L^p THEORY FOR THE MULTIDIMENSIONAL AGGREGATION EQUATION

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Abstract. We consider well-posedness of the aggregation equation $\partial_t u + \mathrm{div}(uv) = 0$, $v = -\nabla K * u$ with initial data in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, in dimensions two and higher. We consider radially symmetric kernels where the singularity at the origin is of order $|x|^\alpha$, $\alpha > 2 - d$, and prove local well-posedness in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for sufficiently large $p > p_s$. In the special case of $K(x) = |x|$, the exponent $p_s = d/(d-1)$ is sharp for local well-posedness, in that solutions can instantaneously concentrate mass for initial data in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $p < p_s$. We also give an Osgood condition on the potential $K(x)$ which guaranties global existence and uniqueness in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$.

1. Introduction

1.1. **Background.** The multidimensional aggregation equation

$$\frac{\partial u}{\partial t} + \mathrm{div}\ (uv) = 0, \tag{1.1}$$

$$v = -\nabla K * u, \tag{1.2}$$

$$u(0) = u_0, \tag{1.3}$$

arises in a number of models for biological aggregation [11, 14, 15, 23, 28, 37, 36, 38, 39] as well as problems in materials science [24, 25] and granular media [3, 17, 18, 31, 40]. The same equation with additional diffusion has been considered in [7, 9, 13, 20, 27, 29, 30, 32] although we do not consider that case in this paper. For the inviscid case, much work has been done recently on the question of finite time blowup in equations of this type, from bounded or smooth initial data [10, 6, 4, 5]. A recent study [16] proves well-posedness of measure solutions for semi-convex kernels. Global existence (but not uniqueness) of measure solutions has been proven in [33, 21] in two space dimension when K is exactly the Newtonian Potential. Moreover, numerical simulations [26], of aggregations involving $K(x) = |x|$, exhibit finite time blowup from bounded data in which the initial singularity remains in L^p for some p rather than forming a mass concentration at the initial blowup time. These facts together bring up the very interesting question of how these equations behave in general when we consider initial data in L^p, that may be locally unbounded but does not involve mass concentration. This work serves to provide a fairly complete theory of the problem in L^p, although some interesting questions remain regarding critical p exponents for general kernels and for data that lives precisely in L^{p_s} for the special kernel $K(x) = |x|$.

The L^p framework adopted in this paper allows us to make two significant advances in the understanding of the aggregation equation. First, it allows us to consider potentials which are more singular than the one which have been considered up to now (with the exception of [33, 21], where they consider the Newtonian potential in 2D). In previous works, the potential K was often required to be at worst Lipschitz singular at the origin, i.e. $K(x) \sim |x|^\alpha$ with $\alpha \geq 1$ (see [28, 6, 5, 16]). In our L^p framework it is possible to consider potentials whose singularity at the origin is of order $|x|^\alpha$ with $\alpha > 2-d$. Such potentials might have a cusp (in 2D) or even blow up (in 3D) at the origin. Interestingly, in dimension $d \geq 3$, $|x|^{2-d}$ is exactly the Newtonian potential. So we can rephrase our result by saying that we prove local existence and uniqueness when the singularity of the potential is "better" than that of the Newtonian potential.

The second important results proven in this paper concerns the specific and biologically relevant potential $K(x) = |x|$. For such a potential, a concept of measure solution is provided in [16]. In the present paper we identify the critical regularity needed on the initial data in order to guaranty that the solution will stay absolutely continuous with respect to the Lebesgue measure at least for short time. To be more specific, we prove that solutions whose initial data are in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ remain in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ at least for short time if $p > d/(d-1)$. Here $\mathcal{P}_2(\mathbb{R}^d)$ denotes probability measure with bounded second moment. On the other hand for any $p < d/(d-1)$ we are able to exhibit initial data in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for which a delta Dirac appears instantaneously in the solution – the solution loses its absolute continuity with respect to the Lebesgue measure instantaneously.

1.2. **Main results of the paper.** Below we state the main results of this paper and how they connect to previous results in the literature.

Theorem 1 (well-posedness)**.** *Consider $1 < q < \infty$ and p its Hölder conjugate. Suppose $\nabla K \in W^{1,q}(\mathbb{R}^d)$ and $u_0 \in L^p(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$ is nonnegative. Then there exists a time $T^* > 0$ and a nonnegative function $u \in C([0,T^*], L^p(\mathbb{R}^d)) \cap C^1([0,T^*], W^{-1,p}(\mathbb{R}^d))$ such that*

$$u'(t) + div\big(\, u(t)\, v(t)\,\big) = 0 \qquad \forall t \in [0,T^*], \tag{1.4}$$

$$v(t) = -u(t) * \nabla K \qquad \forall t \in [0,T^*], \tag{1.5}$$

$$u(0) = u_0. \tag{1.6}$$

Moreover the second moment stays bounded and the L^1 norm is conserved. Furthermore, if ess sup $\Delta K < +\infty$*, then we have global well-posedness.*

Theorem 1 is proved in sections 2 and 3. The fact that $W^{1,q_1}_{loc}(\mathbb{R}^d) \subset W^{1,q_2}_{loc}(\mathbb{R}^d)$ for $q_1 \leq q_2$ allows us to make the following definition:

Definition 2 (critical exponents q_s and p_s)**.** *Suppose $\nabla K(x)$ is compactly supported (or decays exponentially fast as $|x| \to \infty$) and belongs to $W^{1,q}(\mathbb{R}^d)$ for some $q \in (1, +\infty)$. Then there exists an exponent $q_s \in (1, +\infty]$ such that $\nabla K \in W^{1,q}(\mathbb{R}^d)$ for all $q < q_s$ and $\nabla K \notin W^{1,q}(\mathbb{R}^d)$ for all $q > q_s$. The Hölder conjugate of this exponent q_s is denoted p_s.*

The exponent q_s quantifies the singularity of the potential. The more singular the potential, the smaller is q_s. For potentials that behave like a power function at the origin, $K(x) \sim |x|^\alpha$ as $|x| \to 0$, the exponents are easily computed:

$$q_s = \frac{d}{2-\alpha}, \qquad \text{and} \qquad p_s = \frac{d}{d-(\alpha-2)}, \qquad \text{if } 2-d < \alpha < 2, \tag{1.7}$$

$$q_s = +\infty, \qquad \text{and} \qquad p_s = 1, \qquad \text{if } \alpha \geq 2. \tag{1.8}$$

We obtain the following picture for power like potentials:

Theorem 3 (Existence and uniqueness for power potential)**.** *Suppose ∇K is compactly supported (or decays exponentially fast at infinity). Suppose also that $K \in C^2(\mathbb{R}^d \backslash \{0\})$ and $K(x) \sim |x|^\alpha$ as $|x| \to 0$.*

(i) *If $2 - d < \alpha < 2$ then the aggregation equation is locally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for every $p > p_s$. Moreover, it is not globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$.*

(ii) *If $\alpha \geq 2$ then the aggregation equation is globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for every $p > 1$.*

As a consequence we have existence and uniqueness for all potentials which are less singular than the Newtonian potential $K(x) = |x|^{2-d}$ at the origin. In two dimensions this includes potentials with cusp such as $K(x) = |x|^{1/2}$. In three dimensions this includes potentials that blow up such as $K(x) = |x|^{-1/2}$. From [5, 16] we know that the support of compactly supported solutions shrinks to a point in finite time, proving the second assertion in point *(i)* above. The first part of *(i)* and statement (ii) are direct corollary of Theorem 1, Definition 2 and the fact that $\alpha \geq 2$ implies ΔK bounded.

In the case where $\alpha = 1$, i.e. $K(x) \sim |x|$ as $|x| \to 0$, the previous Theorem gives local well posedness in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for all $p > p_s = \frac{d}{d-1}$. The next Theorem shows that it is not possible to obtain local well posedness in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for $p < p_s = \frac{d}{d-1}$.

Theorem 4 (Critical p-exponent to generate instantaneous mass concentration)**.** *Suppose $K(x) = |x|$ in a neighborhood of the origin, and suppose ∇K is compactly supported (or decays exponentially fast at infinity). Then, for any $p < p_s = \frac{d}{d-1}$, there exists initial data in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for which a delta Dirac appears instantaneously in the measure solution.*

In order to make sense of the statement of the previous Theorem, we need a concept of measure solution. The potentials $K(x) = |x|$ is semi-convex, i.e. there exist $\lambda \in \mathbb{R}$ such that $K(x) - \frac{\lambda}{2}|x|^2$ is convex. In [16], Carrillo et al. prove global well-posedness in $\mathcal{P}_2(\mathbb{R}^d)$ of the aggregation equation with semi-convex potentials. The solutions in [16] are weak measure solutions - they are not necessarily absolutely continuous with respect to the Lebesgue measure. Theorems 3 and 4 give a sharp condition on the initial data in order for the solution to stay absolutely continuous with respect to the Lebesgue measure for short time. Theorem 4 is proven in Section 4.

Finally, in section 5 we consider a class of potential that will be referred to as the class of **natural potentials**. A potential is said to be natural if it satisfies that

a) it is a radially symmetric potential, i.e.: $K(x) = k(|x|)$,
b) it is smooth away from the origin and it's singularity at the origin is not worse than Lipschitz,
c) it doesn't exhibit pathological oscillation at the origin,
d) its derivatives decay fast enough at infinity.

All these conditions will be more rigorously stated later. It will be shown that the gradient of natural potentials automatically belongs to $W^{1,q}$ for $q < d$, therefore, using the results from the sections 2 and 3, we have local existence and uniqueness in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > \frac{d}{d-1}$.

A natural potential is said to be repulsive in the short range if it has a local maximum at the origin and it is said to be attractive in the short range if it has a local minimum at the origin. If the maximum (respectively minimum) is strict, the natural potential is said to be strictly repulsive (respectively strictly attractive) at the origin. The main theorem of section 5 is the following:

Theorem 5 (Osgood condition for global well posedness)**.** *Suppose K is a natural potential.*

(i) If K is repulsive in the short range, then the aggregation equation is globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > d/(d-1)$.
(ii) If K is strictly attractive in the short range, the aggregation equation is globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > d/(d-1)$, if and only if

$$r \mapsto \frac{1}{k'(r)} \text{ is not integrable at 0.} \tag{1.9}$$

By globally well posed in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, we mean that for any initial data in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ the unique solution of the aggregation equation will exist for all time and will stay in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for all time.Notice that the exponent $d/(d-1)$ is not sharp in this theorem.

Condition (1.9) will be refered as the Osgood condition. It is easy to understand why the Osgood condition is relevant while studying blowup: the quantity

$$T(d) = \int_0^d \frac{dr}{k'(r)}$$

can be thought as the amount of time it takes for a particle obeying the ODE $\dot{X} = -\nabla K(X)$ to reach the origin if it starts at a distance d from it. For a potential satisfying the Osgood condition, $T(d) = +\infty$, which means that the particle can not reach the origin in finite time. The Osgood condition was already shown in [5] to be necessary and sufficient for global well posedness of L^∞-solutions. Extension to L^p requires L^p estimates rather than L^∞ estimates. See also [43] for an example of the use of the Osgood condition in the context of the Euler equations for incompressible fluid.

The "only if" part of statement (ii) was proven in [5] and [16]. In these two works it was shown that if (1.9) is not satisfied, then compactly supported solutions will collapse into a point mass – and therefore leave L^p – in finite time. In section 5 we prove statement (i) and the 'if' part of statement (ii).

2. Existence of L^p-solutions

In this section we show that if the interaction potential satisfies

$$\nabla K \in W^{1,q}(\mathbb{R}^d), \quad 1 < q < +\infty, \tag{2.10}$$

and if the initial data is nonnegative and belongs to $L^p(\mathbb{R}^d)$ (p and q are Hölder conjugates) then there exists a solution to the aggregation equation. Moreover, either this solution exists for all times, or its L^p-norm blows up in finite time. The duality between L^p and L^q guarantees enough smoothness in the velocity field $v = -\nabla K * u$ to define characteristics. We use the characteristics to construct a solution. The argument is inspired by the existence of L^∞ solutions of the incompressible 2D Euler equations by Yudovich [44] and of L^∞ solutions of the aggregation equation [4]. Section 3 proves uniqueness provided $u \in \mathcal{P}_2$. We prove in Theorem 18 of the present section that if $u_0 \in \mathcal{P}_2$ then the solution stays in $\mathcal{P}_2$. Finally we prove that if in addition to (2.10), we have

$$\text{ess sup } \Delta K < +\infty, \tag{2.11}$$

then the solution constructed exists for all time.

Most of the section is devoted to the proof of the following theorem:

Theorem 6 (Local existence)**.** *Consider $1 < q < \infty$ and p its Hölder conjugate. Suppose $\nabla K \in W^{1,q}(\mathbb{R}^d)$ and suppose $u_0 \in L^p(\mathbb{R}^d)$ is nonnegative. Then there exists a time $T^* > 0$ and a nonnegative function $u \in C([0, T^*], L^p(\mathbb{R}^d)) \cap C^1([0, T^*], W^{-1,p}(\mathbb{R}^d))$ such that*

$$u'(t) + div\big(\, u(t)\, v(t)\,\big) = 0 \qquad \forall t \in [0, T^*], \tag{2.12}$$

$$v(t) = -u(t) * \nabla K \qquad \forall t \in [0, T^*], \tag{2.13}$$

$$u(0) = u_0. \tag{2.14}$$

Moreover the function $t \to \|u(t)\|^p_{L^p}$ is differentiable and satisfies

$$\frac{d}{dt}\left\{\|u(t)\|^p_{L^p}\right\} = -(p-1)\int_{\mathbb{R}^d} u(t,x)^p \; div\, v(t,x)\, dx \qquad \forall t \in [0, T^*]. \tag{2.15}$$

The choice of the space

$$\mathcal{Y}_p := C([0,T^*], L^p(\mathbb{R}^d)) \cap C^1([0,T^*], W^{-1,p}(\mathbb{R}^d))$$

is motivated by the fact that, if $u \in \mathcal{Y}_p$ and $\nabla K \in W^{1,q}$, then the velocity field is automatically C^1 in space and time:

Lemma 7. *Consider $1 < q < \infty$ and p its Hölder conjugate. If $\nabla K \in W^{1,q}(\mathbb{R}^d)$ and $u \in \mathcal{Y}_p$ then*

$$u * \nabla K \in C^1\left([0,T^*] \times \mathbb{R}^d\right) \text{ and } \|u * \nabla K\|_{C^1([0,T^*]\times\mathbb{R}^d)} \leq \|\nabla K\|_{W^{1,q}(\mathbb{R}^d)} \|u\|_{\mathcal{Y}_p} \tag{2.16}$$

where the norm $\|\cdot\|_{C^1([0,T^*]\times\mathbb{R}^d)}$ and $\|\cdot\|_{\mathcal{Y}_p}$ are defined by

$$\|v\|_{C^1([0,T^*]\times\mathbb{R}^d)} = \sup_{[0,T^*]\times\mathbb{R}^d} |v| + \sup_{[0,T^*]\times\mathbb{R}^d} \left|\frac{\partial v}{\partial t}\right| + \sum_{i=1}^d \sup_{[0,T^*]\times\mathbb{R}^d} \left|\frac{\partial v}{\partial x_i}\right|, \tag{2.17}$$

$$\|u\|_{\mathcal{Y}_p} = \sup_{t\in[0,T^*]} \|u(t)\|_{L^p(\mathbb{R}^d)} + \sup_{t\in[0,T^*]} \|u'(t)\|_{W^{-1,p}(\mathbb{R}^d)}. \tag{2.18}$$

Proof. Recall that the convolution between a L^p-function and a L^q-function is continuous and $\sup_{x\in\mathbb{R}^d} |f * g(x)| \leq \|f\|_{L^p} \|g\|_{L^q}$ Therefore, since ∇K and ∇K_{x_i} are in L^q, the mapping

$$f \mapsto \nabla K * f$$

is a bounded linear transformation from $L^p(\mathbb{R}^d)$ to $C^1(\mathbb{R}^d)$, where $C^1(\mathbb{R}^d)$ is endowed with the norm

$$\|f\|_{C^1} = \sup_{x\in\mathbb{R}^d} |f(x)| + \sum_{i=1}^d \sup_{x\in\mathbb{R}^d} |\frac{\partial f}{\partial x_i}(x)|.$$

Since $u \in C([0,T^*], L^p)$ it is then clear that $u*\nabla K \in C([0,T^*], C^1)$. In particular $w(t,x) = (u(t) * \nabla K)(x)$ and $\frac{\partial w}{\partial x_i}(t,x)$ are continuous on $[0,T^*] \times \mathbb{R}^d$. Let us now show that $\frac{\partial w}{\partial t}(t,x)$ exists and is continuous on $[0,T^*] \times \mathbb{R}^d$. Since $u'(t) \in C([0,T^*], W^{-1,p})$ and $\nabla K \in W^{1,q}$, we have

$$\frac{\partial w}{\partial t}(t,x) = -\left(u'(t) * \nabla K\right)(x) = -\langle u'(t), \tau_x \nabla K\rangle$$

where $\langle \ , \ \rangle$ denote the pairing between the two dual spaces $W^{-1,p}(\mathbb{R}^d)$ and $W^{1,q}(\mathbb{R}^d)$, and τ_x denote the translation by x. Since $x \mapsto \tau_x \nabla K$ is a continuous mapping from $\mathbb{R}^d$ to $W^{1,q}$ it is clear that $\frac{\partial w}{\partial t}(t,x)$ is continuous with respect to space. The continuity with respect to time come from the continuity of $u'(t)$ with respect to time. Inequality (2.16) is easily obtained. □

Remark 8. *Let us point out that (2.12) indeed makes sense, when understood as an equality in $W^{-1,p}$. Since $v \in C([0,T^*], C^1(\mathbb{R}^d))$ one can easily check that $uv \in C([0,T^*], L^p(\mathbb{R}^d))$. Also recall that the injection $i\colon L^p(\mathbb{R}^d) \to W^{-1,p}(\mathbb{R}^d)$ and* *the differentiation* $\partial_{x_i} : L^p(\mathbb{R}^d) \to W^{-1,p}(\mathbb{R}^d)$ *are bounded linear operators. Therefore it is clear that both* u *and* $div(uv)$ *belong to* $C([0,T^*], W^{-1,p}(\mathbb{R}^d))$*. Equation* (2.12) *has to be understand as an equality in* $W^{-1,p}$*.*

The rest of this section is organized as follows. First we give the basic a priori estimates in subsection 2.1. Then, in subsection 2.2, we consider a mollified and cutted-off version of the aggregation equation for which we have global existence of smooth and compactly supported solutions. In subsection 2.3 we show that the characteristics of this approximate problem are uniformly Lipschitz continuous on $[0,T^*] \times \mathbb{R}^d$, where $T^* > 0$ is some finite time depending on $\|u_0\|_{L^p}$. In subsection 2.4 we pass to the limit in $C([0,T^*], L^p)$. To do this we need the uniform Lipschitz bound on the characteristics together with the fact that the translation by x, $x \mapsto \tau_x u_0$, is a continuous mapping from $\mathbb{R}^d$ to $L^p(\mathbb{R}^d)$. In subsection 2.5 we prove three theorems. We first prove continuation of solutions. We then prove that L^p-solutions which start in $\mathcal{P}_2$ stay in $\mathcal{P}_2$ as long as they exist. And finally we prove global existence in the case where ΔK is bounded from above.

2.1. **A priori estimates.** Suppose $u \in C^1_c((0,T)\times\mathbb{R}^d)$ is a nonnegative function which satisfies (2.12)-(2.13) in the classical sense. Suppose also that $K \in C^\infty_c(\mathbb{R}^d)$. Integrating by part, we obtain that for any $p \in (1,+\infty)$:

$$\frac{d}{dt}\int_{\mathbb{R}^d} u(t,x)^p dx = -(p-1)\int_{\mathbb{R}^d} u(t,x)^p \text{ div } v(t,x)dx \quad \forall t \in (0,T). \tag{2.19}$$

As a consequence we have:

$$\frac{d}{dt}\|u(t)\|^p_{L^p} \leq (p-1)\|\text{div } v(t)\|_{L^\infty}\|u(t)\|^p_{L^p} \quad \forall t \in (0,T), \tag{2.20}$$

and by Hölder's inequality:

$$\frac{d}{dt}\|u(t)\|^p_{L^p} \leq (p-1)\|\Delta K\|_{L^q}\|u(t)\|^{p+1}_{L^p}. \tag{2.21}$$

We now derive L^∞ estimates for the velocity field $v = -\nabla K * u$ and its derivatives. Hölder's inequality easily gives

$$|v(t,x)| \leq \|u(t)\|_{L^p}\|\nabla K\|_{L^q} \quad \forall (t,x) \in (0,T)\times\mathbb{R}^d, \tag{2.22}$$

$$\left|\frac{\partial v_j}{\partial x_i}(t,x)\right| \leq \|u(t)\|_{L^p}\left\|\frac{\partial^2 K}{\partial x_i dx_j}\right\|_{L^q} \quad \forall (t,x) \in (0,T)\times\mathbb{R}^d. \tag{2.23}$$

Since $\frac{\partial v}{\partial t} = -\nabla K * \frac{\partial u}{\partial t} = \nabla K * \text{div}(uv) = \Delta K * uv$ we have

$$\left|\frac{\partial v}{\partial t}(t,x)\right| \leq \|u(t)v(t)\|_{L^p}\|\Delta K\|_{L^q} \leq \|u(t)\|_{L^p}\|v(t)\|_{L^\infty}\|\Delta K\|_{L^q},$$

which in light of (2.22) gives

$$\left|\frac{\partial v}{\partial t}(t,x)\right| \leq \|u(t)\|^2_{L^p}\|\nabla K\|_{L^q}\|\Delta K\|_{L^q} \quad \forall (t,x) \in (0,T)\times\mathbb{R}^d. \tag{2.24}$$

2.2. **Approximate smooth compactly supported solutions.** In this section we deal with a smooth version of equation (2.12)-(2.14). Suppose $u_0 \in L^p(\mathbb{R}^d)$, $1 < p < +\infty$, and $\nabla K \in W^{1,q}(\mathbb{R}^d)$. Consider the approximate problem

$$u_t + \operatorname{div}(uv) = 0 \qquad \text{in } (0, +\infty) \times \mathbb{R}^d, \tag{2.25}$$

$$v = -\nabla K^\epsilon * u \qquad \text{in } (0, +\infty) \times \mathbb{R}^d, \tag{2.26}$$

$$u(0) = u_0^\epsilon, \tag{2.27}$$

where $K^\epsilon = J_\epsilon K$, $u_0^\epsilon = J_\epsilon u_0$ and J_ϵ is an operator which mollifies and cuts-off, $J_\epsilon f = (f M_{R_\epsilon}) * \eta_\epsilon$ where $\eta_\epsilon(x)$ is a standard mollifier:

$$\eta_\epsilon(x) = \frac{1}{\epsilon^d}\eta\left(\frac{x}{\epsilon}\right),\ \eta \in C_c^\infty(\mathbb{R}^d), \eta \geq 0, \int_{\mathbb{R}^d} \eta(x)dx = 1,$$

and $M_{R_\epsilon}(x)$ is a standard cut-off function: $M_{R_\epsilon}(x) = M(\frac{x}{R_\epsilon})$, $R_\epsilon \to \infty$ as $\epsilon \to 0$,

$$M \in C_c^\infty(\mathbb{R}^d), \qquad \begin{cases} M(x) = 1 & \text{if } |x| \leq 1, \\ 0 < M(x) < 1 & \text{if } 1 < |x| < 2, \\ M = 0 & \text{if } 2 \leq |x|. \end{cases}$$

Let τ_x denote the translation by x, i.e.:

$$\tau_x f(y) := f(y - x).$$

It is well known that given a fixed $f \in L^r(\mathbb{R}^d)$, $1 < r < +\infty$, the mapping $x \mapsto \tau_x f$ from $\mathbb{R}^d$ to $L^r(\mathbb{R}^d)$ is uniformly continuous. In (iv) of the next lemma we show a slightly stronger result which will be needed later.

Lemma 9 (Properties of J_ϵ)**.** *Suppose $f \in L^r(\mathbb{R}^d)$, $1 < r < +\infty$, then*

(i) $J_\epsilon f \in C_c^\infty(\mathbb{R}^d)$,
(ii) $\|J_\epsilon f\|_{L^r} \leq \|f\|_{L^r}$,
(iii) $\lim_{\epsilon \to 0} \|J_\epsilon f - f\|_{L^r} = 0$,
(iv) *The family of mappings $x \mapsto \tau_x J_\epsilon f$ from $\mathbb{R}^d$ to $L^r(\mathbb{R}^d)$ is equicontinuous, i.e.: for each $\delta > 0$, there is a $\eta > 0$ independent of ϵ such that $\|\tau_x J_\epsilon f - \tau_y J_\epsilon f\|_{L^r} \leq \delta$ if $|x - y| \leq \eta$.*

Proof. Statements (i) and (ii) are obvious. If f is compactly supported, one can easily prove (iii) by noting that $f M_{R_\epsilon} = f$ for ϵ small enough. If f is not compactly supported, (iii) is obtained by approximating f by a compactly supported function and by using (ii). Let us now turn to the proof of (iv). Using (ii) we obtain

$$\begin{aligned} \|\tau_x J_\epsilon f - J_\epsilon f\|_{L^r} &\leq \|(\tau_x M_{R_\epsilon})(\tau_x f) - M_{R_\epsilon} f)\|_{L^r} \\ &\leq \|\tau_x M_{R_\epsilon} - M_{R_\epsilon}\|_{L^\infty} \|\tau_x f\|_{L^r} + \|M_{R_\epsilon}\|_{L^\infty} \|\tau_x f - f\|_{L^r}. \end{aligned}$$

Because $x \mapsto \tau_x f$ is continuous, the second term can be made as small as we want by choosing $|x|$ small enough. Since $\|\tau_x M_{R_\epsilon} - M_{R_\epsilon}\|_{L^\infty} \leq \|\nabla M_{R_\epsilon}\|_{L^\infty}|x| \leq \frac{1}{R_\epsilon}\|\nabla M\|_{L^\infty}|x|$, the first term can be made as small as we want by choosing $|x|$ small enough and independently of ϵ. □

Proposition 10 (Global existence of smooth compactly-supported approximates)**.** *Given $\epsilon, T > 0$, there exists a nonnegative function $u \in C_c^1((0,T) \times \mathbb{R}^d)$ which satisfy (2.27) in the classical sense.*

Proof. Since u_0^ϵ and K^ϵ belong to $C_c^\infty(\mathbb{R}^d)$, we can use theorem 3 p. 1961 of [28] to get the existence of a function u^ϵ satisfying

$$u^\epsilon \in L^\infty(0,T;H^k), u_t^\epsilon \in L^\infty,(0,T;H^{k-1}) \text{ for all } k, \tag{2.28}$$

$$u_t^\epsilon + \text{ div } (u^\epsilon(-\nabla K^\epsilon * u^\epsilon)) = 0 \text{ in } (0,T) \times \mathbb{R}^d, \tag{2.29}$$

$$u^\epsilon(0) = u_0^\epsilon, \tag{2.30}$$

$$u^\epsilon(t,x) \geq 0 \text{ for a.e. } (t,x) \in (0,T), \times \mathbb{R}^d. \tag{2.31}$$

Statement (2.28) implies that $u^\epsilon \in C((0,T); H^{k-1})$. Using the continuous embedding $H^{k-1}(\mathbb{R}^d) \subset C^1(\mathbb{R}^d)$ for k large enough we find that u^ϵ and $u^\epsilon_{x_i}$, $1 \leq i \leq d$, are continuous on $(0,T) \times \mathbb{R}^d$. Finally, (2.29) shows that u_t^ϵ is also continuous on $(0,T) \times \mathbb{R}^d$. We have proven that $u^\epsilon \in C^1((0,T) \times \mathbb{R}^d)$. It is then obvious that $v^\epsilon = -\nabla K^\epsilon * u^\epsilon \in C^1((0,T) \times \mathbb{R}^d)$. Note moreover that

$$|v^\epsilon(x,t)| \leq \|u^\epsilon\|_{L^\infty(0,T;L^2)} \|\nabla K^\epsilon\|_{L^2}$$

for all $(t,x) \in (0,T) \times \mathbb{R}^d$. This combined with the fact that v^ϵ is in C^1 shows that the characteristics are well defined and propagate with finite speed. This proves that u^ϵ is compactly supported in $(0,T) \times \mathbb{R}^d$ (because u_0^ϵ is compactly supported in $\mathbb{R}^d$). □

2.3. **Study of the velocity field and the induced flow map.** Note that K^ϵ and u^ϵ are in the right function spaces so that we can apply to them to the a priori estimates derived in section 2.1. In particular we have:

$$\frac{d}{dt}\|u^\epsilon(t)\|_{L^p} \leq (p-1)\|\Delta K\|_{L^q}\|u^\epsilon(t)\|_{L^p}^{p+1},$$
$$\|u^\epsilon(0)\|_{L^p} \leq \|u_0\|_{L^p}.$$

Using Gronwall inequality and the estimate on the supremum norm of the derivatives derived in section 2.1 we obtain:

Lemma 11 (uniform bound for the smooth approximates)**.** *There exists a time $T^* > 0$ and a constant $C > 0$, both independent of ϵ, such that*

$$\|u^\epsilon(t)\|_{L^p} \leq C \qquad \forall t \in [0,T^*], \tag{2.32}$$

$$|v^\epsilon t,x)|,\ |v^\epsilon_{x_i}(t,x)|,\ |v^\epsilon_t(t,x)| \ \leq C \qquad \forall (t,x) \in [0,T^*] \times \mathbb{R}^d. \tag{2.33}$$

From (2.33) it is clear that the family $\{v^\epsilon\}$ is uniformly Lipschitz on $[0,T^*] \times \mathbb{R}^d$, with Lipschitz constant C. We can therefore use the Arzela-Ascoli Theorem to obtain the existence of a continuous function $v(t,x)$ such that

$$v^\epsilon \to v \text{ uniformly on compact subset of } [0,T^*] \times \mathbb{R}^d. \tag{2.34}$$

It is easy to check that this function v is also Lipschitz continuous with Lipschitz constant C. The Lipschitz and bounded vector field v^ϵ generates a flow map $X_\epsilon(t,\alpha),\ t \in [0,T^*],\ \alpha \in \mathbb{R}^d$:

$$\frac{\partial X_\epsilon(t,\alpha)}{\partial t} = v^\epsilon(X_\epsilon(t,\alpha),t),$$
$$X_\epsilon(0,\alpha) = \alpha,$$

where we denote by $X_\epsilon^t : \mathbb{R}^d \to \mathbb{R}^d$ the mapping $\alpha \mapsto X_\epsilon(t,\alpha)$ and by X_ϵ^{-t} the inverse of X_ϵ^t.

The uniform Lipschitz bound on the vector field implies uniform Lipschitz bound on the flow map and its inverse (see for example [4] for a proof of this statement) we therefore have:

Lemma 12 (uniform Lipschitz bound on X_ϵ^t and X_ϵ^{-t})**.** *There exists a constant $C > 0$ independent of ϵ such that:*

(i): *for all $t \in [0,T^*]$ and for all $x_1, x_2 \in \mathbb{R}^d$*

$$|X_\epsilon^t(x_1) - X_\epsilon^t(x_2)| \leq C|x_1 - x_2| \quad \text{and} \quad |X_\epsilon^{-t}(x_1) - x_\epsilon^{-t}(x_2)| \leq C|x_1 - x_2|,$$

(ii): *for all $t_1, t_2 \in [0,T^*]$ and for all $x \in \mathbb{R}^d$*

$$|X_\epsilon^{t_1}(x) - X_\epsilon^{t_2}(x)| \leq C|t_1, t_2| \quad \text{and} \quad |X_\epsilon^{-t_1}(x) - X_\epsilon^{-t_2}(x)| \leq C|t_1 - t_2|.$$

The Arzela-Ascoli Theorem then implies that there exists mapping X^t and X^{-t} such that

$$X_{\epsilon_k}^t(x) \to X^t(x) \qquad \text{uniformly on compact subset of } [0,T^*] \times \mathbb{R}^d,$$
$$X_{\epsilon_k}^{-t}(x) \to X^{-t}(x) \qquad \text{uniformly on compact subset of } [0,T^*] \times \mathbb{R}^d.$$

Moreover it is easy to check that X^t and X^{-t} inherit the Lipschitz bounds of X_ϵ^t and X_ϵ^{-t}.

Since the mapping $X^t : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous, by Rademecher's Theorem it is differentiable almost everywhere. Therefore it makes sense to consider its Jacobian matrix $DX^t(\alpha)$. Because of Lemma 12-(i) we know that there exists a constant C independent of t and ε such that

$$\sup_{\alpha\epsilon\mathbb{R}^d} |\det\, DX^t(\alpha)| \leq C \qquad \text{and} \qquad \sup_{\alpha\in\mathbb{R}^d} |\det\, DX_\epsilon^t(\alpha)| \leq C.$$

By the change of variable we then easily obtain the following Lemma:

Lemma 13. *The mappings $f \mapsto f \circ X^{-t}$ and $f \mapsto f \circ X_\epsilon^{-t},\ t \in [0,T^*],\ \epsilon > 0$, are bounded linear operators from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$. Moreover there exists a constant C^* independent of t and ϵ such that*

$$\|f \circ X^{-t}\|_{L^p} \leq C^* \|f\|_{L^p} \quad \text{and} \quad \|f \circ X_\epsilon^{-t}\|_{L^p} \leq C^* \|f\|_{L^p} \quad \text{for all } f \in L^p(\mathbb{R}^d).$$

Note that Lemma 12-(ii) implies

$$|X_\epsilon^t(\alpha) - \alpha| \leq Ct \text{ for all } (t,\alpha) \in [0,T^*) \times \mathbb{R}^d,$$

and therefore

$$|X^t(\alpha) - \alpha| \leq Ct \text{ for all } (t, \alpha) \in [0, T^*) \times \mathbb{R}^d.$$

This gives us the following lemma:

Lemma 14. *let Ω be a compact subset of $\mathbb{R}^d$, then*

$$X^t_\epsilon(\Omega) \subset \Omega + Ct \qquad \text{and} \qquad X^t(\Omega) \subset \Omega + Ct,$$

where the compact set $\Omega + Ct$ is defined by

$$\Omega + Ct := \{x \in \mathbb{R}^d : \; dist\,(x, \Omega) \leq Ct\}.$$

2.4. **Convergence in $C([0, T^*), L^p)$.** Since u^ϵ and $v^\epsilon = u^\epsilon * \nabla K$ are C^1 functions which satisfy

$$u^\epsilon_t + v^\epsilon \cdot \nabla u^\epsilon = -(\text{div } v^\epsilon) u^\epsilon \qquad \text{and} \qquad u^\epsilon(0) = u^\epsilon_0 \tag{2.35}$$

we have the simple representation formula for $u^\epsilon(t, x),\ t \in [0, T^*),\ x \in \mathbb{R}^d$:

$$u^\epsilon(t, x) = u^\epsilon_0(X^{-t}_\epsilon(x)) e^{-\int_0^t \text{ div } v^\epsilon(s, X^{-(t-s)}_\epsilon(x)) ds} = u^\epsilon_0(X^{-t}_\epsilon(x))\ a^\epsilon(t, x).$$

Lemma 15. *There exists a function $a(t, x) \in C^1([0, T^*) \times \mathbb{R}^d)$ and a sequence $\epsilon_k \to 0$ such that*

$$a^{\epsilon_k}(t, x) \to a(t, x) \text{ uniformly on compact subset of } [0, T^*] \times \mathbb{R}^d. \tag{2.36}$$

Proof. By the Arzela-Ascoli Theorem, it is enough to show that the family

$$b^\epsilon(t, x) := \int_0^t \text{ div } v^\epsilon(s, X^{-(t-s)}_\epsilon(x)) dx$$

is equicontinuous and uniformly bounded. The uniform boundedness simply come from the fact that

$$|\text{div } v^\epsilon| = |u^\epsilon * \Delta K^\epsilon| \leq \|u^\epsilon\|_{L^p} \|\Delta K\|_{L^q}.$$

Let us now prove equicontinuity in space, i.e., we want to prove that for each $\delta > 0$, there is $\eta > 0$ independent of ϵ and t such that

$$|b^\epsilon(t, x_1) - b^\epsilon(t, x_2)| \leq \delta \text{ if } |x_1 - x_2| \leq \eta.$$

First, note that by Hölder's inequality we have

$$|b^\epsilon(t, x_1) - b^\epsilon(t, x_2)| \leq \int_0^t \|u^\epsilon(s)\|_{L^p} \|\tau_\xi J_\epsilon \Delta K - \tau_\zeta J_\epsilon \Delta K\|_{L^q}\ ds$$

where ξ stands for $X^{-(t-s)}_\epsilon(x_1)$ and ζ for $X^{-(t-s)}_\epsilon(x_2)$. Then equicontinuity in space is a consequence of Lemma 9 (iv) together with the fact that

$$|X^{-(t-s)}_\epsilon(x_1) - X^{-(t-s)}_\epsilon(x_2)| \leq C|x_1 - x_2|$$

where C is independent of t, s and ϵ.

Let us finally prove equicontinuity in time. First not that, assuming that $t_1 < t_2$,

$$\begin{aligned} b^\epsilon(t_1,x) - b^\epsilon(t_2,x) &= \int_0^{t_1} \text{div } v^\epsilon(s, X^{-(t_1-s)}(x)) - \text{div } v^\epsilon(s, X^{-(t_2-s)}(x))ds \\ &- \int_{t_1}^{t_2} \text{div } v^\epsilon(s, X_\epsilon^{-(t_2-s)}(x))ds. \end{aligned}$$

Since div v^ϵ is uniformly bounded we clearly have

$$\left| \int_{t_1}^{t_2} \text{div } v^\epsilon(s, X_\epsilon^{-(t_2-s)}(x))ds \right| \le C|t_1 - t_2|.$$

The other term can be treated exactly as before, when we proved equicontinuity in space. ☐

Recall that the function

$$u^\epsilon(t,x) = u_0^\epsilon(X_\epsilon^{-t}(x))\; a^\epsilon(t,x)$$

satisfies the ϵ-problem (2.27). We also have the following convergences:

$$\begin{aligned} u_0^\epsilon &\to u_0 && \text{in } L^p(\mathbb{R}^d), && (2.37)\\ X_{\epsilon_k}^{-t}(x) &\to X^{-t}(x) && \text{unif. on compact subset of } [0,T^*]\times\mathbb{R}^d, && (2.38)\\ a^{\epsilon_k}(t,x) &\to a(t,x) && \text{unif. on compact subset of } [0,T^*]\times\mathbb{R}^d, && (2.39)\\ v^{\epsilon_k}(t,x) &\to v(t,x) && \text{unif. on compact subset of } [0,T^*]\times\mathbb{R}^d. && (2.40) \end{aligned}$$

Define the function

$$u(t,x) := u_0(X^{-t}(x))\; a(t,x). \tag{2.41}$$

Convergence (2.37)-(2.40) together with Lemma 13 and 14 allow us to prove the following proposition.

Proposition 16. *: u, u^ϵ, uv and $u^\epsilon v^\epsilon$ all belong to the space $C([0,T^*), L^p(\mathbb{R}^d))$. Moreover we have:*

$$\begin{aligned} u^{\epsilon_k} &\to u && \text{in } C([0,T^*), L^p), && (2.42)\\ u^{\epsilon_k} v^{\epsilon_k} &\to uv && \text{in } C([0,T^*), L^p). && (2.43) \end{aligned}$$

Proof. Straight forward, see appendix at the end of the paper. ☐

We now turn to the proof of the main theorem of this section.

Proof of Theorem 6. Let $\phi \in C_c^\infty(0,T^*)$ be a scalar test function. It is obvious that u^ϵ and v^ϵ satisfy:

$$-\int_0^{T^*} u^\epsilon(t)\,\phi'(t)\,dt + \int_0^{T^*} \text{div}\big(\, u^\epsilon(t)\, v^\epsilon(t)\,\big)\;\phi(t)\;dt = 0, \tag{2.44}$$

$$v^\epsilon(t,x) = (u^\epsilon(t) * \nabla K^\epsilon)(x) \qquad \text{for all } (t,x)\in[0,T^*]\times\mathbb{R}^d, \tag{2.45}$$

$$u^\epsilon(0) = u_0^\epsilon, \tag{2.46}$$

where the integrals in (2.44) are the integral of a continuous function from $[0, T^*]$ to the Banach space $W^{-1,p}(\mathbb{R}^d)$. Recall that the injection $i : L^p(\mathbb{R}^d) \to W^{-1,p}(\mathbb{R}^d)$ and the differentiation $\partial_{x_i} : L^p(\mathbb{R}^d) \to W^{-1,p}(\mathbb{R}^d)$ are bounded linear operator. Therefore (2.42) and (2.43) imply

$$\begin{aligned} u^{\epsilon_k} \to u \qquad &\text{in } \ C([0,T^*], W^{-1,p}(\mathbb{R}^d)), \\ \text{div } [u^{\epsilon_k} v^{\epsilon_k}] \to \text{ div } [uv] \qquad &\text{in } \ C([0,T^*], W^{-1,p}(\mathbb{R}^d)), \end{aligned} \tag{2.47}$$

hich is more than enough to pass to the limit in relation (2.44). To pass to the limit in (2.45), it is enough to note that for all $(t, x) \in [0, T^*] \times \mathbb{R}^d$ we have

$$\begin{aligned} |(u^\epsilon(t) * \nabla K^\epsilon) - (u(t) * \nabla K)(x)| \leq \|u^\epsilon(t) - u(t)\|_{L^p} \|\nabla K^\epsilon\|_{L^q} \\ + \|u(t)\|_{L^p} \|\nabla K^\epsilon - \nabla K\|_{L^q}, \end{aligned}$$

and finally it is trivial to pass to the limit in relation (2.46).

Equation (2.44) means that the continuous function $u(t)$ (continuous function with values in $W^{-1,p}(\mathbb{R}^d)$) satisfies (2.12) in the distributional sense. But (2.12) implies that the distributional derivative $u'(t)$ is itself a continuous function with value in $W^{-1,p}(\mathbb{R}^d)$. Therefore $u(t)$ is differentiable in the classical sense, i.e, it belongs to $C^1([0, T^*], W^{-1,p}(\mathbb{R}^d))$, and (2.12) is satisfied in the classical sense.

We now turn to the proof of (2.15). The u^ϵ's satisfies (2.19). Integrating over $[0, t]$, $t < T^*$, we get

$$\|u^\epsilon(t)\|_{L^p}^p = \|u_0^\epsilon\|_{L^p}^p - (p-1) \int_0^t \int_{\mathbb{R}^d} u^\epsilon(s,x)^p \text{ div } v^\epsilon(s,x) \, dxdt. \tag{2.48}$$

Proposition 16 together with the general inequality

$$\||f|^p - |g|^p\|_{L^1} \leq 2p \left(\|f\|_{L^p}^{p-1} + \|g\|_{L^p}^{p-1} \right) \|f - g\|_{L^p} \tag{2.49}$$

implies that

$$(u^\epsilon)^p, u^p \in C([0,T^*], \ L^1(\mathbb{R}^d)), \tag{2.50}$$

$$(u^{\epsilon_k})^p \to u^p \in C([0,T^*], \ L^1(\mathbb{R}^d)). \tag{2.51}$$

On the other hand, replacing ∇K by ΔK in (**??**) we see right away that

$$\operatorname{div} v, \ \operatorname{div} v^\epsilon \in C([0,T^*], \ L^\infty(\mathbb{R}^d)), \tag{2.52}$$

$$\operatorname{div} v^{\epsilon_k} \to \operatorname{div} v \ \text{ in } \ C([0,T^*], L^\infty(\mathbb{R}^d)). \tag{2.53}$$

Combining (2.50)-(2.53) we obtain

$$u^p \operatorname{div} v, \ (u^\epsilon)^p \operatorname{div} v^\epsilon \in C([0,T^*], \ L^1(\mathbb{R}^d)), \tag{2.54}$$

$$(u^{\epsilon_k})^p \operatorname{div} v^{\epsilon_k} \to u^p \operatorname{div} v \ \text{ in } \ C([0,T^*], L^1(\mathbb{R}^d)). \tag{2.55}$$

So we can pass to the limit in (2.48) to obtain

$$\|u(t)\|_{L^p}^p = \|u_0\|_{L^p}^p - (p-1) \int_0^t \int_{\mathbb{R}^d} u(s,x)^p \text{ div } v(s,x) \, dxdt.$$

But (2.54) implies that the function $t \to \int_{\mathbb{R}^d} u(t,x)^p \operatorname{div} v(t,x)\, dx$ is continuous, therefore the function $t \to \|u(t)\|^p_{L^p}$ is differentiable and satisfies (2.15). $\square$

2.5. **Continuation and conserved properties.**

Theorem 17 (Continuation of solutions). *The solution provided by Theorem 6 can be continued up to a time $T_{\max} \in (0,+\infty]$. If $T_{\max} < +\infty$, then* $\lim_{t\to T_{\max}} \sup_{\tau\in[0,t]} \|u(\tau)\|_{L^p} = +\infty$

Proof. The proof is standard. One just needs to use the continuity of the solution with respect to time. $\square$

Theorem 18 (Conservation of mass/second moment). *(i) Under the assumption of Theorem 6, and if we assume moreover that $u_0 \in L^1(\mathbb{R}^d)$, then the solution u belongs to $C([0,T^*], L^1(\mathbb{R}^d))$ and satisfies $\|u(t)\|_{L^1} = \|u_0\|_{L^1}$ for all $t \in [0,T^*]$.*

(ii) Under the assumption of Theorem 6, and if we assume moreover that u_0 has bounded second moment, then the second moment of $u(t)$ stays bounded for all $t \in [0,T^*]$.

Proof. We just need to revisit the proof of Proposition 16. Since $u_0 \in L^1 \cap L^p$ it is clear that

$$u_0^\epsilon = J_\epsilon u_0 \to u_0 \text{ in } L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d). \tag{2.56}$$

Using convergences (2.56), (2.38), (2.39) and (2.40) we prove that

$$u^{\epsilon_k} \to u \text{ in } C([0,T^*],\ L^1 \cap L^p). \tag{2.57}$$

The proof is exactly the same than the one of Proposition 16 . Since the aggregation equation is a conservation law, it is obvious that the smooth approximates satisfy $\|u^\epsilon(t)\|_{L^1} = \|u_0^\epsilon\|_{L^1}$. Using (2.57) we obtain $\|u(t)\|_{L^1} = \|u_0\|_{L^1}$.

We now turn to the proof of (ii). Since the smooth approximates u_ϵ have compact support, their second moment is clearly finite, and the following manipulation are justified:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} |x|^2 u_\epsilon(t,x)dx &= 2\int_{\mathbb{R}^d} \vec{x}\cdot v_\epsilon\, du_\epsilon(x) \\
&\le 2\left(\int_{\mathbb{R}^d} |x|^2 u_\epsilon(t,x)dx\right)^{1/2}\left(\int_{\mathbb{R}^d} |v_\epsilon|^2 u_\epsilon(t,x)dx\right)^{1/2} \\
&\le C\left(\int_{\mathbb{R}^d} |x|^2 u_\epsilon(t,x)dx\right)^{1/2}.
\end{aligned} \tag{2.58}$$

Assume now that the second moment of u_0 is bounded. A simple computation shows that if η_ϵ is radially symmetric, then $|x|^2 * \eta_\epsilon = |x|^2 +$second moment of η_ϵ. Therefore

$$\int_{\mathbb{R}^d} |x|^2 u_0^\epsilon(x)dx \le \int_{\mathbb{R}^d} |x|^2 * \eta_\epsilon(x)\ u_0(x)dx$$

$$\begin{aligned} &\leq \int_{\mathbb{R}^d} |x|^2 u_0(x)dx + \int_{\mathbb{R}^d} |x|^2 \eta_\epsilon(x)dx \\ &\leq \int_{\mathbb{R}^d} |x|^2 u_0(x)dx + 1 \quad \text{ for } \epsilon \text{ small enough.} \end{aligned} \tag{2.59}$$

Inequality (2.59) come from the fact that the second moment of η_ϵ goes to 0 as ϵ goes to 0. Estimate (2.58) together with (2.59) provide us with a uniform bound of the second moment of the $u^\epsilon(t)$ which only depends the second moment of u_0. Since u^ϵ converges to u in L^1, we obviously have, for a given R and t:

$$\int_{|x|\leq R} |x|^2 u(t,x)dx = \lim_{\epsilon \longrightarrow 0} \int_{|x|\leq R} |x|^2 u_\epsilon(t,x)dx \leq \limsup_{\epsilon \longrightarrow 0} \int_{\mathbb{R}^d} |x|^2 u_\epsilon(t,x)dx.$$

Since R is arbitrary, this show that the second moment of $u(t,\cdot)$ is bounded for all t for which the solution exists. □

Combining Theorem 17 and 18 together with equality (2.15) we get:

Theorem 19 (Global existence when ΔK is bounded from above)**.** *Under the assumption of Theorem 6, and if we assume moreover that* $u_0 \in L^1(\mathbb{R}^d)$ *and ess sup* $\Delta K < +\infty$*, then the solution* u *exists for all times (i.e.:* $T_{\max} = +\infty$*).*

Proof. Equality (2.15) can be written

$$\frac{d}{dt}\left\{\|u(t)\|_{L^p}^p\right\} = (p-1)\int_{\mathbb{R}^d} u(t,x)^p (u(t) * \Delta K)(x)\, dx. \tag{2.60}$$

Since ΔK is bounded from above we have

$$(u(s) * \Delta K)(x) \leq (\text{ess sup } \Delta K) \int_{\mathbb{R}^d} u(s,x)dx = (\text{ess sup } \Delta K)\, \|u_0\|_{L^1}. \tag{2.61}$$

Combining (2.60), (2.61) and Gronwall inequality gives

$$\|u(t)\|_{L^p}^p \leq \|u_0\|_{L^p}^p\, e^{(p-1)(\text{ess sup } \Delta K)\|u_0\|_{L^1} t},$$

so the L^p-norm can not blow-up in finite time which, because of Theorem 17, implies global existence. □

3. Uniqueness of solutions in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$

In this section we use an optimal transport argument to prove uniqueness of solutions in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, when $\nabla K \in W^{1,q}$ for $1 < q < \infty$ and p its Hölder conjugate. To do that, we shall follow the steps in [19], where the authors extend the work of Loeper [34] to prove, among other, uniqueness of $\mathcal{P}_2 \cap L^\infty$-solutions of the aggregation equation when the interaction potential K has a Lipschitz singularity at the origin.

One can easily check that solutions of the aggregation equation constructed in previous sections are distribution solutions, i.e. they satisfy

$$\int_0^T \int_{\mathbb{R}^N} \left(\frac{\partial \varphi}{\partial t}(t,x) + v(t,x) \cdot \nabla \varphi(t,x) \right) u(t,x)\, dx\, dt = \int_{\mathbb{R}^N} \varphi(0,x) u_0(x)\, dx \tag{3.62}$$

for all $\varphi \in C_0^\infty([0,T^*) \times \mathbb{R}^N))$. A function $u(t,x)$ satisfying (3.62) is said to be a distribution solution to the continuity equation (2.12) with the given velocity field $v(t,x)$ and initial data $u_0(x)$. In fact, it is uniquely characterized by

$$\int_B u(t,x)\, dx = \int_{X^{-t}(B)} u_0(x)\, dx$$

for all measurable set $B \subset \mathbb{R}^d$, see [1]. Here $X^t : \mathbb{R}^d \to \mathbb{R}^d$ is the flow map associated with the velocity field $v(t,x)$ and X^{-t} is its inverse. In the optimal transport terminology this is equivalent to say that X^t transports the measure u_0 onto $u(t)$ ($u(t) = X^t \# u_0$).

We recall, for the sake of completeness, [19, Theorem 2.4], where several results of [2, 35, 22, 1] are put together.

Theorem 20 ([19]). *Let ρ_1 and ρ_2 be two probability measures on $\mathbb{R}^N$, such that they are absolutely continuous with respect to the Lebesgue measure and $W_2(\rho_1, \rho_2) < \infty$, and let ρ_θ be an interpolation measure between ρ_1 and ρ_2, defined as in* [34] *by*

$$\rho_\theta = ((\theta - 1)T + (2 - \theta)\mathbb{I}_{\mathbb{R}^N})_{\#}\rho_1 \tag{3.63}$$

for $\theta \in [1,2]$, where T is the optimal transport map between ρ_1 and ρ_2 due to Brenier's theorem [12] *and $\mathbb{I}_{\mathbb{R}^N}$ is the identity map. Then there exists a vector field $\nu_\theta \in L^2(\mathbb{R}^N, \rho_\theta\, dx)$ such that*

i. $\dfrac{d}{d\theta}\rho_\theta + \operatorname{div}(\rho_\theta \nu_\theta) = 0$ *for all* $\theta \in [1,2]$.
ii. $\displaystyle\int_{\mathbb{R}^N} \rho_\theta |\nu_\theta|^2\, dx = W_2^2(\rho_1, \rho_2)$ *for all* $\theta \in [1,2]$.
iii. *We have the L^p-interpolation estimate*

$$\|\rho_\theta\|_{L^p(\mathbb{R}^N)} \leq \max\left\{ \|\rho_1\|_{L^p(\mathbb{R}^N)}, \|\rho_2\|_{L^p(\mathbb{R}^N)} \right\}$$

for all $\theta \in [1,2]$.

Here, $W_2(f,g)$ is the Euclidean Wasserstein distance between two probability measures $f, g \in \mathcal{P}(\mathbb{R}^n)$,

$$W_2(f,g) = \inf_{\Pi \in \Gamma} \left\{ \iint_{\mathbb{R}^n \times \mathbb{R}^n} |v - x|^2\, d\Pi(v,x) \right\}^{1/2}, \tag{3.64}$$

where Π runs over the set of joint probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals f and g.

Now we are ready to prove the uniqueness of solutions to the aggregation equation.

Theorem 21 (Uniqueness)**.** *Let u_1, u_2 be two bounded solutions of equation* (2.12) *in the interval $[0, T^*]$ with initial data $u_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $1 < p < \infty$ and assume that v is given by $v = -\nabla K * u$, with K such that $\nabla K \in W^{1,q}(\mathbb{R}^N)$, p and q conjugates. Then $u_1(t) = u_2(t)$ for all $0 \le t \le T^*$.*

Proof. Consider two characteristics flow maps, X_1 and X_2, such that $u_i = X_i \# u_0$, $i = 1, 2$. Define the quantity

$$Q(t) := \frac{1}{2} \int_{\mathbb{R}^N} |X_1(t) - X_2(t)|^2 u_0(x)\, \mathrm{d}x, \tag{3.65}$$

From [19, Remark 2.3], we have $W_2^2(u_1(t), u_2(t)) \le 2Q(t)$ which we now prove is zero for all times, implying that $u_1 = u_2$. Now, to see that $Q(t) \equiv 0$ we compute the derivative of Q with respect to time.

$$\begin{aligned}
\frac{\partial Q}{\partial t} &= \int_{\mathbb{R}^N} \langle X_1 - X_2, v_1(x_1) - v_2(x_2) \rangle \rho_0(x) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \langle X_1 - X_2, v_1(x_1) - v_1(x_2) \rangle \rho_0(x) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \langle X_1 - X_2, v_1(x_2) - v_2(x_2) \rangle \rho_0(x) \mathrm{d}x
\end{aligned}$$

where the time variable has been omitted for clarity. The above argument is justified because, due to Lemma 7, the velocity field is C^1 and bounded. Taking into account the Lipschitz properties of v into the first integral and using Hölder inequality in the second one, we can write

$$\begin{aligned}
\frac{\partial Q}{\partial t} &\le CQ(t) + Q(t)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |u_1\left(X_2(t,x)\right) - u_2\left(X_2(t,x)\right)|^2 \rho_0(x) \mathrm{d}x \right)^{\frac{1}{2}} \\
&= CQ(t) + Q(t)^{\frac{1}{2}} I(t)^{\frac{1}{2}}.
\end{aligned} \tag{3.66}$$

Now, in order to estimate $I(t)$, we use that the solutions are constructed transporting the initial data through their flow maps, so we can write it as

$$I(t) = \int_{\mathbb{R}^N} |\nabla K * (u_1 - u_2)\left[X_2(t,x)\right]|^2 u_0(x)\, \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla K * (u_1 - u_2)(x)|^2 u_2(x)\, \mathrm{d}x.$$

Thus, taking an interpolation measure ρ_θ between ρ_1 and ρ_2 and using Hölder inequality and first statement of Theorem 20 we can get a bound for $I(t)$

$$I(t) \le \left(\int_{\mathbb{R}^N} \left| \nabla K * \left(\int_1^2 \partial_\theta u_\theta \right) \right|^{2q} \right)^{1/q} \|u_2(t)\|_{L^p} \tag{3.67}$$

$$\le \int_1^2 \|D_2 K * (\nu_\theta u_\theta)\|^2_{L^{2q}} d\theta \;\; \|u_2(t)\|_{L^p}, \tag{3.68}$$

where $\nu_\theta \in \mathrm{L}^2(\mathbb{R}^N, \upsilon_\theta \mathrm{d}x)$ is a vector field, as described in Theorem 20. Let us work on the first term of the right hand side. Using Young inequality, for α such that $1 + \frac{1}{2q} = 1/q + 1/\alpha$ we obtain

$$\int_1^2 \|D_2 K * (\nu_\theta u_\theta)\|_{L^{2q}}^2 d\theta \le \int_1^2 \|D_2 K\|_{L^q}^2 \|\nu_\theta u_\theta\|_{L^\alpha}^2 d\theta. \tag{3.69}$$

Note that $q \in (1, +\infty)$ implies $\alpha \in (1, 2)$. Therefore we can use Hölder inequality with conjugate exponents $2/(2-\alpha)$ and $2/\alpha$ to obtain

$$\|\nu_\theta u_\theta\|_{L^\alpha}^2 = \left(\int |u_\theta|^{\alpha/2} |u_\theta|^{\alpha/2} |\nu_\theta|^\alpha\right)^{2/\alpha} \le \left(\int |u_\theta|^{\alpha/(2-\alpha)}\right)^{(2-\alpha)/\alpha} \left(\int |u_\theta| |\nu_\theta|^2\right) \tag{3.70}$$

whence, since we can see from simple algebraic manipulations with the exponents that $\frac{\alpha}{2-\alpha} = p$, the conjugate of q,

$$\int_1^2 \|D_2 K * (\nu_\theta u_\theta)\|_{L^{2q}}^2 d\theta \le \|D_2 K\|_{L^q}^2 \int_1^2 \|u_\theta\|_{L^p} \left(\int |u_\theta| |\nu_\theta|^2\right) d\theta. \tag{3.71}$$

Therefore, using statements (ii) and (iii) of Theorem 20 we obtain

$$I(t) \le \|u_2\|_{L^p} \max\{\|u_1\|_{L^p}, \|u_2\|_{L^p}\} \|D_2 K\|_q^2 W_2^2(u_1, u_2) \le CQ(t). \tag{3.72}$$

Finally, going back to (3.66) we see that $\frac{\mathrm{d}Q}{\mathrm{d}t} \le Q(t)$, whence, since $Q(0) = 0$, we can conclude $Q(t) \equiv 0$ and thus $u_1 = u_2$. The limiting case $p = \infty$ is the one studied in [19]. □

Remark 22. *Note that in order to make the above argument rigorous, we need the gradient of the kernel to be at least C^1 when estimating I. It is not the case here, but we can still obtain the estimate using smooth approximations. let us define*

$$I^\epsilon(t) = \int_{\mathbb{R}^N} |\nabla K^\epsilon * (u_1 - u_2)[X_2(t, x)]|^2 u_0(x)\, \mathrm{d}x$$

where $K^\epsilon = J_\epsilon K$ (see section 2.2). Since ∇K^ϵ converges to ∇K in L^q, it is clear that $\nabla K^\epsilon * (u_1 - u_2)$ converges pointwise to $\nabla K * (u_1 - u_2)$. Using the dominated convergence theorem together with the fact that $\|\nabla K^\epsilon * (u_1 - u_2)\|_{L^\infty}$ is uniformly bounded we get that $I^\epsilon(t)$ converges to $I(t)$ for every $t \in (0, T)$.

On the other hand, due to the definition of u_θ we can write the difference $u_2 - u_1$ as the integral between 1 and 2 of $\partial_\theta u_\theta$ with respect to θ. Now, since the equation $\partial_\theta u_\theta + \operatorname{div}(u_\theta \nu_\theta) = 0$ is satisfied in the sense of distribution, and $\nabla K^\epsilon \in C_c^\infty(\mathbb{R}^d)$, we can replace $\partial_\theta u_\theta$ for $\operatorname{div}(\nu_\theta u_\theta)$ and pass the divergence to the other term of the convolution, so that the equality

$$\int_1^2 (D^2 K^\epsilon * \nu_\theta u_\theta)(x) d\theta = \nabla K^\epsilon * (u_2 - u_1)(x)$$

holds for all $x \in \mathbb{R}^d$. The rest of the manipulations performed above are straight forward with K^ϵ. passing to the limit in (3.72) is easy since $D^2 K^\epsilon$ converges to $D^2 K$ in L^q.

4. Instantaneous mass concentration when $K(x) = |x|$

In this section we consider the aggregation equation with an interaction potential equal to $|x|$ in a neighborhood of the origin and whose gradient is compactly supported (or decay exponentially fast at infinity). The Laplacian of this kind of potentials has a $1/|x|$ singularity at the origin, therefore ∇K belongs to $W^{1,q}(\mathbb{R}^d)$ if and only if $q \in [1, d)$. The Hölder conjugate of d is $\frac{d}{d-1}$. Using the theory developed in section 2 and 3 we therefore get local existence and uniqueness of solutions in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for all $p > \frac{d}{d-1}$. Here we study the case where the initial data is in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for $p < \frac{d}{d-1}$.

Given $p < \frac{d}{d-1}$ we exhibit initial data in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for which the solution instantaneously concentrates mass at the origin (i.e. a delta Dirac at the origin is created instantaneously). This shows that the existence theory developed in section 2 and 3 is in some sense sharp. This also shows that it is possible for a solution to lose instantaneously its absolute continuity with respect to the Lebesgue measure.

The solutions constructed in this section have compact support, hence we can simply consider $K(x) = |x|$ without changing the behavior of the solution, given that if the solution has a small enough support, it only feels the part of the potential around the origin.

We build on the work developed in [16] on global existence for measure solutions with bounded second moment:

Theorem 23 (Existence and uniqueness of measure solutions [16])**.** *Suppose $K(x) = |x|$. Given $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, there exists a unique weakly continuous family of probability measures $(\mu_t)_{t\in(0,+\infty)}$ satisfying*

$$\partial_t \mu_t + div(\mu_t v_t) = 0 \text{ in } \mathcal{D}'((0,\infty) \times \mathbb{R}^d), \tag{4.73}$$

$$v_t = -\partial^0 K * \mu_t, \tag{4.74}$$

$$\mu_t \text{ converges weakly to } \mu_0 \text{ as } t \to 0. \tag{4.75}$$

Here $\partial^0 K$ is the unique element of minimal norm in the subdifferential of K. Simply speaking, since $K(x) = |x|$ is smooth away from the origin and radially symmetric, we have $\partial^0 K(x) = \frac{x}{|x|}$ for $x \neq 0$ and $\partial^0 K(0) = 0$, and thus:

$$(\partial^0 K * \mu)(x) = \int_{y\neq x} \frac{x-y}{|x-y|}\, d\mu(y). \tag{4.76}$$

Note that, μ_t being a measure, it is important for $\partial^0 K$ to be defined for every $x \in \mathbb{R}^d$ so that (4.74) makes sense. Equation (4.73) means that

$$\int_0^{+\infty} \int_{\mathbb{R}^d} \Big(\frac{d\psi}{dt}(x,t) + \nabla\psi(x,t) \cdot v_t(x) \Big)\, d\mu_t(x)\, dt = 0, \tag{4.77}$$

for all $\psi \in C_0^\infty(\mathbb{R}^d \times (0, +\infty))$. From (4.76) it is clear that $|v_t(x)| \leq 1$ for all x and t, therefore the above integral makes sense.

The main Theorem of this section is the following:

Theorem 24 (Instantaneous mass concentration). *Consider the initial data*

$$u_0(x) = \begin{cases} \frac{L}{|x|^{d-1+\epsilon}} & \textit{if } |x| < 1, \\ 0 & \textit{otherwise}, \end{cases} \tag{4.78}$$

where $\epsilon \in (0,1)$ *and* $L := \left(\int_{|x|<1} |x|^{-(d-1+\epsilon)} dx\right)^{-1}$ *is a normalizing constant. Note that* $u_0 \in L^p(\mathbb{R}^d)$ *for all* $p \in [1, \frac{d}{d-1+\epsilon})$*. Let* $(\mu_t)_{t\in(0,+\infty)}$ *be the unique measure solution of the aggregation equation with interaction potential* $K(x) = |x|$ *and with initial data* u_0*. Then, for every* $t > 0$ *we have*

$$\mu_t(\{0\}) > 0,$$

i.e., mass is concentrated at the origin instantaneously and the solution is no longer continuous with respect to the Lebesgue measure.

Theorem 24 is a consequence of the following estimate on the velocity field:

Proposition 25. *Let* $(\mu_t)_{t\in(0,+\infty)}$ *be the unique measure solution of the aggregation equation with interaction potential* $K(x) = |x|$ *and with initial data* (4.78)*. Then, for all* $t \in [0, +\infty)$ *the velocity field* $v_t = -\partial^0 K * \mu_t$ *is focussing and there exists a constant* $C > 0$ *such that*

$$|v_t(x)| \geq C|x|^{1-\epsilon} \qquad \textit{for all } t \in [0, +\infty) \textit{ and } x \in B(0, 1). \tag{4.79}$$

By focussing, we mean that the velocity field points inward, i.e. there exists a nonnegative function $\lambda_t : [0, +\infty) \to [0, +\infty)$ such that $v_t(x) = -\lambda_t(|x|)\frac{\vec{x}}{x}$.

4.1. Representation formula for radially symmetric measure solutions.

In this section, we show that for radially symmetric measure solutions, the characteristics are well defined. As a consequence, the solution to (2.12) can be expressed as the push forward of the initial data by the flow map associated with the ODE defining the characteristics.

In the following the unit sphere $\{x \in \mathbb{R}^d, |x| = 1\}$ is denoted by S_d and its surface area by ω_d.

Definition 26. *If* $\mu \in \mathcal{P}(\mathbb{R}^d)$ *is a radially symmetric probability measure, then we define* $\hat{\mu} \in \mathcal{P}([0, +\infty))$ *by*

$$\hat{\mu}(I) = \mu(\{x \in \mathbb{R}^d : |x| \in I\})$$

for all $I \in \mathcal{B}([0, +\infty))$*.*

Remark 27. *If a measure* μ *is radially symmetric, then* $\mu(\{x\}) = 0$ *for all* $x \neq 0$*, and therefore*

$$\int_{\mathbb{R}^d\setminus\{x\}} \nabla K(x - y)\, d\mu(y) = \int_{\mathbb{R}^d} \nabla K(x - y)\, d\mu(y) \qquad \textit{for all } x \neq 0.$$

In other words, for $x \neq 0$, $(\nabla K * \mu)(x)$ is well defined despite the fact that ∇K is not defined at $x = 0$. As a consequence $(\partial^0 K * \mu)(x) = (\nabla K * \mu)(x)$ if $x \neq 0$ and $(\partial^0 K * \mu)(0) = 0$.

Remark 28. *If the radially symmetric measure μ is continuous with respect to the Lebesgue measure and has radially symmetric density $u(x) = \tilde{u}(|x|)$, then $\hat{\mu}$ is also continuous with respect to the Lebesgue measure and has density $\hat{u}$, where*

$$\hat{u}(r) = \omega_d r^{d-1} \tilde{u}(r). \tag{4.80}$$

Lemma 29 (Polar coordinate formula for the convolution)**.** *Suppose $\mu \in \mathcal{P}(\mathbb{R}^d)$ is radially symmetric. Let $K(x) = |x|$, then for all $x \neq 0$ we have:*

$$(\mu * \nabla K)(x) = \left(\int_0^{+\infty} \phi\left(\frac{|x|}{\rho}\right) \, d\hat{\mu}(\rho) \right) \frac{x}{|x|} \tag{4.81}$$

where the function $\phi : [0, +\infty) \to [-1, 1]$ is defined by

$$\phi(r) = \frac{1}{\omega_d} \int_{S_d} \frac{re_1 - y}{|re_1 - y|} \cdot e_1 d\sigma(y). \tag{4.82}$$

Proof. This come from simple algebraic manipulations. These manipulations are shown in [5]. □

In the next Lemma we state properties of the function ϕ defined in (4.82).

Lemma 30 (Properties of the function ϕ)**.**

(i) *ϕ is continuous and non-decreasing on $[0, +\infty)$. Moreover $\phi(0) = 0$, and $\lim_{r\to\infty} \phi(r) = 1$.*

(ii) *$\phi(r)$ is $O(r)$ as $r \to 0$. To be more precise:*

$$\lim_{\substack{r\to 0 \\ r>0}} \frac{\phi(r)}{r} = 1 - \frac{1}{\omega_d} \int_{S_d} (y \cdot e_1)^2 \, d\sigma(y). \tag{4.83}$$

Proof. Consider the function $F : [0, +\infty) \times S_d \to [-1, 1]$ defined by

$$(r, y) \mapsto \frac{re_1 - y}{|re_1 - y|} \cdot e_1. \tag{4.84}$$

Since F is bounded, we have that

$$\phi(r) = \frac{1}{\omega_d} \int_{S_d} F(r, y) d\sigma(y) = \frac{1}{\omega_d} \int_{S_d \setminus \{e_1\}} F(r, y) d\sigma(y).$$

If $y \in S_d \setminus \{e_1\}$ then the function $r \mapsto F(r, y)$ is continuous on $[0, +\infty)$ and C^∞ on $(0, +\infty)$. An explicit computation shows then that

$$\frac{\partial F}{\partial r}(r, y) = \frac{1 - F(r, y)^2}{|re_1 - y|} \geq 0, \tag{4.85}$$

thus ϕ is non-decreasing and, by the Lebesgue dominated convergence, it is easy to see that ϕ is continuous, $\phi(0) = 0$ and $\lim_{r\to\infty} \phi(r) = 1$, which prove (i).

To prove (ii), note that the function $\frac{\partial F}{\partial r}(r,y)$ can be extended by continuity on $[0,+\infty)$. Therefore the right derivative with respect to r of $F(r,y)$ is well defined:

$$\lim_{\substack{r\to 0\\ r>0}} \frac{F(r,y)-F(0,y)}{r} = \frac{1-F(0,y)^2}{|y|} = 1-(y\cdot e_1)^2.$$

and since $\partial F/\partial r$ is bounded on $(0,+\infty)\times S_d\backslash\{e_1\}$, we can now use the Lebesgue dominated convergence theorem to conclude:

$$\begin{aligned}\lim_{\substack{r\to 0\\ r>0}} \frac{\phi(r)}{r} &= \lim_{\substack{r\to 0\\ r>0}} \frac{\phi(r)-\phi(0)}{r}\\ &= \lim_{\substack{r\to 0\\ r>0}} \frac{1}{\omega_d}\int_{S_d} \frac{F(r,y)-F(0,y)}{r} d\sigma(y)\\ &= \frac{1}{\omega_d}\int_{S_d} 1-(y\cdot e_1)^2 d\sigma(y).\end{aligned}$$ $\square$

Remark 31. *Note that for $r>0$ the function $\rho\mapsto\phi\left(\frac{r}{\rho}\right)$ is non increasing and continuous. Indeed, it is equal to 1 when $\rho=0$ and it decreases to 0 as $\rho\to\infty$. In particular, the integral in (4.81) is well defined for any probability measure $\hat{\mu}\in\mathcal{P}([0,+\infty))$.*

Remark 32. *In dimension two, it is easy to check that $\lim_{r\to 1}\phi'(r)=+\infty$ which implies that the derivative of the function ϕ has a singularity at $r=1$ and thus, that the function ϕ is not C^1.*

Proposition 33 (Characteristic ODE)**.** *Let $K(x)=|x|$ and let $(\mu_t)_{t\in[0,+\infty)}$ be a weakly continuous family of radially symmetric probability measures. Then the velocity field*

$$v(x,t)=\begin{cases}-(\nabla K*\mu_t)(x) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases} \tag{4.86}$$

is continuous on $\mathbb{R}^d\backslash\{0\}\times[0+\infty)$. Moreover, for every $x\in\mathbb{R}^d$, there exists an absolutely continuous function $t\to X_t(x)$, $t\in[0,+\infty)$, which satisfies

$$\frac{d}{dt}X_t(x)=v(X_t(x),t) \qquad \text{for a.e. } t\in(0,+\infty), \tag{4.87}$$

$$X_0(x)=x. \tag{4.88}$$

Proof. From formula (4.81), Remark 31, and the weak continuity of the family $(\mu_t)_{t\in[0,+\infty)}$, we obtain continuity in time. The continuity in space simply comes from the continuity and boundedness of the function ϕ together with the Lebesgue dominated convergence theorem.

Since v is continuous on $\mathbb{R}^d\backslash\{0\}\times[0+\infty)$ we know from the Peano theorem that given $x\in\mathbb{R}^d\backslash\{0\}$, the initial value problem (4.87)-(4.88) has a C^1 solution at least for short time. We want to see that it is defined for all time. For that, note that by a continuation argument, the interval given by Peano theorem can be extended as long as the solution stays in $\mathbb{R}^d\backslash\{0\}$. Then, if we denote by T_x the maximum time so that the solution exists in $[0, T_x)$ we have that either $T_x = \infty$ and we are done, or $T_x < +\infty$, in which case clearly $\lim_{t\to T_x} X_t(x) = 0$, and we can extend the function $X_t(x)$ on $[0, +\infty)$ by setting $X_t(x) := 0$ for $t \geq T_x$.

The function $t \to X_t(x)$ that we have just constructed is continuous on $[0, +\infty)$, C^1 on $[0, +\infty)\backslash\{T_x\}$ and satisfies (4.87) on $[0, +\infty)\backslash\{T_x\}$. If $x = 0$, we obviously let $X_t(x) = 0$ for all $t \geq 0$. □

Finally, we present the representation formula, by which we express the solution to (2.12) as a push-forward of the initial data. See [1] or [41] for a definition of the push-forward of a measure by a map.

Proposition 34 (Representation formula)**.** *Let $(\mu_t)_{t\in[0,+\infty)}$ be a radially symmetric measure solution of the aggregation equation with interaction potential $K(x) = |x|$, and let $X_t : \mathbb{R}^d \to \mathbb{R}^d$ be defined by (4.86),* (4.87) *and (4.88). Then for all $t \geq 0$,*

$$\mu_t = X_t\#\mu_0.$$

Proof. In this proof, we follow arguments from [1]. Since for a given x the function $t \mapsto X_t(x)$ is continuous, one can easily prove, using the Lebesgue dominated convergence theorem, that $t \mapsto X_t\#\mu_0$ is weakly continuous. Let us now prove that $\mu_t := X_t\#\mu_0$ satisfies (4.77) for all $\psi \in \mathbb{C}_0^\infty(\mathbb{R}^d \times (0, \infty))$. Given that the test function ψ is compactly supported, there exist $T > 0$ such that $\psi(x, t) = 0$ for all $t \geq T$. We therefore have:

$$0 = \int_{\mathbb{R}^d} \psi(x, T) d\mu_T(x) - \int_{\mathbb{R}^d} \psi(x, 0) d\mu_0(x) = \int_{\mathbb{R}^d} \Big(\psi(X_T(x), T) - \psi(x, 0) \Big) d\mu_0(x). \tag{4.89}$$

If we now take into account that from Proposition 33 the mapping $t \to \phi(X_t(x), t)$ is absolutely continuous, we can rewrite (4.89) as

$$\begin{aligned}
0 &= \int_{\mathbb{R}^d} \int_0^T \Big(\frac{d}{dt}\psi(X_t(x), t) \Big) dt \; d\mu_0(x) && (4.90)\\
&= \int_{\mathbb{R}^d} \int_0^T \Big(\nabla\psi(X_t(x), t) \cdot v(X_t(x), t) + \frac{d\psi}{dt}(X_t(x), t) \Big) dt \; d\mu_0(x) && (4.91)\\
&= \int_0^T \int_{\mathbb{R}^d} \Big(\nabla\psi(X_t(x), t) \cdot v(X_t(x), t) + \frac{d\psi}{dt}(X_t(x), t) \Big) d\mu_0(x) \; dt && (4.92)\\
&= \int_0^T \int_{\mathbb{R}^d} \Big(\nabla\psi(x, t) \cdot v(x, t) + \frac{d\psi}{dt}(x, t) \Big) d\mu_t(x) \; dt.
\end{aligned}$$

The step from (4.91) to (4.92) holds because of the fact that $|v(x, t)| \leq 1$, which justifies the use of the Fubini Theorem. □

Remark 35 (Representation formula in polar coordinates)**.** *Let μ_t and X_t be as in the previous proposition. Let $R_t : [0, +\infty) \to [0, +\infty)$ be the function such that $|X_t(x)| = R_t(|x|)$. Then*

$$\hat{\mu}_t = R_t \# \hat{\mu}_0. \tag{4.93}$$

Remark 36. *Since ϕ is nonnegative (Lemma 30), from* (4.81), (4.86) *and* (4.87) *we see that the function $t \mapsto |X_t(x)| = R_t(|x|)$ is non increasing.*

4.2. **Proof of Proposition 25 and Theorem 24.** We are now ready to prove the estimate on the velocity field and the instantaneous concentration result. We start by giving a frozen in time estimate of the velocity field.

Lemma 37. *Let $K(x) = |x|$, and let $u_0(x)$ be defined by* (4.78) *for some $\epsilon \in (0, 1)$. Then there exist a constant $C > 0$ such that*

$$|(u_0 * \nabla K)(x)| \geq C|x|^{1-\epsilon} \tag{4.94}$$

for all $x \in B(0, 1)\backslash\{0\}$.

Proof. Note that if we do the change of variable $s = \frac{|x|}{\rho}$ in equation (4.81), we find that

$$|(u_0 * \nabla K)(x)| = |x| \int_0^{+\infty} \phi(s)\ \hat{u}_0(\frac{|x|}{s}) \frac{ds}{s^2}. \tag{4.95}$$

On the other hand, using (4.80) we see that the $\hat{u}_0(r)$ corresponding to the $u_0(x)$ defined by (4.78) is

$$\hat{u}_0(r) = \begin{cases} \frac{\omega_d}{r^\epsilon} & \text{if } r < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.96}$$

Then, plugging (4.96) in (4.95) we obtain that for all $x \neq 0$

$$|(u_0 * \nabla K)(x)| = \omega_d |x|^{1-\epsilon} \int_{|x|}^{+\infty} \frac{\phi(\rho)}{\rho^{2-\epsilon}} d\rho. \tag{4.97}$$

In light of statement (ii) of Lemma 30, we see that the previous integral converges as $|x| \to 0$. Hence $|(u_0 * \nabla K)(x)|$ is $O(|x|^{1-\epsilon})$ as $|x| \to 0$ and (4.94) follows. $\square$

Finally, the last piece we need in order to prove Proposition 25 from the previous lemma, is the following comparison principle:

Lemma 38 (Temporal monotonicity of the velocity)**.** *Let $(\mu_t)_{t\in(0,+\infty)}$ be a radially symmetric measure solution of the aggregation equation with interaction potential $K(x) = |x|$. Then, for every $x \in \mathbb{R}^d\backslash\{0\}$ the function*

$$t \mapsto |(\nabla K * \mu_t)(x)|$$

is non decreasing.

Proof. Combining (4.81) and (4.93) we see that

$$|(\mu_t * \nabla K)(x)| = \left(\int_0^{+\infty} \phi(\frac{|x|}{R_t(\rho)})\, d\hat{\mu}_0(\rho) \right). \tag{4.98}$$

Now, by Lemma 30, ϕ is non decreasing and due to Remark 35, $t \mapsto R_t(\rho)$ is non increasing. Henceforth it is clear that (4.98) is itself non decreasing. □

At this point, Proposition 25 follows as a simple consequence of the frozen in time estimate (4.94) together with Lemma 38, and we can give an easy proof for the main result we introduced at the beginning of the section.

Proof of Theorem 24. Using the representation formula (Proposition 34) and the definition of the push forward we get

$$\mu_t(\{0\}) = (X_t \# \mu_0)(\{0\}) = \mu_0(X_t^{-1}(\{0\})).$$

Then, note that the solution of the ODE $\dot{r} = -r^{1-\epsilon}$ reaches zero in finite time. Therefore, from Proposition 25 and 33 we obtain that for all $t > 0$, there exists $\delta > 0$ such that

$$X_t(x) = 0 \qquad \text{for all } |x| < \delta.$$

In other words, for all $t > 0$, there exists $\delta > 0$ such that $B(0, \delta) \subset X_t^{-1}(\{0\})$. Clearly, given our choice of initial condition, we have that $\mu_0(B(0, \delta)) > 0$ if $\delta > 0$, and therefore $\mu_t(\{0\}) > 0$ if $t > 0$. □

4.3. **Remark about the initial data** $1/|x|^{d-1}$**.** An interesting open problem is wether or not there is instantaneous mass concentration when the initial data is defined by (4.78) with $\epsilon = 0$. We right now can not answer this question, but below are some interesting remarks about this case.

Lemma 39. *Let u_0 be defined by (4.78) with $\epsilon = 0$. Then there exists constants $C > 0$ and $\beta > 1$ such that*

$$|(\nabla K * u_0)(x)| \le C|x| \left| \log \frac{|x|}{\beta} \right| \qquad \textit{for all } x \in B(0,1). \tag{4.99}$$

Proof. From Lemma 30 it is clear that there exists a constant $\alpha > 0$ such that $\phi(r)/r \le \alpha$ for all $r \in (0, 1)$. We can then use (4.97) with $\epsilon = 0$ to obtain that, for $|x| < 1$,

$$\begin{aligned}
|(u_0 * \nabla K)(x)| &= \omega_d\, |x| \left(\int_{|x|}^1 \frac{\phi(\rho)}{\rho} \frac{d\rho}{\rho} + \int_1^\infty \frac{\phi(\rho)}{\rho^2} d\rho \right) \\
&\le \omega_d\, |x| \left(\alpha \int_{|x|}^1 \frac{d\rho}{\rho} + \text{cst} \right) = \omega_d\, |x| \left(-\alpha \log |x| + \text{cst} \right) \\
&= -C|x| \log \frac{|x|}{\beta},
\end{aligned}$$

for some constants $C > 0$ and $\beta > 1$. □

Consider the pushforward $S_t(r)$ defined by the ODE

$$\frac{d}{dt}S_t(r) = -CS_t(r)\left|\log\frac{S_t(r)}{\beta}\right|,$$
$$S_0(r) = r.$$

This ODE has an explicit solution: for $r < 1$ we have

$$S_t(r) = \beta\left(\frac{r}{\beta}\right)^{e^{Ct}}$$

The push forward of $\hat{u}_0$ by the map S_t can also be explicitly computed:

$$(S_t\#\hat{u}_0)(r) = \begin{cases} \frac{L(t)}{r^{\alpha(t)}} & \text{if } r < \beta^{1-\exp(Ct)} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{where } \alpha(t) = \frac{e^{Ct}-1}{e^{Ct}}, \quad \text{and } L(t) = \int_0^{\beta^{1-\exp(Ct)}} r^{-\alpha(t)}dr.$$

So the push forward of u_0 by this flow is

$$(X_t\#u_0)(x) = \begin{cases} \frac{L(t)}{|x|^{d-1+\alpha(t)}} & \text{if } |x| < \beta^{1-\exp(Ct)} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{where } \alpha(t) = \frac{e^{Ct}-1}{e^{Ct}}, \quad \text{and } L(t) = \int_{|x|\le\beta^{1-\exp(Ct)}} \frac{1}{|x|^{d-1+\alpha(t)}}dr.$$

5. Osgood condition for global well-posedness

This section considers the global well-posedness of the aggregation equation in $\mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, depending on the potential K. We start by giving a precise definitions of "natural potential", "repulsive in the short range" and "strictly attractive in the short range", and then we prove Theorem 5.

Definition 40. *A* natural potential *is a radially symmetric potential $K(x) = k(|x|)$, where $k : (0,+\infty) \to \mathbb{R}$ is a smooth function which satisfies the following conditions:*

(C1) $\sup_{r\in(0,\infty)} |k'(r)| < +\infty$,

(C2) $\exists\, \alpha > d$ *such that $k'(r)$ and $k''(r)$ are $O(1/r^\alpha)$ as $r \to +\infty$,*

(MN1) $\exists\, \delta_1 > 0$ *such that $k''(r)$ is monotonic (either increasing or decreasing) in $(0, \delta_1)$,*

(MN2) $\exists\, \delta_2 > 0$ *such that $rk''(r)$ is monotonic (either increasing or decreasing) in $(0, \delta_2)$.*

Remark 41. *Note that monotonicity condition (MN1) implies that $k'(r)$ and $k(r)$ are also monotonic in some (different) neighborhood of the origin $(0, \delta)$. Also, note that (C1) and (MN1) imply*

(C3) $\lim_{r\to 0^+} k'(r)$ *exists and is finite.*

Remark 42. *The far field condition (C2) can be dropped when the data has compact support.*

Definition 43. *A natural potential is said to be repulsive in the short range if there exists an interval* $(0, \delta)$ *on which* $k(r)$ *is decreasing. A natural potential is said to be strictly attractive in the short range if there exists an interval* $(0, \delta)$ *on which* $k(r)$ *is strictly increasing.*

We would like to remark that the two monotonicity conditions are not very restrictive as, in order to violate them, a potential would have to exhibit some pathological behavior around the origin, like oscillating faster and faster as $r \to 0$.

5.1. **Properties of natural potentials.** As a last step before proving Theorem 5, let us point out some properties of natural potentials, which show the reason behind the choice of this kind of potentials to work with.

Lemma 44. *If* $K(x) = k(|x|)$ *is a natural potential, then* $k''(r) = o(1/r)$ *as* $r \to 0$.

Proof. First, note that since $k(r)$ is smooth away from 0 we have

$$k'(1) - k'(\epsilon) = \int_\epsilon^1 k''(r)dr.$$

Now, because of (MN1) we know that there exists a neighborhood of zero in which k'' doesn't change sign. Therefore, letting $\epsilon \to 0$ and using (C3) we conclude that k'' is integrable around the origin. A simple integration by part, together with (C3) gives then that

$$\int_0^r k''(s)s\, ds = -\int_0^r k'(s)dr + k'(r)r.$$

Dividing both sides by r and letting $r \to 0$ we obtain

$$\lim_{r\to 0} \frac{1}{r} \int_0^r k''(s)s\, ds = 0.$$

which, combined with (MN2) implies $\lim_{r\to 0} k''(r)r = 0$. □

The next lemma shows that the existence theory developed in the previous section applies to this class of potentials.

Lemma 45. *If* K *is a natural potential then* $\nabla K \in W^{1,q}(\mathbb{R}^d)$ *for all* $1 \leq q < d$. *As a consequence, the critical exponents* p_s *and* q_s *associated to a natural potential satisfy*

$$q_s \geq d \quad and \quad p_s \leq \frac{d}{d-1}.$$

Proof. Recall that

$$\nabla K(x) = k'(|x|)\frac{x}{|x|} \quad \text{and} \quad \frac{\partial^2 K}{\partial x_i \partial x_j}(x) = \left(k''(|x|) - \frac{k'(|x|)}{|x|}\right)\frac{x_i x_j}{|x|^2} + \delta_{ij}\frac{k'(|x|)}{|x|},$$

where δ_{ij} is the Kronecker delta symbol. In order to prove the lemma, it is enough to show that $k'(|x|)$, $k''(|x|)$ and $\frac{k'(|x|)}{|x|}$ belong to $L^q(\mathbb{R}^d)$ for all $1 \le q < d$. To do that, observe that the decay condition (C2) implies that they belong to $L^q(B(0,1)^c)$ for all $q \ge 1$. Then, we take into account that (C3) implies that $\frac{k'(r)}{r} = O(1/r)$ as $r \to 0$ and that we have seen in the previous Lemma that $k''(r) = o(1/r)$ as $r \to 0$. This is enough to conclude, since the function $x \mapsto 1/|x|$ is in $L^q(B(0,1))$ for all $1 \le q < d$. □

The following Lemma together with Theorem 19 gives global existence of solutions for natural potentials which are repulsive in the short range, whence part (i) of Theorem 5 follows.

Lemma 46. *Suppose K is a natural potential which is repulsive in the short range. Then ΔK is bounded from above.*

Proof. We will prove that there is a neighborhood of zero on which $\Delta K \le 0$. This combined with the decay condition (C2) give the desired result. First, recall that $\Delta K(x) = k''(|x|) + (d-1)k'(|x|)|x|^{-1}$. Then, since k is repulsive in the short range, there exists a neighborhood of zero in which $k' \le 0$. Now, we have two possibilities: on one hand, if $\lim_{r\to 0^+} k'(r) = 0$, then given $r \in (0,+\infty)$ there exists $s \in (0,r)$ such that $\frac{k'(r)}{r} = k''(s)$. Together with (MN1), this implies that k'' is also non-positive in some neighborhood of zero. On the other hand if $\lim_{r\to 0^+} k'(r) < 0$, then the fact that $k''(r) = o(1/r)$ implies that $\frac{rk''(r)+(d-1)k'(r)}{r}$ is negative for r small enough. □

Finally, the next Lemma will be needed to prove global existence for natural potentials which are strictly attractive in the short range and satisfy the Osgood criteria.

Lemma 47. *Suppose that K is a natural potential which is strictly attractive in the short range and satisfies the Osgood criteria* (1.9)*. If moreover* $\sup_{x\neq 0} \Delta K(x) = +\infty$ *then the following holds*

- **(Z1):** $\lim_{r\to 0^+} k''(r) = +\infty$ *and* $\lim_{r\to 0^+} \frac{k'(r)}{r} = +\infty$,
- **(Z2):** $\exists \delta_1 > 0$ *such that* $k''(r)$ *and* $\frac{k'(r)}{r}$ *are decreasing for* $r \in (0,\delta_1)$,
- **(Z3):** $\exists \delta_2 > 0$ *such that* $k''(r) \le \frac{k'(r)}{r}$ *for* $r \in (0,\delta_2)$.

Proof. Let us start by proving by contradiction that

$$\lim_{r\to 0^+} \sup_{p\in(0,r)} \frac{k'(r)}{r} = +\infty. \tag{5.100}$$

If we suppose that

$$\frac{k'(r)}{r} < C \qquad \forall r \in (0,1], \tag{5.101}$$

then given a sequence $r_n \to 0^+$ there will exist another sequence $s_n \to 0^+$, $0 < s_n < r_n$, such that

$$\frac{k'(r_n)}{r_n} = k''(s_n) < C.$$

Since $k''(r)$ is monotonic around zero, this implies that k'' is bounded from above, and combining this with (5.101) we see that ΔK must also be bounded from above, which contradicts our assumption. Now, statements (Z1), (Z2), (Z3) follow easily: First note that if $\lim_{r\to 0^+} k'(r) > 0$, then clearly the Osgood condition (1.9) is not satisfied, whence $\lim_{r\to 0^+} k'(r) = 0$. This implies that for all $r > 0$ there exists $s \in (0,r)$ such that

$$\frac{k'(r)}{r} = k''(s). \tag{5.102}$$

Combining (5.102), (5.100) and the monotonicity of k'' we get that $\lim_{r\to 0^+} k''(r) = +\infty$ and k'' is decreasing on some interval $(0,\delta)$, which corresponds with the first part of (Z1) and (Z2). Now, going back to (5.102) we see that if $0 < s < r < \delta$ then $\frac{k'(r)}{r} = k''(s) \geq k''(r)$ which proves (Z3). This implies

$$\frac{d}{dr}\left\{\frac{k'(r)}{r}\right\} = \frac{1}{r}\left(k''(r) - \frac{k'(r)}{r}\right) \leq 0$$

and therefore $k'(r)/r$ decreases on $(0,\delta)$. Thus, (5.100) implies $\lim_{r\to 0^+} \frac{k'(r)}{r} = +\infty$ and the proof is complete. □

5.2. **Global bound of the L^p-norm using Osgood criteria.** We have already proven global existence when the potential is bounded from above. In this section we prove the following proposition, which allows us to prove global existence for potentials which are attractive in the short range, satisfy the Osgood criteria, and whose Laplacian is not bounded from above. From it, second part of Theorem 5 follows readily. This extends prior work on L^∞-solutions [5] to the L^p case.

Proposition 48. *Suppose that K is a natural potential which is strictly attractive in the short range, satisfies the Osgood criteria* (1.9) *and whose Laplacian is not bounded from above (i.e. $\sup_{x\neq 0} \Delta K(x) = +\infty$). Let $u(t)$ be the unique solution of the aggregation equation starting with initial data $u_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, where $p > d/(d-1)$. Define the length scale*

$$R(t) = \left(\frac{\|u(t)\|_{L^1}}{\|u(t)\|_{L^p}}\right)^{q/d}.$$

Then there exists positive constants δ and C such that the inequality

$$\frac{dR}{dt} \geq -C\, k'(R) \tag{5.103}$$

holds for all time t for which $R(t) \leq \delta$.

Remark 49. *$R(t)$ is the natural length scale associated with blow up of L^p norm. Given that mass is conserved, $R(t) > 0$ means that the L^p norm is bounded. The differential inequality (5.103) tell us that $R(t)$ decays slower than the solutions of the ODE $\dot{y} = -Ck'(y)$. Since $k'(r)$ satisfies the Osgood criteria (1.9), solutions of this ODE do not go to zero in finite time. $R(t)$ therefore stays away from zero for all time which provides us with a global upper bound for $\|u(t)\|_{L^p}$.*

Proof. Equality (2.15) can be written

$$\frac{d}{dt}\left\{\|u(t)\|_{L^p}^p\right\} = (p-1)\int_{\mathbb{R}^d} u(t,x)^p (u(t) * \Delta K)(x)\, dx.$$

Using the chain rule we get

$$\frac{d}{dt}\left\{\|u(t)\|_{L^p}^{-\frac{q}{d}}\right\} = -\frac{(p-1)q}{pd}\|u(t)\|_{L^p}^{-\frac{q}{d}-p}\int_{\mathbb{R}^d} u(t,x)^p (u(t) * \Delta K)(x)\, dx \tag{5.104}$$

$$\geq -\frac{(p-1)q}{pd}\|u(t)\|_{L^p}^{-\frac{q}{d}} \sup_{x\in\mathbb{R}^d}\left\{(u(t) * \Delta K)(x)\right\}. \tag{5.105}$$

To obtain (5.103), we now need to carefully estimate $\sup_{x\in\mathbb{R}^d}\left\{(u(t) * \Delta K)(x)\right\}$:

Lemma 50 (potential theory estimate)**.** *Suppose that $K(x) = k(|x|)$ satisfies (C2), (Z1), (Z2) and (Z3). Suppose also that $p > \frac{d}{d-1}$. Then there exists positive constants δ and C such that inequality*

$$\sup_{x\in\mathbb{R}^d}\left(u * \Delta K\right)(x) \leq C\|u\|_{L^1}\frac{k'(R)}{R} \quad \textit{where} \quad R = \left(\frac{\|u\|_{L^1}}{\|u\|_{L^p}}\right)^{q/d} \tag{5.106}$$

holds for all nonnegative $u \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ satisfying $R \leq \delta$.

Remark: By Lemma 47, potentials satisfying the conditions of Proposition 48 automatically satisfy (C2), (Z1), (Z2) and (Z3).

Proof of Lemma 50. Recall that $\Delta K(x) = k''(|x|) + (d-1)\frac{k'(|x|)}{|x|}$ so that (Z3) implies that $\Delta K(x) < d\; k'(|x|)/|x|$ in a neighborhood of zero. So for ϵ small enough we have:

$$\int_{|y|<\epsilon} u(x-y)\Delta K(y)dy \leq d\int_{|y|<\epsilon} u(x-y)\frac{k'(|y|)}{|y|}dy \tag{5.107}$$

$$\leq d\left(\int_{|y|<\epsilon} u(x-y)^p dy\right)^{1/p}\left(\int_{|y|<\epsilon}\left(\frac{k'(|y|)}{|y|}\right)^q dy\right)^{1/q} \tag{5.108}$$

$$\leq d\,\|u\|_{L^p}\left(\int_0^\epsilon\left(\frac{k'(r)}{r}\right)^q r^{d-1}dr\right)^{1/q} \tag{5.109}$$

$$= d\,\|u\|_{L^p}\left(\int_0^\epsilon k'(r)^q r^{d-1-q}dr\right)^{1/q} \tag{5.110}$$

$$\leq d \, \|u\|_{L^p} \, k'(\epsilon) \left(\int_0^\epsilon r^{d-1-q} dr \right)^{1/q} \tag{5.111}$$

$$= \frac{d}{(d-q)^{1/q}} \, \|u\|_{L^p} \, \frac{k'(\epsilon)}{\epsilon} \epsilon^{d/q}. \tag{5.112}$$

To go from (5.107) to (5.108) we have used the fact that k' is nonnegative in a neighborhood of zero (because $\lim_{r\to 0^+} k'(r)/r = +\infty$) and that u is also nonnegative. To go from (5.110) to (5.111) we have used the fact that k' is increasing in a neighborhood of zero (because $\lim_{r\to 0^+} k''(r) = +\infty$). Finally, to go from (5.111) to (5.112) we have used the fact that, since $q < d$, $\int_0^\epsilon r^{d-1-q} dr = \epsilon^{d-q}/(d-q)$. Let us now estimate $\int_{|y|\geq\epsilon} u(x-y)\Delta K(y) dy$. On one hand $k''(r)$ and $k'(r)/r$ go to 0 as $r \to +\infty$. On the other hand $k''(r)$ and $k'(r)/r$ go monotonically to $+\infty$ as $r \to 0^+$. Therefore for ϵ small enough we have

$$\sup_{|x|\geq\epsilon} \Delta K(x) = \sup_{r\geq\epsilon} \left\{ k''(r) + (d-1)\frac{k'(r)}{r} \right\} \leq k''(\epsilon) + (d-1)\frac{k'(\epsilon)}{\epsilon} \leq d\frac{k'(\epsilon)}{\epsilon}$$

which gives, since u is nonnegative,

$$\int_{|y|\geq\epsilon} u(x-y)\Delta K(y) dy \leq d \, \|u\|_{L^1} \, \frac{k'(\epsilon)}{\epsilon}. \tag{5.113}$$

Combining (5.112) and (5.113) we see that for ϵ small enough we have

$$\sup_{x\in\mathbb{R}^d} \left(u * \Delta K\right)(x) \leq c\frac{k'(\epsilon)}{\epsilon} (\epsilon^{d/q} \|u\|_{L^p} + \|u\|_{L^1}) \tag{5.114}$$

where c is a positive constant depending on d and q. To conclude the proof, choose $\epsilon = R = \left(\frac{\|u\|_{L^1}}{\|u\|_{L^p}}\right)^{q/d}$. $\square$

Combining (5.105) and (5.106) allows us to get (5.103), which concludes the proof of Proposition 48. $\square$

6. Conclusions

This paper develops refined analysis for well-posedness of the multidimensional aggregation equation for initial data in L^p. An additional assumption of bounded second moment is needed as a decay condition for uniqueness; fortunately this condition is preserved by the dynamics of the equation. The results connect recent theory developed for L^∞ initial data [4, 5, 6] to recent theory for measure solutions [16]. It turns out that L^p spaces provide a good understanding of the transition from a regular (bounded) solution to a measure solution, which was proved to occur in [5] whenever the potential violates the Osgood condition.

In [5], it is shown that for the special case of $K(x) = |x|$, no 'first kind' similarity solutions exist to describe the blowup to a mass concentration for odd space dimensions larger than one. A subsequent numerical study of blowup [26], for $K = |x|$, illustrates that for dimensions larger than two, there is a 'second kind' self-similar blowup in which no mass concentration occurs at the initial blowup time. Rather, the solution is in L^p for some $p < p_s = d/(d-1)$. The solution has asymptotic structure like the example constructed in Section 4 of this paper, thus we expect after the initial blowup time, that it concentrates mass in a delta. We remark that these results provide an interesting connection to classical results for Burgers equation. Our equation in one space dimension, with $K = |x|$, reduces to a form of Burgers equation by defining $w(x) = \int_0^x u(y)dy$ [10]. Thus, an initial blowup for the aggregation problem is the same as a a singularity in the slope for Burgers equation. Generically, Burgers singularities form by creating a $|x|^{1/3}$ power singularity in w, which gives a $-2/3$ power blowup in u. This corresponds to a blowup in L^p for $p > 3/2$, but does not result in an initial mass concentration. However, as we well know, the Burgers solution forms a shock immediately afterwards, resulting in a delta concentration in u. Thus the scenario described above is a multidimensional analogue of the well-known behavior of how singularities initially form in Burgers equation. The delta-concentrations are analogues of shock formation in scalar conservation laws.

Section 4 constructs an example of a measure solution with initial data in L^p, $p < p_s$, that instantaneously concentrates mass, for the special kernel $K(x) = |x|$ (near the origin). We conjecture that such solutions exist for more general power-law kernels $K(x) = |x|^\alpha$, $2-d < \alpha < 2$. The proof of instantaneous concentration uses some monotonicity properties of the convolution operator $\vec{x}/|x|$ which would need to be proved for the more general case.

Several interesting open problems remain, in addition to proving sharpness of the exponent p_s for more general kernels. For initial data in L^{p_s} local well-posedness is not known.

7. Appendix – Proof of Proposition 16

Since all the convergences (2.38)-(2.40) take place on compact sets, one of the key ideas of this proof will be to approximate $u_0 \in L^p$ by a function with compact support and to use the fact that X^t maps compact sets to compact sets (Lemma 14).

PART I: We will prove that $u(t,x) = u_0(X^{-t}(x))a(t,x)$ belongs to $C([0,T^*), L^p(\mathbb{R}^d))$. Assume first that $u_0 \in C_c(\mathbb{R}^d)$. The function u_0 is then uniformly continuous and, since

$$\sup_{x\in\mathbb{R}^d} |X^{-t}(x) - X^{-s}(x)| \leq C|t-s|,$$

it is clear that the quantity

$$\|u_0(X^{-t}(x)) - u_0(X^s(x))\|_{L^p}$$

can be made as small as we want by choosing $|t-s|$ small enough. We have therefore proven that $u_0(X^{-t}(x)) \in C([0,T^*), L^p(\mathbb{R}^d)$. Assume now that $u_0 \in$

$L^p(\mathbb{R}^d)$. Approximate it by a function $g \in C_c(\mathbb{R}^d)$ and write:

$$\begin{aligned} \|u_0(X^{-t}(x)) - u_0(X^{-s}(x))\|_{L^p} &\le \|u_0(X^{-t}(x)) - g(X^{-t}(x))\|_{L^p} \\ &\quad + \|g(X^{-t}(x)) - g(X^{-s}(x))\|_{L^p} \\ &\quad + \|g(X^{-s}(x)) - u_0(X^{-s}(x))\|_{L^p} \\ &= I + II + III. \end{aligned} \tag{7.115}$$

As we have seen above, the second term can be made as small as we want by choosing $|t-s|$ small enough. Using Lemma 13 we get

$$I, III \le C^* \|u_0 - g\|_{L^p},$$

which can be made as small as we want since $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. We have therefore proven that, if $u_0 \in L^p(\mathbb{R}^d)$, then

$$(t,x) \mapsto u_0(X^{-t}(x)) \in C([0,T^*), L^p(\mathbb{R}^d)). \tag{7.116}$$

Let us now consider the function $u_0(X^{-t}(x))a(t,x)$. Recall that $a(t,x)$ is continuous and bounded on $[0,T^*] \times \mathbb{R}^d$. Write

$$\begin{aligned} \|u_0(X^{-t}(x))a(t,x) - u_0(X^{-s}(x))a(s,x)\|_{L^p} \le \\ \|u_0(X^{-t}(x))\{a(t,x) - a(s,x)\}\|_{L^p} \\ + \|\{u_0(X^{-t}(x)) - u_0(X^{-s}(x))\}a(s,x)\|_{L^p} \\ = I + II. \end{aligned}$$

Since $a(s,x)$ is bounded, (7.116) implies that II can be made as small as we want by choosing $|t-s|$ small enough. Let us now take care of I. Approximate u_0 by a function $g \in C_c(\mathbb{R}^d)$ and write

$$\begin{aligned} I &\le \|\{u_0(X^{-t}(x)) - g(X^{-t}(x))\}\{a(t,x) - a(s,x)\}\|_{L^p} \\ &\quad + \|g(X^{-t}(x))\{a(t,x) - a(s,x)\}\|_{L^p} \\ &= A + B. \end{aligned}$$

From Lemma 13 we have

$$A \le 2\,C^* \|u_0 - g\|_{L^p} \sup_{(t,x)\in[0,T^*]\times\mathbb{R}^d} |a(t,x)|,$$

and since $C_c(\mathbb{R}^d)$ is dense in L^p, A can be made as small as we want. Let Ω denote the compact support of g. Using Lemmas 13 and 14 we obtain:

$$\begin{aligned} B &= \left(\int_{\Omega + Ct} \left| g(X^{-t}(x))\{a(t,x) - a(s,x)\} \right|^p dx \right)^{1/p} \\ &\le \left(\sup_{x\in\Omega+ct} |a(t,x) - a(s,x)| \right) C^* \|g\|_{L^p}. \end{aligned}$$

Since $a(t,x)$ is uniformly continuous on compact subset of $[0,T^*]\times\mathbb{R}^d$, it is clear that by choosing $|t-s|$ small enough we can make B as small as we want. We have proven that

$$u \in C([0,T^*], L^p(\mathbb{R}^d).$$

PART II: In Part I we have proven that $u \in C([0,T^*], L^p)$. The same proof work to show that u^ϵ, uv and $u^\epsilon v^\epsilon$ belong to $C([0,T^*], L^p)$ ($v(t,x)$ is continuous and bounded, so we can handle it exactly like $a(t,x)$.)

PART III: Let us now prove that $u^\epsilon(t,x) = u_0^\epsilon(X_\epsilon^{-t}(x))a^\epsilon(t,x)$ converges to $u(t,x) = u_0(X^{-t}(x))a(t,x)$. For convenience we write ϵ instead of ϵ_k. To do this, we will successively prove:

$$u_0(X_\epsilon^{-t}(x)) \to u_0(X^{-t}(x)) \qquad \text{in } C([0,T^*), L^p), \tag{7.117}$$

$$u_0^\epsilon(X_\epsilon^{-t}(x)) \to u_0(X^{-t}(x)) \qquad \text{in } C([0,T^*), L^p), \tag{7.118}$$

$$u_0^\epsilon(X_\epsilon^{-t}(x))a^\epsilon(t,x) \to u_0(X^{-t}(x))a(t,x) \qquad \text{in } C([0,T^*), L^p). \tag{7.119}$$

To prove (7.117), approximate $u_0 \in L^p(\mathbb{R}^d)$ by a function $g \in C_c(\mathbb{R}^d)$ and write:

$$\begin{aligned}\sup_{t\in[0,T^*)} \|u_0(X_\epsilon^{-t}(x) - u_0(X^{-t}(x))\|_{L^p} &\le \sup_{t\in[0,T^*)} \|u_0(X_\epsilon^{-t}(x)) - g(X_\epsilon^{-t}(x))\|_{L^p} + \\ \sup_{t\in[0,T^*)} \|g(X_\epsilon^{-t}(x)) - g(X^{-t}(x))\|_{L^p} &+ \sup_{t\in[0,T^*)} \|g(X^{-t}(x)) - u_0(X^{-t}(x))\|_{L^p} \\ &= I + II + III.\end{aligned}$$

From Lemma 13 it is clear that I and III can be made as small as we want by choosing an appropriate function g. Using Lemma 14, we see that, if Ω is the support of g

$$II = \sup_{t\in[0,T^*} \left(\int_{\Omega+Ct} \left|g(X_\epsilon^{-t}(x)) - g(X^{-t}(x))\right|^p dx\right)^{1/p}.$$

Using the uniform continuity of g together with the fact that $X_\epsilon^{-t}(x)$ converges uniformly to $X^{-t}(x)$ on $[0,T^*]\times\Omega + CT^*$, we can make II as small as we want by choosing ϵ small enough.

This concludes the proof of (7.117). To prove (7.118), write

$$\begin{aligned}\sup_{t\in[0,T^*)} \|u_0^\epsilon(X_\epsilon^{-t}(x)) - u_0(X^{-t}(x))\|_{L^p} \le& \sup_{t\in[0,T^*)} \|u_0^\epsilon(X_\epsilon^{-t}(x)) - u_0(X_\epsilon^{-t}(x)\|_{L^p} \\ &+ \sup_{t\in[0,T^*)} \|u_0(X_\epsilon^{-t}(x)) - u_0(X^{-t}(x))\|_{L^p} \\ =& \quad I + II.\end{aligned}$$

From (7.117) we know that II can be made as small as we want by choosing ϵ small enough. Using Lemma 13 we obtain

$$I \le C^* \|u_0^\epsilon - u_0\|_{L^p} \to 0 \qquad \text{as } \epsilon \to 0.$$

Let us now prove (7.119). Write

$$\begin{aligned}\sup_{t\in[0,T^*]} &\|u_0^\epsilon(X_\epsilon^{-t}(x))a^\epsilon(t,x) - u_0(X^{-t}(x))a(t,x)\|_{L^p}\\ &\le \sup_{t\in[0,T^*]} \|\{u_0^\epsilon(X_\epsilon^{-t}(x)) - u_0(X^{-t}(x))\}a^\epsilon(t,x)\|_{L^p}\\ &\quad + \sup_{t\in[0,T^*]} \|u_0(X^{-t}(y))\{a^\epsilon(t,x) - a(t,x)\}\|_{L^p}\\ &= I + II.\end{aligned}$$

Since $a^\epsilon(t,x)$ is uniformly bounded on $[0,T^*]\times\mathbb{R}^d$, it is clear from (7.118) that $I\to 0$ as $\epsilon\to 0$. If u_0 is in $C_c(\mathbb{R}^d)$, then it is easy to prove that $II\to 0$ as $\epsilon\to 0$. If $u_0\in L^p(\mathbb{R}^d)$, then approximate it by $g\in C_c(\mathbb{R}^d)$ and proceed as before.

Part IV: To prove that $u^{\epsilon_k}v^{\epsilon_k}$ converges to uv in $C([0,T^*],L^p)$, proceed exactly as in the proof of (7.119).

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