabla_Z(J(X)) - J(\nabla_Z X)$ and $\nabla_Z(K(X)) - K(\nabla_Z X)$ belong to $\Gamma(V) = 0$. Thus

$$\nabla_Z(I(X)) - I(\nabla_Z X) = 0,$$
$$\nabla_Z(J(X)) - J(\nabla_Z X) = 0,$$
$$\nabla_Z(K(X)) - K(\nabla_Z X) = 0,$$

and, therefore, the three almost complex structures are parallel $\nabla I = \nabla J = \nabla K = 0$. Hence, the manifold M is hyperkähler and $\mathrm{Ric}_M \equiv 0$, which means the connection of the Spin^q structure is flat, the Spin^q structure is trivial and [1] applies.

If $\mathcal{F}^\perp = TM$, then at each $x \in M$

$$(T_xM \cdot \psi) \cap (T_xM \cdot \psi \cdot i \oplus T_xM \cdot \psi \cdot j \oplus T_xM \cdot \psi \cdot k) = \{0\}.$$

Thus

$$T_xM \cdot \psi \oplus T_xM \cdot \psi \cdot i \oplus T_xM \cdot \psi \cdot j \oplus T_xM \cdot \psi \cdot k$$

is a direct sum in $\Delta_x^q(M)$ with quaternions multiplying on the right, which implies that

$$T_xM \cdot \psi + iT_xM \cdot \psi + jT_xM \cdot \psi + kT_xM \cdot \psi$$

must also de a direct sum and

$$(TM \cdot \psi) \cap (iTM \cdot \psi \oplus jTM \cdot \psi \oplus kTM \cdot \psi) = \{0\},$$

which contradicts our working assumption that $\mathcal{D} = TM$. □

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Centro de Investigación en Matemáticas
A. P. 402, Guanajuato, Gto.
C.P. 36000, México.
E-mail address: rherrera@cimat.mx