CONNECTIONS BETWEEN ∞–POINCARÉ INEQUALITY, QUASI-CONVEXITY, AND $N^{1,\infty}$

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Abstract. We study a geometric characterization of $\infty-$Poincaré inequality. We show that a path-connected complete doubling metric measure space supports an $\infty-$Poincaré inequality if and only if it is thick quasi-convex. We also prove that these two equivalent properties are also equivalent to the purely analytic property that $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$, where $\mathrm{LIP}^\infty(X)$ is the collection of bounded Lipschitz functions on X and $N^{1,\infty}(X)$ is the Newton-Sobolev space studied in [DJ].

1. $\infty-$Poincaré inequality in metric measure spaces

The classical Poincaré inequality allows one to obtain integral bounds on the oscillation of a function using integral bounds on its derivatives. The idea of Poincaré inequalities make sense in the more general setting of metric measure spaces. Heinonen and Koskela ([HeK1],[HeK2]) introduced a notion of "upper gradients" which serves the role of derivatives in a metric space. A non-negative Borel function g on X is said to be an *upper gradient* for an extended real-valued function u on X if $|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g$ for every rectifiable curve $\gamma\colon [a,b] \to X$. The following Poincaré inequality is now standard in literature on analysis in metric measure spaces.

Definition 1.1. *Let $1 \leq p < \infty$. We say that (X, d, μ) supports a* weak $p-$Poincaré inequality *if there exist constants $C_p > 0$ and $\lambda \geq 1$ such that for every Borel measurable function $u\colon X \to \mathbb{R}$ and every upper gradient $g\colon X \to [0,\infty]$ of u, the pair (u, g) satisfies the inequality*

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C_p\, r \Big(\fint_{B(x,\lambda r)} g^p d\mu \Big)^{1/p}$$

for each $B(x,r) \subset X$. The word weak *refers to the possibility that λ may be strictly greater than* 1.

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Here for arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ we write

$$u_A = \fint_A u = \frac{1}{\mu(A)} \int_A u \, d\mu.$$

It follows from the Hölder inequality that if a space admits a $p-$Poincaré inequality, it admits a $q-$Poincaré inequality for each $q \geq p$. Metric spaces with doubling measure and $p-$Poincaré inequality admit first order differential calculus akin to that in Euclidean spaces, and has strong links to the geometry of the metric measure space. A natural question is what would be the weakest version of the $p-$Poincaré inequality that would still give reasonable information on the geometry of the metric space. One of the goals of this paper is to answer this question using the following version of infinite Poincaré inequality.

Definition 1.2. *We say that (X, d, μ) supports a* weak $\infty-$Poincaré inequality *if there exist constants $C > 0$ and $\lambda \geq 1$ such that for every Borel measurable function $u\colon X \to \mathbb{R}$ and every upper gradient $g\colon X \to [0, \infty]$ of u in $L^\infty(X)$, the pair (u, g) satisfies the inequality*

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C\, r \|g\|_{L^\infty(\lambda B)}$$

for each $B(x,r) \subset X$.

The main result of this paper is Theorem 3.6. A metric measure space is thick quasi-convex if, loosely speaking, every pair of sets that are positive distance apart can be connected by a family of quasi-convex curves such that the $\infty-$modulus of this family of curves is positive. The main aim of this paper is to show that a path-connected complete doubling metric measure space supports a weak $\infty-$Poincaré inequality if and only if it is thick quasi-convex, which is a purely geometric condition. We will also prove that this condition is equivalent to the purely analytic condition that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$, that is, every Lipschitz function belongs to an equivalence class in $N^{1,\infty}(X)$ and every function in any equivalence class in $N^{1,\infty}(X)$ can be modified on a set of measure zero to become a Lipschitz continuous function. See subsequent sections for definitions of the relevant notions.

Remark 1.3. Let us observe that

$$\begin{aligned}\fint_B |u(x) - u_B| \, d\mu(x) &= \fint_B \Big| \fint_B (u(x) - u(y)) d\mu(y) \Big| d\mu(x) \\ &\leq \fint_B \fint_B |u(x) - u(y)| d\mu(y) d\mu(x),\end{aligned}$$

and so, when we want to check that (X, d, μ) supports a weak $\infty-$Poincaré inequality, it is enough to prove that each pair (u, g) satisfies

$$\fint_B \fint_B |u(x) - u(y)| d\mu(y) d\mu(x) \leq C\, r \|g\|_{L^\infty(\lambda B)} \tag{1}$$

for each ball $B \subset X$. On the other hand, the inequality (1) is necessary to verify $\infty-$Poincaré inequality as well. To see this, note that

$$\fint_B \fint_B |u(x) - u(y)| d\mu(y) d\mu(x) \leq \fint_B \fint_B |u(x) - u_B + u_B - u(y)| d\mu(y) d\mu(x)$$
$$\leq 2 \fint_B |u(x) - u_B| d\mu(x).$$

The next example shows that there exist spaces with a weak $\infty-$Poincaré inequality which do not admit a weak $p-$Poincaré inequality for any finite p.

Example 1.4. Let T be a non-degenerate triangular region in $\mathbb{R}^2$ and let T' be an identical copy of T. Let X be the metric space obtained by identifying a vertex V of T with a vertex V' of T' ($V = V' = \{0\}$) and the metric defined by

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \in T \text{ or } x, y \in T', \\ |x - V| + |V' - y| & \text{if } x \in T \text{ and } y \in T'. \end{cases}$$

The space is equipped with the weighted measure μ given by $d_\mu(x) = \omega(x) d\mathscr{L}^2(x)$, where $\omega(x) = e^{-\frac{1}{|x|^2}}$. Note that μ and the Lebesgue measure $\mathscr{L}^2$ have the same zero measure sets. More in general, if $\lambda << \mu$, $L^\infty(\mu) \hookrightarrow L^\infty(\lambda)$ and $\| \cdot \|_{L^\infty(\lambda)} \leq \| \cdot \|_{L^\infty(\lambda)}$. It is already known that this space equipped with the Lebesgue measure $\mathscr{L}^2$ admits a $p-$Poincaré inequality for $p > 2$ (see for example [Sh1]). Let us see that (X, d, μ) does not admit a weak $p-$Poincaré inequality for any finite p but admits a weak $\infty-$Poincaré inequality.

First, let us notice that given a measurable function u in X,

$$\fint_B |u - u_B| \, d\mu \leq 2 \inf_{c \in \mathbb{R}} \fint_B |u - c| d\mu \quad (*),$$

where $u_B = \fint_B u d\mu$. Indeed, let $c \in \mathbb{R}$ and suppose $c > u_B$. Then,

$$\fint_B |c - u_B| \, d\mu = c - u_B = \fint_B c - \fint_B u = \fint_B (c - u) \leq \fint_B |c - u| d\mu.$$

Since $|u(x) - u_B| \leq |u(x) - c| + |c - u_B|$ for each $x \in X$, we have that

$$\fint_B |u - u_B| \, d\mu \leq \fint_B |u - c| \, d\mu + \fint_B |c - u_B| \, d\mu \leq 2 \fint_B |u - c| d\mu.$$

If we take the infimum over c on the right hand of the previous inequality, we get inequality $(*)$. Let us consider an upper gradient g of u.

Now, we obtain the following chain of inequalities by using Hölder's inequality for $p < q$:

$$\fint_B |u - u_B|\, d\mu \overset{(*)}{\leq} 2 \inf_{c\in\mathbb{R}} \fint_B |u - c| d\mu \leq 2 \fint_B |u - u_{B,\mathscr{L}^2}| d\mu$$
$$\leq C_p r \Big(\fint_{5\lambda B} g^p d\mathscr{L}^2 \Big)^{1/p} \leq C_p r \Big(\fint_{5\lambda B} g^q d\mathscr{L}^2 \Big)^{1/q},$$

where $u_{B,\mathscr{L}^2} = \fint_B u d\mathscr{L}^2$. In the third inequality we have applied [HKo, Theorem 5.1]. If we let q tends to infinity we get

$$\fint_B |u - u_B|\, d\mu \leq C_p r \|g\|_{L^\infty(\mathscr{L}^2, 5\lambda B)} = C_p r \|g\|_{L^\infty(\mu, 5\lambda B)},$$

and so, (X, d, μ) admits a weak $\infty-$Poincaré inequality.

Let us see now that (X, d, μ) does not admit a $p-$Poincaré inequality for any finite p. Indeed, consider the function $u = 1$ in T and $u = 0$ in T' and in the vertex. The function $g_\alpha(x) = \frac{\alpha}{|x|}$ is an upper gradient for u for each $\alpha > 0$. One can check that $\fint_X |u - u_B|\, d\mu > 0$ whereas $\int_X g_\alpha^p d\mu$ tends to zero when α tends to zero for $1 < p < \infty$, and so X does not admit a weak $p-$Poincaré inequality for any finite p.

Observe that the measure μ in the above example is *not* doubling. A measure μ is doubling if there is a constant $C_\mu > 0$ such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C_\mu\, \mu(B(x, r)).$$

In a complete metric space X, the existence of a doubling measure which is not trivial and finite on balls implies that X is separable and that closed bounded subsets of X are compact, in particular, X is locally compact

In the rest of this paper, we assume that X is a connected complete metric space which supports a doubling Borel measure μ which is non-trivial and finite on balls.

One of the most useful geometric implications of the $p-$Poincaré inequality for finite p is the fact that if a complete doubling metric measure space supports a $p-$Poincaré inequality then there exists a constant such that each pair of points can be connected with a curve whose length is at most the constant times the distance between the points (see [Se1] or [HKo]), that is, the space is *quasi-convex.* If X is only known to support an $\infty-$Poincaré inequality, the same conclusion holds as demonstrated by Proposition 1.5 below.

Proposition 1.5. *Suppose that (X, d, μ) is a complete metric measure space with μ a doubling measure. If X supports a weak $\infty-$Poincaré inequality, then X is quasi-convex with a constant depending only on the constants of the Poincaré inequality and the doubling constant.*

Proof. Let $\varepsilon > 0$. We say that $x, z \in X$ lie in the same $\varepsilon-$component of X if there exists an $\varepsilon-$chain joining x with z, that is, there exists a finite chain $z_0, z_1, \dots, z_n$ such that $z_0 = x$, $z_n = z$ and $d(z_i, z_{i+1}) \le \varepsilon$ for all $i = 0, \dots, n-1$. If x and y lie in different $\varepsilon-$components, then it is obvious that there does not exist a rectifiable curve joining x and y. Thus, the function $g \equiv 0$ is an upper gradient for the characteristic function of any of the components. Note that for every x in one of the components, the ball $B(x, \varepsilon/2)$ is a subset of that component; that is, each component is open and hence is a measurable set. By applying the weak $\infty-$Poincaré inequality to the characteristic function of any component, it follows that all the points of X lie in the same $\varepsilon-$component.

Now, let us fix $x, y \in X$ and prove that there exists a curve γ joining x and y such that $\ell(\gamma) \le Cd(x, y)$, where C is a constant which depends only on the doubling constant and the constants involved in the Poincaré inequality. We define the $\varepsilon-$distance of x to z to be

$$\rho_{x,\varepsilon}(z) := \inf \sum_{i=0}^{N-1} d(z_i, z_{i+1}),$$

where the infimum is taken over all finite $\varepsilon-$chains $\{z_i\}$. Note that $\rho_{x,\varepsilon}(z) < \infty$ for all $z \in X$. In addition, if $d(z, w) \le \varepsilon$ then $|\rho_{x,\varepsilon}(z) - \rho_{x,\varepsilon}(w)| \le d(z, w)$. Hence, $\rho_{x,\varepsilon}$ is a locally $1-$Lipschitz function, in particular, every point is a Lebesgue point of $\rho_{x,\varepsilon}$ and in addition, for all $\varepsilon > 0$, the function $g \equiv 1$ is an upper gradient of $\rho_{x,\varepsilon}$. Thus, a telescopic argument, together with weak $\infty-$Poincaré inequality give us the following chain of inequalities:

$$\begin{aligned}
|\rho_{x,\varepsilon}(y)| =& |\rho_{x,\varepsilon}(x) - \rho_{x,\varepsilon}(y)| \\
\le& \sum_{i\in\mathbb{Z}} \Big| \fint_{B_i} \rho_{x,\varepsilon} d\mu - \fint_{B_{i+1}} \rho_{x,\varepsilon} d\mu \Big| \\
\le& C_\mu \sum_{i\in\mathbb{Z}} \frac{1}{\mu(B_i)} \int_{B_i} \Big| \rho_{x,\varepsilon} - \fint_{B_{i+1}} \rho_{x,\varepsilon} d\mu \Big| d\mu \\
\le& C_\mu C_p d(x, y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \|g\|_{L^\infty(B_i)} \ \le \ Cd(x, y) \quad (*)
\end{aligned}$$

where $C = 3C_\mu C_p$ is a constant that depends only on X.

Since X is complete, the existence of a non trivial doubling measure implies that closed balls are compact. Using a standard limiting argument, which involves Arzela-Ascoli's theorem and inequality $(*)$, we can construct a $1-$Lipschitz rectifiable curve connecting x and y with length at most $Cd(x, y)$. Since x and y were arbitrary this completes the proof. For further details about the construction of the curve we refer the reader to [Ko, Theorem 3.1]. □

2. $\infty-$MODULUS OF CURVES AND $N^{1,\infty}$

A related generalization of Sobolev spaces to general metric spaces are the so-called Newtonian Spaces $N^{1,p}$ introduced in [Sh1, Sh2]. Its definition is based on the notion of upper gradients of Heinonen and Koskela. In this work, we will focus on the case $p = \infty$.

We denote by $\mathrm{LIP}^{\infty}(X)$ the space of bounded Lipschitz functions. In what follows, $\|\cdot\|_{L^\infty}$ will denote the essential supremum norm, provided we have a measure on X. In addition, $\mathrm{LIP}(\cdot)$ will denote the Lipschitz constant:

$$\mathrm{LIP}(u) = \sup_{x\in X} \sup_{y\in X\setminus\{x\}} \frac{|u(y) - u(x)|}{d(y,x)}.$$

We recall the definition of $\infty-$modulus, an outer measure on the collection of all paths in X. In what follows let $\Upsilon \equiv \Upsilon(X)$ denote the family of all non-constant rectifiable curves in X. It may happen that Υ is empty, but we will be mainly interested in finding out when metric spaces have large enough Υ.

Definition 2.1. *For $\Gamma \subset \Upsilon$, let $F(\Gamma)$ be the family of all Borel measurable functions $\rho\colon X \to [0,\infty]$ such that*

$$\int_\gamma \rho \geq 1 \quad \textit{for all } \gamma \in \Gamma.$$

We define the $\infty-$modulus of Γ *by*

$$\mathrm{Mod}_\infty(\Gamma) = \inf_{\rho\in F(\Gamma)} \{\|\rho\|_{L^\infty}\}.$$

If some property holds for all curves $\gamma \notin \Gamma$, where $\Gamma \subset \Upsilon$ satisfies $\mathrm{Mod}_\infty \Gamma = 0$*, then we say that the property holds for* $\infty-$a.e. curve.

Remark 2.2. It can be easily checked that Mod_∞ is an outer measure as it is for $1 \leq p < \infty$, see for example [H, Theorem 5.2].

Definition 2.3. *A non-negative Borel function g on X is an* $\infty-$weak upper gradient *of an extended real-valued function u on X, if*

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g$$

for $\infty-$a.e. every curve $\gamma \in \Upsilon$.

Let $\widetilde{N}^{1,\infty}(X,d,\mu)$, be the class of all Borel functions $u \in L^\infty(X)$ for which there exists an $\infty-$weak upper gradient g in $L^\infty(X)$. For $u \in \widetilde{N}^{1,\infty}(X,d,\mu)$ we set

$$\|u\|_{\widetilde{N}^{1,\infty}} = \|u\|_{L^\infty} + \inf_g \|g\|_{L^\infty},$$

where the infimum is taken over all $\infty-$weak upper gradients g of u.

Definition 2.4. *We define an equivalence relation in $\widetilde{N}^{1,\infty}$ by $u \sim v$ if and only if $\|u - v\|_{\widetilde{N}^{1,\infty}} = 0$. The space $N^{1,\infty}(X, d, \mu) = N^{1,\infty}(X)$ denotes the quotient $\widetilde{N}^{1,\infty}(X, d, \mu)/\sim$ and it is equipped with the norm*

$$\|u\|_{N^{1,\infty}} = \|u\|_{\widetilde{N}^{1,\infty}}.$$

It was shown in [DJ] that $N^{1,\infty}(X)$ is a Banach space. Note that if $u \in \widetilde{N}^{1,\infty}$ and $v = u$ $\mu-$a.e., then it is not necessarily true[1] that $v \in \widetilde{N}^{1,\infty}$. Nevertheless, in the following lemma we show that if $u, v \in \widetilde{N}^{1,\infty}$, and $v = u$ $\mu-$a.e., then $\|u - v\|_{\widetilde{N}^{1,\infty}} = 0$. Recall here that every rectifiable curve γ admits a parametrization by the arc-length; that is, with $\gamma : [a, b] \to X$, for all $t_1, t_2 \in [a, b]$ with $t_1 \leq t_2$, we have $\ell(\gamma_{|_{[t_1,t_2]}}) = t_2 - t_1$. Hence from now on we only consider curves that are arc-length parametrized.

Lemma 2.5. [DJ, 5.13] *Let $u_1, u_2 \in \widetilde{N}^{1,\infty}(X, d, \mu)$ such that $u_1 = u_2$ $\mu-a.e.$ Then $u_1 \sim u_2$, that is, both functions define exactly the same element in $N^{1,\infty}(X, d, \mu)$.*

The following example shows one of the difficulties in working with $p = \infty$ as opposed to finite values of p.

Example 2.6. Let X be a metric space that supports a doubling Borel measure μ which is non-trivial and finite on balls and suppose that X supports a weak $\infty-$Poincaré inequality. Denote by $\Gamma_{x_0,r,R}$ the family of curves that connect $B(x_0, r)$ to the complement of the ball $B(x_0, R)$ with $0 < r < R/2$. We will prove that if the measure on X is doubling and supports an $\infty-$Poincaré inequality, then there is a constant $C > 0$, independent of R, r and x_0, such that

$$\mathrm{Mod}_\infty(\Gamma_{x_0,r,R}) \geq C/R.$$

To see this, let g be a non-negative Borel measurable function on X such that for all $\gamma \in \Gamma_{x_0,r,R}$, the integral $\int_\gamma g\, ds \geq 1$. We then set

$$\tilde{u}(z) = \inf_{\gamma \text{ path connecting } z \text{ to } B(x_0,r)} \int_\gamma g\, ds,$$

and consider $u = \min\{\tilde{u}, 2\}$. Then it follows that $u = 0$ on $B(x_0, r)$ and by the choice of g, $u \geq 1$ on $X \setminus B(x_0, R)$. By [JJRRS, Corollary 1.10] we know that u is measurable. As in the proof of Proposition 3.7, we see that g is an upper gradient of u.

For each $i \in \mathbb{Z}$, define $B_i = B(x, 2^{1-i}d(x, y))$ if $i \geq 0$, and $B_i = B(y, 2^{1+i}d(x, y))$ if $i \leq -1$. By the weak $\infty-$Poincaré inequality, we get for Lebesgue points

[1] Let $(X = [-1, 1], d, \mu)$ where d denotes the Euclidean distance and μ the Lebesgue measure. Let $u\colon X \to \mathbb{R}$ be the function $u = 1$ and $v\colon X \to \mathbb{R}$ given by $v = 1$ if $x \neq 0$ and $v(x) = \infty$ if $x = 0$. In this case we have that $u = v$ $\mu - a.e.$, $u \in \widetilde{N}^{1,\infty}$ but $v \notin \widetilde{N}^{1,\infty}$.

$x \in B(x_0, r)$ and $y \in X \setminus B(x_0, R)$,

$$\begin{aligned} 1 \le |u(x) - u(y)| &\le \sum_{i\in\mathbb{Z}} \Big| \fint_{B_i} u d\mu - \fint_{B_{i+1}} u d\mu \Big| \\ &\le C_\mu \sum_{i\in\mathbb{Z}} \fint_{B_{i+1}} \Big| u - \fint_{B_{i+1}} u d\mu \Big| d\mu \\ &\le C_\mu C_p d(x,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \|g\|_{L^\infty(B_i)} \\ &\le C d(x,y) \|g\|_{L^\infty(X)}. \end{aligned}$$

Hence

$$\|g\|_{L^\infty(X)} \ge \frac{1}{C\, d(x,y)} \ge \frac{1}{C\,(R-r)} \ge \frac{1}{2CR}.$$

Taking the infimum over all such g we obtain the desired inequality for the $\infty-$Modulus. An analogous statement holds for $\mathrm{Mod}_p(\Gamma_{x_0,r,R})$ if X supports a weak $p-$Poincaré inequality for sufficiently large finite p (that is, with p larger than the lower mass bound exponent obtained from the doubling property of the measure μ). For such finite p, we can approximate test functions g from above and in $L^p(X)$ by lower semi-continuous functions (it follows from Vitali-Caratheodory theorem, see [F, pp. 209–213]), and so we would see that the $p-$Modulus of the collection of all curves that connect x_0 itself to $X \setminus B(x_0, R)$ is positive. Unfortunately such an approximation by lower semi-continuous functions in the $L^\infty-$norm does not hold true, and so we cannot conclude from the above computation that the $\infty-$Modulus of the collection of all curves connecting x_0 to $X \setminus B(x_0, R)$ is positive *if X is only known to support a weak $\infty-$Poincaré inequality.*

The previous example highlights the difficulties when working with the $L^\infty-$norm, namely, the $L^\infty-$norm is insensitive to local changes, and we do not have Vitali-Caratheodory theorem.

Definition 2.7. *Let $E \subset X$. Γ_E^+ is the family of curves γ such that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap E)) > 0$.*

Recall that we only consider curves that are arc-length parametrized.

Lemma 2.8. *Let $E \subset X$. If $\mu(E) = 0$, then $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$.*

Proof. As before, without loss of generality we may assume that E is a Borel set. Let $g = \infty \cdot \chi_E$. For $\gamma \in \Gamma_E^+$, we have that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap E)) > 0$ and so

$$\int_\gamma g ds = \int_{\gamma \cap E} g ds = \infty.$$

Hence, by the definition of modulus

$$\mathrm{Mod}_\infty(\Gamma_E^+) \le \|g\|_{L^\infty(X)} = 0. \qquad \square$$

Lemma 2.9. *Suppose that X supports a weak $\infty-$Poincaré inequality, and let $x, y \in X$ with $x \neq y$. Then for a$0 < \varepsilon < d(x,y)/4$, we have $\mathrm{Mod}_\infty(\Gamma(x,y,\varepsilon)) > 0$, where $\Gamma(x,y,\varepsilon)$ is the collection of all rectifiable curves connecting $B(x,\varepsilon)$ to $B(y,\varepsilon)$.*

Proof. Suppose to the contrary that $\mathrm{Mod}_\infty(\Gamma(x,y,\varepsilon)) = 0$. Then for every $\eta > 0$ there is a non-negative Borel measurable function $\rho_\eta \in L^\infty(X)$ such that $\|\rho_\eta\|_{L^\infty(X)} < \eta$ and for all $\gamma \in \Gamma(x,y,\varepsilon)$, $\int_\gamma \rho_\eta\, ds = \infty$. We use ρ_η to define a function on X as follows:

$$\tilde{u}(z) = \inf_{\gamma \text{ connecting } B(x,\varepsilon) \text{ to } z} \int_\gamma \rho_\eta\, ds,$$

and consider $u = \min\{\tilde{u}, 1\}$. Observe that $u = 0$ on $B(x,\varepsilon)$, $u = 1$ on $B(y,\varepsilon)$ (by the contrary assumption above and the choice of ρ_η), and as in the proof of Proposition 3.7, ρ_η is an upper gradient of u. So $u \in N^{1,\infty}(X)$. Since x and y are therefore Lebesgue points of u, by the weak $\infty-$Poincaré inequality, with $B_i = B(x, 2^{-i}d(x,y))$ if $i \ge 1$, $B_0 = B(x, 2d(x,y))$, and $B_i = B(y, 2^i d(x,y))$ if $i \le -1$, we see that

$$\begin{aligned} 1 = |u(x) - u(y)| \le \sum_{i\in\mathbb{Z}} |u_{B_i} - u_{B_{i+1}}| &\le C \sum_{i\in\mathbb{Z}} \fint_{2B_i} |u - u_{2B_i}|\, d\mu \\ &\le C \sum_{i\in\mathbb{Z}} 2^{1-|i|} d(x,y)\, \|\rho_\eta\|_{L^\infty(2\lambda B_i)} \\ &\le C d(x,y) \|\rho_\eta\|_{L^\infty(X)} < C d(x,y)\eta. \end{aligned}$$

Since the above inequality has to hold true for all $\eta > 0$, we have a contradiction. Thus we conclude that $\mathrm{Mod}_\infty(\Gamma(x,y,\varepsilon)) > 0$. $\square$

3. Geometric characterization of $\infty-$Poincaré inequality

The connection between isoperimetric and Sobolev-type inequalities in the Euclidean setting is well-understood (see [BHo]). In the context of metric spaces supporting a doubling measure, Miranda proved in [M] that a $1-$weak Poincaré inequality implies a relative isoperimetric inequality for sets of finite perimeter. Recently, in [KKo] Kinnunen and Korte gave further characterizations of Poincaré type inequalities in the context of Newtonian spaces in terms of isoperimetric and isocapacitary inequalities.

In what follows, we will prove that $\infty-$Poincaré inequality also has a geometric characterization, namely, it is equivalent to a stronger notion of quasi-convexity, called *thick quasi-convexity* in this paper.

Definition 3.1. *(X, d, μ) is a* thick quasi-convex *space if there exists $C \geq 1$ such that for all $x, y \in X$, $0 < \varepsilon < \frac{1}{4}d(x, y)$, and all measurable sets $E \subset B(x, \varepsilon)$, $F \subset B(y, \varepsilon)$ satisfying $\mu(E)\mu(F) > 0$ we have that*

$$\mathrm{Mod}_\infty(\Gamma(x, y, E, F, \varepsilon, C)) > 0,$$

where $\Gamma(x, y, E, F, \varepsilon, C)$ denotes the set of curves $\gamma_{p,q}$ connecting $p \in B(x, \varepsilon) \cap E$ and $q \in B(y, \varepsilon) \cap F$ with $\ell(\gamma_{p,q}) \leq Cd(p, q)$.

Remark 3.2. Note that every complete thick quasi-convex space X supporting a doubling measure is quasi-convex. Indeed, let $x, y \in X$ and choose a sequence ε_j which tends to zero. Since X is thick quasi-convex, there exists a constant $C \geq 1$ such that for every ε_j there exists $x_j \in B(x, \varepsilon_j)$ and $y_j \in B(y, \varepsilon_j)$ and a curve γ_j connecting x_j to y_j with $\ell(\gamma_j) \leq Cd(x_j, y_j)$. Thus, we obtain a sequence $\{\gamma_j\}$ of curves such that

$$\ell(\gamma_j) \leq Cd(x_j, y_j) \leq 2Cd(x, y),$$

that is, a sequence of curves with uniformly bounded length. Since X is a complete doubling metric space and therefore proper, we may use the Arzela-Ascoli's theorem to obtain a subsequence, also denoted $\{\gamma_j\}$, which converges uniformly to a curve γ which connects x and y with

$$\ell(\gamma) = \lim_{j\to\infty} \ell(\gamma_j) \leq C \lim_{j\to\infty} d(x_j, y_j) = Cd(x, y).$$

Remark 3.3. The space considered in Example 1.4 with a measure that decays very fast to zero at the origin (the point where the two triangluar regions are glued) has quasi-convexity but not thick quasi-convexity. However, this measure is not doubling. In example 3.13 we will give a quasi-convex space endowed with a doubling measure which is not thick quasi-convex.

Remark 3.4. The hypothesis of completeness is not so restrictive. The completion $(\hat{X}, \hat{d})$ of a metric space (X, d) is unique up to isometry. Note that (X, d) is a subspace of $(\hat{X}, \hat{d})$ and X is dense in $\hat{X}$. For our purposes, the crucial observation is that the essential features of X are inherited by $\hat{X}$. Indeed, if X is locally complete and there is a doubling Borel measure μ which is non-trivial and finite on balls, we may extend this measure to $\hat{X}$ so that $\hat{X} \setminus X$ has zero measure and the extended measure has the same properties as the original one. Also, if X supports a weak $p-$Poincaré inequality for some $1 \leq p \leq \infty$, then so does $\hat{X}$.

The following result indicates an advantage of a thick quasi-convex space.

Lemma 3.5. *Suppose that X is a thick quasi-convex doubling metric space. If u is a measurable function on X and g is an upper gradient of u, and if B is a ball* *in X such that $\|g\|_{L^\infty(2CB)} < \infty$, then there is a set $F \subset B$ with $\mu(F) = 0$ such that u is $2C\|g\|_{L^\infty(2CB)}-$Lipschitz continuous on $B \setminus F$. Here C is the constant appearing in the definition of thick quasi-convexity.*

Proof. Let $0 < \varepsilon < \mathrm{rad}(B)/(2C)$ where C is the constant in the thick quasi-convexity property of X. Since X is doubling, we can cover the ball $(1-2\varepsilon)B$ by finitely many balls $B_i = B(x_i, \varepsilon)$ (see for example [Se2, C.30]). Fix B_i in this cover, and let B_j be another ball in this cover such that $d(x_j, x_i) > 4\varepsilon$.

Let $P = \{x \in 2CB : g(x) > \|g\|_{L^\infty(2CB)}\}$; then by assumption, $\mu(P) = 0$, and so it follows from Lemma 2.8 that $\mathrm{Mod}_\infty(\Gamma_P^+) = 0$. So by the thick quasi-convexity property of X, we see that for almost every $x \in B_i$ and almost every $y \in B_j$ there is a curve γ_{xy} connecting x and y such that $\ell(\gamma_{xy}) \le Cd(x,y)$ and $\mathscr{L}^1(\gamma_{xy}^{-1}(\gamma_{xy} \cap P)) = 0$. Let $F_{i,j,\varepsilon}$ be the set of exceptional points in B_i and B_j. Then for all $x \in B_i \setminus F_{i,j,\varepsilon}$ and $y \in B_j \setminus F_{i,j,\varepsilon}$,

$$|u(x) - u(y)| \le \int_{\gamma_{xy}} g\, ds \le \|g\|_{L^\infty(2CB)}\ell(\gamma_{xy}) \le C\|g\|_{L^\infty(2CB)}\, d(x,y).$$

For sufficiently large $k \in \mathbb{N}$ we choose $\varepsilon = 1/k$, and $F_k = \bigcup_{i,j} F_{i,j,1/k}$. We see that $\mu(F_k) = 0$, and by the above argument we know that for all $x, y \in (1-2/k)B \setminus F_k$ with $d(x,y) > 4/k$,

$$|u(x) - u(y)| \le C\|g\|_{L^\infty(2CB)}\, d(x,y).$$

Now taking $F = \bigcup_k F_k$, we see by letting $k \to \infty$ that for all $x, y \in B \setminus F$,

$$|u(x) - u(y)| \le C\|g\|_{L^\infty(2CB)}\, d(x,y);$$

that is, u is $C\|g\|_{L^\infty(2CB)}-$Lipschitz continuous on $B \setminus F$, with $\mu(F) = 0$. $\square$

We are ready to state the main result of this paper.

Theorem 3.6. *Suppose that X is a connected complete metric space supporting a doubling Borel measure μ which is non-trivial and finite on balls. Then the following conditions are equivalent:*

(a) X supports a weak $\infty-$Poincaré inequality.
(b) X is thick quasi-convex.
(c) $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$.
(d) X supports a weak $\infty-$Poincaré inequality for functions in $N^{1,\infty}(X)$.

The result $a \Rightarrow d$ is immediate, and so the proof of Theorem 3.6 will be split in three parts:

- $d \Rightarrow b$: Proposition 3.7.
- $b \Rightarrow c$: Proposition 3.9.
- $c \Rightarrow a$: Proposition 3.11.

Proposition 3.7. *If X supports a weak $\infty-$Poincaré inequality for functions in $N^{1,\infty}(X)$ with upper gradients in $L^\infty(X)$, then X is thick quasi-convex.*

We wish to point out here that $N^{1,\infty}(X)$ consists precisely of functions in $L^\infty(X)$ that have an upper gradient in $L^\infty(X)$.

Proof. Let $x, y \in X$ such that $x \neq y$, and let $0 < \varepsilon < d(x,y)/4$. Fix $n \in \mathbb{N}$ and let Γ_n consist of rectifiable paths γ belonging to the family $\Gamma(x,y,\varepsilon)$ mentioned in Lemma 2.9 such that the length $\ell(\gamma) \le n\, d(x,y)$. Observe that by the choice of ε, if p, q are the end points of γ, then $d(p,q)/4 \le d(x,y) \le 4d(p,q)$.

Suppose that $\mathrm{Mod}_\infty(\Gamma_n) = 0$. By [DJ, Lemma 5.7] there exists a non-negative Borel measurable function $g \in L^\infty(X)$ such that $\|g\|_{L^\infty(X)} = 0$ and for all $\gamma \in \Gamma_n$, the path integral $\int_\gamma g\, ds = \infty$. In this case we define

$$u(z) = \inf_{\gamma \text{ connecting } z \text{ to } B(x,\varepsilon)} \int_\gamma (1+g)\, ds.$$

Observe that $\|1+g\|_{L^\infty(X)} = 1$ and $u = 0$ on $B(x,\varepsilon)$. If $z \in B(y,\varepsilon)$ and γ is a rectifiable curve connecting z to $B(x,\varepsilon)$, then either $\gamma \in \Gamma_n$ in which case $\int_\gamma (1+g)\, ds \ge \int_\gamma g\, ds = \infty$, or else $\gamma \notin \Gamma_n$, in which case $\ell(\gamma) > nd(x,y)$ and so $\int_\gamma (1+g)\, ds \ge \int_\gamma 1\, ds > nd(x,y)$, and so $u(z) \ge n\, d(x,y)$. It follows that the function $v = \min\{u, 2n\, d(x,y)\}$ has the properties that

(1) $v = 0$ on $B(x,\varepsilon)$,
(2) $v \ge nd(x,y)$ on $B(y,\varepsilon)$,
(3) $v \in N^{1,\infty}(X)$,
(4) $1+g$ is an upper gradient of v on X, with $\|g\|_{L^\infty(X)} = 0$.

To see that $1+g$ is an upper gradient of v on X, we argue as follows. Fix $z_1, z_2 \in X$ and β be a rectifiable curve in X connecting z_1 to z_2. There are three possible cases:

(1) $v(z_1) = u(z_1)$ and $v(z_2) = u(z_2)$,
(2) $v(z_1) = u(z_1)$ and $v(z_2) = 2nd(x,y)$,
(3) $v(z_1) = 2nd(x,y) = v(z_2)$.

In the first case, both $u(z_1)$ and $u(z_2)$ are finite. Fix $\varepsilon > 0$; then we can find a rectifiable curve connecting z_1 to $B(x,\varepsilon)$ such that $u(z_1) \ge \int_\gamma (1+g) ds - \varepsilon$, and so

$$u(z_2) - u(z_1) \le \int_{\gamma\cup\beta} (1+g)\, ds - \int_\gamma (1+g)\, ds \ + \varepsilon = \int_\beta (1+g)\, ds \ + \varepsilon,$$

where we can cancel $\int_\gamma (1+g)\,ds$ because it is a finite value. A similar argument gives

$$u(z_1) - u(z_2) \le \int_\beta (1+g)\,ds\ + \varepsilon,$$

and the combination of the above two inequalities followed by letting $\varepsilon \to 0$ gives

$$|v(z_1) - v(z_2)| = |u(z_1) - u(z_2)| \le \int_\beta (1+g)\,ds.$$

In the second case, $u(z_1) = v(z_1) \le v(z_2) \le u(z_2)$. In this case again, $u(z_1)$ is finite. For $\varepsilon > 0$ we can find a rectifiable curve γ connecting z_1 to $B(x,\varepsilon)$ such that $u(z_1) \ge \int_\gamma (1+g)\,ds - \varepsilon$, and so

$$\begin{aligned} |v(z_1) - v(z_2)| &= v(z_2) - v(z_1) \le u(z_2) - u(z_1) \\ &\le \int_{\gamma\cup\beta} (1+g)\,ds - \int_\gamma (1+g)\,ds + \varepsilon \\ &= \int_\beta (1+g)\,ds + \varepsilon, \end{aligned}$$

where again we were able to cancel the term $\int_\gamma (1+g)\,ds \le u(z_1) + \varepsilon$ because it is finite. Letting $\varepsilon \to 0$ we again obtain

$$|v(z_1) - v(z_2)| \le \int_\beta (1+g)\,ds.$$

In the third case we easily obtain the above inequality again, because in this case $v(z_1) - v(z_2) = 0$.

Let $y_0 \in B(y, \varepsilon/2)$ be a Lebesgue point of v; then by using the chain of balls $B_i = B(x, 2^{1-i}d(x,y))$ if $i \ge 0$ and $B_i = B(y_0, 2^{1+i}d(x,y))$ if $i \le -1$ and using the weak $\infty-$Poincaré inequality, we get

$$\begin{aligned} n\,d(x,y) \le v(y_0) = |v(x) - v(y_0)| &\le \sum_{i\in\mathbb{Z}} |v_{B_i} - v_{B_{i+1}}| \\ &\le C \sum_{i\in\mathbb{Z}} \fint_{2B_i} |v - v_{2B_i}|\,d\mu \\ &\le C \sum_{i\in\mathbb{Z}} 2^{-|i|} d(x,y) \|1+g\|_{L^\infty(2\lambda B_i)} \\ &= Cd(x,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \le 2Cd(x,y). \end{aligned}$$

Thus we must have $n \le 2C$, with C depending solely on the doubling constant and the constant of the Poincaré inequality. Hence if $n > 2C$ then the curve family $\Gamma_n = \Gamma(x, y, B(x,\varepsilon), B(y,\varepsilon), \varepsilon, n)$ must have positive $\infty-$Modulus, completing the proof in the simple case that $E = B(x,\varepsilon)$ and $F = B(y,\varepsilon)$. The proof for more general E, F is very similar, where we modify the definition of u by looking at curves that connect z to $B(x,\varepsilon)\cap E$, and then observing that almost every point in $B(x,\varepsilon)\cap E$ and almost every point in $B(y,\varepsilon)\cap F$ are Lebesgue points for the modified function v, with $v=0$ on $B(x,\varepsilon)\cap E$ and $v\ge nd(x,y)$ on $B(y,\varepsilon)\cap F$. This completes the proof of the proposition. □

Remark 3.8. We have already proved in Proposition 1.5 that if X is connected, weak $\infty-$Poincaré inequality for Lipschitz functions implies quasi-convexity. However, in Proposition 3.7 we proved that weak $\infty-$Poincaré inequality for Newtonian functions implies the stronger property of thick quasi-convexity.

Proposition 3.9. *Let X be a thick quasi-convex space. Then* $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$.

Proof. Since we have always that $\mathrm{LIP}^\infty(X)\subset N^{1,\infty}(X)$, it suffices to check is that $N^{1,\infty}(X)\subset \mathrm{LIP}^\infty(X)$. This follows from Lemma 3.5, by exhausting X by balls of large radii and then modifying $f\in N^{1,\infty}(X)$ on the exceptional set of measure zero via McShane extension. □

Lemma 3.10. *Suppose that X is doubling, complete, path-connected, and $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$ in the sense that every function in $N^{1,\infty}(X)$, after modifying on a set of measure zero, is in $\mathrm{LIP}^\infty(X)$, with comparable* energy *norms. Then there exists a constant $C\ge 1$ such that for every $E\subset X$ with $\mu(E)=0$ and for every $x\in X$ and $r>0$ there is a set $F\subset X$ with $\mu(F)=0$ so that whenever $y\in X\setminus(B(x,2r)\cup F)$, there is a rectifiable curve γ_y connecting y to $\overline{B}(x,r)$ such that $\ell(\gamma_y)\le C\,d(x,y)$ and $\mathscr{L}^1(\gamma_y^{-1}(E\cap\gamma_y))=0$.*

Proof. Let $E\subset X$ such that $\mu(E)=0$; since μ is a Borel measure, we may assume (by enlarging E if necessary) that E is a Borel set. Then $\rho=\infty\chi_E\in L^\infty(X)$ is a non-negative Borel measurable function. Let Γ_E^+ be the collection of all rectifiable curves γ for which $\mathscr{L}^1(\gamma^{-1}((\gamma\cap E)))>0$. Then clearly for such curves γ we have $\int_\gamma \rho\,ds=\infty$, and so $\mathrm{Mod}_\infty(\Gamma_E^+)=0$. As before, we define for $r>0$,

$$\tilde u(z)=\inf_{\gamma\text{ connects } z \text{ to } B(x,r)}\int_\gamma (1+\rho)\,ds,$$

where $\|1+\rho\|_{L^\infty(X)}=1$. For positive integers k we set $u_k=\min\{k,\tilde u\}$. then $u_k\in N^{1,\infty}(X)$ with $1+\rho$ as an upper gradient, and $u=0$ on $B(x,r)$. Let F_k be the exceptional set on which u_k has to be modified in order to be Lipschitz continuous; we have $\mu(F_k)=0$. Observe that since $\mathrm{LIP}^\infty(X)=N^{1,\infty}(X)$ is a Banach space with both norms

$$\|f\|_{\mathrm{LIP}}=\{\mathrm{LIP}(f)+\|f\|_\infty\}\quad\text{and}\quad \|f\|_{N^{1,\infty}}=\{\|f\|_{L^\infty}+\inf_g\|g\|_{L^\infty}\},$$

where the infimum is taken over all $\infty-$weak upper gradients g of f, and $\inf_g\|g\|_{L^\infty}\le \mathrm{LIP}(f)$, by the open mapping theorem then there exists a constant $C>0$ such that $\mathrm{LIP}(u_k)\le C[\|f\|_{N^{1,\infty}}-\|f\|_{L^\infty}]$, and hence $\mathrm{LIP}(u_k)\le$

$C\|1+\rho\|_{L^\infty(X)} = C$. Thus for $y \in X \setminus (F_k \cup B(x,2r))$,

$$|u_k(y)| = |u_k(y) - u_k(x_1)| \le C\, d(x,y)$$

for any $x_1 \in B(x,r)\setminus F_k$. If $u_k(y) = k$, then $d(x,y) \ge k/C$, and so if $d(y,x) < k/C$ we see that $u_k(y) = \tilde{u}(y)$. In addition, if γ is a rectifiable curve connecting $B(x,r)$ to y, then

$$\int_\gamma (1+\rho)\, ds \ge \int_\gamma ds = \ell(\gamma) \ge \operatorname{dist}(y, B(x,r)) \ge r > 0.$$

Hence $\tilde{u}(y) > 0$. Thus there is a curve γ_y connecting $B(x,r)$ to y such that $2\tilde{u}(y) \ge \int_{\gamma_y}(1+\rho)\, ds \ge \ell(\gamma_y)$, and so

$$\ell(\gamma_y) \le 2C\, d(x,y).$$

In addition, as $\tilde{u}(y)$ is finite, it follows that $\int_{\gamma_y}(1+\rho)\, ds$ is finite. In particular, $\int_{\gamma_y} \rho\, ds$ is finite, and so $\mathscr{L}^1(\gamma_y^{-1}(\gamma_y \cap E)) = 0$.

Finally let $F = \cup_{k\in\mathbb{N}} F_k$, to complete the proof. $\square$

Proposition 3.11. *Suppose that X is doubling, complete, path-connected, and that $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$ in the sense of Lemma 3.10. Then X supports a weak $\infty-$Poincaré inequality.*

Proof. Let $u \in N^{1,\infty}(X)$ and $g \in L^\infty(X)$ be an upper gradient of u, and fix a ball $B \subset X$. Let $E = \{w \in 2CB : g(w) > \|g\|_{L^\infty(2CB)}\}$, where C is the constant from Lemma 3.10. Then $\mu(E) = 0$. Fix $\varepsilon > 0$. For $x \in B$ such that $\mu(\{x\}) = 0$ (and $\mu-$almost every x is such a point), we can choose $r > 0$ sufficiently small so that

(1) $B(x,2r) \subset B$,
(2) $\mu(B(x,2r)) < \mu(B)/2$,
(3) for all $w \in \overline{B}(x,r)$ we have $|u(w) - u(x)| < \varepsilon$ (possible because u is Lipschitz continuous),
(4) $\int_{\overline{B}(x,2r)} |u - u(x)|\, d\mu < \frac{1}{2}\int_B |u - u(x)|\, d\mu$.

Then,

$$\begin{aligned}\fint_B |u - u(x)|\, d\mu &\le \frac{2}{\mu(B)} \int_{B\setminus B(x,2r)} |u - u(x)|\, d\mu \\ &\le 2 \fint_{B\setminus B(x,2r)} |u(y) - u(x)|\, d\mu(y).\end{aligned}$$

Let $F \subset X$ be the set given by Lemma 3.10 with respect to x and r, and for $y \in B \setminus (F \cup B(x,2r))$ let γ_y be the corresponding curve connecting y to $B(x,r)$. We denote the other end point of γ_y as $w_y \in \overline{B}(x,r)$. By the choice of r, we see that $|u(y) - u(x)| \le |u(y) - u(w_r)| + |u(w_r) - u(x)| < |u(y) - u(w_r)| + \varepsilon$.

It follows that $|u(y) - u(x)| \leq \varepsilon + \int_{\gamma_y} g\, ds \leq \varepsilon + C\|g\|_{L^\infty(2CB)} d(x,y)$, where we used the fact that $\mathscr{L}^1(\gamma_y^{-1}(\gamma_y \cap E)) = 0$. Therefore,

$$\fint_B |u - u(x)|\, d\mu \leq 2 \fint_{B\setminus(F\cup B(x,2r))} (\varepsilon + C\|g\|_{L^\infty(2CB)} d(x,y)) d\mu(y)$$

$$\leq 2 \fint_{B\setminus(F\cup B(x,2r))} (\varepsilon + C\|g\|_{L^\infty(2CB)} \mathrm{rad}(B)) d\mu(y)$$

$$= 2(\varepsilon + C\|g\|_{L^\infty(2CB)} \mathrm{rad}(B)).$$

Now integrating over x, we obtain

$$\fint_B \fint_B |u(y) - u(x)|\, d\mu(y)\, d\mu(x) \leq 2(\varepsilon + C\|g\|_{L^\infty(2CB)} \mathrm{rad}(B)).$$

Letting $\varepsilon \to 0$ we get the inequality

$$\fint_B \fint_B |u(y) - u(x)|\, d\mu(y)\, d\mu(x) \leq 2C \mathrm{rad}(B) \|g\|_{L^\infty(2CB)},$$

which in turn implies the weak $\infty-$Poincaré inequality for the pair f, g. Since the constants are independent of u, g, B, we have that (X, d, μ) supports a weak $\infty-$Poincaré inequality for Newtonian functions. It follows from Proposition 3.7 that X is thick quasi-convex.

To complete the proof, we have to check that (X, d, μ) admits a weak $\infty-$Poincaré inequality for every Borel measurable function $u\colon X \to \mathbb{R}$ and every upper gradient. Let u be a measurable function and let g be a measurable upper gradient for f. Fix B. If $\|g\|_{L^\infty(2CB)} = \infty$ we are done, so let us assume that $\|g\|_{L^\infty(2CB)} < \infty$. Since by above we have X is thick quasi-convex, we can invoke Lemma 3.5 to see that u is Lipschitz in $B \subset X$ up to a set of measure zero. Thus we can repeat the proof above for the pair u and g, with for $x \in B$ we choose $r > 0$ satisfying Conditions 1–4 above, with Condition 3 modified to require that $|u(w) - u(x)| < \varepsilon$ for a.e. $w \in \overline{B}(x,r)$ and x being restricted to the subset of B where u is Lipschitz continuous. Thus the proof is now complete. $\square$

The rest of this section will be devoted to showing that the thick quasi-convexity cannot be replaced with the weaker notion of quasi-convexity. The next lemma is useful in verifying whether a metric space does not support any Poincaré inequality. Its proof it is an adaptation of Lemma [BoP, 4.3] for the case $p = \infty$.

Lemma 3.12. *Let (X, d, μ) be a bounded doubling metric measure space admitting a weak $\infty-$Poincaré inequality, and let $f\colon X \longrightarrow I$ be a surjective $L-$Lipschitz function from X onto an interval $I \subset \mathbb{R}$. Then, $\mathscr{L}^1_{|I} \ll f_{\#}\mu$. Here $f_{\#}\mu$ denotes the pushforward measure of μ under f.*

Proof. Suppose the contrary. Then, there exists a Borel set N in I such that $\mathscr{L}^1(N) > 0$ and $\mu(f^{-1}(N)) = f_{\#}\mu(N) = 0$. On X we consider the function

$$u(x) = \int_0^{f(x)} \chi_N(t) d\mathscr{L}^1(t).$$

This function is $L-$Lipschitz, because for $x, y \in X$ we have

$$\begin{aligned}|u(y) - u(x)| &= \Big| \int_{f(x)}^{f(y)} \chi_N \, d\mathscr{L}^1 \Big| \\ &= \mathscr{L}^1([f(x), f(y)] \cap N) \leq |f(y) - f(x)| \leq L\, d(y, x).\end{aligned}$$

Moreover, $g = L\ \chi_N \circ f$ is an upper gradient of u. Indeed, for each rectifiable curve $\gamma \colon [a, b] \longrightarrow \mathbb{R}$ one has (without loss of generality we assume that $f(\gamma(a)) < f(\gamma(b))$)

$$|u(\gamma(a)) - u(\gamma(b))| = \Big| \int_{f(\gamma(a))}^{f(\gamma(b))} \chi_N(t) d\mathscr{L}^1(t) \Big| = \mathscr{L}^1([f(\gamma(a)), f(\gamma(b))] \cap N),$$

and

$$\int_\gamma g = \int_a^b L \cdot \chi_N \circ f(\gamma(t)) d\mathscr{L}^1(t) = L\, \mathscr{L}^1([a, b] \cap (f \circ \gamma)^{-1}(N)).$$

Because γ is arclength-parametrized, $f \circ \gamma$ is $L-$Lipschitz. It follows that

$$\mathscr{L}^1([a, b] \cap (f \circ \gamma)^{-1}(N)) \geq L^{-1}\, \mathscr{L}^1([f(\gamma(a)), f(\gamma(b))] \cap N),$$

and hence,

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g d\mathscr{L}^1(t)$$

for each rectifiable curve γ in X. However, $\|g\|_\infty = 0$, for $\mu(x \in X : f(x) \in N) = f_{\#}\mu(N) = 0$ by hypothesis, and so $\chi_N \circ f(x) = 0$ $\mu-$a.e. Therefore by the weak $\infty-$Poincaré inequality, $\int_X |u - u_X| \, d\mu = 0$, which means that u is constant $\mu-$almost everywhere on X. Because u is Lipschitz continuous on X, it follows that u is constant on X, which contradicts the fact that u is non-constant on the set $f^{-1}(N)$ (this set is non-empty because f is surjective, and u is not constant here because $\mathscr{L}^1(N) > 0$). □

Example 3.13. Let $Q = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ be the unit square. Divide Q in nine equal squares of sidelength $1/3$ and remove the central one. In this way, we obtain a set Q_1, which is the union of 8 squares of sidelength $1/3$. Repeating this procedure on each square we get a sequence of sets Q_j consisting of 8^j squares of sidelength $1/3^j$. We define the *Sierpinski carpet* to be $S = \bigcap Q_j$. If d is the distance in $\mathbb{R}^2$ given by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

it can be checked that (S, d) is a complete geodesic metric space. Let μ be the Hausdorff measure on (S, d) of dimension s, where s is given by the formula,

$3^s = 8$. It can be checked that μ is a doubling measure and that the metric d defined above is biLipschitz equivalent to the restriction of the Euclidean metric.

The Sierpinski carpet (S, d, μ) is clearly quasi-convex, and so the following corollary demonstrates that the quasi-convexity property is not sufficient to guarantee $\infty-$Poincaré inequality.

Corollary 3.14. *The Sierpinski carpet* (S, d, μ) *does not admit a weak* $\infty-$*Poincaré inequality.*

Proof. Let f be the projection on the horizontal axis. It can be checked that $f_{\#}\mu \perp \mathscr{L}^1$ (see [BoP, 4.5]), and the result follows from Lemma 3.12. $\square$

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