THE BREZIS-NIRENBERG TYPE PROBLEM INVOLVING THE SQUARE ROOT OF THE LAPLACIAN

JINGGANG TAN

Abstract. We establish existence and non-existence results to the Brezis-Nirenberg type problem involving the square root of the Laplacian in a bounded domain with zero Dirichlet boundary condition.

1. Introduction

In this paper we are concerned with positive solutions to the Brezis-Nirenberg type problem involving the square root of the Laplacian operator in a bounded domain with zero Dirichlet boundary condition. Particularly, we are looking for a function u satisfying the nonlinear problem involving the square root of the Laplacian:

$$\begin{cases} A_{1/2}u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $f(u) = u^{\frac{n+1}{n-1}} + \mu u$, $\mu \geq 0$ and Ω is a smooth bounded domain in $\mathbb{R}^n$ and $A_{1/2}$ stands for the square root of the Laplacian operator $-\Delta$ in Ω with zero Dirichlet boundary value on $\partial\Omega$. We establish the existence and nonexistence of positive solutions to this problem.

The fractional powers of the Laplacian, which are called fractional Laplacians and correspond to Lévy stable processes, appear in anomalous diffusion phenomena in physics, biology as well as other areas. They occur in flame propagation, chemical reaction in liquids, population dynamics. Lévy diffusion processes have discontinuous sample paths and heavy tails, while Brownian motion has continuous sample paths and exponential decaying tails. These processes have been applied to American options in mathematical finance for modelling the jump processes of the financial derivatives such as futures, forwards, options, and swaps, see [2] and references therein. Moreover, they play an important role in the study of the quasi-geostrophic equations in geophysical fluid dynamics. Nonlinear heat equation was considered by Varlamov [21]. He proved the existence and uniqueness of a global solution, and constructed the solution in the form of a series of eigenfunctions of the Laplace operator in a ball. Recently the fractional Laplacians attract much interest in nonlinear analysis. Caffarelli and Silvestre [10]

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gave a new formulation of the fractional Laplacians through Dirichlet-Neumann maps. The regularity of the obstacle problem for the fractional powers of the Laplacian operator was proved by Silvestre [19]. Moreover, Caffarelli et al [9],[8] studied a free boundary problem: the Signorini problem involving fractional Laplacians as well as random homogenization of fractional obstacle problems.

Cabré and Solà-Morales [6] studied layer solutions (solutions which are monotone with respect to one variable) of

$$(-\Delta)^{1/2}u = f(u) \quad \text{in } \mathbb{R}^n,$$

where f is of balanced bistable type. That is, if $G(u) = -\int_0^u f(s)\,ds$, then G has two, and only two, absolute minima at the same height. They developed some new ingredients, a nonlocal Modica type estimate, as well as a conserved Hamiltonian quantity for every layer solution. Cabré and Tan [7] established the existence of positive solutions for problem (1.1) with power-type nonlinearities in the subcritical case, the regularity and an L^∞ estimate of Brezis-Kato type for weak solutions, a priori estimates of Gidas-Spruck type and a symmetry result of Gidas-Ni-Nirenberg type.

The purpose of this paper is to look for positive solutions of nonlinear problem (1.1) with critical nonlinearities involving the square root of the Laplacian $A_{1/2}$. Note that $A_{1/2}$ is a nonlocal operator in Ω, but we will realize it through a local problem in $\Omega \times (0,\infty)$. We mention that the half Laplacian in the whole space is a well studied operator. Let u be a smooth function $u \in C_0^\infty(\mathbb{R}^n)$. There is a unique harmonic extension $v \in C^\infty(\mathbb{R}^{n+1}_+)$ of u in a half space such that $D^k v(x,y) \to 0$ as $|(x,y)| \to \infty$, for all $k \geq 0$ and $v(x,0) = u(x)$. It is the solution of the following Laplacian problem:

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v = u & \text{on } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+. \end{cases}$$

Consider the operator $T : u \mapsto -\partial_y v(\cdot,0)$. Since $\partial_y v$ is still a harmonic function, if we apply the operator twice, we obtain

$$T \circ Tu = \partial_{yy} v \mid_{y=0} = -\Delta_x v \mid_{y=0} = -\Delta u \text{ in } \mathbb{R}^n.$$

Thus the operator T that maps the Dirichlet-type data u to the Neumann-type data $-\partial_y v(x,0)$ is actually the half Laplacian.

In the previous paper [7], we introduced an analogue operator but now in a bounded domain $\Omega \subset \mathbb{R}^n$. Consider the harmonic extension v of u in the half-cylinder $\Omega \times (0,\infty)$ vanishing on the lateral boundary $\partial\Omega \times [0,\infty)$. Then since $\partial_y v$ is harmonic and also vanishes on the lateral boundary, as before the Dirichlet-Neumann map of the harmonic extension v on the bottom of the half cylinder is the square root of the Laplacian: $A_{1/2} = B_{1/2}^{-1}$. That is, we have the properties:

$$A_{1/2} \circ A_{1/2} = -\Delta \;\text{ and }\; B_{1/2} \circ B_{1/2} = (-\Delta)^{-1},$$

where $-\Delta$ is the Laplacian in Ω with zero Dirichlet boundary value on $\partial\Omega$. In this way we can study problem (1.1) by variational methods for a local problem. More precisely, we will study the following mixed value boundary problem in a half cylinder:

$$\begin{cases} -\Delta v = 0 & \text{in } \mathcal{C} = \Omega \times (0,\infty), \\ v = 0 & \text{on } \partial_L \mathcal{C} = \partial\Omega \times [0,\infty), \\ \frac{\partial v}{\partial \nu} = f(v) & \text{on } \Omega \times \{0\}, \\ v > 0 & \text{in } \mathcal{C}, \end{cases} \tag{1.2}$$

where ν is the unit outer normal to $\Omega \times \{0\}$. If v satisfies (1.2), then the trace u on $\Omega \times \{0\}$ of the function v will be a solution of problem (1.1). Moveover, we will have that the operator $A_{1/2}$ is self-adjoint and positive definite and that $A_{1/2}$ has a spectral representation in terms of the eigenvalues and the eigenfunctions of $-\Delta$ in Ω with zero Dirichlet boundary values. By studying (1.2), we establish the results for (1.1).

The analogue problem to (1.1) for the Laplacian operator has been investigated widely in the last decades. This is the following problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. If $f(u) = u^p$ in problem (1.3), then there is a sharp contrast between the subcritical case $p < \frac{n+2}{n-2}$, for which the problem admits a solution, and the critical case $p = \frac{n+2}{n-2}$, for which the Sobolev embedding is not compact. Pohozaev discovered that there is no positive solution for the critical or supercritical problem

$$f(u) = u^p \text{ and } p \geq \frac{n+2}{n-2},$$

when Ω is a star-shaped domain.

In the case of $f(u) = u^{\frac{n+2}{n-2}} + \mu u$, $\mu > 0$, the existence of positive solutions of problem (1.3) was studied in a famous paper by Brezis and Nirenberg [5]. For this they studied the minimizing problem:

$$\min \left\{ \int_\Omega (|\nabla u|^2 - \mu |u|^2)\, dx \mid u \in H_0^1(\Omega), \|u\|_{L^{\frac{2n}{n-2}}(\Omega)} = 1 \right\}.$$

The critical points of this constrained functional correspond to weak solutions of problem (1.3). But this energy functional may lose compactness, since the nonlinearity involves the critical exponent. While the functional does not satisfy the Palais-Smale condition globally, some compactness will hold in the range determined by the best constant of the Sobolev inequality. The steps of the proof need a careful analysis introduced by Brezis and Nirenberg about the energy level computed on cut-off functions of the extremal functions for the best constant in the Sobolev inequality. Their technique has been extended to many other situations.

We here build a Pohozaev type formula for problem (1.2). Then by using this identity, we see that there is no positive solution for the critical problem (1.1) or the supercritical case in star-shaped domains.

Theorem 1.1. *Let $n \geq 2$ and $2^\sharp = \frac{2n}{n-1}$. Assume that $f(u) = u^p$ in problem* (1.1)*. If $p \geq 2^\sharp - 1 = \frac{n+1}{n-1}$ and Ω is star-shaped with respect to a point in $\mathbb{R}^n$, then there exists no weak bounded solution of* (1.1)*.*

We also employ the Brézis-Nirenberg technique to build an analogue result for problem (1.1) related to the square root of the Laplacian $A_{1/2}$.

Theorem 1.2. *Let $n \geq 2$ and $2^\sharp = \frac{2n}{n-1}$. Suppose that Ω is a smooth bounded domain in $\mathbb{R}^n$ and $f(u) = u^{2^\sharp - 1} + \mu u = u^{\frac{n+1}{n-1}} + \mu u$, and that $\mu_1 = \sqrt{\lambda_1}$ is the first eigenvalue of $A_{1/2}$, where λ_1 is the first eigenvalue of the Laplacian $-\Delta$ in Ω with zero Dirichlet boundary condition.*

Then, for every $\mu \in (0, \mu_1)$, there exists at least one $C^{2,\alpha}(\overline{\Omega})$ solution of (1.1)*. Furthermore, there exists no bounded weak solution of* (1.1) *for $\mu \geq \mu_1$.*

There is another approach to the Brézis-Nirenberg result, based on a careful study of the compactness properties for Palais-Smale sequences of the functional $\Phi(u)$ associated to problem (1.3):

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{n-2}{2n}\int_\Omega |u|^{\frac{2n}{n-2}}\, dx - \frac{\mu}{2}\int_\Omega |u|^2\, dx.$$

Both approaches are completely equivalent. However, the second approach brings out the peculiarities of the limiting case more clearly, which gives the energy estimate of Palais-Smale sequences. We will prove the existence of positive solutions of (1.1) via both procedures. The steps of our proofs need a careful analysis about the energy level computed on cut-off functions of the extremal functions for the best constant in the Sobolev trace inequality.

We point out that Chipot, Chlebík, Fila and Shafrir [11] studied the problem:

$$\begin{cases} -\Delta v = g(v) & \text{in} \quad B_R^+ = \{z \in \mathbb{R}^{n+1} \mid |z| \leq R, z_{n+1} > 0\}, \\ v = 0 & \text{on} \quad \partial B_R^+ \cap \{z_{n+1} > 0\}, \\ \frac{\partial v}{\partial \nu} = f(v) & \text{on} \quad \partial B_R^+ \cap \{z_{n+1} = 0\}, \\ v > 0 & \text{in} \quad B_R^+, \end{cases} \tag{1.4}$$

where $f, g \in C^1(\mathbb{R})$ and ν is the unit outer normal. They proved existence, non-existence and axial symmetry results for solutions of (1.4).

The paper is organized as follows. In Section 2, we recall the definition of the square root of the Laplacian operator and of the appropriate function spaces given by Cabré and Tan [7]. The proof of Theorem 1.1 is given in Section 3.

The proof of Theorem 1.2 is in Sections 4 and 5. The Palais-Smale sequences is studued in Section 5.

2. Preliminaries

In this section, we collect preliminary facts for future reference. First of all, let us write the standard notations which we will use in this paper.

$$\mathbb{R}^{n+1}_+ = \{z = (x, y) = (x_1, \cdots, x_n, y) \in \mathbb{R}^{n+1} \mid y > 0\}.$$

Denote by $H^s(U) = W^{s,2}(U)$ the fractional Sobolev space in a domain U of $\mathbb{R}^n$ or $\mathbb{R}^{n+1}_+$.

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Denote the half cylinder with base Ω by

$$\mathcal{C} = \Omega \times (0, \infty)$$

and its lateral boundary by

$$\partial_L \mathcal{C} = \partial\Omega \times [0, \infty).$$

To treat the nonlocal problem (1.1), we will study a corresponding extension problem in one more dimension, which allows us to investigate problem (1.1) by studying a local problem via classical nonlinear variational methods. We define a Sobolev space of functions whose traces vanish on $\partial_L \mathcal{C}$:

$$H^1_{0,L}(\mathcal{C}) = \{v \in H^1(\mathcal{C}) \mid v = 0 \text{ a.e. on } \partial_L \mathcal{C}\}, \tag{2.1}$$

equipped with the norm

$$\|v\| = \left(\int_{\mathcal{C}} |\nabla v|^2 \, dxdy\right)^{1/2}. \tag{2.2}$$

We denote by tr_Ω the trace operator on $\Omega \times \{0\}$ for functions in $H^1_{0,L}(\mathcal{C})$:

$$\mathrm{tr}_\Omega v := v(x, 0), \text{ for } v \in H^1_{0,L}(\mathcal{C}).$$

We have that $\mathrm{tr}_\Omega v \in H^{1/2}(\Omega)$, since it is well known that traces of H^1 functions are $H^{1/2}$ functions on the boundary.

Now we can state some results of the function space, the operator $A_{1/2}$. For the convenience, we sketch the proofs here. At the end of this section, we will state some regularity results. For more details, see Cabré and Tan [7].

Proposition 2.1. [7] *Let $\mathcal{V}_0(\Omega)$ be the space of all traces on $\Omega \times \{0\}$ of functions in $H^1_{0,L}(\mathcal{C})$. Then we have the following properties:*

$$\begin{aligned} \mathcal{V}_0(\Omega) &:= \{u = \mathrm{tr}_\Omega v \mid v \in H^1_{0,L}(\mathcal{C})\} \\ &= \{u \in L^2(\Omega) \mid u = \sum_{k=1}^{\infty} b_k \varphi_k \textit{ satisfying } \sum_k b_k^2 \lambda_k^{1/2} < +\infty\}, \end{aligned}$$

where λ_k, φ_k is the spectral decomposition of $-\Delta$ in Ω as above, with $\{\varphi_k\}$ an orthonormal basis in $L^2(\Omega)$.

Proposition 2.2. [7] *If $u \in \mathcal{V}_0(\Omega)$, then there exists a unique harmonic extension v in $\mathcal{C}$ of u such that $v \in H^1_{0,L}(\mathcal{C})$. In particular, if the expansion of u is written by $u(x) = \sum_{k=1}^{\infty} b_k \varphi_k(x) \in \mathcal{V}_0(\Omega)$, then*

$$v(x,y) = \sum_{k=1}^{\infty} b_k \varphi_k(x) \exp(-\lambda_k^{1/2} y) \in H^1_{0,L}(\mathcal{C}),$$

where λ_k, φ_k is the spectral decomposition of $-\Delta$ in Ω as above, with $\{\varphi_k\}$ an orthonormal basis in $L^2(\Omega)$. Let us define the operator $A_{1/2} : \mathcal{V}_0(\Omega) \to \mathcal{V}_0^*(\Omega)$ by

$$A_{1/2} u := \frac{\partial v}{\partial \nu} |_{\Omega \times \{0\}},$$

where $\mathcal{V}_0^*(\Omega)$ is the dual space of $\mathcal{V}_0(\Omega)$. Then

$$A_{1/2} u = \sum_{k=1}^{\infty} b_k \lambda_k^{1/2} \varphi_k,$$

and $A_{1/2}^2$ is equal to $-\Delta$ in Ω with zero Dirichlet boundary value on $\partial\Omega$. More precisely, the inverse $B_{1/2} := A_{1/2}^{-1}$ is the unique square root of the inverse Laplacian $(-\Delta)^{-1}$ in Ω with zero Dirichlet boundary value on $\partial\Omega$.

Let us give some properties of the space $H^1_{0,L}(\mathcal{C})$. Denote the closure of the set of smooth functions compactly supported in $\overline{\mathbb{R}^{n+1}_+}$, by $\mathcal{D}^{1,2}(\mathbb{R}^{n+1}_+)$, with respect to the norm of

$$\|w\|_{\mathcal{D}^{1,2}(\mathbb{R}^{n+1}_+)} = \Big(\int_{\mathbb{R}^{n+1}_+} |\nabla w|^2 \, dxdy \Big)^{1/2}.$$

The well known Sobolev trace inequality [18] states that for $w \in \mathcal{D}^{1,2}(\mathbb{R}^{n+1}_+)$,

$$\Big(\int_{\mathbb{R}^n} |w(x,0)|^{2n/(n-1)} dx \Big)^{(n-1)/2n} \le C \Big(\int_{\mathbb{R}^{n+1}_+} |\nabla w(x,y)|^2 dxdy \Big)^{1/2}, \tag{2.3}$$

where C depends only on n.

Denote for $n \ge 2$,

$$2^\sharp = \frac{2n}{n-1} \quad \text{and} \quad 2^\sharp - 1 = \frac{n+1}{n-1}.$$

We say that p is subcritical if $1 < p < 2^\sharp - 1 = \frac{n+1}{n-1}$ for $n \ge 2$, and $1 < p < \infty$ for $n = 1$. We also say that p is critical if $p = 2^\sharp - 1 = \frac{n+1}{n-1}$ for $n \ge 2$, and that p is supercritical if $p > 2^\sharp - 1 = \frac{n+1}{n-1}$ for $n \ge 2$.

Consider

$$S_0 = \inf \left\{ \frac{\int_{\mathbb{R}^{n+1}_+} |\nabla w(x,y)|^2 dxdy}{(\int_{\mathbb{R}^n} |w(x,0)|^{2^\sharp} dx)^{2/2^\sharp}} \mid w \in \mathcal{D}^{1,2}(\mathbb{R}^{n+1}_+) \right\}. \tag{2.4}$$

It is known [14] that S_0 is achieved by the extremal functions

$$U_\varepsilon(x,y) = \frac{\varepsilon^{(n-1)/2}}{|(x, y+\varepsilon)|^{n-1}}, \tag{2.5}$$

where $\varepsilon > 0$ is arbitrary.

For $v \in H^1_{0,L}(\mathcal{C})$, its extension by zero in $\mathbb{R}^{n+1}_+ \setminus \mathcal{C}$ can be approximated by functions compactly supported in $\overline{\mathbb{R}^{n+1}_+}$. Thus the Sobolev trace inequality (2.3) and Hölder inequality lead to:

Lemma 2.3. (i) *Let $n \geq 2$ and $2^\sharp = \frac{2n}{n-1}$. There exists a constant C, depending only on n, such that, for all $v \in H^1_{0,L}(\mathcal{C})$,*

$$\Big(\int_\Omega |v(x,0)|^{2^\sharp} dx\Big)^{1/2^\sharp} \leq C\Big(\int_{\mathcal{C}} |\nabla v(x,y)|^2 dxdy\Big)^{1/2}. \tag{2.6}$$

(ii) *Let $1 \leq q \leq 2^\sharp$ for $n \geq 2$. Then we have that for all $v \in H^1_{0,L}(\mathcal{C})$,*

$$\Big(\int_\Omega |v(x,0)|^{q} dx\Big)^{1/q} \leq C\Big(\int_{\mathcal{C}} |\nabla v(x,y)|^2 dxdy\Big)^{1/2}, \tag{2.7}$$

where C depends only on n, q and the measure of Ω. Moreover, (2.7) holds if $1 \leq q < \infty$ for $n = 1$.

(iii) *Let $1 \leq q < 2^\sharp = \frac{2n}{n-1}$ for $n \geq 2$ and $1 \leq q < \infty$ for $n = 1$. Then $\mathrm{tr}_\Omega(H^1_{0,L}(\mathcal{C}))$ is compactly embedded in $L^q(\Omega)$.*

Recall that the fractional Sobolev space $H^{1/2}(\Omega)$ is a Banach space with the norm

$$\|u\|_{H^{1/2}(\Omega)} = \Big(\int_\Omega\int_\Omega \frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{n+1}}\,dxd\bar{x} + \int_\Omega |u(x)|^2\,dx\Big)^{1/2}. \tag{2.8}$$

Note that the closure $H^{1/2}_0(\Omega)$ of smooth functions with compact support $C^\infty_c(\Omega)$ in $H^{1/2}(\Omega)$ is all the space $H^{1/2}(\Omega)$, by Theorem 11.1 in [17]; that is, $C^\infty_c(\Omega)$ is dense in $H^{1/2}(\Omega)$. Denote by $\mathcal{V}_0(\Omega)$ the space of traces on $\Omega \times \{0\}$ of functions in $H^1_{0,L}(\mathcal{C})$:

$$\mathcal{V}_0(\Omega) := \{u = \mathrm{tr}_\Omega v \mid v \in H^1_{0,L}(\mathcal{C})\} \subset H^{1/2}(\Omega), \tag{2.9}$$

endowed with the norm of $H^{1/2}(\Omega)$. The dual space of $\mathcal{V}_0(\Omega)$ is denoted by $\mathcal{V}^*_0(\Omega)$, equipped with the norm

$$\|g\|_{\mathcal{V}^*_0} = \sup\{\langle u, g\rangle \mid u \in \mathcal{V}_0(\Omega), \|u\|_{H^{1/2}(\Omega)} \leq 1\}.$$

Next we give the first characterization of the space $\mathcal{V}_0(\Omega)$.

Now we consider, for a function $u \in \mathcal{V}_0(\Omega)$ on $\Omega \subset \mathbb{R}^n$, the minimizing problem:

$$\inf\{\int_{\mathcal{C}} |\nabla v|^2\,dxdy, \mid v \in H^1_{0,L}(\mathcal{C}), v = u \text{ on } \Omega\}.$$

Note that the set of functions v where we minimize is non empty by the definition of $\mathcal{V}_0(\Omega)$ and the fact that $u \in \mathcal{V}_0(\Omega)$. By lower weak semi-continuity and by Lemma 2.3, we will see next that there is a minimizer of J.

We call v a *weak solution* of the problem

$$\begin{cases} -\Delta v = 0 & \text{in } \mathcal{C}, \\ v = 0 & \text{on } \partial_L \mathcal{C}, \\ v = u & \text{on } \Omega \times \{0\}. \end{cases} \tag{2.10}$$

The existence of the minimizer is proved as follows:

Lemma 2.4. [7] *If $u \in \mathcal{V}_0(\Omega)$, then there exists a unique minimizer $v \in H^1_{0,L}(\mathcal{C})$ of $J(v)$. The function v is the harmonic extension of u (in the weak sense) to $\mathcal{C}$ and vanishing on $\partial_L \mathcal{C}$.*

Proof. By the definition of $\mathcal{V}_0(\Omega)$, we have that, for every $u \in \mathcal{V}_0(\Omega)$, there exists at least one $w \in H^1_{0,L}(\mathcal{C})$ such that $\mathrm{tr}_\Omega(w) = u$. Then the standard minimization argument gives (using lower semi-continuity and Lemma 2.3) the existence of a minimizer. The uniqueness follows automatically from the identity of the parallelogram used for two possible minimizers v_1 and v_2:

$$0 \le J(\frac{v_1 - v_2}{2}) = \frac{1}{2} J(v_1) + \frac{1}{2} J(v_2) - J(\frac{v_1 + v_2}{2}) \le 0,$$

where $J(v) = \int_{\mathcal{C}} |\nabla v|^2 \, dxdy$. □

By Lemma 2.4, there exists a function $v \in H^1_{0,L}(\mathcal{C})$, which is the harmonic extension of u in $\mathcal{C}$ vanishing on $\partial_L \mathcal{C}$, denoted by

$$v := \text{h-ext}(u).$$

It is easy to see that for every $\xi \in C^\infty$ and $\xi \equiv 0$ on $\partial_L \mathcal{C}$,

$$\int_{\mathcal{C}} \nabla v \nabla \xi \, dxdy = \int_\Omega \frac{\partial v}{\partial \nu} \xi \, dx. \tag{2.11}$$

Since the h-ext operator is bijective from $\mathcal{V}_0(\Omega)$ to $H^1_{0,L}(\mathcal{C})$, by using the trace theorem we can deduce the following.

Definition 2.5. *Define the operator $A_{1/2} : \mathcal{V}_0(\Omega) \to \mathcal{V}_0^*(\Omega)$ by*

$$A_{1/2} u := \frac{\partial v}{\partial \nu} \, |_{\Omega \times \{0\}}, \tag{2.12}$$

where $v = $ h-ext$(u) \in H^1_{0,L}(\mathcal{C})$. It is clear that $A_{1/2}$ is linear and bounded from $\mathcal{V}_0(\Omega)$ to $\mathcal{V}_0^*(\Omega)$.

Recall the well known spectral theory of the Laplacian $-\Delta$ in a smooth bounded domain Ω with zero Dirichlet boundary value. We repeat each eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary condition according to its (finite) multiplicity:

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \cdots \to \infty, \quad \text{as } k \to \infty$$

and we denote by $\varphi_k \in H_0^1(\Omega)$ an eigenfunction corresponding to λ_k for $k = 1, 2, \cdots$. Namely,

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } \Omega, \\ \varphi_k = 0 & \text{on } \Omega. \end{cases} \tag{2.13}$$

We can take them to form an orthonormal basis $\{\varphi_k\}$ of $L^2(\Omega)$, in particular,

$$\int_\Omega \varphi_k^2 \, dx = 1,$$

and to belong to $C_0(\overline{\Omega}) \cap C^2(\overline{\Omega})$ by regularity theory.

We now give the spectral representation of $A_{1/2}$ and the corresponding structure of the space $\mathcal{V}_0(\Omega)$.

Lemma 2.6. [7] (i) *Let $\{\varphi_k\}$ be an orthonormal basis of $L^2(\Omega)$ giving an spectral decomposition of $-\Delta$ in Ω with Dirichlet boundary conditions as in* (2.13)*. Then we have*

$$\mathcal{V}_0(\Omega) = \left\{ u = \sum_{k=1}^{\infty} b_k\varphi_k \in L^2(\Omega) \mid \sum_{k=1}^{\infty} b_k^2 \lambda_k^{1/2} < +\infty \right\}.$$

(ii) *Let $u \in \mathcal{V}_0(\Omega)$. Then we have, for $u = \sum_{k=1}^{\infty} b_k\varphi_k$,*

$$A_{1/2}u = \sum_{k=1}^{\infty} b_k \lambda_k^{1/2} \varphi_k.$$

Proof. Let $u \in \mathcal{V}_0(\Omega)$, which is contained in $L^2(\Omega)$. Then its expansion is written by $u(x) = \sum_{k=1}^{\infty} b_k\varphi_k(x)$. Consider the smooth function for $y > 0$,

$$v(x, y) = \sum_{k=1}^{\infty} b_k\varphi_k(x) \exp(-\sqrt{\lambda_k} y). \tag{2.14}$$

Observe that $v(x, 0) = u(x)$ in Ω and, for $y > 0$,

$$\Delta v(x, y) = \sum_{k=1}^{\infty} b_k\{-\lambda_k\varphi_k \exp(-\sqrt{\lambda_k} y) + \lambda_k\varphi_k \exp(-\sqrt{\lambda_k} y)\} = 0.$$

So v is a harmonic extension. We will see that $v = \text{h-ext}(u)$ by uniqueness once we find the condition on $\{b_k\}$ for v to belong to $H_{0,L}^1(\mathcal{C})$. But such condition is simple: using (2.14) and that $\{\varphi_k\}$ are eigenfunctions of $-\Delta$ and orthonormal in $L^2(\Omega)$. We have

$$\begin{aligned} \int_0^\infty \int_\Omega |\nabla v|^2 \, dxdy &= \int_0^\infty \int_\Omega \{|\nabla_x v|^2 + |\partial_y v|^2\} \, dxdy \\ &= 2\sum_{k=1}^{\infty} b_k^2 \lambda_k \int_0^\infty \exp(-2\lambda_k^{1/2} y) \, dy \\ &= 2\sum_{k=1}^{\infty} b_k^2 \lambda_k \frac{1}{2\lambda_k^{1/2}} = \sum_{k=1}^{\infty} b_k^2 \lambda_k^{1/2}. \end{aligned}$$

This means that $v \in H^1_{0,\,L}(\mathcal{C})$ if and only if $\sum_{k=1}^{\infty} b_k^2 \lambda_k^{1/2} < \infty$. Therefore, this condition on $\{b_k\}$ is equivalent to $u \in \mathcal{V}_0(\Omega)$.

Assertion (ii) follows directly from (2.14). □

Definition 2.7. *Define the operator* $B_{1/2} : \mathcal{V}_0^*(\Omega) \to \mathcal{V}_0(\Omega)$, *by* $g \mapsto \mathrm{tr}_\Omega v$, *where* v *is found by solving the following problem:*

$$\begin{cases} -\Delta v = 0 & in \quad \mathcal{C}, \\ v = 0 & on \quad \partial_L \mathcal{C}, \\ \frac{\partial v}{\partial \nu} = g(x) & on \quad \Omega \times \{0\}, \end{cases} \tag{2.15}$$

as we indicate next.

Note that $B_{1/2} : L^2(\Omega) \to L^2(\Omega)$ is a self-adjoint operator. In fact, since for $v_1, v_2 \in H^1_{0,L}(\mathcal{C})$,

$$\int_{\mathcal{C}} (v_2 \Delta v_1 - v_1 \Delta v_2)\, dxdy = \int_\Omega (v_2 \frac{\partial v_1}{\partial \nu} - v_1 \frac{\partial v_2}{\partial \nu})\, dx,$$

we see

$$\int_\Omega B_{1/2} g_2 \cdot g_1 \, dx = \int_\Omega B_{1/2} g_1 \cdot g_2 \, dx$$

and

$$\int_\Omega v_2(x,0) A_{1/2} v_1(x,0)\, dx = \int_\Omega v_1(x,0) A_{1/2} v_2(x,0)\, dx.$$

On the other hand, by using Lemma 2.3, we obtain that $B_{1/2}$ is a positive compact operator in $L^2(\Omega)$. Hence by the operator theory of compact, self adjoint operators we have that all the eigenvalues of $B_{1/2}$ are real, positive, and there are corresponding eigenfunctions which make up an orthonormal basis of $L^2(\Omega)$. Furthermore, such basis and eigenvalues are explicit in terms of those of the Laplacian with Dirichlet boundary conditions, since we have the expression of $A_{1/2}$ given in Lemma 2.6 (ii). Summarizing:

Proposition 2.8. [7] *Let* $\{\varphi_k\}$ *be an orthonormal basis of* $L^2(\Omega)$ *giving an spectral decomposition of* $-\Delta$ *in* Ω *with Dirichlet boundary conditions as in* (2.13). *Then for all* $k \geq 1$,

$$\begin{cases} A_{1/2} \varphi_k = \lambda_k^{1/2} \varphi_k & in\, \Omega, \\ \varphi_k = 0 & on\, \partial\Omega. \end{cases} \tag{2.16}$$

In particular, $\{\varphi_k\}$ *is also a basis of eigenfunctions of* $A_{1/2}$, *with eigenvalues* $\lambda_k^{1/2}$.

We state regularity result of weak solutions for the nonlinear problem:

$$\begin{cases} A_{1/2} u = f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on}\, \partial\Omega, \end{cases} \tag{2.17}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. The precise meaning for (2.17) is that $v \in H^1_{0,L}(\mathcal{C})$, $v(x,0) = u$, and v is a weak solution of

$$\begin{cases} -\Delta v = 0 & \text{in} \quad \mathcal{C}, \\ v = 0 & \text{on} \quad \partial_L \mathcal{C}, \\ \frac{\partial v}{\partial \nu} = f(v(\cdot,0)) & \text{on} \quad \Omega \times \{0\}. \end{cases} \tag{2.18}$$

We proved this result in [7] by using reflections.

Proposition 2.9. [7] *Let $\alpha \in (0,1)$, Ω be a $C^{2,\alpha}$ bounded domain of $\mathbb{R}^n$, f be a $C^{1,\alpha}$ function such that $f(0) = 0$. If u is a bounded weak solution of* (2.17)*, and thus $v \in H^1_{0,\,L}(\mathcal{C}) \cap L^\infty(\mathcal{C})$ is a weak solution of* (2.18)*, then $u \in C^{2,\alpha}(\overline{\Omega}) \cap C_0(\overline{\Omega})$. In addition, $v \in C^{2,\alpha}(\overline{\mathcal{C}})$.*

3. Pohozaev type formula and nonexistence of solutions

Next we prove a Pohozaev type formula for the problem

$$\begin{cases} -\Delta v = 0 & \text{in} \quad \mathcal{C} = \Omega \times (0,\infty), \\ v = 0 & \text{on} \quad \partial_L \mathcal{C} = \partial\Omega \times [0,\infty), \\ \frac{\partial v}{\partial \nu} = f(v) & \text{on} \quad \Omega \times \{0\}. \end{cases} \tag{3.1}$$

Lemma 3.1. *Assume that f is a C^1 function, $f(0) = 0$, with primitive $F(s) = \int_0^s f(t)\,dt$ and that v is a weak solution of* (3.1) *in $H^1_{0,L}(\mathcal{C}) \cap L^\infty(\mathcal{C})$. Then v satisfies a Pohozaev type identity:*

$$\frac{1}{2}\int_{\partial_L \mathcal{C}} |\nabla v|^2 (x,\nu)\, d\sigma - \int_{\Omega\times\{0\}} nF(v)\,dx + \frac{n-1}{2}\int_{\Omega\times\{0\}} vf(v)\,dx = 0. \tag{3.2}$$

Proof. Let v be a weak bounded solution of (3.1) and $z = (x,y)$. Then we know by Proposition 2.9 that $v \in C^2(\overline{\mathcal{C}})$. The following identity is known:

$$\operatorname{div}\left\{(z,\nabla v)\nabla v - z\frac{|\nabla v|^2}{2}\right\} + (\frac{n+1}{2} - 1)|\nabla v|^2 = (z,\nabla v)\Delta v.$$

Thus, by (3.1), we know that in $\mathcal{C}$,

$$\operatorname{div}\left\{(z,\nabla v)\nabla v - z\frac{|\nabla v|^2}{2}\right\} + (\frac{n+1}{2} - 1)|\nabla v|^2 = 0.$$

Integrating the above equation over $\Omega \times (0,R)$, by the divergence theorem and using that $v \equiv 0$ on $\partial\Omega \times [0,\infty)$, we see

$$\begin{aligned} \frac{1}{2}\int_{\partial\Omega\times(0,R)} |\nabla v|^2(x,\nu)\,d\sigma &+ \int_{\Omega\times\{y=0\}} (x,\nabla_x v)(\nabla v,\nu)\,dx \\ &+ (\frac{n+1}{2} - 1)\int_{\Omega\times(0,R)} |\nabla v|^2\,dxdy \\ &+ \int_{\Omega\times\{y=R\}} \left\{((x,\nabla_x v) + R\partial_y v)\partial_y v - R\frac{|\nabla v|^2}{2}\right\} dx = 0. \end{aligned} \tag{3.3}$$

We have

$$\int_{\Omega\times\{0\}}(x,\nabla_x v)(\nabla v,\nu)\,dx=\int_{\Omega\times\{0\}}(x,\nabla_x v)f(v)\,dx$$
$$=\int_{\Omega\times\{0\}}(x,\nabla_x F(v))\,dx=-\int_{\Omega\times\{0\}}nF(v)\,dx,$$

since $F(0)=0$ and $v\equiv 0$ on $\partial\Omega\times\{0\}$. Next we claim that there exists a sequence $R_m\to\infty$ such that

$$\lim_{m\to\infty}\int_{\Omega\times\{y=R_m\}}\left\{((x,\nabla_x v)+R_m\partial_y v)\partial_y v-R_m\frac{|\nabla v|^2}{2}\right\}dx=0.$$

To prove this, note first that

$$\left|\int_{\Omega\times\{y=R\}}\left\{((x,\nabla_x v)+R\partial_y v)\partial_y v-R\frac{|\nabla v|^2}{2}\right\}dx\right|$$
$$\leq C(\mathrm{diam}(\Omega)+1)\int_{\Omega\times\{y=R\}}R|\nabla v|^2\,dx.$$

If

$$\liminf_{R\to\infty}\int_{\Omega\times\{y=R\}}R|\nabla v|^2\,dx=c_0>0,$$

then there exists R_0 such that, for all $R_1\geq R_0$,

$$\int_{R_0}^{R_1}\int_{\Omega\times\{y=R\}}|\nabla v|^2\,dxdR\geq\frac{c_0}{2}\int_{R_0}^{R_1}\frac{1}{R}dR=\frac{c_0}{2}\log\frac{R_1}{R_0}.$$

Letting $R_1\uparrow\infty$, this contradicts that $v\in H^1_{0,\,L}(\mathcal{C})$. The only remanding term in our equality (3.3) is $\frac{n-1}{2}\int_{\Omega\times(0,R)}|\nabla v|^2\,dxdy$. Integrating by parts, the integral is equal to $\int_{\Omega\times\{0\}}vf(v)\,dx+\int_{\Omega\times\{R\}}v\partial_y v\,dx$. The second integral goes to zero for a sequence $\{R_m\}$ by the previous argument and since $v\in L^\infty$.

Thus, we obtain by taking $R=R_m\to\infty$

$$\frac{1}{2}\int_{\partial_L\mathcal{C}}|\nabla v|^2(x,\nu)\,d\sigma-\int_{\Omega\times\{0\}}nF(v)\,dx+\frac{n-1}{2}\int_{\Omega\times\{0\}}vf(v)\,dx=0.\qquad\square$$

Proof of Theorem 1.1. Since $f(v)=|v|^{p-1}v$ has primitive $F(v)=\frac{1}{p+1}|v|^{p+1}$, from Lemma 3.1, we have

$$\frac{1}{2}\int_{\partial_L\mathcal{C}}|\nabla v|^2(x,\nu)\,d\sigma=(\frac{n}{p+1}-\frac{n-1}{2})\int_{\Omega\times\{0\}}|v|^{p+1}\,dx.$$

If $p\geq\frac{n+1}{n-1}$, then the right hand side is nonpositive, but the left hand side is positive if after a translation in $\mathbb{R}^n$ we take Ω star-shaped with respect to the origin. This gives a contradiction. $\square$

4. Small perturbation from the critical nonlinearity

Now, we establish the Brezis-Nirenberg type result in Theorem 1.2 concerning the nonlinearity $f(u) = u^{2^\sharp-1} + \mu u$, where $\mu > 0$ and $n \geq 2$. Namely we look for positive solutions of problem (1.1). Equivalently, we consider the following problem:

$$\begin{cases} -\Delta v = 0 & \text{in } \mathcal{C} = \Omega \times (0,\infty), \\ v = 0 & \text{on } \partial_L \mathcal{C} = \partial\Omega \times [0,\infty), \\ \frac{\partial v}{\partial \nu} = v^{2^\sharp-1} + \mu v & \text{on } \Omega \times \{0\}, \\ v > 0 & \text{in } \mathcal{C}. \end{cases} \tag{4.1}$$

Let

$$\mu_1 = \inf\left\{ \int_{\mathcal{C}} |\nabla v(x,y)|^2 dxdy \mid v \in H^1_{0,L}(\mathcal{C}), \int_\Omega |v(x,0)|^2 dx = 1 \right\}.$$

Note that, by Proposition 2.8, $\mu_1 = \lambda_1^{1/2}$ is the first eigenvalue of $A_{1/2}$ and a minimizer $\phi_1 > 0$ of the above minimization problem is an $H^1_{0,L}(\mathcal{C})$ harmonic function with $\frac{\partial \phi_1}{\partial \nu} = \mu_1 \phi_1$ on $\Omega \times \{0\}$. Its trace $\varphi_1 = \phi_1(\cdot, 0)$ is the first eigenfunction of $A_{1/2}$ in Proposition 2.8.

Lemma 4.1. *If $\mu_1 \leq \mu$, then* (4.1) *does not admit* (*a positive*) *solution.*

Proof. Assume that the problem admits a positive solution v. Since

$$0 = \int_{\mathcal{C}} [v\Delta\phi_1 - \phi_1 \Delta v]\, dxdy = \int_{\Omega\times\{0\}} \left[v\frac{\partial \phi_1}{\partial \nu} - \phi_1 \frac{\partial v}{\partial \nu}\right] dx,$$

we have

$$\begin{aligned} \int_{\Omega\times\{0\}} \mu_1 v\phi_1 \, dx &= \int_{\Omega\times\{0\}} \phi_1 \frac{\partial v}{\partial \nu}\, dx \\ &= \int_{\Omega\times\{0\}} \left[v^{2^\sharp-1}\phi_1 + \mu v\phi_1 \right] dx > \int_{\Omega\times\{0\}} \mu v\phi_1 \, dx. \end{aligned}$$

Then

$$\mu_1 > \mu.$$

□

In order to prove Theorem 1.2, we need to estimate the functional:

$$Q_\mu(v) = \frac{\int_{\mathcal{C}} |\nabla v|^2 \, dxdy - \mu \int_{\Omega\times\{0\}} |v|^2 \, dx}{\left(\int_{\Omega\times\{0\}} |v|^{2^\sharp} \, dx \right)^{2/2^\sharp}}.$$

Proposition 4.2. *Let $\mu \in (0, \mu_1)$ and denote by S_0 the best constant for the Sobolev trace inequality defined by* (2.4). *Then we have*

$$S_\mu = S_\mu(\Omega) := \inf\{ Q_\mu(v) \mid v \in H^1_{0,L}(\mathcal{C})\} < S_0.$$

Proof. Let U_ε be the minimizers for S_0, which are given by expression (2.5). We see that

$$\int_{\mathbb{R}^n\times\{0\}} |U_\varepsilon|^{2^\sharp}\,dx = \int_{\mathbb{R}^n} \frac{\varepsilon^n}{(\varepsilon^2+|x|^2)^n}\,dx$$
$$= \omega_n \int_0^\infty \frac{r^{n-1}}{(1+r^2)^n}\,dr =: K_1,$$

where ω_n is the surface area of unit sphere in $\mathbb{R}^n$. Denote

$$B_\rho^+ = \{(x,y) \mid |(x,y)| < \rho, \text{ and } y>0\}.$$

Let $\eta \in C^\infty(\overline{\mathcal{C}})$, $0 \le \eta \le 1$ and for small fixed ρ,

$$\eta(x,y) = \begin{cases} 1 & (x,y) \in B^+_{\rho/2}, \\ 0 & (x,y) \notin \overline{B^+_\rho}, \end{cases} \qquad |\nabla \eta| \le \text{Const.}/\rho. \tag{4.2}$$

We take ρ small enough so that $\overline{B^+_\rho} \subset \mathcal{C} \cup (\Omega \times \{0\})$. Thus the function $\eta U_\varepsilon \in H^1_{0,L}(\mathcal{C})$ and we will use it as test function v in the expression for Q_μ above. In what follows, $O(\varepsilon^a)$ will have usual meaning, with constants that may depend on ρ, which is fixed.

We have

$$\int_{\Omega\times\{0\}} |\eta U_\varepsilon|^{2^\sharp}\,dx = \int_{\mathbb{R}^n} \frac{\varepsilon^n \eta^{2^\sharp}(x,0)}{(\varepsilon^2+|x|^2)^n}\,dx$$
$$= K_1 + \int_{\mathbb{R}^n} \frac{\varepsilon^n(\eta^{2^\sharp}(x,0)-1)}{(\varepsilon^2+|x|^2)^n}\,dx$$
$$= K_1 + \varepsilon^n \int_{\mathbb{R}^n\setminus B_{\rho/2}} \frac{\eta^{2^\sharp}(x,0)-1}{(\varepsilon^2+|x|^2)^n}\,dx$$
$$= K_1 + O(\varepsilon^n).$$

Hence we see

$$\Big(\int_{\mathbb{R}^n} |\eta U_\varepsilon|^{2^\sharp}\,dx\Big)^{2/2^\sharp} = K_1^{2/2^\sharp} + O(\varepsilon^n).$$

Let now

$$K_2 := \int_{\mathbb{R}^{n+1}_+} |\nabla U_\varepsilon|^2\,dxdy.$$

Since U_ε are minimizers for S_0, we have that

$$\frac{K_2}{K_1^{2/2^\sharp}} = S_0. \tag{4.3}$$

We see

$$\int_{\mathcal{C}} |\nabla(\eta U_\varepsilon)|^2\,dxdy = \int_{\mathcal{C}} \eta^2 |\nabla U_\varepsilon|^2\,dxdy + O(\varepsilon^{n-1}),$$

since in the second term all integrals are computed in $B_\rho^+ \setminus B_{\rho/2}^+$. Thus, by uniform integrability of $\varepsilon^{1-n}|\nabla U_\varepsilon|^2$ in $\mathbb{R}_+^{n+1} \setminus B_{\rho/2}^+$, we deduce

$$\begin{aligned}\int_{\mathcal{C}} |\nabla(\eta U_\varepsilon)|^2\,dxdy &= \int_{\mathbb{R}_+^{n+1}} |\nabla U_\varepsilon|^2\,dxdy + O(\varepsilon^{n-1})\\ &= K_2 + O(\varepsilon^{n-1}).\end{aligned}$$

On the other hand, we have for all $\varepsilon < \rho/2$

$$\begin{aligned}\int_{\Omega\times\{0\}} |\eta U_\varepsilon|^2\,dx &= \int_{\Omega\times\{0\}} \frac{\varepsilon^{n-1}\eta^2(x,0)}{(|x|^2+\varepsilon^2)^{n-1}}\,dx\\ &\geq \int_{\{|x|<\rho/2\}} \frac{\varepsilon^{n-1}}{(|x|^2+\varepsilon^2)^{n-1}}\,dx\\ &\geq \int_{\{|x|<\varepsilon\}} \frac{\varepsilon^{n-1}}{(2\varepsilon^2)^{n-1}}\,dx + \int_{\{\varepsilon<|x|<\rho/2\}} \frac{\varepsilon^{n-1}}{(2|x|^2)^{n-1}}\,dx\\ &= c_1\varepsilon + c_2\varepsilon^{n-1}\int_\varepsilon^\rho r^{1-n}\,dr\\ &= \begin{cases} c_3\varepsilon + O(\varepsilon^{n-1}) & \text{for } n \geq 3,\\ c_4\varepsilon\ln(1/\varepsilon) + O(\varepsilon) & \text{for } n = 2,\end{cases}\end{aligned}$$

where c_1, c_2, c_3 and c_4 are positive constants.

We compute, using the above,

$$Q_\mu(\eta U_\varepsilon) = \frac{\int_{\mathcal{C}} |\nabla(\eta U_\varepsilon)|^2\,dxdy - \mu\int_{\Omega\times\{0\}} |\eta U_\varepsilon|^2\,dx}{(\int_{\Omega\times\{0\}} |\eta U_\varepsilon|^{2^\sharp}\,dx)^{2/2^\sharp}}.$$

In the case $n \geq 3$, we have, recalling (4.3),

$$\begin{aligned}Q_\mu(\eta U_\varepsilon) =& \frac{K_2 - \mu c_3\varepsilon + O(\varepsilon^{n-1})}{K_1^{2/2^\sharp} + O(\varepsilon^n)}\\ =& \frac{S_0 - \mu c_3 K_1^{-2/2^\sharp}\varepsilon + O(\varepsilon^{n-1})}{1 + O(\varepsilon^n)}.\end{aligned}$$

Then

$$Q_\mu(\eta U_\varepsilon) = S_0 - \mu\frac{c_3\varepsilon}{K_1^{2/2^*}} + O(\varepsilon^{n-1}) < S_0,$$

if we take $\varepsilon > 0$ small enough.

In the case $n=2$, we see

$$\begin{aligned} Q_\mu(\eta U_\varepsilon) =& \frac{K_2 - \mu c_4 \varepsilon \ln(1/\varepsilon) + O(\varepsilon)}{K_1^{2/2^\sharp} + O(\varepsilon^2)} \\ =& \frac{S_0 - \mu c_4 K_1^{-2/2^\sharp} \varepsilon \ln(1/\varepsilon) + O(\varepsilon)}{1 + O(\varepsilon^2)} \\ =& S_0 - \mu \frac{c_4 \varepsilon \ln(1/\varepsilon)}{K_1^{2/2^*}} + O(\varepsilon) < S_0, \end{aligned}$$

for ε small enough. □

We will prove that $\inf\{Q_\mu(v) \mid v \in H^1_{0,L}(\mathcal{C})\}$ is achieved. Recall the Brezis-Lieb Lemma:

Lemma 4.3. [4] *Let U be an open subset of $\mathbb{R}^n$ and let $\{v_m\} \subset L^q(U)$, $2 \le q < \infty$. Assume that* (i) *v_m weakly converges to v in $L^q(U)$;* (ii) *$v_m \to v$ a.e.* (*almost everywhere*) *in U, as $m \to \infty$. Then we have*

$$\lim_{m\to\infty} (\|v_m\|^q_{L^q(U)} - \|v_m - v\|^q_{L^q(U)}) = \|v\|^q_{L^q(U)}.$$

Proposition 4.4. *For $\mu \in (0, \mu_1)$, we have that*

$$S_\mu = S_\mu(\Omega) := \inf\{Q_\mu(v) \mid v \in H^1_{0,L}(\mathcal{C})\}$$

is achieved.

Proof. Note that first that $Q_\mu \ge 0$ since $\mu \le \mu_1$. Let $\{v_m\} \subset H^1_{0,L}(\mathcal{C})$ be a minimizing sequence for S_μ. Normalize it to satisfy $\|v_m\|_{L^{2^\sharp}(\Omega\times\{0\})} = 1$. Replacing v_m by $|v_m|$, we may assume $v_m \ge 0$. Since the $\|v_m\|_{L^{2^\sharp}(\Omega\times\{0\})}$ is bounded, the minimizing property leads to

$$\int_{\mathcal{C}} |\nabla v_m|^2 \, dxdy \le C.$$

Then, by Lemma 2.3, we extract a subsequence, still denoted by $\{v_m\}$, such that, as $m \to \infty$,

$$\begin{aligned} & v_m \rightharpoonup v \quad \text{weakly in } \ H^1_{0,L}(\mathcal{C}), \\ & v_m(\cdot,0) \to v(\cdot,0) \quad \text{strongly in } \ L^q(\Omega), 2 \le q < 2^\sharp, \\ & v_m(x,0) \to v(x,0) \quad \text{a.e. in } \ \Omega. \end{aligned}$$

Developing the square and by weak convergence, we have

$$\int_{\mathcal{C}} |\nabla(v_m - v)|^2 \, dxdy = \int_{\mathcal{C}} |\nabla v_m|^2 \, dxdy - \int_{\mathcal{C}} |\nabla v|^2 \, dxdy + o(1).$$

On the other hand, by Lemma 4.3, we have

$$\|v_m(\cdot,0) - v(\cdot,0)\|^{2^\sharp}_{L^{2^\sharp}(\Omega)} = \|v_m(\cdot,0)\|^{2^\sharp}_{L^{2^\sharp}(\Omega)} - \|v(\cdot,0)\|^{2^\sharp}_{L^{2^\sharp}(\Omega)} + o(1).$$

Therefore, we see

$$\begin{aligned}
Q_\mu(v_m) =& \int_{\mathcal{C}} |\nabla(v_m - v)|^2\, dxdy + \int_{\mathcal{C}} |\nabla v|^2\, dxdy \\
&- \mu \int_\Omega |v(x,0)|^2\, dx + o(1) \\
\geq& S_0 \|v_m - v\|^2_{L^{2^\sharp}(\Omega\times\{0\})} + S_\mu \|v\|^2_{L^{2^\sharp}(\Omega\times\{0\})} + o(1) \\
\geq& S_0 \|v_m - v\|^{2^\sharp}_{L^{2^\sharp}(\Omega\times\{0\})} + S_\mu \|v\|^{2^\sharp}_{L^{2^\sharp}(\Omega\times\{0\})} + o(1) \\
=& (S_0 - S_\mu)\|v_m - v\|^{2^\sharp}_{L^{2^\sharp}(\Omega\times\{0\})} + S_\mu \|v_m\|^{2^\sharp}_{L^{2^\sharp}(\Omega\times\{0\})} + o(1).
\end{aligned}$$

Hence we have

$$S_\mu \geq (S_0 - S_\mu)\|v_m - v\|^{2^\sharp}_{L^{2^\sharp}(\Omega\times\{0\})} + S_\mu + o(1).$$

This implies, since $S_0 - S_\mu > 0$ by Proposition 4.2, that

$$v_m(\cdot,0) \to v(\cdot,0) \quad \text{in } L^{2^\sharp}(\Omega).$$

Hence, by lower semi-continuity, we see that $v \geq 0$ is a minimizer for Q_μ. □

Proof of Theorem 1.2. It follows from Proposition 4.4. Indeed, let $v \geq 0$ be the minimizer for Q_μ of Proposition 4.4. Since $\mu < \mu_1$, we deduce that $S_\mu > 0$. Now compute the first variation of Q_μ, we see that a positive multiple of v a solution of (4.1). The $C^{2,\alpha}(\overline{\Omega})$ regularity of the solution follows from Proposition 2.9. Finally Lemma 4.1 gives the nonexistence of solution for $\mu \geq \mu_1$. □

5. Palais-Smale sequences

We now also give the second approach for Theorem 1.2, based on a careful study of the compactness properties for Palais-Smale sequences of the free functional $I_\mu(v)$ defined as follows:

$$I_\mu(v) = \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2\, dxdy - \frac{\mu}{2}\int_{\Omega\times\{0\}} |v|^2\, dx - \frac{1}{2^\sharp}\int_{\Omega\times\{0\}} |v|^{2^\sharp}\, dx. \tag{5.4}$$

Both approaches are completely equivalent. However, the second approach will bring out the peculiarities of the limiting case more clearly. We begin from the following lemma which gives the energy estimates of Palais-Smale sequences.

Lemma 5.1. *Assume that the functional $I_\mu(v)$ is defined as in (5.4) and that Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$. Then for every $\mu \in \mathbb{R}$, every sequence $\{v_m\}$ in $H^1_{0,L}(\mathcal{C})$ such that, as $m \to \infty$,*

$$I_\mu(v_m) \to \beta < \frac{1}{2n} S_0^n, \quad I'_\mu(v_m) \to 0, \tag{5.5}$$

is relatively compact in $H^1_{0,L}(\mathcal{C})$.

Proof. It is easy to check that $\{v_m\}$ is bounded in $H^1_{0,L}(\mathcal{C})$. In fact,

$$\frac{1}{2n}S_0^n + o(1)(1+\|v_m\|) \geq I_\mu(v_m) - \frac{1}{2}\langle I'_\mu(v_m), v_m\rangle$$
$$=(\frac{1}{2}-\frac{1}{2^\sharp})\int_{\Omega\times\{0\}}|v_m|^{2^\sharp}\,dx \geq C_1\left(\int_{\Omega\times\{0\}}|v_m|^2\,dx\right)^{2^\sharp/2}.$$

So

$$\|v_m\|^2 = 2I_\mu(v_m) + \mu\int_{\Omega\times\{0\}}|v_m|^2\,dx + \frac{2}{2^\sharp}\int_{\Omega\times\{0\}}|v_m|^{2^\sharp}\,dx$$
$$\leq C_2 + o(1)\|v_m\|,$$

where $o(1)\to 0$ as $m\to\infty$. Thus it follows that $\{v_m\}$ is bounded in $H^1_{0,L}(\mathcal{C})$.

By Lemma 2.3, we can extract a subsequence, still denoted by $\{v_m\}$, such that, as $m\to\infty$,

$$v_m \rightharpoonup v \quad \text{weakly in } \ H^1_{0,L}(\mathcal{C}),$$
$$v_m(\cdot,0)\to v(\cdot,0) \quad \text{strongly in } \ L^q(\Omega), 2\leq q<2^\sharp,$$
$$v_m(x,0)\to v(x,0) \quad \text{a.e. in } \ \Omega.$$

In particular, for every $\varphi\in H^1_{0,L}(\mathcal{C})$, we obtain that, as $m\to\infty$,

$$\langle I'_\mu(v_m),\varphi\rangle = \int_{\mathcal{C}}\nabla v_m\nabla\varphi\,dxdy - \int_{\Omega\times\{0\}}[\mu v_m\varphi + |v_m|^{2^\sharp-2}v_m\varphi]\,dx$$
$$\to \int_{\mathcal{C}}\nabla v\nabla\varphi\,dxdy - \int_{\Omega\times\{0\}}[\mu v\varphi + |v|^{2^\sharp-2}v\varphi]\,dx = \langle I'_\mu(v),\varphi\rangle.$$

Since by hypothesis $I'_\mu(v_m)\to 0$, we deduce $\langle I'_\mu(v),\varphi\rangle = 0$. Thus, $v\in H^1_{0,L}(\mathcal{C})$ solves the three first equations of problem (4.1). We have by choosing $\varphi = v$,

$$\langle I'_\mu(v), v\rangle = \int_{\mathcal{C}}|\nabla v|^2\,dxdy - \int_{\Omega\times\{0\}}[\mu v^2 + |v|^{2^\sharp}]\,dx = 0.$$

Hence, we see

$$I_\mu(v) = \left(\frac{1}{2}-\frac{1}{2^\sharp}\right)\int_{\Omega\times\{0\}}|v|^{2^\sharp}\,dx = \frac{1}{2n}\int_{\Omega\times\{0\}}|v|^{2^\sharp}\,dx \geq 0.$$

Moreover, Lemma 4.3 leads to

$$\int_{\Omega\times\{0\}}|v_m|^{2^\sharp}\,dx = \int_{\Omega\times\{0\}}|v_m - v|^{2^\sharp}\,dx + \int_{\Omega\times\{0\}}|v|^{2^\sharp}\,dx + o(1).$$

Notice that, developing the square and by weak convergence,

$$\int_{\mathcal{C}}|\nabla v_m|^2\,dxdy = \int_{\mathcal{C}}|\nabla v_m - \nabla v|^2\,dxdy + \int_{\mathcal{C}}|\nabla v|^2\,dxdy + o(1).$$

Thus, we have

$$I_\mu(v_m) = I_\mu(v) + I_0(v_m - v) + o(1). \tag{5.6}$$

Furthermore,

$$\begin{aligned}
&\int_{\Omega\times\{0\}} (|v_m|^{2^\sharp-1}v_m - |v|^{2^\sharp-2}v)(v_m - v)\,dx \\
&\qquad = \int_{\Omega\times\{0\}} (|v_m|^{2^\sharp} - |v_m|^{2^\sharp-2}v_m v)\,dx + o(1) \\
&\qquad = \int_{\Omega\times\{0\}} (|v_m|^{2^\sharp} - |v|^{2^\sharp})\,dx + o(1) = \int_{\Omega\times\{0\}} (|v_m - v|^{2^\sharp})\,dx + o(1).
\end{aligned}$$

It gives

$$\begin{aligned}
o(1) = \langle I'_\mu(v_m), v_m - v\rangle &= \langle I'_\mu(v_m) - I'_\mu(v), v_m - v\rangle \\
&= \int_{\mathcal{C}} |\nabla(v_m - v)|^2\,dxdy - \int_{\Omega\times\{0\}} |v_m - v|^{2^\sharp}\,dx + o(1).
\end{aligned}$$

Then we obtain

$$I_0(v_m - v) = \frac{1}{2n}\int_{\mathcal{C}} |\nabla(v_m - v)|^2\,dxdy + o(1).$$

On the other hand, by (5.6) and since $I_\mu(v) \ge 0$, we see that there is a large $m_0 > 0$ such that, for $m \ge m_0$,

$$\begin{aligned}
I_0(v_m - v) &= I_\mu(v_m) - I_\mu(v) + o(1) \\
&\le I_\mu(v_m) + o(1) < \frac{1}{2n}S_0^n.
\end{aligned}$$

Therefore, we have the following inequality

$$\|v_m - v\|^2 < S_0^n.$$

This implies

$$\int_{\mathcal{C}} |\nabla(v_m - v)|^2\,dxdy < S_0^n \le \left(\frac{\int_{\mathcal{C}} |\nabla(v_m - v)|^2\,dxdy}{\left(\int_{\Omega\times\{0\}} |v_m - v|^{2^\sharp}\,dx\right)^{(n-1)/n}}\right)^n.$$

It gives

$$1 > c_5 \ge \frac{\int_{\Omega\times\{0\}} |v_m - v|^{2^\sharp}\,dx}{\int_{\mathcal{C}} |\nabla(v_m - v)|^2\,dxdy},$$

for all $m \geq m_0$. Then we obtain that, as $m \to \infty$,

$$(1-c_5)\|v_m - v\|^2 \leq \|v_m - v\|^2 \left(1 - \frac{\int_{\Omega\times\{0\}} |v_m - v|^{2^\sharp}\, dx}{\int_{\mathcal{C}} |\nabla(v_m - v)|^2\, dxdy}\right)$$

$$\leq \int_{\mathcal{C}} |\nabla(v_m - v)|^2\, dxdy - \int_{\Omega\times\{0\}} |v_m - v|^{2^\sharp}\, dx = o(1),$$

establishing that $v_m \to v$ strongly in $H^1_{0,L}(\mathcal{C})$. □

Before giving the second proof of Theorem 1.2, let us recall the Mountain Pass Theorem which was developed in [1].

Lemma 5.2. *Let E be a real Banach space with its dual space E^* and suppose that $I \in C^1(E, R)$ satisfies the condition*

$$\max\{I(0), I(u_1)\} \leq \alpha < \beta \leq \inf_{\|u_1\|=\rho} I(u),$$

for some $\beta > \alpha, \rho > 0$ and $u_1 \in E$ with $\|u_1\| > \rho$. Let $c \geq \beta$ be characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq\tau\leq 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0, \gamma(1) = u_1\}$ is the set of continuous paths joining 0 and u_1. Then, there exists a sequence $\{u_m\} \subset E$ such that, as $m \to \infty$,

$$I(u_m) \to c \geq \beta \text{ and } I'(u_m)\mid_{E^*} \to 0.$$

Let

$$\Sigma = \{v \in H^1_{0,\,L}(\mathcal{C}) \setminus \{0\} \mid \langle I'_\mu(v), v\rangle = 0\}.$$

We define critical values for the functionals as follows:

$$c^* = \inf_{v\in\Sigma} I_\mu(v),$$
$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I_\mu(\gamma(t)),$$
$$c^{**} = \inf_{v\in H^1_{0,\,L}(\mathcal{C})\setminus\{0\}} \sup_{t\geq 0} I_\mu(tv),$$

where $\Gamma := \{\gamma \in C([0,1], H^1_{0,\,L}(\mathcal{C})) \mid \gamma(0) = 0, I_\mu(\gamma(1)) < 0\}$. We have the following relations, whose proofs are standard.

Lemma 5.3.

$$c = c^* = c^{**}.$$

Proof of Theorem 1.2. It is sufficient to prove that I_μ satisfies the condition (5.5). Considering the functional I_μ, for every $v \in H^1_{0,L}(\mathcal{C})$ and $t \geq 0$, we have

$$I_\mu(tv) = \frac{A_1 t^2}{2} - \frac{A_2 t^{2^\sharp}}{2^\sharp},$$

where

$$A_1 = \int_{\mathcal{C}} |\nabla v|^2 \, dxdy - \mu \int_{\Omega\times\{0\}} |v|^2 \, dx$$

and

$$A_2 = \int_{\Omega\times\{0\}} |v|^{2^\sharp} \, dx.$$

We see that $I_\mu(tv)$ has its maximum at $t_0 = (\frac{A_1}{A_2})^{1/(2^\sharp-2)} = (\frac{A_1}{A_2})^{(n-1)/2}$. Hence, we obtain

$$\sup_{t\geq 0} I_\mu(tv) = \max_{t\geq 0} I_\mu(tv) = I_\mu(tv)|_{t=t_0} = \frac{1}{2n}\left(\frac{A_1}{A_2^{2/2^\sharp}}\right)^n.$$

This implies

$$\inf_{0\neq v\in H^1_{0,L}(\mathcal{C})} \sup_{t\geq 0} I_\mu(tv) \leq \frac{1}{2n}\left(\inf_{0\neq v\in H^1_{0,L}(\mathcal{C})} Q_\mu(v)\right)^n$$

$$< \frac{1}{2n} S_0^n,$$

by Proposition 4.2. Then by using Lemma 5.2, 5.3 and 5.1, we obtain

$$c^{**} = \inf_{0\neq v\in H^1_{0,L}(\mathcal{C})} \sup_{t\geq 0} I_\mu(tv)$$

is a critical value of I_μ. Finally we complete the proof of regularity of the solution by Proposition 2.9. □

Remark 5.4. *By minimizing the functional $I_\mu(v)$ on the Nehari manifold $\Sigma = \{v \in H^1_{0,L}(\mathcal{C}) \mid \langle I'_\mu(v), v\rangle = 0\}$, one can get another proof of Theorem* 1.2.

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Departamento de Ingeniería Matemática
Universidad de Chile
and Departamento de Matemática
Universidad Técnica Federico Santa María
Avda. España 1680, Valparaíso, Chile
E-mail address: `jinggang.tan@usm.cl`