

DISCONTINUOUS GALERKIN METHODS FOR THE ONE-DIMENSIONAL VLASOV-POISSON SYSTEM

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ABSTRACT. We construct a new family of semi-discrete numerical schemes for the approximation of the one-dimensional periodic Vlasov-Poisson system. The methods are based on the coupling of discontinuous Galerkin approximation to the Vlasov equation and several finite element (conforming, non-conforming and mixed) approximations for the Poisson problem. We show optimal error estimates for the all proposed methods in the case of smooth compactly supported initial data. The issue of energy conservation is also analyzed for some of the methods.

1. INTRODUCTION

The Vlasov-Poisson system is one of the basic and simplest models in the mesoscopic description of large ensembles of interacting particles. In one-space dimension and in dimensionless variables, the Vlasov equation reads

$$(1) \quad \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \Phi_x \frac{\partial f}{\partial v} = 0 \quad (x, v, t) \in \Omega_x \times \mathbb{R} \times [0, \infty),$$

where the electrostatic field, $-\Phi_x(x, t)$, derives from a potential $\Phi(x, t)$ that satisfies:

$$(2) \quad -\Phi_{xx} = \rho(x, t) - 1 \quad (x, t) \in \Omega_x \times [0, \infty),$$

with $\rho(x, t)$ being the charge density which is defined by

$$(3) \quad \rho(x, t) = \int_{\mathbb{R}} f(x, v, t) dv \quad \text{for all } (x, t) \in \Omega_x \times [0, \infty).$$

The above system describes the evolution of a collisionless plasma of charged particles (electrons and ions) in the case where the only interaction (between particles) considered relevant is the mean-field force created through electrostatic effects, hence neglecting the electromagnetic effects. $f(x, v, t)$ is the electron distribution, which is a non-negative function depending on the position: $x \in \Omega_x \subset \mathbb{R}$; the velocity: $v \in \mathbb{R}$, and the time: $t \in \mathbb{R}$, with Ω_x denoting the spatial domain where the plasma is confined. As ions are much heavier than

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electrons, it is assumed that their distribution is uniform and since the plasma should be neutral, one has

$$(4) \quad \int_{\Omega_x} \rho(x, t) dx = \int_{\Omega_x} \int_{\mathbb{R}} f(x, v, t) dv dx = 1 \quad \text{for all } t \in [0, \infty).$$

We refer to the surveys [41, 12, 34] for good account on the state of the art in the mathematical analysis and properties of the solutions of the Cauchy problem for the Vlasov-Poisson system.

Many efforts have been dedicated to the numerical approximation of the Vlasov-Poisson system with either probabilistic or deterministic solvers. Since the beginnings of numerical plasma simulations in the 60's, Particle methods [11] have been often preferred because of their lower computational complexity. For these methods, the motion of the plasma is approximated by a finite number of macroparticles in the physical space that follow backward the characteristics of the Vlasov equation. Several works have also analyzed their convergence in one [30, 59] and higher dimensions [58]. However, a well known drawback of these methods is their inherent numerical noise which prevents from getting an accurate description of the distribution function in the phase space for many applications. To overcome this lack of precision, eulerian solvers, methods discretizing the Vlasov equation on a mesh of the phase space, have been also considered. Their design has been explored by many authors and with many different techniques: finite volumes [36, 37]; Fourier-Fourier transform [44]; finite elements [60, 61], splitting schemes [33, 17]; and semi-lagrangian methods [39, 31, 16]. All these methods present different pros and cons and we refer to [38] and the references therein for a discussion. Finite volumes are a simple and inexpensive option, but in general, are low order. Fourier-Fourier transform schemes suffer from Gibbs phenomena if other than periodic boundary conditions are imposed. Semi-lagrangian schemes can achieve high order allowing also for time integration with larger time steps. However, they require high order interpolation to compute the origin of the characteristics, which in turn destroys the local character of the reconstruction. Standard Finite Element methods suffer from numerical oscillations when approximating the Vlasov equation. In contrast, Discontinuous Galerkin (DG) finite elements are particularly well suited for hyperbolic problems and their application to non-linear conservation laws has already shown their usefulness [26, 25, 28].

Based on a totally discontinuous finite element spaces, DG methods are extremely versatile and have numerous attractive features: local conservation properties; can easily handle irregularly refined meshes and variable approximation degrees (*hp*-adaptivity), weak approximation of boundary conditions and built-in parallelism which permits coarse-grain parallelization. In addition, DG mass matrices are block-diagonal and can be inverted at a very low computational cost, giving rise to very efficient time-stepping algorithms in the context of time-dependent problems, as it is the case here. Pioneering research on discontinuous

Galerkin methods was pursued in [51, 47, 35, 56, 3]. We refer to [24, 4] for a detailed historical overview and for more recent developments to [54, 49, 14, 5] and references therein. However, although DG methods can deal robustly with partial differential equations of almost any kind, their application in the realm of numerical approximation of kinetic models has been considered only very recently. In [20] and [9] the authors study, respectively, the use of DG for the Boltzmann-Poisson system in semiconductors and for water-bag approximations of the Vlasov-Poisson system. In [40], an L^1 -analysis is carried out in for a simplified linear Vlasov-Boltzmann equation with a given confining force field.

Despite the numerical performance of all these eulerian solvers has been extensively studied, to our knowledge, the issue of their convergence and error analysis for the Vlasov-Poisson system, has not been tackled till very recently, and only for the one-dimensional periodic case. The convergence and error analysis for a low order finite volume scheme is contained in [37]. More recently, semi-lagrangian schemes have been analyzed; of first order in [7] and high order is considered in [8, 10]. In these works the authors have also proved a-priori error bounds in different norms for both the distribution function and the electrostatic field. We also mention that for other kinetic models, finite differences [52] and spectral methods [46, 45] have been also considered and analyzed.

The present paper is concerned with the design and analysis of discontinuous Galerkin approximation for the one-dimensional periodic Vlasov-Poisson system. We introduce a new whole family of eulerian schemes, based on the combination of DG approximation to the Vlasov equation with various different finite element (conforming and nonconforming) approximations to the electrostatic field. The first one is a direct conforming approximation obtained by taking advantage of the explicit integration of the Poisson equation in one dimension. Such approximation is equivalent to what most authors, if not all, have usually considered for this system. However, in spite of its simplicity, it might not be the most appropriate scheme in view of the possible extension/adaptation of the numerical scheme to higher dimensions and to more complex kinetic models. For this reason, in the present paper we also examine a different approach: since the coupling in the Vlasov-Poisson system is through the electrostatic field, the main interest in the Poisson problem is the approximation to Φ_x rather than to Φ , and therefore mixed finite element methods seem to be the right choice. We explore Raviart-Thomas and several mixed DG approximations for the Poisson problem.

We also deal with the convergence and error analysis for the proposed DG methods for the case of smooth compactly supported solutions. We derive optimal error bounds in the L^2 -norm for both the distribution function and the electrostatic field, for high order methods, namely $k \geq 1$, k being the polynomial degree of the DG approximation for the distribution function. The analysis for piecewise constant approximation ($k = 0$) is different and will be carried out somewhere else. Although Vlasov equation might be seen as a simple transport equation, its coupling with Poisson, brings into play in such equation, a

non-linear (quadratic) and non-local term. This generates some difficulties in the error analysis, precluding a straightforward extension of other works. A key ingredient is the construction of some projection operators, inspired in those introduced in [47, 53, 23, 62], but specially designed for the Vlasov-Poisson system. These special projections allow for avoiding the loss of half order, typical of the standard error analyses of finite element methods for hyperbolic problems. We have focused on semi-discrete schemes; discussion on suitable time integrators and design and analysis of fully discretized schemes is outside the scope of this paper and will be the subject of future research.

Finally, we wish to note that while developing the methods, we have taken special care in ensuring that physical properties of the continuous system are preserved. The DG approximation for the Vlasov equation ensures in an easy way that the total charge of the system is preserved (4). We also discuss the conservation of the total energy for the proposed schemes. In particular, we propose a full DG method (DG for Vlasov equation and a particular Local discontinuous Galerkin (LDG) for Poisson problem), that preserves the total discrete energy of the system. To the best of our knowledge this is the first scheme proposed in literature for which energy conservation can be shown. Our proof however requires a technical assumption on the polynomial degree for the DG methods, namely $k \geq 2$. Whether this restriction is really necessary or not, will be the subject of future research. For many others full DG schemes presented in the paper, we provide a bound on the energy dissipated by the system.

Extension to higher dimensions, numerical validation of the results presented here and numerical performance of the presented numerical schemes in challenging questions such as the Landau damping of Langmuir waves [63] or the Raman scattering instability [9] will be carried out somewhere else.

The outline of the paper is as follows. In section 2 we describe the main properties of the continuous problem, we introduce the notations and revise some basic results we need for the description and analysis of the numerical methods. In section 3 we present the numerical methods proposed to approximate the one dimensional periodic Vlasov-Poisson system. The error analysis for the presented method is detailed in section 4. We discuss the energy conservation properties of the schemes in section 5. The paper is completed with two appendix, Appendix A and B, containing some proofs of technical and auxiliary lemmas used in the convergence analysis.

2. PRELIMINARIES, NOTATION AND AUXILIARY RESULTS

Throughout this paper, we use the standard notation for Sobolev spaces [1]. For a bounded domain $B \subset \mathbb{R}^2$, we denote by $H^m(B)$ the L^2 -Sobolev space of order $m \geq 0$ and by $\|\cdot\|_{m,B}$ and $|\cdot|_{m,B}$ the usual Sobolev norm and seminorm, respectively. For $m = 0$, we write $L^2(B)$ instead of $H^0(B)$. We shall denote by $H^m(B)/\mathbb{R}$ the quotient space consisting of equivalence classes of elements of $H^m(B)$ differing by constants; for $m = 0$ it is denoted by $L^2(B)/\mathbb{R}$. We shall indicate by $L_0^2(B)$ the space of $L^2(B)$ functions having zero average over B . This

notation will also be used for periodic Sobolev spaces without any other explicit reference to periodicity to avoid cumbersome notations.

2.1. Continuous Problem: The 1D periodic Vlasov-Poisson System. In the rest of the paper we take $\Omega_x = [0, 1]$ in (1)-(2)-(3)-(4). Let f_0 denote a given initial distribution $f(x, v, 0) = f_0(x, v)$ in $(x, v) \in [0, 1] \times \mathbb{R}$. We impose periodic boundary conditions on x for the transport equation (1),

$$f(0, v, t) = f(1, v, t) \quad \text{for all } (v, t) \in \mathbb{R} \times [0, \infty).$$

and also for the Poisson equation (2),

$$(5) \quad \Phi(0, t) = \Phi(1, t) \quad \text{and} \quad \Phi_x(0, t) = \Phi_x(1, t) \quad \text{for all } t \in [0, \infty).$$

Notice that (4) is coherent with the 1-periodicity of Φ_x . Let us also emphasize that the correct way of including periodic boundary conditions is to assume that f and Φ are the restriction to $[0, 1]$ of periodic functions defined in \mathbb{R} in the right spaces. To guarantee the uniqueness of the solution Φ (otherwise is determined only up to a constant), we fix the value of Φ at a point. We set

$$(6) \quad \Phi(0, t) = 0 \quad \text{for all } t \in [0, \infty).$$

However, notice that since the Poisson equation (2) is one-dimensional it could be directly integrated. More precisely, by using twice the Fundamental Theorem of Calculus, it follows that Φ is defined for all $t \in [0, \infty)$ as

$$(7) \quad \Phi(x, t) = D + C_E x + \frac{x^2}{2} - \int_0^x \int_0^s \rho(z, t) dz ds \quad \forall x \in [0, 1],$$

where D and C_E are integration constants determined from (6) and (5);

$$(8) \quad D = 0, \quad C_E = \int_0^1 \int_0^z \rho(s, t) ds dz - \frac{1}{2} \quad \forall t \in [0, T].$$

Denoting then by $E(x, t) = \Phi_x(x, t)$, and differentiating (7) with respect to x , we find

$$(9) \quad E(x, t) = \Phi_x(x, t) = C_E + x - \int_0^x \rho(s, t) ds \quad \forall x \in [0, 1],$$

with C_E defined as in (8). Throughout the paper, E will be referred as the electrostatic field. Although the physical one is indeed $-E$, we shall use this abuse in the notation to follow the standard notation for the Poisson solvers in the Discontinuous Galerkin community. Observe that (5) implies that the electrostatic field has zero average in agreement with the charge neutrality.

In order to perform our error analysis we restrict our attention to smooth compactly supported solutions f in a fixed time interval $[0, T]$ for all $T > 0$. Given a distribution function $f(x, v, t)$, we will denote by

$$Q(t) = 1 + \sup\{|v| : \exists x \in [0, 1] \text{ and } \tau \in [0, t] \text{ such that } f(x, v, \tau) \neq 0\},$$

for all $t \in [0, \infty)$ as a measure of the support of the distribution function. The following result is essentially contained in [29, 57, 41].

Theorem 2.1 (Well-posedness Continuous 1DVP). *Given $f_0 \in C^1(\mathbb{R}_x \times \mathbb{R}_v)$, 1-periodic in x and compactly supported in v , $Q(0) \leq Q_0$ with $Q_0 > 0$. Then the periodic Vlasov-Poisson system (1)-(2) has a unique classical solution (f, E) , $f \in C^1(0, T; C^1(\mathbb{R}_x \times \mathbb{R}_v))$ and $E \in C^1(0, T; C^1(\mathbb{R}_x))$ that is 1-periodic in x for all time t in $[0, T]$ for all $T > 0$, such that:*

- i) *Regularity: If in addition $f_0 \in C^m(\mathbb{R}_x \times \mathbb{R}_v)$, $m \geq 2$, then, the distribution function f belongs to $C^m(0, T; C^m(\mathbb{R}_x \times \mathbb{R}_v))$ and the force field $E \in C^m(0, T; C^m(\mathbb{R}_x))$.*
- ii) *Control of Support: There exists a constant C depending on Q_0 and f_0 such that $Q(T) \leq CT$ for all $T > 0$.*

In the rest of this work, we will assume that the initial data f_0 satisfies the hypotheses in Theorem 2.1, and thus, the unique classical solution to the periodic Vlasov-Poisson system (1)-(2) satisfies that there exists $L > 0$ depending on f_0 , T and Q_0 such that $\text{supp}(f(t)) \subseteq \Omega$ for all $t \in [0, T]$, where we have defined $\Omega = \mathcal{I} \times \mathcal{J}$, with $\mathcal{I} = [0, 1]$ and $\mathcal{J} = [-L, L]$. The Vlasov transport equation (1) is regarded as a transport equation in $\Omega_T := \Omega \times [0, T]$. Taking into account the boundary conditions, the weak formulation of the continuous problem (1) reads: find (f, E) such that

$$(10) \quad \iint_{\Omega} f_t \phi \, dx \, dv - \iint_{\Omega} v f \phi_x \, dx \, dv + \iint_{\Omega} E f \phi_v \, dx \, dv = 0 \quad \forall \phi \in C^\infty(\Omega).$$

It is well known [41, 12, 34] that the continuous solution of (1)-(2) satisfies four important properties:

- *Positivity: $f(t, x, v) \geq 0$, for all $(x, v, t) \in \Omega_T$.*
- *Charge conservation: as given in (4).*
- *L^p -conservation:*

$$(11) \quad \|f(t)\|_{L^p(\Omega)} = \|f_0\|_{L^p(\Omega)} \quad 1 \leq p \leq \infty, \quad \forall t \in [0, T].$$

- *Conservation of the total Energy:*

$$(12) \quad \frac{d}{dt} \left(\int_{\Omega} |v|^2 f(x, v, t) \, dx \, dv + \int_{\mathcal{I}} |E(x, t)|^2 \, dx \right) = 0.$$

In deriving numerical methods for (1)-(2), we will try to ensure that the resulting schemes will be able to produce approximate solutions, enjoying some of these properties. As usual with high-order schemes for hyperbolic problems, we cannot expect to preserve positivity of the scheme. However, we will be able to conserve the total energy for particular method, see section 5.

2.2. Discontinuous Galerkin Approximation: Basic Notations. Let $\{\mathcal{T}_h\}$ be a family of partitions of our computational/physical domain $\Omega = \mathcal{I} \times \mathcal{J} = [0, 1] \times [-L, L]$, which we assume to be regular [21] and made of rectangles. Each

cartesian mesh \mathcal{T}_h is defined as $\mathcal{T}_h := \{T_{ij} = I_i \times J_j, 1 \leq i \leq N_x, 1 \leq j \leq N_v\}$ where

$$I_i = [x_{i-1/2}, x_{i+1/2}] \quad \forall i = 1, \dots, N_x; \quad J_j = [v_{j-1/2}, v_{j+1/2}] \quad \forall j = 1, \dots, N_v,$$

and the mesh sizes h_x and h_v relative to the partition are defined as

$$0 < h_x = \max_{1 \leq i \leq N_x} h_i^x := x_{i+1/2} - x_{i-1/2}, \quad 0 < h_v = \max_{1 \leq j \leq N_v} h_j^v := v_{i+1/2} - v_{i-1/2},$$

being h_i^x and h_j^v the cell lengths of I_i and J_j , respectively. The mesh size of the partition is defined as $h = \max(h_x, h_v)$. For simplicity in the exposition we also assume that $v = 0$ corresponds to a node, $v_{j-1/2} = 0$ for some j , of the partition of $[-L, L]$. The set of all vertical edges is denoted by Γ_x , and respectively, we will refer to Γ_v as the set of all horizontal edges;

$$\Gamma_x := \bigcup_{i,j} \{x_{i-1/2}\} \times J_j, \quad \Gamma_v := \bigcup_{i,j} I_i \times \{v_{j-1/2}\}, \quad \Gamma_h = \Gamma_x \cup \Gamma_v.$$

By $\{\mathcal{I}_h\}$ we shall denote the family of partitions of the interval \mathcal{I} ;

$$\mathcal{I}_h := \{I_i : 1 \leq i \leq N_x\} \quad \gamma_x := \bigcup_i \{x_{i-1/2}\}.$$

Next, for $k \geq 0$, we define the discontinuous finite element spaces V_h^k and \mathcal{Z}_h^k and a conforming finite element space, W_h^{k+1} ,

$$\begin{aligned} V_h^k &= \{\psi \in L^2(\mathcal{I}) : \psi \in \mathbb{P}^k(I_i), \forall x \in I_i, i = 1, \dots, N_x\}, \\ \mathcal{Z}_h^k &:= \{z \in L^2(\Omega) : z \in \mathbb{Q}^k(T_{ij}), \forall (x, v) \in T_{ij} = I_i \times J_j, \forall i, j\}, \\ W_h^{k+1} &= \{\chi \in \mathcal{C}^0(\mathcal{I}) : \chi \in \mathbb{P}^{k+1}(I_i), \forall x \in I_i, i = 1, \dots, N_x\} \cap L^2(\mathcal{I})/\mathbb{R}, \end{aligned}$$

where $\mathbb{P}^k(I_i)$ is the space of polynomials (in one dimension) of degree up to k , and $\mathbb{Q}^k(T_{ij})$ is the space of polynomials of degree at most k in each variable.

Trace Operators: We denote by $(\varphi_h)_{i+1/2,v}^+$ and $(\varphi_h)_{i+1/2,v}^-$ the values of φ_h at $(x_{i+1/2}, v)$ from the right cell $I_{i+1} \times J_j$ and from the left cell $I_i \times J_j$, respectively;

$$(\varphi_h)_{i+1/2,v}^\pm = \lim_{\varepsilon \downarrow 0} \varphi_h(x_{i+1/2} \pm \varepsilon, v), \quad (\varphi_h)_{x,j+1/2}^\pm = \lim_{\varepsilon \downarrow 0} \varphi_h(x, v_{j+1/2} \pm \varepsilon),$$

for all $(x, v) \in \mathcal{I} \times \mathcal{J}$ or in short-hand notation

$$(13) \quad (\varphi_h)_{i+1/2,v}^\pm = \varphi_h(x_{i+1/2}^\pm, v), \quad (\varphi_h)_{x,j+1/2}^\pm = \varphi_h(x, v_{j+1/2}^\pm),$$

for all $(x, v) \in I_i \times J_j$. The jump $[[\cdot]]$ and average $\{\cdot\}$ trace operators of φ_h at $(x_{i+1/2}, v)$, $\forall v \in J_j$ are defined by

$$(14) \quad \begin{aligned} [[\varphi_h]]_{i+1/2,v} &:= (\varphi_h)_{i+1/2,v}^+ - (\varphi_h)_{i+1/2,v}^- & \forall \varphi_h \in \mathcal{Z}_h^k, \\ \{\varphi_h\}_{i+1/2,v} &:= \frac{1}{2} [(\varphi_h)_{i+1/2,v}^+ + (\varphi_h)_{i+1/2,v}^-] & \forall \varphi_h \in \mathcal{Z}_h^k. \end{aligned}$$

2.3. Technical Tools. We start defining the space

$$H^m(\mathcal{T}_h) := \{\varphi \in L^2(\Omega) : \varphi|_{T_{ij}} \in H^m(T_{ij}) \forall T_{ij} \in \mathcal{T}_h\} \quad m \geq 0.$$

In our error analysis, since we consider a non-conforming approximation, we shall employ the following seminorm and norms,

$$\begin{aligned} |\varphi|_{1,h}^2 &= \sum_{i,j} |\varphi|_{1,T_{ij}}^2 & \|\varphi\|_{m,\mathcal{T}_h}^2 &:= \sum_{i,j} \|\varphi\|_{m,T_{ij}}^2 \quad \forall \varphi \in H^m(\mathcal{T}_h), m \geq 0 \\ \|\varphi\|_{0,\infty,\mathcal{T}_h} &= \sup_{T_{ij} \in \mathcal{T}_h} \|\varphi\|_{0,\infty,T_{ij}} & \|\varphi\|_{L^p(\mathcal{T}_h)}^p &:= \sum_{i,j} \|\varphi\|_{L^p(T_{ij})}^p \quad \forall \varphi \in L^p(\mathcal{T}_h), \end{aligned}$$

for all $1 \leq p < \infty$. We also introduce the following norms over the skeleton of the finite element partition,

$$\|\varphi\|_{0,\Gamma_x}^2 := \sum_{i,j} \int_{J_j} |(\varphi)_{i+1/2,v}|^2 dv, \quad \|\varphi\|_{0,\Gamma_v}^2 := \sum_{i,j} \int_{I_i} |(\varphi)_{x,j+1/2}|^2 dx \quad \forall \varphi \in H^1(\mathcal{T}_h).$$

Then, we define $\|\varphi\|_{0,\Gamma_h}^2 = \|\varphi\|_{0,\Gamma_x}^2 + \|\varphi\|_{0,\Gamma_v}^2$. We notice that all the above definitions apply also for the partition \mathcal{I}_h with the obvious changes in the notation.

Projection operators: For $k \geq 0$, we denote by $P^k : L^2(\mathcal{I}) \rightarrow V_h^k$ the standard L^2 -projection onto the finite element space V_h^k defined locally, i.e., for each $1 \leq i \leq N_x$,

$$(15) \quad \int_{I_i} (P^k(w) - w) q_h dx = 0 \quad \forall q_h \in \mathbb{P}^k(I_i).$$

This projection is stable in $L^p(\mathcal{I})$ for all p [32], i.e.,

$$(16) \quad \|P^k(w)\|_{L^p(\mathcal{I}_h)} \leq C \|w\|_{L^p(\mathcal{I})} \quad \forall w \in L^p(\mathcal{I}), \quad 1 \leq p \leq \infty.$$

We next introduce two more refined projections (see [53]), which we denote by π^\pm , that can be defined only for more regular functions, say $w \in H^{1/2+\epsilon}(I_i)$ for all i . The projections $\pi^+(w)$ and $\pi^-(w)$ are the unique polynomials of degree at most $k \geq 1$, that satisfy for each $1 \leq i \leq N_x$

$$(17) \quad \int_{I_i} (\pi^\pm(w) - w) q_h dx = 0, \quad \forall q_h \in \mathbb{P}_h^{k-1}(I_i),$$

together with the matching conditions;

$$(18) \quad \pi^+(w(x_{i-1/2}^+)) = w(x_{i-1/2}^+); \quad \pi^-(w(x_{i+1/2}^-)) = w(x_{i+1/2}^-).$$

Provided w enjoys enough regularity, say $w \in H^{k+1}(I_i)$, the following error estimates can be easily shown for all these projections:

$$(19) \quad \left. \begin{aligned} \|w - P^k(w)\|_{0,I_i} \\ \|w - \pi^\pm(w)\|_{0,I_i} \end{aligned} \right\} \leq Ch^{k+1} |w|_{k+1,I_i} \quad \forall w \in H^{k+1}(I_i).$$

where C is a constant depending only on the shape-regularity of the mesh and the polynomial degree [21, 53]. For the standard L^2 -projection we will also need estimates in the L^∞ -norm [55],

$$(20) \quad \|w - P^k(w)\|_{0,\infty,\mathcal{I}} \leq Ch^{k+1}|w|_{k+1,\infty,\mathcal{I}}.$$

Let $k \geq 0$ and let $\mathcal{P}_h : L^2(\Omega) \rightarrow \mathcal{Z}_h^k$ be the standard L^2 -projection (in the two-dimensional case) defined by $\mathcal{P}_h(w) = (P_x^k \otimes P_v^k)(w)$; i.e., for all i and j ,

$$(21) \quad \int_{I_i} \int_{J_j} (\mathcal{P}_h(w(x,v)) - w(x,v)) \varphi_h(x,v) dv dx = 0 \quad \forall \varphi_h \in \mathbb{P}^k(I_i) \otimes \mathbb{P}^k(J_j).$$

From its definition, it follows immediately its L^2 -stability, but it can be shown to be stable in L^p for all p [32],

$$(22) \quad \|\mathcal{P}_h(w)\|_{L^p(\mathcal{T}_h)} \leq C\|w\|_{L^p(\Omega)} \quad \forall w \in L^p(\Omega), \quad 1 \leq p \leq \infty.$$

3. THE SUGGESTED NUMERICAL METHODS

In this section we formulate the numerical schemes we propose to approximate the Vlasov-Poisson system. The first one is a scheme where the DG approximation for the transport equation is coupled with a simple conforming approximation of higher degree for the electrostatic field. The second scheme results by combining mixed finite element approximation for the Poisson problem together with DG approximation to the transport equation. Last approach is based on fully DG approximation for both variables the electron distribution f and the electrostatic field.

Due to the special structure of the transport equation: v is independent of x and E is independent of v ; for all methods the DG approximation for the electron distribution function is done exactly in the same way. Therefore we start by introducing the DG method for the transport equation (1), and in what follows, we denote by E_h^i the restriction to I_i of the finite element approximation E_h to be defined later on.

Let $f_h(0) = \mathcal{P}_h(f_0)$ be the approximation to the initial data. The numerical method reads: find $(E_h, f_h) : [0, T] \rightarrow (\mathcal{W}_h, \mathcal{Z}_h^k)$ such that

$$(23) \quad \sum_{i=1}^{N_x} \sum_{j=1}^{N_v} \mathcal{B}_{ij}^h(E_h; f_h, \varphi_h) = 0 \quad \forall \varphi_h \in \mathcal{Z}_h^k,$$

where the bilinear form $\mathcal{B}_{ij}^h(E_h; f_h, \varphi_h)$ is defined for each i, j and $\varphi_h \in \mathcal{Z}_h^k$ as:

$$\begin{aligned}
\mathcal{B}_{ij}(E_h; f_h, \varphi_h) &= \int_{T_{ij}} \frac{\partial f_h}{\partial t} \varphi_h \, dv \, dx - \int_{T_{ij}} v f_h \frac{\partial \varphi_h}{\partial x} \, dv \, dx + \int_{T_{ij}} E_h^i f_h \frac{\partial \varphi_h}{\partial v} \, dv \, dx \\
(24) \quad &+ \int_{J_j} \left[((\widehat{v f_h}) \varphi_h^-)_{i+1/2, v} - ((\widehat{v f_h}) \varphi_h^+)_{i-1/2, v} \right] \, dv \\
&- \int_{I_i} \left[((\widehat{E_h^i f_h}) \varphi_h^-)_{x, j+1/2} - ((\widehat{E_h^i f_h}) \varphi_h^+)_{x, j-1/2} \right] \, dx,
\end{aligned}$$

where we have used the short hand notation given in (13). Notice that the expression $\mathcal{B}_{ij}^h(E_h; f_h, \varphi_h)$ is in fact a bilinear form. E_h is used only to emphasise the nonlinear dependence on it. Here, the boundary terms are the so-called numerical fluxes, which are nothing but the approximation of the functions $v f$ and $E f$ at the vertical and horizontal boundaries Γ_x and Γ_v , respectively. By specifying them, the DG method is completely determined. The design of these numerical fluxes is the key issue to ensure the stability of the numerical scheme. We consider the following upwind choice:

$$(25) \quad \widehat{v f_h} = \begin{cases} v f_h^- & \text{if } v \geq 0, \\ v f_h^+ & \text{if } v < 0, \end{cases} \quad \widehat{E_h^i f_h} = \begin{cases} E_h^i f_h^+ & \text{if } E_h^i \geq 0, \\ E_h^i f_h^- & \text{if } E_h^i < 0. \end{cases}$$

We define the numerical fluxes at the boundary $\partial\Omega$ by

$$(\widehat{v f_h})_{1/2, v} = (\widehat{v f_h})_{N_x+1/2, v}, \quad (\widehat{E_h^i f_h})_{x, 1/2} = (\widehat{E_h^i f_h})_{x, N_v+1/2} = 0, \quad \forall (x, v) \in \mathcal{I} \times \mathcal{J},$$

so that the periodicity in x and the compactness in v are reflected. The discrete density, denoted by $\rho_h(x, t)$, is given by

$$(26) \quad \rho_h(x, t) = \sum_j \int_{J_j} f_h(x, v, t) \, dv \quad \forall x \in \mathcal{I}, \quad \forall t \in [0, T].$$

Note that from the definitions (3) and (26) of ρ and ρ_h , respectively, and using Cauchy-Schwartz's inequality it is straightforward to see that

$$(27) \quad \|\rho(t) - \rho_h(t)\|_{0, \mathcal{I}}^2 \leq 2L \|f_h(t) - f(t)\|_{0, \mathcal{T}_h}^2 \quad \forall t \in [0, T].$$

One of the nice features of the DG approximation for the transport is that charge conservation is ensured by construction, as the following result shows:

Lemma 3.1. Particle or Mass Conservation: *Let $k \geq 0$ and let $f_h \in \mathcal{C}^1([0, T]; \mathcal{Z}_h^k)$ be the DG approximation to f , satisfying (23)-(24). Then,*

$$(28) \quad \sum_{i, j} \int_{T_{ij}} f_h(t) \, dv \, dx = \sum_{i, j} \int_{T_{ij}} f_h(0) \, dv \, dx = \sum_{i, j} \int_{T_{ij}} f_0 \, dv \, dx = 1 \quad \forall t \in [0, T].$$

Proof. Note that since $f_h(0) = \mathcal{P}_h(f_0)$, from the definition of the L^2 -projection (21) (with $\varphi_h = 1$) together with (4) we have

$$(29) \quad \sum_{i,j} \int_{T_{ij}} f_h(0) dv dx = \sum_{i,j} \int_{T_{ij}} \mathcal{P}_h(f_0) dv dx = \sum_{i,j} \int_{T_{ij}} f_0 dv dx = 1.$$

We now fix an arbitrary T_{ij} and take in (24) the test function $\varphi_h = 1$ in T_{ij} ; $\varphi_h = 0$ elsewhere. Noting that such a test function verifies $(\varphi_h)_{i+1/2,v}^- = (\varphi_h)_{i-1/2,v}^+ = 1$, we have

$$\begin{aligned} \mathcal{B}_{ij}(E_h : f_h, 1) &= \frac{d}{dt} \int_{T_{ij}} f_h dv dx + \int_{J_j} [(\widehat{v f_h})_{i+1/2,v} - (\widehat{v f_h})_{i-1/2,v}] dv \\ &\quad - \int_{I_i} [(\widehat{E_h^i f_h})_{x,j+1/2} - (\widehat{E_h^i f_h})_{x,j-1/2}] dx . \end{aligned}$$

Moreover, note that since the choice of T_{ij} was done arbitrarily, last identity holds true for all i, j . By summing last identity over all i and j , the flux terms telescope and there is no boundary term left because of the periodic (for i) and compactly supported (for j) boundary conditions. Hence, taking into account (23) we have,

$$0 = \sum_{i,j} \mathcal{B}_{ij}(E_h; f_h, 1) = \frac{d}{dt} \sum_{i,j} \int_{T_{ij}} f_h dv dx = 0,$$

and so integration in time together with (29) lead to (28). \square

We next deal with the approximation to the electrostatic field $E(x, t) = \Phi_x(x, t)$. The discrete Poisson problem reads,

$$(30) \quad (\Phi_h)_{xx} = 1 - \rho_h \quad x \in [0, 1], \quad \Phi_h(1, t) = \Phi_h(0, t).$$

The well posedness of the above discrete problem is guaranteed by (28) from Lemma 3.1 which in particular implies

$$(31) \quad (\Phi_h)_x(1, t) = (\Phi_h)_x(0, t).$$

To ensure the uniqueness of the solution we set $\Phi_h(0, t) = 0$. To get the solution of the discrete Poisson problem at least two possible approaches arise:

- i)* Direct integration of the discrete Poisson problem (30),
- ii)* approximation of (2) with some mixed finite element method; possibly discontinuous.

We next consider in detail these approaches.

3.1. Conforming approximation to the Electrostatic potential. Reasoning as in section 2.1, direct integration of the discrete Poisson problem (30) together with $\Phi_h(0, t) = 0$ gives

$$(32) \quad \Phi_h(x, t) = C_E^h x + \frac{x^2}{2} - \int_0^x \int_0^s \rho_h(z, t) dz ds \quad \forall x \in [0, 1],$$

where C_E^h is determined from the boundary conditions in (30),

$$(33) \quad C_E^h = \int_0^1 \int_0^z \rho_h(s, t) ds dz - \frac{1}{2} \quad \forall t \in [0, T].$$

Then, differentiation w.r.t x in (32) leads to

$$(34) \quad E_h(x, t) = C_E^h + x - \int_0^x \rho_h(s, t) ds \quad \forall x \in [0, 1].$$

Observe that since $\rho_h \in V_h^k$, E_h turns out to be a continuous polynomial of degree $k + 1$; so E_h is conforming. Its restriction to I_i is given by

$$(35) \quad E_h^i(x, t) = E_h^{i-1}(x_{i-1/2}, t) + (x - x_{i-1/2}) - \int_{x_{i-1/2}}^x \int_{\mathcal{J}} f_h(s, \chi, t) d\chi ds \quad \forall x \in I_i,$$

and $E_h^i(x, t) = 0$ for all $x \in \mathcal{I} \setminus [x_{i-1/2}, x_{i+1/2}]$. The boundary condition (31) reads

$$(36) \quad E_h^0(x_{1/2}, t) = E_h^{N_x}(x_{N_x+1/2}, t) \quad \forall t \in [0, T].$$

To show that E_h indeed belongs to W_h^{k+1} we have to verify that it has zero average. From (33) it follows straightforwardly

$$\sum_i \int_{I_i} E_h(x) dx = E_h(x_{1/2}, t) \sum_i h_i + \sum_i \frac{x_{i+1/2}^2 - x_{i-1/2}^2}{2} - \sum_i \int_{I_i} \int_{x_{1/2}}^x \rho_h(x) dx = 0.$$

Finally, we state a Lemma that relates the error committed in the approximation to E , with the error in accumulated in the approximation to f . This result will be used in our subsequent analysis and its proof is given in Appendix A.

Lemma 3.2. *Let $k \geq 0$ and let $(E_h, f_h) \in \mathcal{C}^0([0, T]; W_h^{k+1}) \times \mathcal{C}^1([0, T]; \mathcal{Z}_h^k)$ be the conforming-DG approximation to the solution of Vlasov-Poisson system (E, f) , solution of (23)–(24)–(34). Then,*

$$(37) \quad \|E(t) - E_h(t)\|_{0, \mathcal{I}} \leq C_1 \|f(t) - f_h(t)\|_{0, \mathcal{T}_h} \quad \forall t \in [0, T],$$

where $2L = \text{meas}(\mathcal{J})$ and $C_1 = (4L(1 + h_x))^{1/2}$. Furthermore, if the force field $E \in \mathcal{C}^0([0, T]; H^1(\mathcal{I}))$, the following estimates also hold for all $t \in [0, T]$,

$$(38) \quad \|E(t) - E_h(t)\|_{0, \infty, \mathcal{I}} \leq C_2 \|f(t) - f_h(t)\|_{0, \mathcal{T}_h} \quad \text{with} \quad C_2 = ((2L)^{1/2} + C_1),$$

and

$$(39) \quad \|E_h(t)\|_{0, \infty, \mathcal{I}} \leq C_2 \|f(t) - f_h(t)\|_{0, \mathcal{T}_h} + \|E(t)\|_{1, \mathcal{I}}.$$

3.2. Mixed Finite Element Approximation for the Poisson problem. We rewrite problem (30) as a first order system:

$$(40) \quad E_h = \frac{\partial \Phi_h}{\partial x} \quad x \in [0, 1]; \quad -\frac{\partial E_h}{\partial x} = \rho_h - 1 \quad x \in [0, 1]$$

with boundary condition $\Phi_h(0, t) = \Phi_h(1, t) = 0$. In this section, we consider a mixed approximation to (40), with the one-dimensional version of Raviart-Thomas elements, RT_k $k \geq 0$ [50, 15]. In 1D the mixed finite element spaces turn out to be the (W_h^{k+1}, V_h^k) -finite element spaces. Note that in particular, $\frac{d}{dx}(W_h^{k+1}) = V_h^k$. For $k \geq 0$ the scheme reads: find $(E_h, \Phi_h) \in W_h^{k+1} \times V_h^k$ such that for all i

$$(41) \quad \int_{\mathcal{I}} E_h z \, dx + \int_{\mathcal{I}} \Phi_h z_x \, dx = 0 \quad \forall z \in W_h^{k+1},$$

$$(42) \quad -\int_{\mathcal{I}} (E_h)_x p \, dx = \int_{\mathcal{I}} (\rho_h - 1)p \, dx \quad \forall p \in V_h^k.$$

We refer to [50, 15] for the stability and error analysis of the method for linear second order problems (see also [6] for the 1D-version of the scheme in the lowest order case $k = 0$). However, in our case, the Poisson problem is “non linear” since the source term in (2) depends on the solution through ρ . Therefore in the error analysis a *consistency error* appears. We have the following result, whose proof can be found in Appendix A.

Lemma 3.3. *Let $k \geq 0$ and let $(E_h, \Phi_h) \in \mathcal{C}^0([0, T]; W_h^{k+1} \times V_h^k)$ be the RT_k approximation to the Poisson problem (40). Then, the following estimates hold for all $t \in [0, T]$:*

$$(43) \quad \|E(t) - E_h(t)\|_{0,\mathcal{I}} + |E(t) - E_h(t)|_{1,\mathcal{I}} \leq Ch^{k+1} \|E(t)\|_{k+1,\mathcal{I}} + \sqrt{2L} \|f(t) - f_h(t)\|_{0,\mathcal{T}_h},$$

$$\|E(t) - E_h(t)\|_{0,\infty,\mathcal{I}} \leq Ch^{k+1} \|E(t)\|_{k+1,\mathcal{I}} + (2L)^{1/2} \|f(t) - f_h(t)\|_{0,\mathcal{T}_h},$$

$$\|E_h(t)\|_{0,\infty,\mathcal{I}} \leq |E(t)|_{1,\mathcal{I}} + (2L)^{1/2} \|f(t) - f_h(t)\|_{0,\mathcal{T}_h} + Ch \|E(t)\|_{1,\mathcal{I}},$$

3.3. DG approximation for the Poisson problem. Consider the DG approximation to the first order system (40): find $(E_h, \Phi_h) \in \mathcal{V}_h^r \times \mathcal{V}_h^r$ such that for all i :

$$(44) \quad \int_{I_i} E_h z \, dx = -\int_{I_i} \Phi_h z_x \, dx + [(\widehat{\Phi}_h z^-)_{i+1/2} - (\widehat{\Phi}_h z^+)_{i-1/2}] \quad \forall z \in \mathcal{V}_h^r,$$

$$(45) \quad \int_{I_i} E_h p_x \, dx - [(\widehat{E}_h p^-)_{i+1/2} - (\widehat{E}_h p^+)_{i-1/2}] = \int_{I_i} (\rho_h - 1)p \, dx \quad \forall p \in \mathcal{V}_h^r,$$

where $(\widehat{\Phi}_h)_{i-1/2}$ and $(\widehat{E}_h)_{i-1/2}$ are the numerical fluxes. In this work we focus on the following family of DG-schemes (see however remark 3.5):

$$(46) \quad \begin{cases} (\widehat{\Phi}_h)_{i-1/2} &= \{\Phi_h\}_{i-1/2} - c_{12}[[\Phi_h]]_{i-1/2} + c_{22}[[E_h]]_{i-1/2} , \\ (\widehat{E}_h)_{i-1/2} &= \{E_h\}_{i-1/2} + c_{12}[[E_h]]_{i-1/2} + c_{11}[[\Phi_h]]_{i-1/2} , \end{cases}$$

where the parameters c_{11} , c_{12} and c_{22} depend solely on $x_{i-1/2} \forall i$, and are still at our disposal. At the boundary nodes due to periodicity in x we impose

$$(\widehat{\Phi}_h)_{1/2} = (\widehat{\Phi}_h)_{N_x+1/2}, \quad (\widehat{E}_h)_{1/2} = (\widehat{E}_h)_{N_x+1/2}.$$

Following [18] we define

$$\begin{aligned} a(E_h, z) &:= \sum_i \int_{I_i} E_h z dx + \sum_i c_{22}[[E_h]]_{i-1/2}[[z]]_{i-1/2} , \\ b(\Phi_h, z) &:= \sum_i \int_{I_i} \Phi_h z_x dx + \sum_i (\{\Phi_h\} - c_{12}[[\Phi_h]])[[z]]_{i-1/2} , \\ c(\Phi_h, p) &:= \sum_i c_{11}[[\Phi_h]]_{i-1/2}[[p]]_{i-1/2} , \end{aligned}$$

and

$$\mathcal{A}((E_h, \Phi_h); (z, p)) = a(E_h, z) + b(\Phi_h, z) - b(p, E_h) + c(\Phi_h, p) .$$

Thus, problem (44)-(45) can be rewritten as: find $(E_h, \Phi_h) \in V_h^r \times V_h^r$ such that

$$(47) \quad \mathcal{A}((E_h, \Phi_h); (z, p)) = \sum_i \int_{I_i} (\rho_h - 1)p dx \quad \forall (z, p) \in V_h^r \times V_h^r .$$

Note that $\mathcal{A}(\cdot, \cdot)$ induces the following semi-norm $\forall (z, p) \in H^1(\mathcal{I}_h) \times H^1(\mathcal{I}_h)$:

$$(48) \quad |(z, p)|_{\mathcal{A}}^2 := \mathcal{A}((z, p); (z, p)) = \|z\|_{0, \mathcal{I}_h}^2 + \|c_{22}[[z]]\|_{0, \gamma_x}^2 + \|c_{11}[[p]]\|_{0, \gamma_x}^2 .$$

We also define the norm for all $r \geq 0$

$$(49) \quad |||(E, \Phi)|||_{r+1, \mathcal{I}}^2 := \|E\|_{r+1, \mathcal{I}}^2 + \|\Phi\|_{r+2, \mathcal{I}}^2 \quad \forall (E, \Phi) \in H^{r+1}(\mathcal{I}) \times H^{r+2}(\mathcal{I}).$$

We next describe the specific choices of the methods we consider (by specifying the parameters in (46)). We restrict ourselves to $k \geq 1$, k being the order of approximation used for f_h .

(i) Local Discontinuous Galerkin (LDG) method: we take $r = k + 1$ so the spaces are $\mathcal{V}_h^r = V_h^{k+1}$ and we set $c_{22} = 0$ and $c_{11} = ch^{-1}$ with c a strictly positive constant. This method was first introduced in [27] for a time dependent convection diffusion problem (with $c_{11} = O(1)$). In this paper we take $c_{11} = ch^{-1}$ with c a positive constant, and $|c_{12}| = 1/2$; that is:

$$(50) \quad \begin{cases} (\widehat{E}_h)_{i-1/2} &= \{E_h\}_{i-1/2} - c_{12}[[E_h]]_{i-1/2} + ch^{-1}[[\Phi_h]]_{i-1/2} , \\ (\widehat{\Phi}_h)_{i-1/2} &= \{\Phi_h\}_{i-1/2} + c_{12}[[\Phi_h]]_{i-1/2} \end{cases} \quad |c_{12}| = \frac{1}{2} .$$

For the approximation of linear problems, it has been proved (see [27],[18]) convergence of order $r + 1$ and r for Φ_h and E_h , respectively.

(ii) Minimal dissipation LDG and DG methods (MD-LDG and MD-DG): we set $r = k$ and the spaces are taken as $\mathcal{V}_h^r = V_h^k$. For the MD-LDG method, the numerical fluxes are defined by taking in (46) $c_{22} = 0$, $c_{12} = 1/2$ and $c_{11} = 0$ except at a boundary node, that is,

$$(51) \quad \begin{cases} \widehat{(\Phi_h)}_{i-1/2} = (\Phi_h)_{i-1/2}^- \\ \widehat{(E_h)}_{i-1/2} = (E_h)_{i-1/2}^+ + c_{11} \llbracket \Phi_h \rrbracket_{i-1/2}, \end{cases} \quad c_{11} = \begin{cases} 0 & i \leq N_x - 1, \\ crh^{-1} & i = N_x. \end{cases}$$

For the MD-DG method the same choice applies except for $\widehat{(\Phi_h)}_{i-1/2} = (\Phi_h)_{i-1/2}^- + c_{22} \llbracket E_h \rrbracket_{i-1/2}$ with $c_{22} = ch/r$. For the approximation of linear problems, the MD-LDG method was first introduced for the 2D case in [23] but with $c_{11} = O(1)$ rather than $O(h^{-1})$ at the boundary. The analysis in the one-dimensional case for both the MD-LDG and the MD-DG can be found in [19], where the authors show that the approximation to E , with both methods, superconverges with order $r + 1$.

(iii) General DG & Hybridized LDG method: we set $r = k$ so that the spaces are taken as $\mathcal{V}_h^r = V_h^k$, and we take the numerical fluxes as in (46) with:

$$c_{11}, c_{22} > 0 \quad |c_{12}| \text{ bounded} \quad c_{11} \sim \frac{1}{c_{22}}.$$

Superconvergence results are proved in [22] (for dimension $d \geq 2$) for the approximation of linear problems. Another option which also provides superconvergence and could be efficiently implemented, is the Hybridized LDG method (see [22]) in which the numerical fluxes can be recast in the form (46) by setting:

$$\begin{cases} \widehat{(E_h)}_{i-1/2} = \left(\frac{\tau^-}{\tau^+ + \tau^-} \right) (E_h)_{i-1/2}^+ + \left(\frac{\tau^+}{\tau^+ + \tau^-} \right) (E_h)_{i-1/2}^- + \left(\frac{\tau^- \tau^+}{\tau^+ + \tau^-} \right) \llbracket \Phi_h \rrbracket_{i-1/2}, \\ \widehat{(\Phi_h)}_{i-1/2} = \left(\frac{\tau^+}{\tau^+ + \tau^-} \right) (\Phi_h)_{i-1/2}^+ + \left(\frac{\tau^-}{\tau^+ + \tau^-} \right) (\Phi_h)_{i-1/2}^- + \left(\frac{1}{\tau^+ + \tau^-} \right) \llbracket E_h \rrbracket_{i-1/2}, \end{cases}$$

where τ^\pm are non-negative constants. To achieve superconvergence, it is enough to take in each interval I_i one $\tau \neq 0$ at one end and at the other end we set $\tau = 0$. Superconvergence can be shown by following the analysis in [22] but using the special projections defined through (17)-(18).

As it happened with RT_k approximation, our poisson problem is nonlinear and therefore the estimates shown in [18],[19] and [22] are not directly applicable. However, we have the following result, whose proof can be found in Appendix A.

Lemma 3.4. *Let $k \geq 1$ and let $(E_h, \Phi_h) \in \mathcal{C}^0([0, T]; V_h^r \times V_h^r)$ be the DG approximation to the Poisson problem (40) solution of (44)-(45)-(46), with any of the three choices (i), (ii) or (iii). Then, the following estimate hold for all $t \in [0, T]$,*

$$(52) \quad \|E(t) - E_h(t)\|_{0, \mathcal{I}_h}^2 \leq Ch^{2(k+1)} \| (E(t), \Phi(t)) \|_{r+1, \mathcal{I}}^2 + 2L \|f(t) - f_h(t)\|_{0, \mathcal{I}_h}^2,$$

where r is the order of polynomials of \mathcal{V}_h^r as given in **(i)**, **(ii)**, **(iii)**. Furthermore, it also holds

$$|(E(t) - E_h(t), \Phi(t) - \Phi_h(t))|_{\mathcal{A}}^2 \leq Ch^{2(k+1)} \| (E(t), \Phi(t)) \|_{r+1, \mathcal{I}}^2 + 2L \| f(t) - f_h(t) \|_{0, \mathcal{T}_h}^2.$$

where $r = k + 1$ for **(i)** and $r = k$ for **(ii)** and the H -LDG in **(iii)**.

Remark 3.5. Since $k \geq 1$, one might consider any of the (consistent and stable) DG methods that fit in the framework given in [4] for approximating the Poisson problem (2). Most of the results shown in this paper for the general LDG discretization (with general c_{12}), hold (with minor changes in the proofs) for any of the resulting methods. For the sake of conciseness, the details are omitted.

4. ERROR ANALYSIS

We start by showing a cell-entropy inequality [43] for the proposed DG schemes (23), which guarantees their L^2 -stability. We then derive the error equation and give some auxiliary results that are used in the proofs of the main results, which are given at the end of the section.

4.1. Stability. Next Proposition shows that the above selection of the numerical fluxes is enough to preserve the L^2 -stability of numerical solution of (23)-(24), for all $k \geq 0$.

Proposition 4.1 (L^2 -stability). *Let $k \geq 0$ and let $f_h \in \mathcal{Z}_h^k$ be the approximation of problem (1), solution of (23)-(24), with the numerical fluxes as in (25). Then*

$$(53) \quad \|f_h(t)\|_{0, \mathcal{T}_h} \leq \|f_h(0)\|_{0, \mathcal{T}_h} \quad \forall t \in [0, T].$$

Proof. By setting $\varphi_h = f_h$ in (24) we have

$$\begin{aligned} \mathcal{B}_{i,j}^h(E_h; f_h, f_h) &= \frac{1}{2} \int_{I_i} \int_{J_j} \frac{\partial(f_h^2)}{\partial t} dv dx - \frac{1}{2} \int_{J_j} \int_{I_i} v \frac{\partial(f_h^2)}{\partial x} dv dx \\ &+ \frac{1}{2} \int_{I_i} \int_{J_j} E_h^i \frac{\partial(f_h^2)}{\partial v} dv dx + \int_{J_j} \left[(\widehat{v f_h f_h^-})_{i+1/2, v} - (\widehat{v f_h f_h^+})_{i-1/2, v} \right] dv \\ &- \int_{I_i} \left[(\widehat{E_h^i f_h f_h^-})_{x, j+1/2} - (\widehat{E_h^i f_h f_h^+})_{x, j-1/2} \right] dx. \end{aligned}$$

Taking into account that E_h depends only on x (through f_h) while v is independent of x , integration of the second and third volume terms leads to

$$(54) \quad \begin{aligned} \mathcal{B}_{i,j}^h(E_h; f_h, f_h) &= \frac{1}{2} \frac{d}{dt} \|f_h\|_{0, \mathcal{T}_{ij}}^2 + \left[\widehat{F}_{i+1/2, j} - \widehat{F}_{i-1/2, j} \right] + \Theta_{i-1/2, j}^F \\ &+ \left[\widehat{G}_{i, j+1/2} - \widehat{G}_{i, j-1/2} \right] + \Theta_{i, j-1/2}^G, \end{aligned}$$

where $\widehat{F}_{i+1/2,j}, \widehat{G}_{i,j+1/2}$ are defined for all i, j , as

$$\begin{aligned}\widehat{F}_{i+1/2,j} &= - \int_{J_j} \left[\frac{v}{2} (f_h^2)^- - v \widehat{f_h f_h^-} \right]_{i+1/2,v} dv \\ \widehat{G}_{i,j+1/2} &= \int_{I_i} \left[\frac{E_h^i}{2} (f_h^2)^- - \widehat{E_h^i f_h f_h^-} \right]_{x,j+1/2} dx,\end{aligned}$$

and

$$\begin{aligned}\Theta_{i-1/2,j}^F &= - \int_{J_j} \left[\frac{v}{2} (f_h^2)^- - v \widehat{f_h f_h^-} \right]_{i-1/2,v} dv + \int_{J_j} \left[\frac{v}{2} (f_h^2)^+ - v \widehat{f_h f_h^+} \right]_{i-1/2,v} dv, \\ \Theta_{i,j-1/2}^G &= \int_{I_i} \left[\frac{E_h^i}{2} (f_h^2)^- - \widehat{E_h^i f_h f_h^-} \right]_{x,j-1/2} dx - \int_{I_i} \left[\frac{E_h^i}{2} (f_h^2)^+ - \widehat{E_h^i f_h f_h^+} \right]_{x,j-1/2} dx.\end{aligned}$$

We next show that the choice (25) ensures that both $\Theta_{i-1/2,j}^F$ and $\Theta_{i,j-1/2}^G$, for all i and j , are non-negative. By rewriting our choice of the numerical fluxes (25) as:

$$(55) \quad (\widehat{v f_h}) = v \{f_h\} - \frac{|v|}{2} \llbracket f_h \rrbracket, \quad (\widehat{E_h^i f_h}) = E_h^i \{f_h\} + \frac{|E_h^i|}{2} \llbracket f_h \rrbracket,$$

and using that $\llbracket f_h^2 \rrbracket = 2\{f_h\}\llbracket f_h \rrbracket$, it can be easily seen that $\Theta_{i-1/2,j}^F$ and $\Theta_{i,j-1/2}^G$ become

$$(56) \quad \Theta_{i-1/2,j}^F = \int_{J_j} \left[\frac{v}{2} \llbracket f_h^2 \rrbracket - v \widehat{f_h} \llbracket f_h \rrbracket \right]_{i-1/2,v} dv = \int_{J_j} \frac{|v|}{2} \llbracket f_h \rrbracket_{i-1/2,v}^2 dv,$$

$$(57) \quad \Theta_{i,j-1/2}^G = \int_{I_i} \left[\widehat{E_h^i f_h} \llbracket f_h \rrbracket - \frac{E_h^i}{2} \llbracket f_h^2 \rrbracket \right]_{x,j-1/2} dx = \int_{I_i} \frac{|E_h^i|}{2} \llbracket f_h \rrbracket_{x,j-1/2}^2 dx.$$

Therefore, $\Theta_{i-1/2,j}^F \geq 0$ and $\Theta_{i,j-1/2}^G \geq 0$ for all i and j and so substitution in (54) leads to

$$\frac{1}{2} \frac{d}{dt} \int_{T_{i,j}} f_h^2 dv dx + \left[\widehat{F}_{i+1/2,j} - \widehat{F}_{i-1/2,j} \right] + \left[\widehat{G}_{i,j+1/2} - \widehat{G}_{i,j-1/2} \right] \leq 0,$$

By summing in the above inequality over i and j , the flux terms telescope and there is no boundary term left because of the periodic (for i) and compactly supported (for j) boundary conditions. Hence,

$$(58) \quad \frac{1}{2} \frac{d}{dt} \sum_{i,j} \int_{T_{i,j}} f_h^2 dv dx = \frac{1}{2} \frac{d}{dt} \|f_h\|_{0,\mathcal{T}_h}^2 \leq 0,$$

and therefore, integration in time of the above inequality yields to (53). \square

Remark 4.2. *By carefully revising the proof one realise that in fact inequality (58) is replaced by the identity*

$$(59) \quad \frac{1}{2} \left(\frac{d}{dt} \|f_h\|_{0,\mathcal{T}_h}^2 + \||v|^{1/2} \llbracket f_h \rrbracket\|_{0,\Gamma_x}^2 + \||E_h|^{1/2} \llbracket f_h \rrbracket\|_{0,\Gamma_v}^2 \right) = 0.$$

Therefore, by defining the norm

(60)

$$\|f_h(t)\|^2 := \|f_h(t)\|_{0,\mathcal{T}_h}^2 + \int_0^t \| |v|^{1/2} [f_h(s)] \|_{0,\Gamma_x}^2 ds + \int_0^t \| |E_h|^{1/2} [f_h(s)] \|_{0,\Gamma_v}^2 ds ,$$

the thesis of Proposition 4.1 can be reformulated as:

$$\|f_h(t)\|^2 = \|f_h(0)\|_{0,\mathcal{T}_h}^2 \leq \|f_h(0)\|^2 \quad \text{for all } t \in [0, T].$$

Finally, we note that for the convergence and error analysis of numerical schemes for non-linear problems, one usually needs to assume/prove that some a-priori estimate on the approximate solution holds for all time. In fact, what is generally done is to assume that there exists some $C_\kappa > 0$ such that,

$$\|f - f_h\|_{*,\mathcal{T}_h} \leq C_\kappa, \quad \forall t \in [0, T],$$

where $\|\cdot\|_{*,\mathcal{T}_h}$ usually refer to a stronger norm than the one for which the error analysis is carried out. For instance $\|\cdot\|_{*,\mathcal{T}_h} = \|\cdot\|_{0,\infty,\mathcal{T}_h}$ if the error analysis is carried out in the L^2 or energy norm, see [48]. We wish to stress that in the present work, due to the structure of the continuous problem, such type of assumption is not required. The main reason is that although our L^2 -error analysis requires a bound on $\|E_h\|_{0,\infty,\mathcal{I}}$, such an estimate would depend ultimately on ρ_h (*zero order moment* of f_h), which in general is *more regular* than f_h itself. In the end, this fact allows for getting a bound for $\|E_h\|_{0,\infty,\mathcal{I}}$ depending on the L^2 -error $\|f - f_h\|_{0,\mathcal{T}_h}$, for which we can easily guarantee that there exists $c_\kappa > 0$ such that,

$$(61) \quad \|f - f_h\|_{0,\mathcal{T}_h} \leq c_\kappa, \quad \forall t \in [0, T].$$

Estimate (61) follows from the L^2 -conservation property of the continuous solution (11) and the L^2 -stability of its approximation f_h given in Proposition 4.1, together with triangle inequality and the L^2 -stability of the standard L^2 -projection, (22) with $p = 2$,

$$\begin{aligned} \|f(t) - f_h(t)\|_{0,\mathcal{T}_h}^2 &\leq 2(\|f(t)\|_{0,\mathcal{T}_h}^2 + \|f_h(t)\|_{0,\mathcal{T}_h}^2) \leq 2\|f_0\|_{0,\Omega}^2 + 2\|\mathcal{P}_h^k(f_0)\|_{0,\mathcal{T}_h}^2 \\ &\leq 2(1 + C)\|f_0\|_{0,\mathcal{T}_h}^2 = c_\kappa . \end{aligned}$$

Let us point out that this result allows us to obtain error estimates that hold for every h and not only in the asymptotic regime.

4.2. Error Equation and Special Projection. To derive the error equation the weak formulation (10) is of little use, since we should take the test function in \mathcal{Z}_h . Hence, by allowing the test function to be discontinuous we find that the true solution satisfies the variational formulation:

$$(62) \quad \sum_{i=1}^{N_x} \sum_{j=1}^{N_v} \mathcal{B}_{i,j}(E; f, \varphi_h) = 0 \quad \forall \varphi_h \in \mathcal{Z}_h^k,$$

where

$$(63) \quad \begin{aligned} \mathcal{B}_{i,j}(E; f, \varphi_h) &= \int_{T_{i,j}} \frac{\partial f}{\partial t} \phi_h \, dv \, dx - \int_{T_{i,j}} v f \frac{\partial \phi_h}{\partial x} \, dv \, dx + \int_{T_{i,j}} E^i f \frac{\partial \phi_h}{\partial v} \, dv \, dx \\ &+ \int_{J_j} \left[(v f \varphi_h)_{i+1/2,v}^- - (v f \varphi_h)_{i-1/2,v}^+ \right] dv - \int_{I_i} \left[(E^i f \varphi_h)_{x,j+1/2}^- - (E^i f \varphi_h)_{x,j-1/2}^+ \right] dx, \end{aligned}$$

E^i being the restriction of the electrostatic field E to I_i ; i.e., $E^i = E|_{I_i}$. Subtracting (23) from (62) we obtain the error equation,

$$(64) \quad \begin{aligned} 0 &= \sum_{i,j} \mathcal{B}_{i,j}(E; f, \varphi_h) - \mathcal{B}_{i,j}^h(E_h^i; f_h, \varphi_h) \\ &= \sum_{i,j} a_{i,j}(f - f_h, \varphi_h) + \sum_{i,j} \mathcal{N}_{i,j}(E; f, \varphi_h) - \mathcal{N}_{i,j}^h(E_h; f_h, \varphi_h) \quad \forall \varphi_h \in \mathcal{Z}_h. \end{aligned}$$

where the bilinear form $a(\cdot, \cdot) = \sum_{i,j} a_{i,j}(\cdot, \cdot)$ gathers all linear terms:

$$\begin{aligned} a_{i,j}(f_h, \varphi_h) &= \int_{J_j} \int_{I_i} \left[\frac{\partial f_h}{\partial t} \varphi_h - v f_h \frac{\partial \varphi_h}{\partial x} \right] dv \, dx \\ &+ \int_{J_j} \left[(\widehat{v f_h \varphi_h^-})_{i+1/2,v} - (\widehat{v f_h \varphi_h^+})_{i-1/2,v} \right] dv \end{aligned}$$

and $\mathcal{N}_{i,j}^h(E_h; \cdot, \cdot)$ (resp. $\mathcal{N}_{i,j}(E; \cdot, \cdot)$) carries the nonlinear part;

$$\begin{aligned} \mathcal{N}_{i,j}^h(E_h^i; f_h, \varphi_h) &= \int_{I_i} \int_{J_j} E_h^i f_h \frac{\partial \varphi_h}{\partial v} \, dv \, dx \\ &- \int_{I_i} \left[(\widehat{E_h^i f_h \varphi_h^-})_{x,j+1/2} - (\widehat{E_h^i f_h \varphi_h^+})_{x,j-1/2} \right] dx. \end{aligned}$$

Notice that due to the nonlinearity, the true solution f does not satisfy the equations defining the numerical scheme (23)–(24). In fact we have a consistency error: $\mathcal{N}^h(E; f, \varphi_h) - \mathcal{N}(E; f, \varphi_h)$ for all $\varphi_h \in \mathcal{Z}_h$, which is "hidden" in the nonlinear error $\mathcal{N}(E; f, \varphi_h) - \mathcal{N}^h(E_h; f_h, \varphi_h)$.

Special Projection: We next introduce the 2-dimensional projection operator $\Pi_h : \mathcal{C}^0(\Omega) \rightarrow \mathcal{Z}_h^k$ which is defined in the following way. Let $T_{i,j} = I_i \times J_j$ be an arbitrary element of \mathcal{T}_h and let $w \in \mathcal{C}^0(\overline{T_{i,j}})$. The restriction of $\Pi_h(w)$ to $T_{i,j}$ is defined by

$$(65) \quad \Pi_h(w) = \begin{cases} \tilde{\pi}_x \otimes \tilde{\pi}_v(w), & \text{if } \text{sign}(E^i) = \text{constant}, \\ P^k \otimes \tilde{\pi}_v(w), & \text{if } \text{sign}(E^i) \neq \text{constant}, \end{cases}$$

where P_x^k denotes the standard L^2 -projection onto $\mathbb{P}^k(I_i)$ defined in (15) and $\tilde{\pi}_x, \tilde{\pi}_v$ are defined by

$$(66) \quad \tilde{\pi}_x(w) = \begin{cases} \pi_x^+(w) & \text{if } E^i > 0, \\ \pi_x^-(w) & \text{if } E^i < 0, \end{cases} \quad \tilde{\pi}_v(w) = \begin{cases} \pi_v^-(w) & \text{if } v > 0, \\ \pi_v^+(w) & \text{if } v < 0, \end{cases}$$

with $\pi_x^\pm : C^0(I_i) \rightarrow V_h^k$ and $\pi_v^\pm : C^0(J_j) \rightarrow V_h^k$ being the special projection operators in the x and v direction respectively, defined as in (17)-(18). The definition of projection Π_h is inspired in those considered in [47, 23] and that introduced in [62] for the analysis of Runge-Kutta methods for conservation laws, see Remark 4.4. Note that taking into account (65)-(66) together with (17)-(18), it is straightforward to see that $\Pi_h(w)$ is uniquely defined. Next Lemma although elementary provides the several approximation results needed for our analysis.

Lemma 4.3. *Let $w \in H^{s+2}(T_{i,j})$, $s \geq 0$ and let Π_h be the projection operator defined through (65)-(66). Then,*

$$(67) \quad \begin{aligned} \|w - \Pi_h(w)\|_{0,T_{ij}} &\leq Ch^{\min(s+2,k+1)} \|w\|_{s+1,T_{ij}}, \\ \|w - \Pi_h(w)\|_{0,e} &\leq Ch^{\min(s+\frac{3}{2},k+\frac{1}{2})} \|w\|_{s+1,T_{ij}}, \quad \forall e = I_i, J_j \subset \partial T_{ij}. \end{aligned}$$

Proof. From the definition (65) we distinguish two cases. If T_{ij} is an element such that $\text{sign}(E^i(x))$ is constant $\forall x \in T_{ij}$, the proof is the same as [18, Lemma 3.2]. If on the contrary, T_{ij} is such that $\exists x \in T_{ij}$ for which $E^i(x) = 0$, we have $\Pi_h(w) = P^k \otimes \tilde{\pi}_v(w)$. But still, since Π_h is a polynomial preserving and linear operator, estimates (67) follow also in this case from Bramble-Hilbert lemma, trace Theorem and standard scaling arguments. Details are omitted for the sake of conciseness. \square

Summing estimates (67) from Lemma 4.3, over elements of the partition \mathcal{T}_h ,

$$(68) \quad \|w - \Pi_h(w)\|_{0,\Omega} + h^{-1/2} \|w - \Pi_h(w)\|_{0,\Gamma} \leq Ch^{k+1} \|w\|_{k+1,\Omega} \quad \forall w \in H^{k+1}(\Omega).$$

Now, denoting by

$$(69) \quad \omega^h = \Pi_h(f) - f_h, \quad \omega^e = \Pi_h(f) - f,$$

we can write

$$(70) \quad f - f_h = [\Pi_h(f) - f_h] - [\Pi_h(f) - f] = \omega^h - \omega^e.$$

Then, by taking as test function $\varphi_h = \omega^h \in \mathcal{Z}_h^k$, the error equation (64) becomes

$$(71) \quad \sum_{i,j} [a(\omega^h - \omega^e, \omega^h) + \mathcal{N}_{i,j}(E^i; f, \omega^h) - \mathcal{N}_{i,j}^h(E_h^i; f_h, \omega^h)] = 0.$$

We next define

$$(72) \quad \mathcal{K}^1(v, f, \omega^h) = \sum_{i,j} \mathcal{K}_{i,j}^1(v, \omega^e, \omega^h), \quad \mathcal{K}^2(E_h, f, \omega^h) = \sum_{i,j} \mathcal{K}_{i,j}^2(E_h, f, \omega^h),$$

where

$$(73) \quad \begin{aligned} \mathcal{K}_{i,j}^1(v, f, \omega^h) &= \int_{T_{i,j}} v \omega^e \omega_x^h dv dx \\ &- \int_{J_j} [(\widehat{v\omega^e}(\omega^h)^-)_{i+1/2,v} - (\widehat{v\omega^e}(\omega^h)^+)_{i-1/2,v}] dv, \end{aligned}$$

$$(74) \quad \begin{aligned} \mathcal{K}_{i,j}^2(E_h, f, \omega^h) &= \int_{T_{i,j}} E_h \omega^e \omega_v^h dv dx \\ &- \int_{I_i} [(\widehat{E_h \omega^e}(\omega^h)^-)_{x,j+1/2} - (\widehat{E_h \omega^e}(\omega^h)^+)_{x,j-1/2}] dx. \end{aligned}$$

Next two Lemmas provide estimates for the terms defined in (72). Both lemmas extend and generalize [23, Lemma 3.6] to the case of variable coefficients and nonlinear problems, respectively. To keep the readability flow of the paper, the proofs of these technical Lemmas are postponed till Appendix B.

Remark 4.4. *We wish to note that the definition (65) of Π_h is done in terms of E (and v), while the definition of the numerical fluxes is done in terms of E_h (and v). This is due to the non-linearity of the problem and it is inspired in the ideas used in [62]. By defining Π_h in terms of E rather than E_h and using the regularity of the solution, we will be able to estimate optimally the expression \mathcal{K}^2 without any further assumption on the mesh partition \mathcal{T}_h .*

Lemma 4.5. *Let \mathcal{T}_h be a cartesian mesh of Ω , $k \geq 1$ and let $f_h \in \mathcal{Z}_h^k$ be the approximate distribution function satisfying (23)-(24). Let $f \in \mathcal{C}^0([0, T]; H^{k+2}(\Omega))$ and let \mathcal{K}^1 be defined as in (72). Assume that the partition \mathcal{T}_h is constructed so that $v = 0$ corresponds to a node of the partition. Then, the following estimate holds true*

$$(75) \quad |\mathcal{K}^1(v, f, \omega^h)| \leq Ch^{k+1} (\|f\|_{k+1, \Omega} + CL\|f\|_{k+2, \Omega}) \|\omega^h\|_{0, \mathcal{T}_h}.$$

Lemma 4.6. *Let \mathcal{T}_h be a cartesian mesh of Ω , $k \geq 1$ and let $(E_h, f_h) \in \mathcal{W}_h \times \mathcal{Z}_h^k$ be the solution to (23)-(24) with \mathcal{W}_h a finite element space, conforming or non-conforming, of at least first order ($\mathcal{W}_h = W_h^{k+1}$ or $W_h = V_h^r$). Let $(E, f) \in \mathcal{C}^0([0, T]; W^{1, \infty}(\mathcal{I}) \times H^{k+2}(\Omega))$ and let \mathcal{K}^2 be defined as in (72). Then, the following estimate holds*

$$(76) \quad \begin{aligned} |\mathcal{K}^2(E_h, f, \omega^h)| &\leq Ch^k \|E - E_h\|_{0, \infty, \mathcal{I}} \|f\|_{k+1, \Omega} \|\omega^h\|_{0, \mathcal{T}_h} \\ &+ Ch^{k+1} (\|f\|_{k+2, \Omega} \|E\|_{0, \infty} + \|f\|_{k+1, \Omega} \|E\|_{1, \infty, \mathcal{I}}) \|\omega^h\|_{0, \mathcal{T}_h}. \end{aligned}$$

4.3. Auxiliary Results. We next prove two Lemmas that are needed for the proofs of the main Theorems 4.9, 4.13, and 4.11. The first one reduces the expression for the linear part of the error equation (71):

Lemma 4.7. *Let $f \in \mathcal{C}^0(\Omega)$ and let $f_h \in \mathcal{Z}_h^k$ with $k \geq 1$. Then, the following equality holds*

$$a(f - f_h, \omega^h) = \sum_{i,j} \int_{T_{i,j}} (\omega_t^h - \omega_t^e) \omega^h dx dv + \sum_{i,j} \int_{J_j} \frac{|v|}{2} \llbracket \omega^h \rrbracket_{i-1/2,v}^2 dv + \mathcal{K}^1(v, f, \omega^h).$$

Proof. From (70) we get $a(f - f_h, \omega^h) = a(\omega^h, \omega^h) - a(\omega^e, \omega^h)$. Arguing as for (56) in the proof of Proposition 4.1 (note that $\omega^h \in \mathcal{Z}_h$), we have for the first term

$$(77) \quad a(\omega^h, \omega^h) = \sum_{i,j} \int_{I_i} \int_{J_j} \omega_t^h \omega^h dx dv + \sum_{i,j} \int_{J_j} \frac{|v|}{2} \llbracket \omega^h \rrbracket_{i-1/2,v}^2 dv.$$

The definition (73) of \mathcal{K}_1 , the continuity of f and the numerical fluxes (25) imply

$$\begin{aligned} a(\omega^e, \omega^h) &= \sum_{i,j} \int_{I_i} \int_{J_j} \omega_t^e \omega^h dx dv - \int_{J_j} \int_{I_i} v \omega^e \omega_x^h dx dv - \sum_{i,j} \int_{J_j} [\widehat{v \omega^e} \llbracket \omega^h \rrbracket]_{i-1/2,v} dv \\ &= \sum_{i,j} \int_{I_i} \int_{J_j} \omega_t^e \omega^h dx dv - \mathcal{K}^1(v, f, \omega^h). \end{aligned}$$

which together with (77) completes the proof. \square

The other auxiliary Lemma deals with the error coming from the nonlinear term:

Lemma 4.8. *Let $E \in \mathcal{C}^0(I)$, $f \in \mathcal{C}^0(\Omega)$ and $f_h \in \mathcal{Z}_h^k$ with $k \geq 1$. Then, the following identity holds*

$$(78) \quad \begin{aligned} &\sum_{i,j} [\mathcal{N}_{i,j}(E; f; \omega^h) - \mathcal{N}_{i,j}^h(E_h; f_h, \omega^h)] = \\ &= \sum_{i,j} \int_{I_i} \frac{|E_h^i|}{2} \llbracket \omega^h \rrbracket_{x,j-1/2}^2 dx - \sum_{i,j} \int_{T_{i,j}} [E^i - E_h^i] \frac{\partial f}{\partial v} \omega^h dv dx - \mathcal{K}^2(E_h, f, \omega^h). \end{aligned}$$

Proof. Subtracting the nonlinear terms in (63) and (24) we have

$$(79) \quad \begin{aligned} \mathcal{N}_{i,j}(E; f; \omega^h) - \mathcal{N}_{i,j}^h(E_h; f_h, \omega^h) &= - \int_{I_i} \int_{J_j} [E^i f - E_h^i f_h] \frac{\partial \omega^h}{\partial v} dv dx \\ &\quad - \int_{I_i} \left[([E^i f - \widehat{E_h^i f_h}] \omega^h)_{x,j-1/2}^- - ([E^i f - \widehat{E_h^i f_h}] \omega^h)_{x,j-1/2}^+ \right] dx. \end{aligned}$$

Notice that the integrand of the volume part above, can be decomposed as

$$(80) \quad [E^i f - E_h^i f_h] \pm E_h^i f = [E^i - E_h^i] f + E_h^i (f - f_h),$$

and so substituting into (79) we find

$$(81) \quad \mathcal{N}_{i,j}(E; f; \omega^h) - \mathcal{N}_{i,j}^h(E_h; f_h, \omega^h) = T_1 + T_2 + T_3,$$

where

$$\begin{aligned} T_1 &= \int_{I_i} \int_{J_j} [E^i - E_h^i] f \omega_v^h dv dx, & T_2 &= \int_{I_i} \int_{J_j} E_h^i [f - f_h] \omega_v^h dv dx, \\ T_3 &= \int_{I_i} \left[\left([(E^i f)^+ - \widehat{E_h^i f_h}] (\omega^h)^+ \right)_{x,j-1/2} - \left([(E^i f)^- - \widehat{E_h^i f_h}] (\omega^h)^- \right)_{x,j+1/2} \right] dx. \end{aligned}$$

Since neither E nor E_h depend on v , integration by parts of T_1 gives $T_1 = T_{1a} + T_{1b}$:

$$T_1 = - \int_{I_i} \int_{J_j} [E^i - E_h^i] \frac{\partial f}{\partial v} \omega^h dv dx + \int_{I_i} (E^i - E_h^i) \left[(f \omega^h)^-_{x,j+1/2} - (f \omega^h)^+_{x,j-1/2} \right] dx.$$

Summing now over j and taking into account the continuity of f we find for T_{1b} ,

$$(82) \quad \sum_j T_{1b} = - \sum_j \int_{I_i} (E^i - E_h^i) (f \llbracket \omega^h \rrbracket)_{x,j-1/2} dx.$$

We next deal with T_2 . From the splitting (70) we have

$$T_2 = \int_{I_i} E_h^i \int_{J_j} \omega^h \omega_v^h dv dx - \int_{I_i} E_h^i \int_{J_j} \omega^e \omega_v^h dv dx = T_{2a} + T_{2b},$$

and so, integrating the first term and summing over j we easily get

$$(83) \quad \sum_j T_{2a} = \sum_j \frac{1}{2} \int_{I_i} E_h^i \int_{J_j} \frac{\partial (\omega^h)^2}{\partial v} dv dx = - \sum_j \int_{I_i} \frac{E_h^i}{2} \llbracket (\omega^h)^2 \rrbracket_{x,j-1/2} dx.$$

We finally deal with the boundary terms collected in T_3 . Summation over j and the continuity of E and f gives

$$\sum_j T_3 = \sum_j \int_{I_i} [E^i f - \widehat{E_h^i f_h}]_{x,j-1/2} \llbracket \omega^h \rrbracket_{x,j-1/2} dx.$$

Then, reasoning as in (80), we deduce for all i that

$$\left(E^i f - \widehat{E_h^i f_h} \right) \pm E_h^i f = (E^i - E_h^i) f + \left(E_h^i f - \widehat{E_h^i f_h} \right) = (E^i - E_h^i) f + \widehat{E_h^i (\omega^h)} - \widehat{E_h^i (\omega^e)},$$

where in the last step we have used the continuity of f together with the consistency of the numerical flux $\widehat{E_h^i f_h}$. Thus, substituting back into T_3 , we infer

$$\begin{aligned} \sum_j T_3 &= \sum_j \int_{I_i} \left((E^i - E_h^i) f \llbracket \omega^h \rrbracket + \widehat{E_h^i \omega^h} \llbracket \omega^h \rrbracket - \widehat{E_h^i \omega^e} \llbracket \omega^h \rrbracket \right)_{x,j-1/2} dx \\ &= \sum_j T_{3a} + \sum_j T_{3b} + \sum_j T_{3c}. \end{aligned}$$

Then, for the first term, T_{3a} , recalling the expression (82), we get

$$(84) \quad \sum_j [T_{1b} + T_{3a}] = 0.$$

Next, summing T_{3b} and T_{2a} from (83) and arguing as for (57) in the proof of Proposition 4.1, we find

$$(85) \quad \sum_j [T_{2a} + T_{3b}] = \sum_j \frac{|E_h^i|}{2} \llbracket \omega^h \rrbracket_{x,j-1/2}^2 dx .$$

Finally, recalling the definition (74) of \mathcal{K}_2 and adding up T_{3c} with T_{2b} we get

$$\sum_j [T_{2b} + T_{3c}] = -\mathcal{K}^2(E_h, f, \omega^h).$$

Thus, substituting the above identity together (85) and the expression for T_{1a} into the equation (81) we reach (78) and so the proof is complete. \square

4.4. Approximation. We next show the main convergence results of this work proving a-priori error estimates for the electron distribution f , for all the proposed methods. In each case, as a byproduct result, we also get the corresponding convergence results for the electrostatic field E . The section is closed with some remarks about the comparison with the convergence of other methods. We start with the result for the Conforming-DG method:

Theorem 4.9 (Conforming-DG method). *Let $k \geq 1$ and consider the unique compactly supported solution of the Vlasov-Poisson system (1)-(2) given by Theorem 2.1 with $f \in \mathcal{C}^1([0, T]; H^{k+2}(\Omega))$ and $E \in \mathcal{C}^0([0, T]; W^{1,\infty}(\mathcal{I}))$. Let $(E_h, f_h) \in \mathcal{C}^0([0, T]; W_h^{k+1}) \times \mathcal{C}^1([0, T]; \mathcal{Z}_h^k)$ be the conforming-DG approximation, solution of (23), (24) and (34). Then,*

$$\|f(t) - f_h(t)\|_{0, \mathcal{T}_h} \leq C_0 h^{k+1} \quad \forall t \in [0, T],$$

where C_0 depends on the time t , the polynomial degree k , the shape regularity of the partition and depends also on f and on E through the norms

$$C_0 = C_0(\|f(t)\|_{k+2, \Omega}, \|f_t(t)\|_{k+1, \Omega}, L, \|E(t)\|_{1, \infty, \mathcal{I}}) .$$

Proof. Recalling the error equation (71)

$$a(\omega^h - \omega^e, \omega^h) + \mathcal{N}(E^i; f, \omega^h) - \mathcal{N}^h(E_h^i; f_h, \omega^h) = 0,$$

and using Lemmas 4.7 and 4.8, we have

$$(86) \quad \begin{aligned} & \sum_{i,j} \int_{T_{i,j}} \omega_i^h \omega^h dv dx + \sum_{i,j} \int_{J_j} \frac{|v|}{2} \llbracket \omega^h \rrbracket_{i+1/2, v}^2 dv + \sum_{i,j} \int_{I_i} \frac{|E_h^i|}{2} \llbracket \omega^h \rrbracket_{x, j+1/2}^2 dx \\ &= \sum_{i,j} \int_{T_{i,j}} \omega_i^e \omega^h dv dx + \sum_{i,j} \int_{T_{i,j}} [E^i - E_h^i] \frac{\partial f}{\partial v} \omega^h dv dx - \mathcal{K}^1(v, f, \omega^h) + \mathcal{K}^2(E_h, f, \omega^h) \\ &= T_1 + T_2 - \mathcal{K}^1 + \mathcal{K}^2 . \end{aligned}$$

Notice that the left hand side of the above equation, is exactly what results after summation over i and j in (54) from Proposition (4.1), see also (59). Then, it

is enough to estimate the terms on the right hand side of the above equation. The first term is directly estimated by using Cauchy-Schwarz and the arithmetic-geometric inequalities together with the interpolation property (68)

$$(87) \quad |T_1| \leq \frac{C}{2} (\|\omega_t^e\|_{0,\mathcal{T}_h}^2 + \|\omega^h\|_{0,\mathcal{T}_h}^2) \leq Ch^{2k+2} \|f_t\|_{H^{k+1}(\Omega)} + C\|\omega^h\|_{0,\mathcal{T}_h}^2.$$

The second term on the rhs of (86), is readily estimated by using Hölder inequality together with estimate (38) from Lemma 3.2, the splitting (70), the arithmetic-geometric inequality and the interpolation estimate (68),

$$(88) \quad \begin{aligned} |T_2| &\leq C\|E - E_h\|_{0,\infty,\mathcal{I}} \|f_v\|_{0,\Omega} \|\omega^h\|_{0,\mathcal{T}_h} \leq CC_2 \|f - f_h\|_{0,\mathcal{T}_h} \|f_v\|_{0,\Omega} \|\omega^h\|_{0,\mathcal{T}_h} \\ &\leq CC_2 (\|\omega^e\|_{0,\mathcal{T}_h} + \|\omega^h\|_{0,\mathcal{T}_h}) \|f_v\|_{0,\Omega} \|\omega^h\|_{0,\mathcal{T}_h} \\ &\leq CC_2 h^{2k+2} \|f\|_{k+1,\Omega}^2 \|f_v\|_{0,\Omega} + C_2 \|f_v\|_{0,\Omega} \|\omega^h\|_{0,\Omega}^2, \end{aligned}$$

where $C_2 \approx L^{1/2}$ is the constant in Lemma 3.2. Estimate (75) from Lemma 4.5 and the arithmetic-geometric inequality give for the third term,

$$(89) \quad |\mathcal{K}^1| \leq Ch^{2k+2} L^2 \|f\|_{k+2,\Omega}^2 + C\|\omega^h\|_{0,\mathcal{T}_h}^2.$$

Last term is bounded by using estimate (76) from Lemma 4.6 and arguing similarly as for T_2 ; using estimate (38) from Lemma 3.2, the splitting (70), the arithmetic-geometric inequality and the interpolation estimate (68),

$$(90) \quad \begin{aligned} |\mathcal{K}^2| &\leq Ch^k \|f\|_{k+1,\Omega} (\|\omega^e\|_{0,\mathcal{T}_h} + \|\omega^h\|_{0,\mathcal{T}_h}) \|\omega^h\|_{0,\mathcal{T}_h} + Ch^{k+1} \|f\|_{k+2,\Omega} \|E\|_{1,\infty,\mathcal{I}} \|\omega^h\|_{0,\mathcal{T}_h} \\ &\leq Ch^{2k+2} (\|f\|_{k+2,\Omega}^2 \|E\|_{1,\infty,\mathcal{I}}^2 + C_2 h^k \|f\|_{k+1,\Omega}^3) + C(1 + h^k \|f\|_{k+1,\Omega}) \|\omega^h\|_{0,\mathcal{T}_h}^2. \end{aligned}$$

Then, by substituting the above estimate together with (87), (88) and (89) into the error equation (86), we conclude

$$\frac{d}{dt} \|\omega^h(t)\|_{0,\mathcal{T}_h}^2 \leq A(t) \|\omega^h(t)\|_{0,\mathcal{T}_h}^2 + h^{2k+2} B(t)$$

with $A(t) = (C + L^{1/2} \|f_v\|_{0,\Omega} + CL^{1/2} h^k \|f\|_{k+1,\Omega})$ and

$$B(t) = C \|f\|_{k+2,\Omega}^2 (L^2 + \|E\|_{1,\infty,\mathcal{I}}^2) + \|f_t\|_{k+1,\Omega}^2 + CL^{1/2} \|f\|_{k+1,\Omega}^2 (\|f_v\|_{0,\Omega}^2 + h^k \|f\|_{k+1,\Omega}).$$

Therefore, integration in time of the above inequality and a standard application of Gronwall's inequality gives the error estimate,

$$(90) \quad \|\omega^h(t)\|_{0,\mathcal{T}_h}^2 \leq C_0^2 h^{2k+2},$$

where C_0 is as stated in the claim. Hence, Theorem 4.9 follows from the triangle inequality and the interpolation property (68). \square

As a direct consequence of Theorem 4.9 together with estimates (37) and (38) of Lemma 3.2, we obtain the following result on the error of the electrostatic field.

Corollary 4.10. *Under the hypothesis of Theorem 4.9, the following error estimates hold*

$$\begin{aligned} \|E(t) - E_h(t)\|_{0,\mathcal{I}} &\leq C_0 C_1 h^{k+1} \quad \forall t \in [0, T], \\ \|E(t) - E_h(t)\|_{\infty,\mathcal{I}} &\leq C_0 C_2 h^{k+1} \quad \forall t \in [0, T], \end{aligned}$$

where C_1 and C_2 are given in (37) and (38), respectively and C_0 in Theorem 4.9.

Next result establishes the convergence for the RT_k -DG method:

Theorem 4.11 (RT_k -DG method). *Let $k \geq 1$ and consider the unique compactly supported solution of the Vlasov-Poisson system (1)-(2) given by Theorem 2.1 with $f \in \mathcal{C}^1([0, T]; H^{k+2}(\Omega))$ and $E \in \mathcal{C}^0([0, T]; H^{k+1}(\mathcal{I}))$. Let $((E_h, \Phi_h), f_h) \in \mathcal{C}^0([0, T]; (W_h^{k+1} \times V_h^k)) \times \mathcal{C}^1([0, T]; \mathcal{Z}_h^k)$ be the RT_k -DG approximation solution of (23), (24), (41), and (42). Then,*

$$\|f(t) - f_h(t)\|_{0,\Omega} \leq C_4 h^{k+1} \quad \forall t \in [0, T],$$

where C_4 depends on the time t , the polynomial degree k , the shape regularity of the partition and depends also on f and on E through the norms

$$C_4 = C_4(\|f(t)\|_{k+2,\Omega}, \|f_t(t)\|_{k+1,\Omega}, L, \|E(t)\|_{k+1,\mathcal{I}}),$$

Proof. The proof follows exactly the same lines as the proof of Theorem 4.9. In this case, to bound the error $\|E - E_h\|_{0,\infty,\mathcal{I}}$ that appears in the estimates for T_2 and \mathcal{K}_2 one has to use estimate (43) from Lemma 3.3. We omit the details for the sake of conciseness. \square

Corollary 4.12. *Under the hypothesis of Theorem 4.11, the following error estimates hold*

$$\begin{aligned} \|E(t) - E_h(t)\|_{0,\mathcal{I}} + |E(t) - E_h(t)|_{1,\mathcal{I}} &\leq 2C_4 L^{1/2} h^{k+1} + C h^{k+1} \|E\|_{k+1,\mathcal{I}} \\ \|E(t) - E_h(t)\|_{0,\infty,\mathcal{I}} &\leq C_4 L^{1/2} h^{k+1} + C h^{k+1} \|E\|_{k+1,\mathcal{I}} \end{aligned}$$

for all $t \in [0, T]$, where C_4 is the constant of Theorem 4.11.

Finally, we show the convergence for the full DG approximation:

Theorem 4.13 (DG-DG method). *Let $r \geq k \geq 1$ and consider the unique compactly supported solution of the Vlasov-Poisson system (1)-(2) given by Theorem 2.1 with $f \in \mathcal{C}^1([0, T]; H^{k+2}(\Omega))$ and $E \in \mathcal{C}^0([0, T]; H^{r+1}(\mathcal{I}))$. Let $((E_h, \Phi_h), f_h) \in \mathcal{C}^0([0, T]; \mathcal{V}_h^r \times \mathcal{V}_h^r) \times \mathcal{C}^1([0, T]; \mathcal{Z}_h^k)$ be the DG-DG approximation that satisfies (23), (24), (44), and (45) with any of the three choices (i), (ii) or (iii). Then,*

$$\|f(t) - f_h(t)\|_{0,\Omega} \leq C_5 h^{k+1} \quad \forall t \in [0, T],$$

where C_5 depends on time t , the polynomials degrees k and r , the shape regularity of the partition and depends also on f and on (E, Φ) through the norms

$$C_5 = C_5(\|f(t)\|_{k+2,\Omega}, \|f_t(t)\|_{k+1,\Omega}, L, \|(E, \Phi)\|_{r+1,\mathcal{I}}).$$

Proof. The proof follows essentially the same lines as the proof of Theorems 4.9 and 4.11, but dealing with T_2 we use estimate (52) from Lemma 3.4;

$$\begin{aligned}
 |T_2| &\leq C \|E - E_h\|_{0,\mathcal{I}} \|f_v\|_{0,\infty,\Omega} \|\omega^h\|_{0,\mathcal{I}_h} \\
 &\leq [Ch^{k+1} \| |(E, \Phi)| \|_{r+1,\mathcal{I}} + (2L)^{1/2} \|f - f_h\|_{0,\mathcal{I}_h}] \|f_v\|_{0,\infty,\Omega} \|\omega^h\|_{0,\mathcal{I}_h} \\
 &\leq Ch^{2k+2} (\| |(E, \Phi)| \|_{r+1,\mathcal{I}}^2 + 2L \|f\|_{k+1,\Omega}^2) \|f_v\|_{0,\infty,\Omega} \\
 (91) \quad &+ (C + (2L)^{1/2} \|f_v\|_{0,\infty,\Omega}) \|\omega^h\|_{0,\Omega}^2.
 \end{aligned}$$

Also, to bound for \mathcal{K}^2 we first note that $E_h = P^{k+1}(E_h)$ since $E_h \in V_h^r$ (and $r = k+1$ or $r = k$), so that inverse inequality, estimate (20) and the L^∞ -stability of the L^2 -projection give

$$\begin{aligned}
 \|E - E_h\|_{0,\infty,\mathcal{I}_h} &\leq \|E - P^{k+1}(E)\|_{0,\infty,\mathcal{I}_h} + Ch^{-1/2} \|P^{k+1}(E) - E_h\|_{0,\mathcal{I}_h} \\
 (92) \quad &\leq Ch^{k+1} \|E\|_{k+1,\infty,\mathcal{I}} + Ch^{-1/2} \|E - E_h\|_{0,\mathcal{I}_h}.
 \end{aligned}$$

Then, using estimate (76) from Lemma 4.6 together with the above estimate and the L^2 -bound for the error $E - E_h$ given in Lemma 3.4, we get

$$\begin{aligned}
 |\mathcal{K}^2| &\leq C(1 + L^{1/2}h^{k-1/2} \|f\|_{k+1,\Omega}) \|\omega^h\|_{0,\mathcal{I}_h}^2 \\
 &+ Ch^{2k+2} (\|E\|_{1,\infty,\mathcal{I}}^2 \|f\|_{k+2,\Omega}^2 + \| |(E, \Phi)| \|_{r+1,\mathcal{I}}^2 \|f\|_{k+1,\Omega}^2 + h^{k-1/2} \|f\|_{k+1,\Omega}^3),
 \end{aligned}$$

where we have neglected high order terms of order $O(h^{4k-1/4})$. Noting that $k \geq 1$, the proof can now be completed by arguing as in the proof of Theorem 4.9. We omit the details for the sake of brevity. \square

Remark 4.14. Taking into account the definition (60) of the norm $\| |\cdot| \|$ (see Remark 4.2), observe that in the proof of Theorems 4.9, 4.11 and 4.13, similarly as how it is obtained the error estimate (90), we also get

$$(93) \quad \| |\omega^h(t)| \| ^2 \leq C_s^2 h^{2k+2} \quad s = 0, 4, 5.$$

As a direct consequence of Theorem 4.13 and Lemma 3.4 we have the following Corollary whose proof is omitted.

Corollary 4.15. Under the hypothesis of Theorem 4.13, the following error estimates hold for all $t \in [0, T]$

$$\|E(t) - E_h(t)\|_{0,\mathcal{I}}^2 \leq Ch^{2k+2} \| |(E(t), \Phi(t))| \|_{r+1,\mathcal{I}}^2 + C_5^2 L h^{2k+2}$$

where C_5 is the constant of Theorem 4.13, and

$$\|E(t) - E_h(t)\|_{0,\mathcal{I}}^2 + c_{11} \| |\Phi_h(t)| \|_{0,\gamma_x}^2 + c_{22} \| |E_h(t)| \|_{0,\gamma_x}^2 \leq C_6 h^{2k+2},$$

with $C_6 = C_5^2 L + C \| |(E(t), \Phi(t))| \|_{r+1,\mathcal{I}}^2$ where $r = k+1$ for **(i)** and $r = k$ for **(ii)** and the H-LDG in **(iii)**.

Remark 4.16 (Order of convergence attained by other methods). *As noted in the introduction, there are very few works dealing with the convergence and error analysis of eulerian solvers for the (periodic) Vlasov-Poisson system. High order schemes have been only analyzed in the context of semi-lagrangian methods [7, 8, 10]. Although, it is difficult to compare their results with ours, since these analysis deal with fully discrete schemes, we just mention briefly what one can expect to achieve with these methods in the case of a constant Courant-Friederichs-Levy CFL ($\nu = dt/h = \text{constant}$) and in the case where the time step dt were taken the largest possible. In [7], error estimates in L^∞ of first order (for CFL=constant) and slightly better than first order (at most of order $4/3$ for the largest possible time step), are shown assuming the initial data is of class C^2 . High order schemes, by using polynomials of degree k in the reconstruction, are considered in [8, 10]. There, the authors prove error bounds for the distribution function and the electrostatic field in L^2 and L^∞ , respectively, of at most order k (if CFL=constant) and of order $2(k+1)/3$ if the largest possible time step wants to be used. These works typically require the technical assumption $f \in W^{k+1,\infty}(\Omega)$.*

5. ENERGY CONSERVATION

In this section we discuss the issue of energy conservation (12) for the proposed numerical schemes. We start by showing that for a particular choice of the LDG approximation to the Poisson-problem (2), the resulting LDG-DG method for the Vlasov system possess such conservation property, under a technical restriction the on the degree of the polynomial spaces; namely we require $k \geq 2$. However, we wish to note that such restriction is rather natural since we want to use v^2 as test function, as it is done in the proof of (12) for the continuous problem. We close the section with two results that provide (under the same restriction) an energy inequality for others full DG methods considered in this paper.

Theorem 5.1 (Energy conservation). *Let $k \geq 2$ and let $((E_h, \Phi_h), f_h)$ be the LDG-DG approximation belonging to $C^1([0, T]; (V_h^k \times V_h^k) \times \mathcal{Z}_h^k)$ of the Vlasov-Poisson system (1)-(2), solution of (23), (24), (44), and (45), with the numerical fluxes (25) for the approximate electron distribution. Let $(E_h, \Phi_h) \in V_h^k \times V_h^k$ be the corresponding LDG approximation to the associated Poisson problem, solution of (44)-(45) with numerical fluxes:*

$$(94) \quad \begin{cases} (\widehat{E}_h)_{i-1/2} &= \{E_h\}_{i-1/2} - \frac{\text{sign}(v)}{2} \llbracket E_h \rrbracket_{i-1/2} + c_{11} \llbracket \Phi_h \rrbracket_{i-1/2} , \\ (\widehat{\Phi}_h)_{i-1/2} &= \{\Phi_h\}_{i-1/2} + \frac{\text{sign}(v)}{2} \llbracket \Phi_h \rrbracket_{i-1/2} , \end{cases}$$

where $c_{11} > 0$ and $c_{22} = 0$ at all nodes. Then, the following identity holds true

$$(95) \quad \frac{d}{dt} \left(\sum_{i,j} \int_{T_{ij}} v^2 f_h(t) dv dz + \sum_i \int_{I_i} E_h(t)^2 dx + c_{11} \sum_i \llbracket \Phi_h(t) \rrbracket_{i-1/2}^2 \right) = 0 .$$

Remark 5.2. *Prior to give the proof of the above Proposition, we wish to point out that we are making an abuse of notation by saying that $(E_h, \Phi_h) \in V_h^k \times V_h^k$ is the solution with numerical fluxes (94). Actually, we should talk about two solutions, one for each sign of v . Such two solutions (one for $v > 0$ the other for $v < 0$) enter in the Vlasov equation, when it comes to evaluate the fluxes in the v -direction (i.e., $(\widehat{E_h^i f_h})$).*

Proof. To simplify the notation, throughout the proof, we drop the sub/super indexes h from the finite element functions. The proof is carried out in several steps.

First step:

We start by noting that since $f \in \mathcal{Z}_h^k$, for each fixed $v \in \mathcal{J}$, $f(\cdot, v) \in V_h^k$ (as a polynomial in x). Hence, we can set $z = f$ in (44)

$$\int_{I_i} E f dx = - \int_{I_i} \Phi f_x dx + [(\widehat{\Phi} f^-)_{i+1/2} - (\widehat{\Phi} f^+)_{i-1/2}].$$

Then, multiplying the above equation by v and integrating over \mathcal{J} , we find

$$\int_{\mathcal{J}} \int_{I_i} v E f dv dx = - \int_{\mathcal{J}} \int_{I_i} v \Phi f_x dv dx + \int_{\mathcal{J}} v [(\widehat{\Phi} f^-)_{i+1/2} - (\widehat{\Phi} f^+)_{i-1/2}] dv .$$

Integration by parts of the volume term on the right hand side above, gives

$$(96) \quad \int_{\mathcal{J}} \int_{I_i} v E f dv dx = \int_{\mathcal{J}} \int_{I_i} v f \Phi_x dv dx + \int_{\mathcal{J}} v [(\widehat{\Phi} f - f \Phi)_{i+1/2}^- - (\widehat{\Phi} f - f \Phi)_{i-1/2}^+] dv .$$

Next, we set $\varphi_h = \Phi \in V_h^k \subset \mathcal{Z}_h^k$ in (24) (Φ as a polynomial in \mathcal{Z}_h^k is constant in v)

$$\begin{aligned} \sum_{i,j} \int_{T_{ij}} f_t \Phi dv dx - \int_{T_{ij}} v f (\Phi)_x dv dx + \int_{J_j} [((\widehat{v f}) \Phi^-)_{i+1/2,v} - ((\widehat{v f}) \Phi^+)_{i-1/2,v}] dv \\ + \int_{T_{ij}} E f (\Phi)_v dv dx - \int_{I_i} \Phi [(\widehat{E_h^i f})_{x,j+1/2} - (\widehat{E_h^i f})_{x,j-1/2}] dx = 0. \end{aligned}$$

Then, note that last two terms in the above equation vanish; the volume part cancels since Φ does not depend on v , and the sum of the boundary terms telescope, due to the consistency of the numerical flux $\widehat{E_h^i f}$, and no boundary term is left due to the zero boundary conditions in v . Thus we have,

$$(97) \quad \sum_{i,j} \int_{T_{ij}} f_t \Phi dv dx = \sum_{i,j} \int_{T_{ij}} v f \Phi_x dv dx - \int_{J_j} [(\widehat{v f} \Phi^-)_{i+1/2,v} - (\widehat{v f} \Phi^+)_{i-1/2,v}] dv .$$

Combining then the above equation with (96) and using the periodicity of the boundary conditions in x we get,

$$(98) \quad \sum_{i,j} \int_{T_{ij}} f_t \Phi \, dv \, dx = \sum_{i,j} \left[\int_{J_j} \left(\widehat{vf}[\Phi] + v(\widehat{\Phi}[f] - [f\Phi]) \right)_{i-1/2,v} dv + \int_{T_{ij}} v E f \, dv \, dx \right].$$

Second step:

Now, we differentiate with respect to time the first order system (40) and consider its DG approximation. The second equation (45) reads,

$$\int_{I_i} E_t p_x \, dx - \left[(\widehat{E_t p^-})_{i+1/2} - (\widehat{E_t p^+})_{i-1/2} \right] = \int_{I_i} \rho_t p \, dx \quad \forall p \in V_h^k,$$

where the definition for $\widehat{E_t}$ corresponds to that chosen for \widehat{E} but with (E, Φ) replaced by (E_t, Φ_t) . By setting $p = \Phi$ and replacing ρ_t by its definition (26), we have

$$(99) \quad \int_{I_i} E_t \Phi_x \, dx - \left[(\widehat{E_t \Phi^-})_{i+1/2} - (\widehat{E_t \Phi^+})_{i-1/2} \right] = \int_{I_i} \int_{\mathcal{J}} f_t \Phi \, dv \, dx \quad \forall p \in V_h^k.$$

Now, taking $z = E_t$ in (44) and integrating by parts the volume term on the right hand side of that equation, we find

$$\int_{I_i} E E_t \, dx = \int_{I_i} \Phi_x E_t \, dx - [(\Phi E_t)_{i+1/2}^- - (\Phi E_t)_{i-1/2}^+] + [(\widehat{\Phi}(E_t)^-)_{i+1/2} - (\widehat{\Phi}(E_t)^+)_{i-1/2}].$$

Then, combining (99) with the above equation, summing over i , and using the periodic boundary conditions for the Poisson problem, we get

$$(100) \quad \sum_i \int_{I_i} E E_t \, dx = \sum_i \int_{I_i} \int_{\mathcal{J}} f_t \Phi \, dv \, dx + \sum_i \left[[\Phi E_t] - (\widehat{\Phi}[E_t] + \widehat{E_t}[\Phi]) \right]_{i-1/2}.$$

Third step:

We now proceed as in the proof for the continuous case, for instance see [34], and we take $\varphi = \frac{v^2}{2}$ in (23)-(24),

$$\begin{aligned} & \sum_{i,j} \left(\int_{T_{ij}} f_t \frac{v^2}{2} \, dv \, dx - \int_{T_{ij}} v f \left(\frac{v^2}{2} \right)_x \, dv \, dx + \int_{J_j} \frac{v^2}{2} \left[(\widehat{vf})_{i+1/2,v} - (\widehat{vf})_{i-1/2,v} \right] dv \right) \\ & + \sum_{i,j} \left(\int_{T_{ij}} E f v \, dv \, dx - \int_{I_i} \frac{v^2}{2} \left[(\widehat{E_h^i f})_{x,j+1/2} - (\widehat{E_h^i f})_{x,j-1/2} \right] dx \right) = 0. \end{aligned}$$

Then, using the consistency of the numerical fluxes (\widehat{vf}) and $(\widehat{E_h^i f})$, the boundary terms telescope and no boundary term is left due to the periodic in x and zero

in v boundary conditions. Hence, we simply get

$$(101) \quad \sum_{i,j} \left(\int_{T_{ij}} f_t \frac{v^2}{2} dv dx + \int_{T_{ij}} E f v dv dx \right) = 0.$$

Next, we use equation (98) to substitute the last term in (101),

$$0 = \sum_{i,j} \left(\int_{T_{ij}} f_t \frac{v^2}{2} dv dx + \int_{T_{ij}} f_t \Phi dv dx - \int_{J_j} \left(\widehat{v} f [[\Phi]] - v [[\Phi f]] + v \widehat{\Phi} [[f]] \right)_{i-1/2,v} dv \right).$$

Finally, we substitute the second volume term above by means of (100),

$$(102) \quad \begin{aligned} 0 = & \sum_{i,j} \int_{T_{ij}} f_t \frac{v^2}{2} dv dx + \sum_i \int_{I_i} E E_t dx - \sum_i \left[[[\Phi E_t]] - (\widehat{\Phi} [[E_t]] + \widehat{E_t} [[\Phi]]) \right]_{i-1/2} \\ & - \sum_{i,j} \int_{J_j} \left(\widehat{v} f [[\Phi]] - v [[\Phi f]] + v \widehat{\Phi} [[f]] \right)_{i-1/2,v} dv. \end{aligned}$$

We next define for all i ,

$$(103) \quad \begin{aligned} \Theta_{i-1/2}^H &= \widehat{\Phi} [[E_t]] - [[\Phi E_t]] + \widehat{E_t} [[\Phi]], \\ \Theta_{i-1/2,v}^F &= -\widehat{v} f [[\Phi]] + v [[\Phi f]] - v \widehat{\Phi} [[f]], \end{aligned}$$

so that (102) can be rewritten as

$$(104) \quad \sum_{i,j} \int_{T_{ij}} f_t \frac{v^2}{2} dv dx + \sum_i \int_{I_i} E E_t dx + \sum_i \Theta_{i-1/2}^H + \sum_{i,j} \int_{J_j} \Theta_{i-1/2,v}^F dv = 0.$$

Thus, we only need to show that $\Theta_{i-1/2}^H$ and $\Theta_{i-1/2,v}^F$ are, for all i , either zero or the time derivative of a non-negative function. From the definition of the numerical fluxes (94), and using that

$$(105) \quad [[ab]] = a^+ b^+ - a^- b^- = \{a\} [[b]] + [[a]] \{b\}, \quad \forall a, b \in V_h^k,$$

we find

$$\Theta_{i-1/2}^H = \{E_t\} [[\Phi]] + \{\Phi\} [[E_t]] + c_{11} [[\Phi_t]] [[\Phi]] - [[\Phi E_t]] = c_{11} [[\Phi_t]] [[\Phi]].$$

Therefore since (E, Φ) is \mathcal{C}^1 in time,

$$(106) \quad \Theta_{i-1/2}^H = c_{11} [[\Phi_t]] [[\Phi]] = \frac{1}{2} \frac{d}{dt} (c_{11} [[\Phi]]^2).$$

Similarly from (25) and (105), we get

$$\begin{aligned} \Theta_{i-1/2,v}^F &= -v \{f\} [[\Phi]] - v \{\Phi\} [[f]] + \frac{|v|}{2} [[f]] [[\Phi]] + v \cdot c_{12} [[f]] [[\Phi]] + v [[\Phi f]] \\ &= \frac{|v|}{2} [[f]] [[\Phi]] + v \cdot c_{12} [[f]] [[\Phi]]. \end{aligned}$$

Now, recalling that $c_{12} = -\text{sign}(v)/2$ and noting that $v \cdot \text{sign}(v) = |v|$, we also have that $\Theta_{i-1/2,v}^F = 0$ for all i and so substituting the above result together with (106) into (104) we reach (95). \square

5.1. Energy inequalities. The energy conservation property given in last Theorem heavily relies on the choice of the approximation for the Poisson problem and more precisely on the definition of the numerical fluxes, which somehow accounts for the coupling of the transport equation with the Poisson problem. Nevertheless, for full DG approximation of the Vlasov-Poisson system with other choices of numerical fluxes as given in Section 3.3, we can prove some energy inequality measuring the error in energy committed in terms of h at time t . For all $t \in [0, T]$, we define the discrete energy as

$$(107) \quad \mathcal{E}_h(t) := \sum_{i,j} \int_{T_{ij}} f_h(t) v^2 dv dx + \sum_i \int_{I_i} |E_h(t)|^2 dx + \sum_i (c_{11} [\Phi^h(t)]_{i+1/2}^2 + c_{22} [E_h(t)]_{i+1/2}^2).$$

We next state two results: the former, Proposition 5.3, requires smoothness of the solution; the latter, Proposition 5.4, establishes a decay of order $O(h)$ for the energy, provided $h < L$, without any further regularity assumption on the solution. The proof of both Propositions can be found in Appendix C.

Proposition 5.3. *Let $m \geq k \geq 2$ and consider the unique compactly supported solution of the Vlasov-Poisson system (1)-(2) given by Theorem 2.1 with $f \in \mathcal{C}^1([0, T]; H^{k+1}(\Omega))$ and $E \in \mathcal{C}^0([0, T]; H^{m+1}(\mathcal{I}))$. Let the DG-DG approximation of the Vlasov-Poisson problem (1)-(2) be $((E_h, \Phi_h), f_h) \in \mathcal{C}^1([0, T]; (V_h^k \times V_h^k) \times \mathcal{Z}_h^k)$, solution of (23), (24), (44), and (45), with the numerical fluxes (25) for the approximate density and (46) for the DG approximation of the Poisson problem. Then,*

$$|\mathcal{E}_h(t) - \mathcal{E}_h(0)| \leq h^{2\min(k+1, m)} \mathfrak{R}_0 + h^{\min(2k+1, 2m)} (c_{22} + c_{11}^{-1}) \mathfrak{R}_1,$$

where $m = k$ for any LDG (50) and the general DG (46); and $m = k + 1$ for the Hybridized LDG method (iii). The constants \mathfrak{R}_0 and \mathfrak{R}_1 depend on

$$\mathfrak{R}_0 = \mathfrak{R}_0 \left(\int_0^t (\|E_h(s)\|_{m+1, \mathcal{I}} + \|\Phi_h(s)\|_{m+2, \mathcal{I}})^2 ds, C_5 \right),$$

$$\mathfrak{R}_1 = \mathfrak{R}_1 \left(L \int_0^t \|f(s)\|_{k+1, \Omega}^2 ds, C_5 \right).$$

Proposition 5.4. *Let $m \geq k \geq 2$ and consider the unique compactly supported solution of the Vlasov-Poisson system (1)-(2) given by Theorem 2.1 with $f \in \mathcal{C}^1([0, T]; H^{k+1}(\Omega))$ and $E \in \mathcal{C}^0([0, T]; H^{m+1}(\mathcal{I}))$. Let the LDG-DG approximation of the Vlasov-Poisson problem (1)-(2) be $((E_h, \Phi_h), f_h) \in \mathcal{C}^1([0, T]; (V_h^k \times V_h^k) \times \mathcal{Z}_h^k)$, solution of (23), (24), (44), and (45), with the numerical fluxes*

(25) for the approximate density and (46), with $c_{11} = ch^{-1}$ and $c_{22} = 0$ for the LDG approximation of the Poisson problem. Then, for $h < 1/L$,

$$|\mathcal{E}_h(t) - \mathcal{E}_h(0)| \leq chL \left[\sum_i c_{11} \|\Phi^h(t)\|_{i+1/2}^2 + chLtF_0 \right],$$

where F_0 is defined as

$$F_0 := \|[\mathcal{P}_h(f_0)]^{1/2}v\|_{0,\mathcal{I}_h}^2 + \|\mathcal{P}_h(f_0)\|_{0,\mathcal{I}_h}^2 + \|E_0^i\|_{0,\mathcal{I}_h}^2 + \|c_{11}^{1/2}[\Phi_0]\|_{0,\Gamma_x}^2.$$

APPENDIX A. PROOFS OF THE ERROR ESTIMATES OF THE ELECTROSTATIC FIELD

In this appendix we provide the proofs of all the Lemmas stated in section 3 related to the consistency error in the approximation to the Electrostatic field.

A.1. Conforming approximation to the Electrostatic Potential.

Proof of Lemma 3.2. We first show (37). Since both E and E_h have zero average over \mathcal{I} , we deduce $\|E - E_h\|_{L_0^2(\mathcal{I})} = \|E - E_h\|_{0,\mathcal{I}}$. From the definitions (9) and (35) of the electric field E and its approximation E_h , we find for all $x \in I_i$

$$|E(x) - E_h^i(x)|^2 \leq 2|E(x_{1/2}) - E_h^0(x_{1/2})|^2 + 2 \left| \int_{x_{1/2}}^x [\rho_h(s) - \rho(s)] ds \right|^2 = 2(T_0 + T_1).$$

The term T_0 can be readily estimated from (8) and (33) and Hölder inequality

$$T_0 = |E(x_{1/2}) - E_h^0(x_{1/2})|^2 = |C_E - C_E^h|^2 = \left| \int_0^1 \int_0^z [\rho_h(s) - \rho(s)] ds dz \right|^2 \leq \|\rho_h - \rho\|_{0,\mathcal{I}}^2.$$

Holder's inequality yields $T_1 \leq \|\rho_h - \rho\|_{0,\mathcal{I}}^2$. Hence, integration over I_i and summation over i and Cauchy-Schwarz inequality, gives $\|E - E_h\|_{0,\mathcal{I}}^2 \leq 4\|\rho_h - \rho\|_{0,\mathcal{I}}^2$, and so by using (27), estimate (37) follows.

To prove (38), from the conformity of the approximation ($E_h \in W_h^{k+1}$), Sobolev imbeddings together with triangle inequality, we find

$$\|E - E_h\|_{0,\infty,\mathcal{I}} \leq \|E - E_h\|_{1,\mathcal{I}} \leq \sqrt{2}(\|E_h - E\|_{0,\mathcal{I}} + |E_h - E|_{1,\mathcal{I}}).$$

The first term above has been already estimated. For the second, note that

$$\frac{\partial}{\partial x}[E - E_h] = \rho_h(x, t) - \rho(x, t), \quad \forall x \in (x_{i-1/2}, x_{i+1/2}) \quad \forall i,$$

and so,

$$|E_h - E|_{1,\mathcal{I}}^2 = \sum_i \int_{I_i} \left| \frac{\partial}{\partial x}[E - E_h] \right|^2 dx = \sum_i \int_{I_i} |\rho_h(x, t) - \rho(x, t)|^2 dx = \|\rho - \rho_h\|_{0,\mathcal{I}_h}^2.$$

Hence, from (27) and (37) and substituting above we reach (39). The proof for the uniform estimate (39) follows immediately.

A.2. Mixed Finite Element Approximation for the Poisson Problem.

Proof of Lemma 3.3. The proof of estimate (43) would follow from the a-priori estimate for linear problems together with an “application” of a version of Strang’s Lemma for mixed methods. We briefly sketch it for the sake of completeness. In one dimension, we only need to show (43) due to Sobolev’s imbedding $H^1(\mathcal{I}) \subset L^\infty(\mathcal{I})$. Using (41)-(42), we get

$$(108) \quad - \int_I (E - E_h)z \, dx + \int_I (\Phi - \Phi_h)z_x \, dx = 0 \quad \forall z \in W_h^{k+1},$$

$$(109) \quad \int_I (E - E_h)_x p \, dx = \int_I (\rho - \rho_h)p \, dx \quad \forall p \in V_h^k.$$

being (E, Φ) the continuous solution to the Poisson problem. The term on the right hand side of equation (109) is the consistency error. Next, let $\mathcal{R}_h: H^1(\mathcal{I}) \rightarrow W_h^{k+1}$ be the projection operator defined by:

$$\begin{cases} \int_{I_i} (z - \mathcal{R}_h(z))q \, dx = 0, & \forall q \in \mathbb{P}^{k-1}(I_i), \\ \mathcal{R}_h(z)(x_{i-1/2}) = z(x_{i-1/2}), & \mathcal{R}_h(z)(x_{i+1/2}) = z(x_{i+1/2}), \end{cases} \quad \forall i.$$

For $k = 0$ the definition of \mathcal{R}_h reduces to that of the standard conforming interpolant. It is easy to verify that \mathcal{R}_h corresponds to the one-dimensional Raviart-Thomas projection. In particular it satisfies the approximation property

$$(110) \quad \|z - \mathcal{R}_h(z)\|_{0,\mathcal{I}} \leq Ch^{k+1}\|z\|_{k+1,\mathcal{I}} \quad \forall z \in H^{k+1}(\mathcal{I}).$$

From the definition of \mathcal{R}_h it is straightforward to verify that

$$(111) \quad \int_{\mathcal{I}} (z - \mathcal{R}_h(z))_x p \, dx = 0, \quad \forall z \in H^1(\mathcal{I}), \quad \forall p \in V_h^k,$$

which express the commuting property of the projection $\mathcal{R}_h: \frac{d}{dx}(\mathcal{R}_h(z)) = P^k(z_x)$, for all $z \in W_h^{k+1}$, P^k being the L^2 -standard projection. Combining (111) with $z = E$ and equation (109), this last equation becomes

$$(112) \quad \int_I [\mathcal{R}_h(E) - E_h]_x p \, dx = \int_I (\rho - \rho_h)p \, dx \quad \forall p \in V_h^k,$$

and so, by setting in $p = (\mathcal{R}_h(E) - E_h)_x \in \frac{d}{dx}W_h^{k+1} = V_h^k$, we have

$$(113) \quad \int_I |(\mathcal{R}_h(E) - E_h)_x|^2 \, dx = \int_I (\rho - \rho_h)[\mathcal{R}_h(E) - E_h]_x \, dx.$$

Hence, denoting by $\eta_E = \mathcal{R}_h(E) - E_h$, Cauchy Schwarz gives

$$(114) \quad |\eta_E|_{1,\mathcal{I}} = |\mathcal{R}_h(E) - E_h|_{1,\mathcal{I}} \leq C\|\rho - \rho_h\|_{0,\mathcal{I}}.$$

We next get the L^2 -error estimate. We take $z = \eta_E$ in (108) and decompose $\Phi - \Phi_h = [\Phi - P^k(\Phi)] + [P^k(\Phi) - \Phi_h]$ and $E - E_h = [E - \mathcal{R}_h(E)] + [\mathcal{R}_h(E) - E_h]$.

Then, from the definition of the standard L^2 -projection P^k , we find

$$\|\eta_E\|_{0,\mathcal{I}}^2 = \int_I |\mathcal{R}_h(E) - E_h|^2 dx = - \int_I [E - \mathcal{R}_h(E)]\eta_E dx + \int_I [P^k(\Phi) - \Phi_h][\eta_E]_x dx .$$

Note that from (108) and the definition of the L^2 -projection, we have

$$\int_I (\Phi_h - P^k(\Phi))z_x dx = - \int_I (E - E_h)z dx \quad \forall z \in W_h^{k+1} ,$$

and thus, we can apply this to $z = \eta_E$. By setting $p = (P^k(\Phi) - \Phi_h)$ in (112) and substituting the result above we get,

$$(115) \quad \int_I |\eta_E|^2 dx = \int_I [\mathcal{R}_h(E) - E]\eta_E dx + \int_I (\rho_h - \rho)(P^k(\Phi) - \Phi_h) dx .$$

Then, summing (113) to the above equation and using Cauchy-Schwarz together with the interpolation estimate (110), we find

$$\begin{aligned} \|\eta_E\|_{0,\mathcal{I}}^2 + |\eta_E|_{1,\mathcal{I}}^2 &\leq \|E - \mathcal{R}_h(E)\|_{0,\mathcal{I}}\|\eta_E\|_{0,\mathcal{I}} + \|\rho - \rho_h\|_{0,\mathcal{I}}(|\eta_E|_{1,\mathcal{I}} + \|P^k(\Phi) - \Phi_h\|_{0,\mathcal{I}}) \\ &\leq Ch^{k+1}\|E\|_{k+1,\mathcal{I}}\|\eta_E\|_{0,\mathcal{I}} + \|\rho - \rho_h\|_{0,\mathcal{I}}(|\eta_E|_{1,\mathcal{I}} + \|P^k(\Phi) - \Phi_h\|_{0,\mathcal{I}}). \end{aligned}$$

To conclude we need a bound for $\|P^k(\Phi) - \Phi_h\|_{0,\mathcal{I}}$. Now, taking $z \in W_h^{k+1}$ such that $z_x = \Phi_h - P^k(\Phi)$ we obtain

$$\|P^k(\Phi) - \Phi_h\|_{0,\mathcal{I}}^2 \leq C\|E - E_h\|_{0,\mathcal{I}}\|P^k(\Phi) - \Phi_h\|_{0,\mathcal{I}}$$

where in the last step we have used Poincarè's inequality ($\|z\|_{0,\mathcal{I}} \leq C\|z_x\|_{0,\mathcal{I}}$). Hence, plugging it into the previous estimate, we get

$$\begin{aligned} \|\eta_E\|_{0,\mathcal{I}}^2 + |\eta_E|_{1,\mathcal{I}}^2 &\leq (Ch^{k+1}\|E\|_{k+1,\mathcal{I}} + \|\rho - \rho_h\|_{0,\mathcal{I}})\|\eta_E\|_{0,\mathcal{I}} \\ &\quad + \|\rho - \rho_h\|_{0,\mathcal{I}}(Ch^{k+1}\|E\|_{k+1,\mathcal{I}} + |\eta_E|_{1,\mathcal{I}}) \\ &\leq Ch^{2k+2}\|E\|_{k+1,\mathcal{I}}^2 + C'\|\rho - \rho_h\|_{0,\mathcal{I}}^2 + \frac{1}{4}(\|\eta_E\|_{0,\mathcal{I}}^2 + |\eta_E|_{1,\mathcal{I}}^2) \\ &\quad + C\|\rho - \rho_h\|_{0,\mathcal{I}}h^{k+1}\|E\|_{k+1,\mathcal{I}} , \end{aligned}$$

from which by a ‘‘kick-back’’ argument we get,

$$\|\eta_E\|_{0,\mathcal{I}}^2 + |\eta_E|_{1,\mathcal{I}}^2 \leq \frac{4}{3}(Ch^{k+1}\|E\|_{k+1,\mathcal{I}} + C'\|\rho - \rho_h\|_{0,\mathcal{I}})^2 ,$$

that together with the interpolation estimate (110) and estimate (27) yields (43).

A.3. DG approximation for the Poisson problem.

Proof of Lemma 3.4. The result follows by adapting the proofs in [18, 23] and [19, 22] for the cases **(i)**; **(ii)**; **(iii)**, respectively, so that they account for the consistency error. Notice also that for **(i)**,**(ii)** estimate (52) follows from (3.4). Hence, for the sake of completeness, we sketch the proof of this last estimate in

some detail for the LDG method **(i)** and the general DG **(iii)**. Using (47) and that (E, f) is the continuous solution, we get the following error equation:

$$(116) \quad \mathcal{A}((E - E_h, \Phi - \Phi_h); (z, p)) = \sum_i \int_{I_i} (\rho - \rho_h) p \, dx, \quad \forall (z, p) \in V_h^r \times V_h^r.$$

The term on the right hand side is the consistency error, which is the only novelty in the proof w.r.t those in the above-mentioned works. We decompose $E - E_h = \eta^h - \eta^e$ where $\eta^e = P^r(E) - E$ and $\eta^h = P^r(E) - E_h$, and analogously $\Phi - \Phi_h = \xi^h - \xi^e$ where $\xi^e = P^r(\Phi) - \Phi$ and $\xi^h = P^r(\Phi) - \Phi_h$. Then, [18, Lemma 3.3] gives

$$(117) \quad \begin{aligned} |(E - E_h, \Phi - \Phi_h)|_{\mathcal{A}} &\leq |(\eta^e, \xi^e)|_{\mathcal{A}} + |(\eta^h, \xi^h)|_{\mathcal{A}} \\ &\leq K_{a0} h^{r+1/2} \| (E, \Phi) \|_{r, \mathcal{I}} + |(\eta^h, \xi^h)|_{\mathcal{A}}, \end{aligned}$$

where $\| \cdot \|_{\mathcal{A}}$ is the semi-norm defined in (48) and $K_{a0}^2 \approx C(h + c_{22} + c_{11})$. To estimate the second term by setting $(z, p) = (\eta^h, \xi^h)$ in the error equation (116) and using the definition of $\mathcal{A}(\cdot, \cdot)$, that of the semi-norm (48) and the approximation properties of the standard L^2 -projection (19), we find

$$(118) \quad \begin{aligned} |(\eta^h, \xi^h)|_{\mathcal{A}}^2 &= \mathcal{A}((\eta^h, \xi^h), (\eta^h, \xi^h)) \leq \|\rho - \rho_h\|_{0, \mathcal{I}_h} \|\xi^h\|_{0, \mathcal{I}_h} + |\mathcal{A}((\eta^e, \xi^e), (\eta^h, \xi^h))| \\ &\leq \|\rho - \rho_h\|_{0, \mathcal{I}_h} (\|\xi^e\|_{0, \mathcal{I}_h} + \|\Phi - \Phi_h\|_{0, \mathcal{I}_h}) + |\mathcal{A}((-\eta^h, \xi^h), (-\eta^e, \xi^e))| \\ &\leq \|\rho - \rho_h\|_{0, \mathcal{I}_h} (Ch^{r+1} \|\Phi\|_{r+1, \mathcal{I}} + \|\Phi - \Phi_h\|_{0, \mathcal{I}_h}) \\ &\quad + |(\eta^h, \xi^h)|_{\mathcal{A}} C_{K_{b0}} h^{r+1/2} \| (E, \Phi) \|_{r, \mathcal{I}}, \end{aligned}$$

where in the last step we have used [18, Lemma 3.6] together with [18, assumption (2.21)] and $K_{b0}^2 \approx C(c_{11}^{-1} + c_{22} + c_{11})$. To conclude we need an estimate for $\|\Phi - \Phi_h\|_{0, \mathcal{I}_h}$ that will be obtained by duality. Let $u \in H^2(\mathcal{I})$ be the solution of the dual problem, $-u_{xx} = \Phi - \Phi_h$ in \mathcal{I} with $u(0) = u(1) = 0$, and let $q = u_x$. Then, it is easy to verify

$$(119) \quad \mathcal{A}((q, u); (z, p)) = (\Phi - \Phi_h, p), \quad \forall (z, p) \in H^1(\mathcal{I}_h) \times H^1(\mathcal{I}_h).$$

Thus by setting $(z, p) = (E_h - E, \Phi - \Phi_h)$ in the above equation, using the definition of $\mathcal{A}(\cdot, \cdot)$ together with (116), the H^1 -stability of the standard L^2 -projection [13] and denoting by $\theta_q := q = P^r(q)$ and $\theta_u := u - P^r(u)$ we get

$$\begin{aligned} \|\Phi - \Phi_h\|_{0, \mathcal{I}_h}^2 &= \mathcal{A}((-q, u); (E_h - E, \Phi - \Phi_h)) = \mathcal{A}((E - E_h, \Phi - \Phi_h); (q, u)) \\ &= \mathcal{A}((E - E_h, \Phi - \Phi_h); (q - P^r(q), u - P^r(u))) + \int_{\mathcal{I}} (\rho - \rho_h) P^r(u) \, dx \\ &\leq |\mathcal{A}((\eta^h, \xi^h); (\theta_q, \theta_u))| + |\mathcal{A}((\eta^e, \xi^e); (\theta_q, \theta_u))| + \|\rho - \rho_h\|_{-1, \mathcal{I}_h} |P^r(u)|_{1, \mathcal{I}_h} \\ &\leq Ch^{1/2} \| (q, u) \|_{0, \mathcal{I}} + K_{b1} (|(\eta^h, \xi^h)|_{\mathcal{A}} + K_{a1} h^{r+1/2} \| (E, \Phi) \|_{r, \mathcal{I}}) \\ &\quad + C \|\rho - \rho_h\|_{-1, \mathcal{I}_h} |u|_{1, \mathcal{I}}. \end{aligned}$$

where the first two terms have been estimated by using Lemmas 3.6 and 3.3 from [18], respectively, and the constants are defined by:

$$K_{b1}^2 \approx C(c_{11}^{-1} + c_{22} + h^2 c_{11}), \quad K_{a1}^2 \approx C(c_{22} + h + c_{11} h^2)(1 + h + c_{22} + c_{11})$$

Appealing now to the a-priori estimates for the dual problem (119)

$$\|u\|_{m+2, \mathcal{I}} + \|q\|_{m+1, \mathcal{I}} \leq C \|\Phi - \Phi_h\|_{m, \mathcal{I}} \quad m = -1, 0,$$

together with the inclusion $L^2(\mathcal{I}) \subset H^{-1}(\mathcal{I})$, we finally get

$$\|\Phi - \Phi_h\|_{0, \mathcal{I}_h} \leq Ch^{1/2} [K_{b1} |(\eta^h, \xi^h)|_{\mathcal{A}} + h^{r+1/2} K_{a1} \| |(E, \Phi) \|_{r, \mathcal{I}}] + \|\rho - \rho_h\|_{0, \mathcal{I}_h}.$$

Substituting the above estimate in (118) and using the Young's inequality,

$$\begin{aligned} |(\eta^h, \xi^h)|_{\mathcal{A}}^2 &\leq \|\rho - \rho_h\|_{0, \mathcal{I}_h}^2 (1 + 4K_{b1}^2 h) + \frac{1}{2} |(\eta^h, \xi^h)|_{\mathcal{A}}^2 + Ch^{2r+2} \|\Phi\|_{r+1, \mathcal{I}}^2 \\ &\quad + Ch^{2r+1} \| |(E, \Phi) \|_{r, \mathcal{I}}^2 (K_{b0}^2 + K_{a1}^2 h), \end{aligned}$$

and so by a ‘‘kick-back argument’’ and taking square roots we get

$$\frac{1}{2} |(\eta^h, \xi^h)|_{\mathcal{A}} \leq C \|\rho - \rho_h\|_{0, \mathcal{I}_h} + (K_{b0}^2 + K_{a1}^2 h)^{1/2} h^{r+1/2} \| |(E, \Phi) \|_{r, \mathcal{I}} + Ch^{r+1} \|\Phi\|_{r+1, \mathcal{I}}.$$

Substituting this estimate in (117), and taking into account the values of the parameters c_{11} and c_{22} selected, we reach (3.4) which in particular implies (52).

For the MD-LDG (ii) one adapts easily this proof taking into account the values for c_{11} and c_{22} and replaces the L^2 -projection by the special projection defined through (17)-(18). For the H-LDG (considered in (iii)), the easiest way to prove (52) is to introduce an auxiliary approximation, say (E_h^*, Φ_h^*) , to the continuous Poisson problem. The error estimates in the L^2 norm for $E - E_h$ are decomposed in two parts: the error $E - E_h^*$ estimated in [22] and the consistency error $E_h^* - E_h$ dealt with the ideas in this proof. We omit the details for the sake of conciseness.

APPENDIX B. PROOFS OF LEMMAS 4.5 AND 4.6

Proof of Lemma 4.5. We shall first estimate each term $\mathcal{K}_{i,j}^1(v, f, \omega^h)$ for fixed i, j and then sum over i, j . So let i, j be fixed and denote $T = T_{i,j}$, $I = I_i$ and $J = J_j$. The boundary of the element T consist of two vertical and two horizontal edges; $\partial T = J^{i-1/2} \cup J^{i+1/2} \cup I^{j-1/2} \cup I^{j+1/2}$ where we have denoted by $J^{i+1/2} := \{x_{i+1/2}\} \times J$ and $I^{j+1/2} := I \times \{v_{j+1/2}\}$. Notice that the definitions of both, the numerical fluxes (25) and the projection Π_h , depend on the sign of v . However, since $v = 0$ is a node of the partition, v as a function does not change sign inside any element $T_{i,j} \in \mathcal{T}_h$. Hence, denoting by $v_{\pm} = \max\{\pm v, 0\}$ the positive and negative parts of v , the term \mathcal{K}^1 can be rewritten as $\mathcal{K}^1(v, f, \omega^h) =$

$\mathcal{K}^{1,+}(v, f, \omega^h) - \mathcal{K}^{1,-}(v, f, \omega^h)$ with

$$\mathcal{K}^{1,\pm}(v, f, \omega^h) = \sum_{i,j} \mathcal{K}_{i,j}^1(v_{\pm}, f, \omega^h) .$$

We can reduce ourselves to show the result for the case of v_+ since in the case v_- is treated analogously. Since $v > 0$ on T , from the definition of the numerical fluxes (25), the definition of Π_h (65) and noting that $\Pi_h|_J = \tilde{\pi}_v$, this term reads

$$(120) \quad \begin{aligned} \mathcal{K}_{i,j}^{1,+}(v, f, \omega^h) &= \int_T v \omega^e(\omega^h)_x dx dv - \int_{J^{i+1/2}} v (f - \pi_v^- f)^-(\omega^h)^- dv \\ &+ \int_{J^{i-1/2}} v (f - \pi_v^- f)^-(\omega^h)^+ dv . \end{aligned}$$

Observe (120) is independent of the $\text{sign}(v)$. Let $\bar{v} := P^0(v)$ denote the local projection of v onto the constants on J . Then, summing and subtracting \bar{v} in $\mathcal{K}_{i,j}^{1,+}$, we have

$$\mathcal{K}_{i,j}^{1,+}(v, f, \omega^h) = \mathcal{K}_{i,j}^{1,+}(v - \bar{v}, f, \omega^h) + \mathcal{K}_{i,j}^{1,+}(\bar{v}, f, \omega^h) .$$

The last term is estimated exactly as in [18, Lemma 3.6] (see also [47]), giving

$$(121) \quad |\mathcal{K}_{i,j}^{1,+}(\bar{v}, f, \omega^h)| \leq Ch_T^{k+1} |v| \|f\|_{k+2, T_{i,j}} \|\omega^h\|_{0, T_{i,j}}$$

where we have also used the stability of the L^2 -projection (16). We wish to stress that the properties of the special projections Π_h and π_v^{\pm} are essential for the proof of the above estimate. We next estimate the remaining term in the expression for $\mathcal{K}^{1,+}$. From the definition in (120), using Hölder inequality, trace inequality [2] and inverse inequality [21] together with the error estimates (20) and (67), we find

$$\begin{aligned} |\mathcal{K}_{i,j}^{1,+}(v - \bar{v}, f, \omega^h)| &\leq \left| \int_T (v - \bar{v}) \int_I \omega^e \omega_x^h dv dx \right| + \left| \int_{J^{i\pm 1/2}} [v - \bar{v}] (f - \pi_v^- f)^-(\omega^h)^{\mp} dv \right| \\ &\leq C \|v - \bar{v}\|_{0, \infty, J} \|\omega^e\|_{0, T} \|(\omega^h)_x\|_{0, T} + C \sum_{m=i\pm 1/2} \|v - \bar{v}\|_{0, \infty, J^m} \|\omega^e\|_{0, J^m} \|\omega^h\|_{0, J^m} \\ &\leq Ch_v h_T^{k+1} \|f\|_{k+1, T} h_v^{-1} \|\omega^h\|_{0, T} + Ch_v h_T^{k+1/2} \|f\|_{k+1, T} h_v^{-1/2} \|\omega^h\|_{0, T} \\ &\leq Ch^{k+1} \|f\|_{k+1, T} \|\omega^h\|_{0, T} . \end{aligned}$$

Then, using the above estimate together with (121) and summing over i and j we get

$$|\mathcal{K}^{1,+}(v, f, \omega^h)| \leq Ch^{k+1} \|\omega^h\|_{0, T} (\|f\|_{k+1, T} + L \|f\|_{k+2, T}) ,$$

giving the desired estimate (75).

Proof of Lemma 4.6. We follow the notation of the previous proof. We start by noting that we cannot directly argue as in the proof of Lemma 4.5 since now the

definition of the numerical fluxes depend on the sign of E_h while the definition of the projection depend on the sign of E . We first write

$$(122) \quad \mathcal{K}_{i,j}^2(E_h^i, f, \omega^h) = \mathcal{K}_{i,j}^{2a}(E_h^i, f, \omega^h) + \mathcal{K}_{i,j}^{2b}(E_h^i, f, \omega^h)$$

with

$$\begin{aligned} \mathcal{K}_{i,j}^{2a}(E_h, f, \omega^h) &= \int_{T_{i,j}} E_h \omega^e \omega_v^h dv dx \\ \mathcal{K}_{i,j}^{2b}(E_h, f, \omega^h) &= - \int_{I_i} \left[(\widehat{E_h \omega^e}(\omega^h)^-)_{x,j+1/2} - (\widehat{E_h \omega^e}(\omega^h)^+)_{x,j-1/2} \right] dx, \end{aligned}$$

and we shall consider a further splitting of each of the above expressions. For the first one, we set

$$(123) \quad \mathcal{K}_{i,j}^{2a}(E_h, f, \omega^h) = \mathcal{K}_{i,j}^{2a}(E_h - E, f, \omega^h) + \mathcal{K}_{i,j}^{2a}(E, f, \omega^h).$$

Then, Hölder inequality together with inverse inequality and estimate (67) give

$$(124) \quad \begin{aligned} |\mathcal{K}_{i,j}^{2a}(E_h - E, f, \omega^h)| &\leq \|E_h - E\|_{0,\infty,I_i} \|f - \Pi_h(f)\|_{0,T_{i,j}} \|(\omega^h)_v\|_{0,T_{i,j}} \\ &\leq Ch^{k+1} h_v^{-1} \|E_h - E\|_{0,\infty,I_i} \|f\|_{k+1,T_{i,j}} \|\omega^h\|_{0,T_{i,j}}. \end{aligned}$$

Now, we deal with the boundary term $\mathcal{K}_{i,j}^{2b}$ in (122). Since the definition of the numerical flux (25) on Γ_v depends on the sign of E_h at $(x, v_{j\pm 1/2})$,

$$(\widehat{E_h^i \omega^e})_{x,j-1/2} = (E_h^i(x))_+ [f - \Pi_h(f)]_{x,j-1/2}^+ - (E_h^i(x))_- [f - \Pi_h(f)]_{x,j-1/2}^-, \quad \forall x \in I_i$$

where $(E_h^i(x))_{\pm} = \max(\pm E_h^i(x), 0)$ denotes respectively, the positive and negative parts of $E_h^i(x)$. Hence, the above splitting induces a further decomposition of \mathcal{K}^{2b} :

$$\mathcal{K}_{i,j}^{2b}(E_h, f, \omega^h) = \mathcal{A}_{i,j}^+((E_h^i)_+, f, \omega^h) + \mathcal{A}_{i,j}^-((E_h^i)_-, f, \omega^h),$$

where \pm in \mathcal{A}^{\pm} refers to the side (from the left or from the right in the v -direction) from which the term $f - \Pi_h(f)$ is evaluated, that is:

$$\begin{aligned} \mathcal{A}_{i,j}^{\pm}((E_h^i)_{\pm}, f, \omega^h) &= - \int_{I_i \cap \{x: \pm E_h^i > 0\}} E_h^i \left\{ ([f - \Pi_h(f)]^{\pm}(\omega^h)^-)_{x,j+1/2} \right. \\ &\quad \left. - ([f - \Pi_h(f)]^{\pm}(\omega^h)^+)_{x,j-1/2} \right\} dx. \end{aligned}$$

Notice now that $\Pi_h(f)|_{I_j^{\pm 1/2}}$ is a projection on the x -direction, and so independent on v . Thus, this observation together the continuity of f implies that,

$$[f - \Pi_h(f)]_{x,j+1/2}^- = [f - \Pi_h(f)]_{I_j^+}^- = [f - \Pi_h(f)]_{I_j^+}^+ = [f - \Pi_h(f)]_{x,j+1/2}^+, \quad \forall x \in I_i, \quad \forall j.$$

Hence, $\mathcal{K}_{i,j}^{2b}(E_h, f, \omega^h)$ can be rewritten as

$$\mathcal{K}_{i,j}^{2b} = - \int_{I_i} E_h^i \left\{ ([f - \Pi_h(f)]^*(\omega^h)^-)_{x,j+1/2} - ([f - \Pi_h(f)]^*(\omega^h)^+)_{x,j-1/2} \right\} dx,$$

where $*$ can be taken either as $+$ or $-$, without changing the final result. Note that the above expression does not depend any more on the sign of E_h^i . Hence, adding and subtracting E^i to the above expressions, $\mathcal{K}_{i,j}^{2b}(E_h, f, \omega^h)$ becomes,

$$(125) \quad \mathcal{K}_{i,j}^{2b}(E_h, f, \omega^h) = \mathcal{K}_{i,j}^{2b}(E_h - E, f, \omega^h) + \mathcal{K}_{i,j}^{2b}(E, f, \omega^h).$$

The first term is easily bounded by using Hölder inequality together with trace and inverse inequalities and the approximation result (67),

$$(126) \quad \begin{aligned} |\mathcal{K}_{i,j}^{2b}(E_h - E, f, \omega^h)| &\leq \sum_{m=j\pm 1/2} \|E_h^i - E^i\|_{0,\infty,I^m} \|f - \Pi_h(f)\|_{0,I^m} \|\omega^h\|_{0,I^m} \\ &\leq Ch^k \|E_h^i - E^i\|_{0,\infty,I_i} \|f\|_{k+1,T_{i,j}} \|\omega^h\|_{0,T_{i,j}}. \end{aligned}$$

To estimate the last term in (125), recalling the splitting in (123), we define

$$(127) \quad \mathcal{K}^3(E^i, f, \omega^h) = \sum_{i,j} (\mathcal{K}_{i,j}^{2a}(E^i, f, \omega^h) + \mathcal{K}_{i,j}^{2b}(E, f, \omega^h)).$$

Observe now that $\mathcal{K}_{i,j}^3(E^i, f, \omega^h)$ is a term “similar” to \mathcal{K}^1 from (73), in the sense that the definition of the projection Π_h depends of the sign of E^i on each I_i . Therefore, we argue similarly as in Lemma 4.5 to rewrite the term \mathcal{K}^3 as

$$(128) \quad \mathcal{K}^3(E^i, f, \omega^h) = \sum_{i,j} \mathcal{K}_{i,j}^{3,+}(E^i, f, \omega^h) + \mathcal{K}_{i,j}^{3,-}(E^i, f, \omega^h) + \mathcal{K}_{i,j}^{3,0}(E^i, f, \omega^h)$$

where $\mathcal{K}_{i,j}^{3,\pm}(E^i, f, \omega^h)$ are the contributions coming from those elements where E^i is either positive or negative in the whole I_i and the term $\mathcal{K}_{i,j}^{3,0}(E^i, f, \omega^h)$ corresponds to the contribution of those elements where E restricted to I_i changes sign. Therefore, the estimates for $\mathcal{K}_{i,j}^{3,\pm}(E^i, f, \omega^h)$ are done similarly as for $\mathcal{K}^{1,\pm}(v, f, \omega^h)$, so we just sketch the procedure. Adding and subtracting $P^0(E)$ we have

$$\mathcal{K}_{i,j}^{3,\pm}(E^i, f, \omega^h) = \mathcal{K}_{i,j}^{3,\pm}(E^i - P^0(E^i), f, \omega^h) + \mathcal{K}_{i,j}^{3,\pm}(P^0(E^i), f, \omega^h).$$

The last term is bounded as in [18, Lemma 3.6]. As for estimate (121), the special properties of the projections Π_h and π_x^\pm are heavily used in this proof. Using the stability of the L^2 -projection (16),

$$(129) \quad |\mathcal{K}_{i,j}^{3,\pm}(P^0(E), f, \omega^h)| \leq Ch_T^{k+1} \|E^i\|_{0,\infty,I_i} \|f\|_{k+2,T_{i,j}} \|\omega^h\|_{0,T_{i,j}}.$$

To estimate the first term $\mathcal{K}_{i,j}^{3,+} := \mathcal{K}_{i,j}^{3,+}(E - P^0(E), f, \omega^h)$ notice that $\Pi_h|_{I_i} = \pi_x^+|_{I_i}$ and since f is continuous $[\pi_x^+(f)]^+ = [\pi_x^+(f)]^-$. Then, using the L^∞ -estimate for the L^2 -projection (20), Hölder inequality, trace and inverse inequalities together with the approximation estimate (67), we deduce

$$\begin{aligned}
 |\mathcal{K}_{i,j}^{3,+}| &\leq \left| \int_T [E_h - P^0(E)] \omega^e(\omega^h)_v dx dv \right| + \sum_{m=j\pm 1/2} \left| \int_{I^m} [E - P^0(E)] (f - \pi_x^+ f)(\omega^h)^\mp dx \right| \\
 &\leq C \|E - P^0(E)\|_{0,\infty,I_i} \left(\|\omega^e\|_{0,T_{i,j}} h_x^{-1} \|\omega^h\|_{0,T_{i,j}} + \sum_{m=j\pm 1/2} \|\omega^e\|_{0,I^m} h_x^{-1/2} \|\omega^h\|_{0,T_{i,j}} \right) \\
 (130) \quad &\leq Ch^{k+1} \|E\|_{1,\infty,I_i} \|f\|_{k+1,T_{i,j}} \|\omega^h\|_{0,T_{i,j}} .
 \end{aligned}$$

We finally estimate the term $\mathcal{K}_{i,j}^{3,0} := \mathcal{K}_{i,j}^{3,0}(E, f, \omega^h)$. Note that now $\Pi_h|_{I_i} = P^k|_{I_i}$. Then, Hölder inequality, estimates (67) together with inverse and trace inequalities gives

$$\begin{aligned}
 |\mathcal{K}_{i,j}^{3,0}| &\leq \|E^i\|_{0,\infty,I_i} \left(\|f - \Pi_h(f)\|_{0,T_{i,j}} \|(\omega^h)_v\|_{0,T_{i,j}} + \|f - P^k(f)\|_{0,I^m} \|\omega^h\|_{0,I^m} \right) \\
 (131) \quad &\leq C \|E^i\|_{0,\infty,I_i} h^k \|f\|_{k+1,T_{i,j}} \|\omega^h\|_{0,T_{i,j}} .
 \end{aligned}$$

Thus, to conclude we only need to provide an estimate for $\|E^i\|_{0,\infty,I_i}$. Note that since E changes sign inside I_i there exists some $x^* \in I_i$ such that $E(x^*) = 0$. Using mean value theorem together with the regularity of E we have

$$(132) \quad \|E^i\|_{0,\infty,I_i} = \sup_{x \in I_i} |E(x) - E(x^*)| = \sup_{x \in I_i} \left| \int_{x^*}^x E_x(s) ds \right| \leq Ch_x |E|_{1,\infty,I_i} .$$

Substituting it into the bound for $\mathcal{K}_{i,j}^{3,0}$ and summing over elements, we finally get

$$|\mathcal{K}^{3,0}(E, f, \omega^h)| \leq Ch^{k+1} |E|_{1,\infty,\mathcal{I}} \|f\|_{k+1,\Omega} \|\omega^h\|_{0,\mathcal{T}_h} ,$$

Then, summing over all the elements of the partition estimates (124), (126), (129) and (130) concludes the proof of the Lemma.

APPENDIX C. PROOFS OF THE ENERGY INEQUALITIES

Proof of Proposition 5.3. The first part of the proof follows exactly the same steps as the proof of Proposition 5.1, till one reaches equation (104), which we can write as

$$(133) \quad \frac{1}{2} \frac{d}{dt} \left(\sum_{i,j} \int_{T_{i,j}} f_h v^2 dv dx + \sum_i \int_{I_i} (E_h)^2 dx \right) + \sum_i \Theta_{i-1/2}^H + \sum_{i,j} \int_{J_j} \Theta_{i-1/2,v}^F dv = 0,$$

with $\Theta_{i-1/2}^H$ and $\Theta_{i-1/2,v}^F$ as defined in (103):

$$\begin{aligned}
 \Theta_{i-1/2}^H &= \widehat{\Phi}[(E_h)_t] - [\Phi_h(E_h)_t] + \widehat{E}_t[\Phi_h] , \\
 \Theta_{i-1/2,v}^F &= -\widehat{v}f[\Phi_h] + v[\Phi_h f_h] - v\widehat{\Phi}[f_h] .
 \end{aligned}$$

Then, using (105) and the definition of the numerical fluxes (46), we get for $\Theta_{i-1/2}^H$

$$\begin{aligned}\Theta_{i-1/2}^H &= \{(E_h)_t\}[\Phi_h] + \{\Phi_h\}[(E_h)_t] + c_{11}[(\Phi_h)_t][\Phi_h] \\ &\quad + c_{22}[E_h][(E_h)_t] - [\Phi_h(E_h)_t] = c_{11}[(\Phi_h)_t][\Phi_h] + c_{22}[E_h][(E_h)_t] \\ &= \frac{1}{2} \frac{d}{dt} (c_{11}[\Phi_h]^2 + c_{22}[E_h]^2) ,\end{aligned}$$

where in last step we have used that (E_h, Φ_h) is \mathcal{C}^1 in time. Arguing similarly, and one easily gets for $\Theta_{i-1/2,v}^F$;

$$\begin{aligned}\Theta_{i-1/2,v}^F &= -v\{f_h\}[\Phi_h] + \frac{|v|}{2} [f_h][\Phi_h] - v\{\Phi_h\}[f_h] + vc_{12}[f_h][\Phi_h] \\ &\quad - vc_{22}[f_h][E_h] + v[\Phi_h f_h] \\ &= \left(\frac{|v|}{2} + vc_{12} \right) [f_h][\Phi_h] - vc_{22}[f_h][E_h] = \tilde{v}[f_h][\Phi_h] - vc_{22}[f_h][E_h],\end{aligned}$$

where we have denoted by $\tilde{v} = (|v|/2 + vc_{12})$. Then, substituting into (133) we have

$$\begin{aligned}&\frac{1}{2} \frac{d}{dt} \sum_{i,j} \left(\int_{T_{i,j}} f_h \frac{v^2}{2} dv dx + \int_{I_i} (E_h)^2 dx + c_{22}[E_h]_{i-1/2}^2 + c_{11}[\Phi_h]_{i-1/2}^2 \right) = \\ &= \sum_{i,j} c_{22}[E_h]_{i-1/2} \int_{J_j} v[f_h]_{i-1/2,v} dv - \sum_{i,j} [\Phi_h]_{i-1/2} \int_{J_j} \tilde{v}[f_h]_{i-1/2,v} dv ,\end{aligned}$$

and therefore integrating in time from 0 up to time t both sides, taking the absolute value and using triangle inequality, we get

$$\begin{aligned}&\frac{1}{2} \left| \int_0^t \frac{d}{dt} \sum_{i,j} \left(\int_{T_{i,j}} f_h \frac{v^2}{2} dv dx + \int_{I_i} (E_h)^2 dx + c_{22}[E_h]_{i-1/2}^2 + c_{11}[\Phi_h]_{i-1/2}^2 \right) ds \right| \\ &\leq \left| \int_0^t \sum_{i,j} c_{22}[E_h]_{i-1/2} \int_{J_j} v[f_h]_{i-1/2,v} dv ds \right| \\ (134) \quad &+ \left| \int_0^t \sum_{i,j} [\Phi_h]_{i-1/2} \int_{J_j} \tilde{v}[f_h]_{i-1/2,v} dv ds \right| .\end{aligned}$$

We next bound the last two terms. For the first term, from the arithmetic-geometric inequality we get

$$\begin{aligned}2 \left| \int_0^t \sum_{i,j} c_{22}[E_h]_{i-1/2} \int_{J_j} v[f_h]_{i-1/2,v} dv ds \right| &\leq \\ &\leq L \int_0^t \left\| c_{22}^{1/2}[E_h(s)] \right\|_{0,\gamma_x}^2 ds + \int_0^t c_{22} \| |v|^{1/2}[f_h(s)] \|_{0,\Gamma_x}^2 ds .\end{aligned}$$

For the other term, using that c_{12} is bounded ($|c_{12}| \leq c$) we can simply use the bound $|\tilde{v}| \leq c|v|$. Then, from the arithmetic-geometric inequality, we have

$$\begin{aligned} & 2 \left| \sum_{i,j} \llbracket \Phi_h \rrbracket_{i-1/2} \int_{J_j} \tilde{v} \llbracket f_h \rrbracket_{i-1/2,v} dv \right| \leq \\ & \leq cL \int_0^t \left\| c_{11}^{1/2} \llbracket \Phi_h(s) \rrbracket \right\|_{0,\gamma_x}^2 ds + Cc_{11}^{-1} \int_0^t \left\| |v|^{1/2} \llbracket f_h(s) \rrbracket \right\|_{0,\Gamma_x}^2 ds. \end{aligned}$$

Therefore, substituting back into (134) and taking into account the definition (107) of the discrete energy we have

$$\begin{aligned} (135) \quad |\mathcal{E}_h(t) - \mathcal{E}_h(0)| & \leq L \int_0^t \left(\left\| c_{11}^{1/2} \llbracket \Phi_h(s) \rrbracket \right\|_{0,\gamma_x}^2 + \left\| c_{22}^{1/2} \llbracket E_h(s) \rrbracket \right\|_{0,\gamma_x}^2 \right) ds \\ & + C(c_{22} + c_{11}^{-1}) \int_0^t \left\| |v|^{1/2} \llbracket f_h(s) \rrbracket \right\|_{0,\Gamma_x}^2 ds. \end{aligned}$$

Now we observe that the first sum on the right hand side is part of the energy norm of the DG approximation (E_h, Φ_h) . Thus, from Corollary 4.15 and taking into account the regularity of the continuous solution we have

$$\begin{aligned} (136) \quad \int_0^t \sum_i (c_{11} \llbracket \Phi_h(s) \rrbracket_{i-1/2}^2 + c_{22} \llbracket E_h(s) \rrbracket_{i-1/2}^2) ds & \leq \\ & \leq h^{2\min(k+1,m)} \left(C_5 + \int_0^t \left\| (E_h(s), \Phi_h(s)) \right\|_{k,\mathcal{I}}^2 ds \right). \end{aligned}$$

We next bound the second term in (135). Observe that since $f \in \mathcal{C}^0(\Omega)$,

$$\llbracket f_h \rrbracket = \llbracket f_h - f \rrbracket = \llbracket f_h - \Pi_h(f) \rrbracket + \llbracket \Pi_h(f) - f \rrbracket.$$

Thus, in view of the definition of the norm (60) and remark 4.14 we have for the first term above

$$(137) \quad \int_0^t \left\| |v| \llbracket f_h(s) - \Pi_h(f(s)) \rrbracket \right\|_{\Gamma_x}^2 ds \leq \left\| f_h(t) - \Pi_h(f(t)) \right\|^2 \leq C_5^2 h^{2(\min(k+1,m))}.$$

For the other term, using the interpolation estimate (68) together with a trace inequality [2], we get

$$\int_0^t \left\| |v| \llbracket \Pi_h(f(s)) - f(s) \rrbracket \right\|_{\Gamma_x}^2 ds \leq CLh^{2k+1} \int_0^t \|f(s)\|_{k+1,\Omega}^2 ds.$$

Hence, this estimate together with (137) finally give

$$\int_0^t \left\| |v|^{1/2} \llbracket f_h(s) \rrbracket \right\|_{0,\Gamma_x}^2 ds \leq (C_5 h^{2(\min(k+1,m))} + CLh^{2k+1} \int_0^t \|f(s)\|_{k+1,\Omega}^2 ds)$$

and so by substituting the above estimate and estimate (136) into (135), the proof is complete.

Proof of Proposition 5.4. The first part of the proof follows exactly the same steps as the proof of Proposition 5.1, till one reaches equation (104),

$$(104) \quad \sum_{i,j} \int_{T_{i,j}} f_t \frac{v^2}{2} dv dx + \sum_i \int_{I_i} E E_t dx + \sum_i \Theta_{i-1/2}^H + \sum_{i,j} \int_{J_j} \Theta_{i-1/2,v}^F dv = 0,$$

with $\Theta_{i-1/2}^H$ and $\Theta_{i-1/2,v}^F$ as defined in (103):

$$\begin{aligned} \Theta_{i-1/2}^H &= \widehat{\Phi} [E_t] - [\Phi E_t] + \widehat{E}_t [\Phi], \\ \Theta_{i-1/2,v}^F &= -v \widehat{f} [\Phi] + v [\Phi f] - v \widehat{\Phi} [f]. \end{aligned}$$

Then, using the definition of the numerical fluxes and (105) it is easy to verify that while $\Theta_{i-1/2}^H$ is still given by (106), for $\Theta_{i-1/2,v}^F$ one gets a term for that might change sing

$$(138) \quad \Theta_{i-1/2,v}^F = -v \{f\} [\Phi] - v \{\Phi\} [f] + \frac{|v|}{2} [f] [\Phi] - \frac{v}{2} [f] [\Phi] + v [\Phi f] = -v_- [\Phi] [f].$$

Then, substituting the above result together with (106) into (104) we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{i,j} \left(\int_{T_{i,j}} f \frac{v^2}{2} dv dx + \int_{I_i} (E)^2 dx + c_{22} [E]_{i-1/2}^2 + c_{11} [\Phi]_{i-1/2}^2 \right) \\ - \sum_{i,j} [\Phi]_{i-1/2} \int_{J_j} v_- [f]_{i-1/2,v} dv = 0. \end{aligned}$$

Next, we add equation (59) (resulting from the L^2 -stability; Proposition 4.1) to the above equation, to get

$$(139) \quad \frac{1}{2} \frac{d}{dt} \left[\sum_{i,j} \left(\int_{T_{i,j}} f v^2 dv dx + \int_{T_{i,j}} f^2 dx dv \right) + \sum_i \int_{I_i} (E)^2 dx + \sum_i c_{11} [\Phi]_{i-1/2}^2 \right]$$

$$(140) \quad + \sum_{i,j} \int_{I_i} \frac{|E|}{2} [f]_{x,j-1/2}^2 dx + \sum_{i,j} \int_{J_j} \frac{|v|}{2} [f]_{i-1/2,v}^2 dv$$

$$(140) \quad - \sum_{i,j} [\Phi]_{i-1/2} \int_{J_j} v_- [f]_{i-1/2,v} dv = 0.$$

Then, from the obvious inequality $ab \geq -|ab|$ and the arithmetic-geometric inequality, we have

$$\sum_{i,j} [\Phi]_{i-1/2} \int_{J_j} v_+ [f]_{i-1/2,v} dv \geq -\frac{L}{2} \sum_i [\Phi]_{i-1/2}^2 - \frac{1}{2} \sum_{i,j} \int_{J_j} |v| [f]_{i-1/2,v}^2 dv$$

and so substituting back into (140) and neglecting the strictly non-negative terms, we find

$$\frac{d}{dt} \left[\sum_{i,j} \left(\int_{T_{i,j}} [fv^2 + f^2] dx dv + \int_{I_i} (E)^2 dx + \sum_i c_{11} [\Phi]_{i-1/2}^2 \right) \right] - L \sum_i [\Phi]_{i-1/2}^2 \leq 0.$$

or equivalently

$$(141) \quad \frac{1}{2} \frac{d}{dt} \sum_{i,j} \left(\int_{T_{i,j}} [fv^2 + f^2] dx dv + \int_{I_i} (E)^2 dx + c_{11} [\Phi]_{i-1/2}^2 \right) \leq \frac{L}{2} \sum_i [\Phi]_{i-1/2}^2 ,$$

Now, let us define

$$(142) \quad \begin{aligned} F_0 &:= \sum_{i,j} \left(\int_{T_{i,j}} [\mathcal{P}_h(f_0)v^2 + (\mathcal{P}_h(f_0))^2] dx dv + \int_{I_i} (E_0)^2 + c_{11} [\Phi_0]_{i-1/2}^2 \right) \\ &= \|[\mathcal{P}_h(f_0)]^{1/2}|v|\|_{0,\mathcal{T}_h}^2 + \|\mathcal{P}_h(f_0)\|_{0,\mathcal{T}_h}^2 + \|E_0\|_{0,\mathcal{T}_h}^2 + \|c_{11}^{1/2}[\Phi_0]\|_{0,\Gamma_x}^2. \end{aligned}$$

Then integration in time from time 0 up to time t in (141), yields to

$$\begin{aligned} &\sum_{i,j} \left(\int_{T_{i,j}} [f(t)v^2 + f^2(t)] dx dv + \int_{I_i} (E(t))^2 dx + c_{11} [\Phi(t)]_{i-1/2}^2 \right) \\ &\leq F_0 + \frac{L}{2} \int_0^t \sum_i [\Phi(z)]_{i-1/2}^2 dz, \end{aligned}$$

which in particular implies,

$$0 \leq \sum_i c_{11} [\Phi(t)]_{i-1/2}^2 \leq F_0 + \frac{Lc_{11}^{-1}}{2} \int_0^t \sum_i c_{11} [\Phi(z)]_{i-1/2}^2 dz ,$$

and therefore, standard application of Gronwall's inequality (see [42]) gives,

$$\frac{d}{dt} \left[\sum_i c_{11} [\Phi(t)]_{i-1/2}^2 \right] \leq F_0 e^{\frac{L}{2c_{11}} t} .$$

which implies the a-priori estimate

$$\sum_i c_{11} [\Phi(t)]_{i-1/2}^2 \leq \left[\sum_i c_{11} [\Phi(0)]_{i-1/2}^2 \right] + F_0 \left(e^{\frac{L}{2c_{11}} t} - 1 \right) .$$

Then, substitution of the above estimate into (141), leads to the thesis of the Proposition.

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REFERENCES

- [1] R. A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.
- [2] S. Agmon. *Lectures on elliptic boundary value problems*. Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Van Nostrand Mathematical Studies, No. 2. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965.
- [3] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19(4):742–760, 1982.
- [4] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779 (electronic), 2001/02.
- [5] B. Ayuso and L. D. Marini. Discontinuous Galerkin methods for advection-diffusion-reaction problems. *SIAM J. Numer. Anal.*, 47(2):1391–1420, 2009.
- [6] I. Babuška and R. Narasimhan. The Babuška-Brezzi condition and the patch test: an example. *Comput. Methods Appl. Mech. Engrg.*, 140(1-2):183–199, 1997.
- [7] N. Besse. Convergence of a semi-Lagrangian scheme for the one-dimensional Vlasov-Poisson system. *SIAM J. Numer. Anal.*, 42(1):350–382 (electronic), 2004.
- [8] N. Besse. Convergence of a high-order semi-Lagrangian scheme with propagation of gradients for the one-dimensional Vlasov-Poisson system. *SIAM J. Numer. Anal.*, 46(2):639–670, 2008.
- [9] N. Besse, F. Berthelin, Y. Brenier, and P. Bertrand. The multi-water-bag equations for collisionless kinetic modeling. *Kinet. Relat. Models*, 2(1):39–80, 2009.
- [10] N. Besse and M. Mehrenberger. Convergence of classes of high-order semi-Lagrangian schemes for the Vlasov-Poisson system. *Math. Comp.*, 77(261):93–123 (electronic), 2008.
- [11] C. K. Birdsall and A. B. Langdon. *Plasma Physics Via Computer Simulation*. McGraw-Hill, New York, 1985.
- [12] F. Bouchut, F. Golse, and M. Pulvirenti. *Kinetic equations and asymptotic theory*, volume 4 of *Series in Applied Mathematics (Paris)*. Gauthier-Villars, Éditions Scientifiques et Médicales Elsevier, Paris, 2000. Edited and with a foreword by Benoît Perthame and Laurent Desvillettes.
- [13] J. H. Bramble, J. E. Pasciak, and O. Steinbach. On the stability of the L^2 projection in $H^1(\Omega)$. *Math. Comp.*, 71(237):147–156 (electronic), 2002.

- [14] F. Brezzi, B. Cockburn, L. D. Marini, and E. Süli. Stabilization mechanisms in discontinuous Galerkin finite element methods. *Comput. Methods Appl. Mech. Engrg.*, 195(25-28):3293–3310, 2006.
- [15] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.
- [16] M. Campos Pinto and M. Mehrenberger. Convergence of an adaptive semi-Lagrangian scheme for the Vlasov-Poisson system. *Numer. Math.*, 108(3):407–444, 2008.
- [17] J. A. Carrillo and F. Vecil. Nonoscillatory interpolation methods applied to Vlasov-based models. *SIAM J. Sci. Comput.*, 29(3):1179–1206 (electronic), 2007.
- [18] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau. An a priori error analysis of the local discontinuous Galerkin method for elliptic problems. *SIAM J. Numer. Anal.*, 38(5):1676–1706 (electronic), 2000.
- [19] F. Celiker and B. Cockburn. Superconvergence of the numerical traces of discontinuous Galerkin and hybridized methods for convection-diffusion problems in one space dimension. *Math. Comp.*, 76(257):67–96 (electronic), 2007.
- [20] Y. Cheng, I. Gamba, A. Majorana, and C.-W. Shu. A discontinuous galerkin solver for boltzmann poisson systems in nano devices. *Comput. Methods Appl. Mech. Engrg.*, 198(34-40):3130–3150, 2009.
- [21] P. G. Ciarlet. Basic error estimates for elliptic problems. In *Handbook of numerical analysis, Vol. II*, Handb. Numer. Anal., II, pages 17–351. North-Holland, Amsterdam, 1991.
- [22] B. Cockburn, J. Guzmán, and H. Wang. Superconvergent discontinuous Galerkin methods for second-order elliptic problems. *Math. Comp.*, To appear.
- [23] B. Cockburn, G. Kanschat, I. Perugia, and D. Schötzau. Superconvergence of the local discontinuous Galerkin method for elliptic problems on Cartesian grids. *SIAM J. Numer. Anal.*, 39(1):264–285 (electronic), 2001.
- [24] B. Cockburn, G. E. Karniadakis, and C.-W. Shu. The development of discontinuous Galerkin methods. In *Discontinuous Galerkin methods (Newport, RI, 1999)*, volume 11 of *Lect. Notes Comput. Sci. Eng.*, pages 3–50. Springer, Berlin, 2000.
- [25] B. Cockburn and C.-W. Shu. TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. II. General framework. *Math. Comp.*, 52(186):411–435, 1989.
- [26] B. Cockburn and C.-W. Shu. The Runge-Kutta local projection P^1 -discontinuous-Galerkin finite element method for scalar conservation laws. *RAIRO Modél. Math. Anal. Numér.*, 25(3):337–361, 1991.
- [27] B. Cockburn and C.-W. Shu. The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM J. Numer. Anal.*, 35(6):2440–2463 (electronic), 1998.
- [28] B. Cockburn and C.-W. Shu. The Runge-Kutta discontinuous Galerkin method for conservation laws. V. Multidimensional systems. *J. Comput. Phys.*, 141(2):199–224, 1998.
- [29] J. Cooper and A. Klimas. Boundary value problems for the Vlasov-Maxwell equation in one dimension. *J. Math. Anal. Appl.*, 75(2):306–329, 1980.
- [30] G.-H. Cottet and P.-A. Raviart. Particle methods for the one-dimensional Vlasov-Poisson equations. *SIAM J. Numer. Anal.*, 21(1):52–76, 1984.
- [31] N. Crouseilles, G. Latu, and E. Sonnendrücker. A parallel Vlasov solver based on local cubic spline interpolation on patches. *J. Comput. Phys.*, 228(5):1429–1446, 2009.
- [32] M. Crouzeix and V. Thomée. The stability in L_p and W_p^1 of the L_2 -projection onto finite element function spaces. *Math. Comp.*, 48(178):521–532, 1987.
- [33] C. C.Z. and G. Knorr. The integration of the Vlasov equation in configuration space. *J. Computational Phys.*, 22:330–351, 1976.

- [34] J. Dolbeault. An introduction to kinetic equations: the Vlasov-Poisson system and the Boltzmann equation. *Discrete Contin. Dyn. Syst.*, 8(2):361–380, 2002. Current developments in partial differential equations (Temuco, 1999).
- [35] J. Douglas, Jr. and T. Dupont. Interior penalty procedures for elliptic and parabolic Galerkin methods. In *Computing methods in applied sciences (Second Internat. Sympos., Versailles, 1975)*, pages 207–216. Lecture Notes in Phys., Vol. 58. Springer, Berlin, 1976.
- [36] E. Fijalkow. A numerical solution to the Vlasov equation. *Comput. Phys. Comm.*, 116(2-3):319–328, 1999.
- [37] F. Filbet. Convergence of a finite volume scheme for the Vlasov-Poisson system. *SIAM J. Numer. Anal.*, 39(4):1146–1169 (electronic), 2001.
- [38] F. Filbet and E. Sonnendrücker. Comparison of Eulerian Vlasov solvers. *Comput. Phys. Comm.*, 150(3):247–266, 2003.
- [39] F. Filbet, E. Sonnendrücker, and P. Bertrand. Conservative numerical schemes for the Vlasov equation. *J. Comput. Phys.*, 172(1):166–187, 2001.
- [40] I. Gamba and J. Proft. Stable discontinuous galerkin schemes for linear vlasov-boltzmann transport equations. *Preprint ICES-Texas 07-25*, 2007.
- [41] R. T. Glassey. *The Cauchy problem in kinetic theory*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.
- [42] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.
- [43] G. S. Jiang and C.-W. Shu. On a cell entropy inequality for discontinuous Galerkin methods. *Math. Comp.*, 62(206):531–538, 1994.
- [44] A. J. Klimas and W. M. Farrell. A splitting algorithm for Vlasov simulation with filamentation filtration. *J. Comput. Phys.*, 110(1):150–163, 1994.
- [45] D. J. Knezevic and E. Süli. A heterogeneous alternating-direction method for a micro-macro dilute polymeric fluid model. *M2AN Math. Model. Numer. Anal.*, 43(6):1117–1156, 2009.
- [46] D. J. Knezevic and E. Süli. Spectral Galerkin approximation of Fokker-Planck equations with unbounded drift. *M2AN Math. Model. Numer. Anal.*, 43(3):445–485, 2009.
- [47] P. Lasaint and P.-A. Raviart. On a finite element method for solving the neutron transport equation. In *Mathematical aspects of finite elements in partial differential equations (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1974)*, pages 89–123. Publication No. 33. Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York, 1974.
- [48] J.-G. Liu and C.-W. Shu. A high-order discontinuous Galerkin method for 2D incompressible flows. *J. Comput. Phys.*, 160(2):577–596, 2000.
- [49] Y. Liu, C.-W. Shu, E. Tadmor, and M. Zhang. L^2 stability analysis of the central discontinuous Galerkin method and a comparison between the central and regular discontinuous Galerkin methods. *M2AN Math. Model. Numer. Anal.*, 42(4):593–607, 2008.
- [50] P.-A. Raviart and J. M. Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975)*, pages 292–315. Lecture Notes in Math., Vol. 606. Springer, Berlin, 1977.
- [51] W. H. Reed and T. Hill. Triangular mesh methods for the neutron transport equation. Technical Report LA-UR-73-479, Los Alamos Scientific Laboratory, 1973.
- [52] J. Schaeffer. Convergence of a difference scheme for the Vlasov-Poisson-Fokker-Planck system in one dimension. *SIAM J. Numer. Anal.*, 35(3):1149–1175 (electronic), 1998.
- [53] D. Schötzau and C. Schwab. Time discretization of parabolic problems by the hp -version of the discontinuous Galerkin finite element method. *SIAM J. Numer. Anal.*, 38(3):837–875 (electronic), 2000.

- [54] C.-W. Shu. Discontinuous Galerkin methods: general approach and stability. In *Numerical solutions of partial differential equations*, Adv. Courses Math. CRM Barcelona, pages 149–201. Birkhäuser, Basel, 2009.
- [55] L. B. Wahlbin. *Superconvergence in Galerkin finite element methods*, volume 1605 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.
- [56] M. F. Wheeler. An elliptic collocation-finite element method with interior penalties. *SIAM J. Numer. Anal.*, 15(1):152–161, 1978.
- [57] S. Wollman. Results on existence and uniqueness of solutions to the Vlasov equation. In *Fluids and plasmas: geometry and dynamics (Boulder, Colo., 1983)*, volume 28 of *Contemp. Math.*, pages 251–267. Amer. Math. Soc., Providence, RI, 1984.
- [58] S. Wollman. On the approximation of the Vlasov-Poisson system by particle methods. *SIAM J. Numer. Anal.*, 37(4):1369–1398 (electronic), 2000.
- [59] S. Wollman and E. Ozizmir. Numerical approximation of the one-dimensional Vlasov-Poisson system with periodic boundary conditions. *SIAM J. Numer. Anal.*, 33(4):1377–1409, 1996.
- [60] S. I. Zaki and L. R. T. Gardner. A finite element code for the simulation of one-dimensional vlasov plasmas. i. theory. *J. Comput. Phys.*, 79(1):184–199, 1988.
- [61] S. I. Zaki, B. T.J.M., and L. R. T. Gardner. A finite element code for the simulation of one-dimensional vlasov plasmas. ii. applications. *J. Comput. Phys.*, 79(1):200–208, 1988.
- [62] Q. Zhang and C.-W. Shu. Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin methods for scalar conservation laws. *SIAM J. Numer. Anal.*, 42(2):641–666 (electronic), 2004.
- [63] T. Zhou, Y. Guo, and C.-W. Shu. Numerical study on landau damping. *Physica D*, 157:322–333, 2001.

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