TWO WEIGHT INEQUALITIES FOR MAXIMAL TRUNCATIONS OF DYADIC CALDERÓN-ZYGMUND OPERATORS

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Abstract. Let σ and ω be positive Borel measures on $\mathbb{R}^n$ and $1 < p < \infty$. For a class of dyadic Calderón-Zygmund operators T, we characterize the two weight inequalities

$$\|\mathrm{T}_\natural(f\sigma)\|_{L^p(w)} \lesssim \|f\|_{L^p(\sigma)}\,,$$

where $\mathrm{T}_\natural$ denotes the maximal truncations of the operator T. The characterization is given in terms of testing conditions, identifiable as a variant of the T 1 theorem of David and Journé.

1. Introduction

We are interested in two weight inequalities for dyadic Calderón-Zygmund operators. Indeed, our main results, Theorem 1.19 and Theorem 1.22 below characterize the weak and strong type (p, p) two weight inequalities for such operators. The characterization holds for all $1 < p < \infty$, and for *individual* operators. Let $\mathcal{Q}$ denote the class of dyadic cubes in $\mathbb{R}^d$, specifically,

$$\mathcal{Q} = \left\{ 2^k(j + [0,1)^d) \ : \ k \in \mathbb{Z}\,, j \in \mathbb{Z}^d \right\}.$$

We consider dyadic operators, of Calderón-Zygmund type, but our definition is unconventional, in that we will permit the operators be unbounded on $L^2(dx)$.

Definition 1.1. Specifically, let $\mathcal{Q}$ denote dyadic cubes on $\mathbb{R}^d$. Consider an operator

$$\mathrm{T}\, f(x) := \sum_{Q \in \mathcal{Q}} \langle f, h'_Q \rangle \cdot h_Q \tag{1.2}$$

The function h_Q is assumed to satisfy

$$\|h_Q\|_\infty \le |Q|^{-1/2}, \tag{1.3}$$

$$h_Q \text{ is supported on } Q, \tag{1.4}$$

$$h_Q \text{ is constant on dyadic subcubes } Q' \text{ of } Q \text{ with } |Q'| \le 2^{-\zeta d}|Q|. \tag{1.5}$$

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In the last condition, $\zeta \geq 1$ is a fixed positive integer. These are discrete analogs of the usual size and smoothness conditions placed upon a Calderón-Zygmund kernel. The assumptions on h'_Q are the same, and we assume that $h_Q \equiv 0$ if and only if $h'_Q \equiv 0$. We work with maximal truncations of singular integrals.

$$\mathrm{T}_\natural\, f(x) := \sup_{0<\epsilon<\rho<\infty} \lvert T_{\epsilon,\upsilon} f(x)\rvert\,,$$

$$\mathrm{T}_{\epsilon,\upsilon}\, f := \sum_{\substack{Q\in\mathcal{Q}\\ \epsilon\le\ell(Q)|<\rho}} \langle f, h'_Q\rangle \cdot h_Q\,. \tag{1.6}$$

Here, and throughout the paper, $\ell(Q) = |Q|^{1/d}$ is the side length of the cube Q. For a positive $0<\tau<1$, let us define a (much simpler) maximal operator by

$$\mathrm{U}_\tau\, f(x) := \sup_{Q\in\mathcal{U}_\tau} \lvert\langle f, h'_Q\rangle h_Q(x)\rvert \tag{1.7}$$

$$\mathcal{U}_\tau := \{Q\in\mathcal{Q}\ :\ \inf_{x\in Q} \min\bigl\{|h_Q(x)|\,,\ |h'_Q(x)|\bigr\}\sqrt{|Q|} < \tau\}\,. \tag{1.8}$$

That is, the collection of $\mathcal{U}_\tau$ consists of those Q for which $h_Q(x)$ is 'small' at some point $x\in Q$.

There are relevant examples of $L^2(dx)$ bounded operators that fall within the scope of this definition. We will leave it to the reader to write down an example of an operator which fits this definition but is not bounded on $L^2(dx)$.

Example 1.9 (Martingale Transforms)**.** Let h_Q be a Haar function. That is it satisfies (1.3), is constant on dyadic strict subcubes of Q, and has integral zero. A *martingale transform* would be

$$\mathrm{T}\, f := \sum_{Q\in\mathcal{Q}} \varepsilon_Q \langle f, h_Q\rangle h_Q\,, \qquad \varepsilon_Q\in\{\pm1\}\,.$$

Here, we assume that h_Q satisfy (1.3)—(1.5) and satisfy $\int h_Q\ dx = 0$ for all Q. The Beurling operator can be obtained from the averages of such in two dimensions, [3].

Example 1.10 (Haar Shift)**.** The Haar shift in one dimension would be

$$\mathrm{T}\, f := \sum_I \langle f, h_I\rangle (h_{I_{\text{left}}} + h_{I_{\text{right}}})$$

Here, I denotes a dyadic intervals in one dimension, h_I is the L^2-normalized Haar function supported on I, and I_{left} is the left-half of I, likewise for I_{right}. These operators were introduced in [9], and are shown in this reference that the Hilbert transform can be obtained as an appropriate average of these operators. Also see [11] for higher dimensional Haar shift operators and the Riesz transforms.

Example 1.11 (Paraproduct Operators)**.** Let h_Q be a Haar function. That is it satisfies (1.3), is constant on dyadic strict subcubes of Q, and has integral zero. Let b be a function, and define

$$\mathrm{P}(b, f) := \sum_{Q \in \mathcal{Q}} \langle b, h_Q \rangle \cdot \mathbb{E}_Q f \cdot h_Q \tag{1.12}$$

$$\mathbb{E}_Q f := |Q|^{-1} \int_Q f \, dx \,. \tag{1.13}$$

Here, we impose the condition on b that $|\langle b, h_Q \rangle| \le |Q|^{1/2}$ for all dyadic Q, (a weaker condition than $b \in \mathrm{BMO}$, which is equivalent to $\mathrm{P}(b, \cdot)$ being a bounded operator on $L^2(dx)$). Holding b fixed, the linear operator $\mathrm{P}(b, \cdot)$ satisfies the our Definition 1.1. These operators are commonly called dyadic paraproduct operators.

Note that the for $0 < \tau < 1$, the collection $\mathcal{U}_\tau$ in (1.8) will be empty for all of these examples. In those examples where it is not empty, it is dominated by the maximal function, which is easy to see from (1.3).

Part of the interest in this class of dyadic operators is that significant advances of our understanding of weighted estimates have come from analysis specialized to these cases. We refer to just two of Nazarov-Treil-Volberg series of innovative papers on weighted inequalities [7, 8]; the work of Wittwer [17] addressing A_2 estimates for martingale transforms and [18] for the continuous square function; the work of Petermichl and Volberg [12] which proved the sharp A_2 inequality for the Beurling operator, answering a question of Astala; Petermichl's proof of the (much harder) sharp A_p inequality for the Hilbert transform [10], and the Riesz transforms [12]; Beznosova's sharp A_p inequality for discrete paraproducts [1]; and the recent work of Lacey-Petermichl-Reguera [4] giving, with a single argument, the sharp A_p inequality for all Haar shifts. (The Beurling, Hilbert and Riesz are in the convex hull of Haar shifts. The papers [10, 12, 12, 17] proved weighted estimates for the associated Haar shift, the proof depending upon the particular Haar shift being used.)

We recall this dyadic variant of the $\mathrm{T}\,1$ theorem of David and Journé [2].

The $\mathrm{T}\,1$ Theorem of David and Journé 1.14. *An operator T be as in Definition 1.1 extends to a bounded operator on $L^2(\mathbb{R}^d)$ if and only if these conditions are met. For all cubes Q, R with $Q^{(\zeta)} \supset R$ or $R^{(\zeta)} \supset Q$*

$$|\langle \mathrm{T}\, \mathbf{1}_Q, \mathbf{1}_R \rangle| \lesssim \sqrt{|Q| \cdot |R|} \qquad \text{(Wkly Bounded)}$$

$$\int_Q |\mathrm{T}\, \mathbf{1}_Q|^2 \, dx \lesssim |Q| \qquad (\mathrm{T}\,1 \in BMO)$$

$$\int_Q |\mathrm{T}^* \mathbf{1}_Q|^2 \, dx \lesssim |Q| \qquad (\mathrm{T}^*\,1 \in BMO) \tag{1.15}$$

The primary focus of this paper is extensions of this Theorem to the two weight setting. These considerations are motivated in part by a well developed theory of two weight estimates for positive operators. These Theorems have formulations strikingly similar to the T 1 Theorem, which theory encompasses the Theorems due to Sawyer concerning two weight, both strong and weak type, for the maximal operator [15] and fractional integral operators [13, 14]. There is also the bilinear embedding inequality of Nazarov-Treil-Volberg [7]. We refer the reader to [6] for a discussion of these results. While directly relevant to the considerations of this paper, developing this theme will unnecessarily lengthen this introduction.

There is a beautiful result of Nazarov-Treil-Volberg [8], a two-weight version of the T 1 theorem, with one notable caveat, namely (1.17) below. A subcase of their result is as follows.

Nazarov-Treil-Volberg Two weight T 1 **Theorem. 1.16.** *Let* T *be as in Definition 1.1, with the additional assumption that*

$$\int h_Q \, dx = \int h'_Q \, dx = 0 \qquad Q \in \mathcal{Q}\,. \tag{1.17}$$

Let σ, μ *be two positive measures. The* L^2 *inequality*

$$\|\mathrm{T}(\sigma f)\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\sigma)}$$

holds iff the following three conditions hold. For all cubes $Q^{(\zeta)} \supset R$ *or* $R^{(\zeta)} \supset Q$*,*

$$\left|\langle \mathrm{T}(\sigma \mathbf{1}_Q), \mathbf{1}_R \mu\rangle\right| \lesssim \sqrt{\sigma(Q)\mu(R)} \qquad \text{(Weak Bnded)}$$

$$\int |\mathrm{T}(\sigma \mathbf{1}_Q)|^2 \, \mu(dx) \lesssim \sigma(Q) \qquad (T1 \in BMO)$$

$$\int |\mathrm{T}^*(\mu \mathbf{1}_Q)|^2 \sigma(dx) \lesssim \mu(Q) \qquad (T^*1 \in BMO) \tag{1.18}$$

It is essential to note that this result requires (1.17)—it does apply to paraproduct operators in (1.12), unlike the results below. Its proof is also fundamentally restricted to the case of $p = 2$, whereas $1 < p < \infty$ is arbitrary below. We will prove a characterization of a two-weight inequalities for the pair of operators of the maximal function and $\mathrm{T}_\natural$.

Weak Type Inequalities for $\mathrm{T}_\natural$**. 1.19.** *Let* T *be as in Definition 1.1, and let* $0 < \tau < 1$*. These two conditions are equivalent.*

$$\|\mathrm{T}_\natural(f\sigma)\|_{L^{p,\infty}(w)} \lesssim \|f\|_{L^p(\sigma)}\,, \tag{1.20}$$

if and only if

$$\begin{cases} \|\mathrm{U}_\tau(f\sigma)\|_{L^{p,\infty}(w)} \lesssim \|f\|_{L^p(\sigma)}\,, \\ \int_Q \mathrm{T}_\natural(\sigma f \mathbf{1}_Q) \ \omega(dx) \lesssim \|f\|_{L^p(\sigma)} \omega(Q)^{1/p'} \end{cases} \tag{1.21}$$

For the strong type, we have this characterization.

Strong Type Inequalities for $\mathrm{T}_\natural$. 1.22. *Let* T *be as in Definition 1.1, and let* $0 < \tau < 1$. *We have the equivalence*

$$\|\mathrm{T}_\natural(f\sigma)\|_{L^p(w)} \lesssim \|f\|_{L^p(\sigma)}\,, \tag{1.23}$$

if and only if

$$\begin{cases} \|\mathrm{U}_\tau(f\sigma)\|_{L^p(w)} \lesssim \|f\|_{L^p(\sigma)}\,, \\ \int_Q \mathrm{T}_\natural(\sigma f \mathbf{1}_Q)\ \omega(dx) \lesssim \|f\|_{L^p(\sigma)} \omega(Q)^{1/p'}\,, & \forall\ Q \in \mathcal{Q}\,, \\ \|\mathbf{1}_Q\, \mathrm{T}_\natural(f\sigma\mathbf{1}_Q)\|_{L^p(w)} \lesssim \sigma(Q)^{1/p} & \forall\ \|f\|_\infty \le 1\,,\ Q \in \mathcal{Q}\,. \end{cases} \tag{1.24}$$

The conditions involving the operator U_τ, as defined in (1.7), are not satisfactory in that they require the corresponding norm inequality, but keep in mind that this operator is *much smaller and simpler* than $\mathrm{T}_\natural$, and in many relevant examples, this operator *will in fact be zero.*

The testing condition appearing in the weak-type characterization

$$\int_Q \mathrm{T}_\natural(\sigma f \mathbf{1}_Q)\ \omega(dx) \lesssim \|f\|_{L^p(\sigma)} \omega(Q)^{1/p'}$$

looks rather complicated, with the appearance of $f \in L^p(\sigma)$ in it. It is however a close relative of the '$\mathrm{T}^*\,1 \in BMO$' conditions of (1.15) and (1.18). This is discussed in § 2.2 below, see in particular (2.7).

The dual testing condition

$$\|\mathbf{1}_Q\, \mathrm{T}_\natural(f\sigma\mathbf{1}_Q)\|_{L^{p,\infty}(w)} \lesssim \sigma(Q)^{1/p} \tag{1.25}$$

has the complication of involving an arbitrary function f bounded by one on the left-hand side, though f does *not appear* on the right hand-side of the inequality. (There is a similar difficulty in (2.7).) Despite this difficulty, we are not as of yet aware of a situation where the more natural testing condition below holds, but the one above does not.

$$\|\mathbf{1}_Q\, \mathrm{T}_\natural(\sigma\mathbf{1}_Q)\|_{L^{p,\infty}(w)} \lesssim \sigma(Q)^{1/p}\,. \tag{1.26}$$

Nor are we aware of a setting in which we can verify (1.26) but not (1.25).

The method of proof is an extension of that of Sawyer's approach to the two weight fractional integrals [14], but also [6]. A significant variant of this argument arises from the *multi-height* Calderón-Zygmund decompositions in § 6.1. This argument follows the outlines of the proof in [5], which proves variants of Theorem 1.19 and Theorem 1.22 for smooth Calderón-Zygmund operators. The current argument is, naturally, much easier while retaining the essential ideas and techniques of [5]. (The reader can also compare the arguments of this paper to those of [6].) We think the main results of this paper are interesting in their own right, as well the proof should be a guide to its much more complicated variant [5].

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2. Generalities of the Proof

2.1. **Universal Maximal Function, and its Consequences.** A fundamental tool is derived from (the usual) general maximal function estimates that hold for any measure. In particular, for weight w we define

$$\mathrm{M}_w f(x) := \sup_{Q \in \mathcal{Q}} \mathbf{1}_Q \mathbb{E}_Q^w |f| \,,$$
$$\mathbb{E}_Q^w f := \omega(Q)^{-1} \int_Q f \;\; \omega(dx) \,.$$

Here we are extending the definition in (1.13) to *arbitrary* weights. It is a basic fact, proved by exactly the same methods that prove the non-weighted inequality, that we have

Theorem 2.1. *We have the inequalities*

$$\|\mathrm{M}_w f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \,, \qquad 1 < p < \infty \,. \tag{2.2}$$

This will place variants of the Calderón-Zygmund Decomposition at our disposal. Indeed, we will use Calderón-Zygmund Decompositions *at all heights simultaneously.*

2.2. **Linearizing Maximal Operators.** We use the method of linearizing maximal operators. This is familiar in the context of the maximal function, and we make a comment about it here. Let $\{E(Q) \; : \; Q \in \mathcal{Q}\}$ be any selection of measurable disjoint sets $E(Q) \subset Q$ indexed by the dyadic cubes. Define corresponding linear operator N by

$$\mathrm{N}\,\phi := \sum_{Q \in \mathcal{Q}} \mathbf{1}_{E(Q)} \mathbb{E}_Q^w f \,. \tag{2.3}$$

Then, (2.2) is equivalent to the bound $\|\mathrm{N}\, f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}$ with implied constant independent of w and the sets $\{E(Q) \; : \; Q \in \mathcal{Q}\}$. This estimate will be used repeatedly below.

There is a related way to linearize $\mathrm{T}_\natural$, which deserves careful comment as we would like, at different points, to treat $\mathrm{T}_\natural$ as a linear operator. While it is not a linear operator, $\mathrm{T}_\natural$ is a pointwise supremum of the linear truncation operators $\mathrm{T}_{\varepsilon,\upsilon}$, and as such, the supremum can be linearized with measurable selection of the truncation parameters.

Definition 2.4. We say that L is a linearization of $\mathrm{T}_\natural$ if there are measurable functions $\epsilon(x), \upsilon(x) \in (0,\infty)$ and $\vartheta(x) \in [0, 2\pi)$ such that, using (1.6), we have

$$\mathrm{L}\, f(x) = e^{i\vartheta(x)}\, \mathrm{T}_{\epsilon(x),\upsilon(x)}\, f(x) \geq 0, \qquad x \in \mathbb{R}^d. \tag{2.5}$$

(The requirement that $\mathrm{L}\, f(x) \geq 0$ then defines $\vartheta(x)$ everywhere except when $\mathrm{T}_{\epsilon(x),\upsilon(x)}\, f(x) = 0$.) For fixed f we can always choose a linearization L so that $\mathrm{T}_\natural\, f(x) \leq 2\, \mathrm{L}\, f(x)$ for all x.

A key advantage of L is that it is a linear operator, and as such it has an adjoint, given by the formal expression

$$\mathrm{L}^*\, \nu(y) = \sum_{Q\in\mathcal{Q}} \tau_Q h'_Q(y) \int_{Q\cap\{\varepsilon(x)\leq|Q|\leq\upsilon(x)\}} e^{i\vartheta(x)} f(x) h_Q(x)\, dx\,. \tag{2.6}$$

The testing condition in (1.24) has a more convincing formulation in the linearizations. It is equivalent to

$$\|\mathrm{L}^*(\mathbf{1}_Q g w)\|_{L^{p'}(\sigma)} \leq \mathfrak{T}\omega(Q)^{1/p'}\,, \qquad \|g\|_\infty \leq 1\,, \tag{2.7}$$

$$\mathfrak{T} := \sup_Q \sup_{\|f\|_{L^p(\sigma)}\leq 1} \int_Q \mathrm{T}_\natural(f\sigma\mathbf{1}_Q)\, \omega(dx)\,. \tag{2.8}$$

This holds uniformly over all choices of linearizations, which fact is referred to repeatedly below.

The operators L^* have a certain 'smoothness property' that is fundamental for us.

Lemma 2.9. *Suppose for measure ν and cube Q_0 we have $|\nu|(Q_0) = 0$. Then, $\mathrm{L}^*\, \nu(\cdot)$ is constant on subcubes $Q' \subset Q_0$ with $|Q'| \leq 2^{-\zeta d}$.*

Proof. The sum (2.6) defining the adjoint operator becomes

$$\mathrm{L}^*\, \nu(y) = \sum_{\substack{Q\in\mathcal{Q}\\ Q_0\subset Q}} h'_Q(y) \int_{Q\cap\{\epsilon(x)\leq|Q|\leq\upsilon(x)\}} e^{i\vartheta(x)} h_Q(x)\, \nu(dx)\,.$$

The sum is restricted to a sum over cubes $Q \supset Q_0$, so the Lemma follows by assumption on the functions h'_Q. □

2.3. **Whitney Decompositions.** We make general remarks about the sets

$$\Omega_k = \{\mathrm{T}_\natural(f\sigma) > 2^k\} \tag{2.10}$$

where f is a finite linear combination of indicators of dyadic cubes. The assumptions we will have will show that Ω_k will be an open set with compact closure.

Let $Q^{(1)}$ denote the parent of Q, and inductively define $Q^{(j+1)} = (Q^{(j)})^{(1)}$. For an integer $\rho \geq 2$, we should choose collections $\mathcal{Q}_k$ of disjoint dyadic cubes so that these several conditions are met.

$$\Omega_k = \dot{\bigcup}_{Q\in\mathcal{Q}_k} Q \qquad \text{(disjoint cover)} \tag{2.11}$$

$$Q^{(\rho)} \subset \Omega_k \,, \; Q^{(\rho+1)} \cap \Omega_k^c \neq \emptyset \qquad \text{(Whitney condition)} \tag{2.12}$$

$$\sum_{Q \in \mathcal{Q}_k} \mathbf{1}_{Q^{(\rho)}} \lesssim \mathbf{1}_{\Omega_k} \qquad \text{(finite overlap)} \tag{2.13}$$

$$\sup_{Q \in \mathcal{Q}_k} \sharp\{Q' \in \mathcal{Q}_k \;:\; Q' \cap Q^{(\rho)}\} \lesssim 1 \,, \qquad \text{(crowd control)} \tag{2.14}$$

$$Q \in \mathcal{Q}_k \,, \; Q' \in \mathcal{Q}_l \,, \; Q \subsetneqq Q' \quad \text{implies} \quad k > l \,. \qquad \text{(nested property)} \tag{2.15}$$

We will apply the Whitney decompositions with $\rho = \zeta + 1$, where ζ is the constant in (1.5).

Proof. Take $\mathcal{Q}_k$ to be the maximal dyadic cubes $Q \subset \Omega_k$ which satisfy (2.12). Then (2.11) holds. As the sets Ω_k are themselves nested, (2.15) holds.

Let us show that (2.13) holds. Note that holding the volume of the cubes constant we have

$$\sum_{|Q|=1} \mathbf{1}_{Q^{(\rho)}} \leq 2^{\rho d}$$

where d is the dimension. So if we take an integer ρ, and assume that for some k and $x \in \mathbb{R}^d$

$$\sum_{Q \in \mathcal{Q}_k} \mathbf{1}_{Q^{(\rho+1)}}(x) \geq 8 \cdot 2^{(\rho+1)n} \,,$$

then we can choose $Q, R \in \mathcal{Q}_k$ with $x \in Q^{(\rho)} \cap R^{(\rho)}$ and the side-length of R satisfies $|R|^{1/d} \leq 2^{-3}|Q|^{1/d}$. But then it will follow that $R^{(\rho+1)} \subset Q^{(\rho)}$. We thus see that $R^{(\rho+1)}$ does not meet Ω_k^c, which is a contradiction.

Let us see that (2.14) holds. Fix $Q \in \mathcal{Q}_k$. If we had $Q' \supsetneqq Q^{(\rho)}$ for any $Q' \in \mathcal{Q}_k$, we would violate (2.12). Thus, we must have $Q' \subset Q^{(\rho)}$. The cubes Q' are disjoint. Suppose that there were more than $2^{\rho+2}$ in number. Then, there would have to be a $Q' \subset Q^{(\rho)}$ with $|Q'| \leq 2^{-\rho-1}|Q^{(\rho)}|$. That is, $(Q')^{(\rho+1)} \subset Q^{(\rho)}$, violating the Whitney condition (2.12). □

2.4. **Maximum Principle.** A fundamental tool is the use of what we term here as 'maximum principle'. (We could also use the term 'good-λ technique'): Subject to the assumption that a maximal function is of small size, we will be able to see that the maximal truncations are large due to the restriction of the function to a local cube. This leads to an essential 'localization' of the singular integrals.

Maximum Principle. 2.16. *For any cube $Q \in \mathcal{Q}_k$ as above we have the pointwise inequality*

$$\sup_{\ell(Q)+\zeta+1<\epsilon<\upsilon} |\mathrm{T}_{\epsilon,\upsilon}(f\sigma)(x)| \lesssim 2^k + \sup_{\substack{\zeta+2 \leq r \leq 2\zeta+2 \\ Q^{(r)} \in \mathcal{U}_\tau}} |\langle f\sigma, h'_{Q^{(r)}} \rangle h_{Q^{(r)}}(x)| \,, \quad x \in Q \,. \tag{2.17}$$

In particular if $\mathcal{U}_\tau$*, given in* (1.8) *is empty, we have*

$$\sup_{\ell(Q)+\zeta+1<\epsilon<\upsilon} |\mathrm{T}_{\epsilon,\upsilon}(f\sigma)(x)| \lesssim 2^k\,, \qquad x\in Q\,. \tag{2.18}$$

Remark 2.19. One can give a more familiar upper bound in (2.17). By the support and size conditions on h_Q and h'_Q, namely (1.4) and (1.3), we have

$$\sum_{r=\zeta+2}^{2\zeta+2} |\langle f\sigma, h'_Q\rangle h_Q(x)| \lesssim \mathbb{E}_{Q^{(2\zeta+2)}}|f|\sigma\,.$$

The last term is obviously dominated by the maximal function. But our point here is to obtain the smallest possible terms here, so we prefer (2.17) as written.

Proof. Take $x\in Q$. By the Whitney condition in (2.12), there is a point $\overline{x}\in Q^{(\zeta+2)}\cap\Omega_k^c$. Consider the differences

$$h_{Q^{(r)}}(x)\langle f\sigma, h'_{Q^{(r)}}\rangle - h_{Q^{(r)}}(\overline{x})\langle f\sigma, h'_{Q^{(r)}}\rangle\,.$$

Note that by (1.5), if $r\ge\zeta+\rho+1=2\zeta+2$, then $h_{Q^{(r)}}(x)=h_{Q^{(r)}}(\overline{x})$, and so the difference above is 0. Thus,

$$\sup_{x\in Q}\sup_{\ell(Q)+2<\epsilon<\upsilon}|\mathrm{T}_{\epsilon,\upsilon}(f\sigma)(x)| \le \mathrm{T}_\natural\, f\sigma(\overline{x}) + \sum_{r=\zeta+2}^{2\zeta+2}|h_{Q^{(r)}}(x)\langle f\sigma, h'_{Q^{(r)}}\rangle|\,.$$

Consider $\zeta+2\le r\le 2\zeta+2$. If $Q^{(r)}\in\mathcal{U}_\tau$, we have $|h_{Q^{(r)}}(x)\langle f\sigma, h'_{Q^{(r)}}\rangle|\le \mathrm{U}_\tau\, f\sigma(x)$. Otherwise, we have, upon combining (1.3) and (1.8),

$$|h_{Q^{(r)}}(x)\langle f\sigma, h'_{Q^{(r)}}\rangle| \le \tau^{-1}|h_{Q^{(r)}}(\overline{x})\langle f\sigma, h'_{Q^{(r)}}\rangle| \le \tau^{-1}\,\mathrm{T}_\natural\, f\sigma(\overline{x})\,.$$

As $\mathrm{T}_\natural\, f\sigma(\overline{x})<2^k$, this proves (2.17). (The parameters τ and ζ enter into the implied constant, but we do not attempt to track this dependence throughout the proof.) □

3. Proof of the Weak-Type Inequality

We prove (1.19), the characterization of the weak-type inequalities. The necessity of the conditions follows immediately from standard considerations.

We turn to the reverse implication, namely the inequality (2.8). Specifically, we will show that

$$\|\mathrm{T}_\natural(\sigma\cdot)\|_{L^p(\sigma)\mapsto L^{p,\infty}(\omega)} \lesssim \mathfrak{T}+\mathfrak{U} \tag{3.1}$$

$$\mathfrak{U} := \|\mathrm{U}_\tau(\sigma\cdot)\|_{L^p(\sigma)\mapsto L^{p,\infty}(\omega)}\,, \tag{3.2}$$

where $\mathfrak{T}$ expression is defined in (2.8).

Let us first note that the testing condition (2.8) gives us a two weight A_p type condition. By taking $f = h'_Q$, we can estimate as follows provided Q *is not in* the collection $\mathcal{U}_\tau$ in (1.8).

$$\begin{aligned}\frac{\sigma(Q)\omega(Q)}{|Q|^{3/2}} &\le \tau^{-2}\int_Q |\langle h'_Q\sigma, h'_Q\rangle| \frac{\omega(dx)}{\sqrt{|Q|}} \\ &\le \tau^{-3}\int \mathrm{T}_\natural(h'_Q)\omega(dx)\,, \\ &\le \tau^{-3}\mathfrak{T}\frac{\sigma(Q)^{1/p}}{\sqrt{|Q|}}\omega(Q)^{1/p'}\,.\end{aligned}$$

This yields the condition

$$\sup_{Q\in\mathcal{Q}-\mathcal{U}_\tau} \frac{\sigma(Q)^{1/p'}\omega(Q)^{1/p}}{|Q|} \lesssim \mathfrak{T}\,. \tag{3.3}$$

Let us note that if f is a finite linear combination of indicators of dyadic cubes, that we then have for any integer k,

$$\sup_k 2^{kp}\omega(\{\mathrm{T}_\natural(f\sigma) > 2^k\}) < \infty\,. \tag{3.4}$$

Indeed, we can cover the support of f by a union of at most 2^d cubes. Let E denote the union of the doubles of these cubes. For cubes Q that meet E, but are not contained in it, we can use a combination of (3.3) and the boundedness of the maximal operator U_τ to see that (3.4) holds. (At first read, one is free to set $\mathrm{U}_\tau = 0$!)

Now, we will argue that for an absolute constant m and $0 < \eta < 1$ we have

$$\begin{aligned}2^{kp}\omega(\{\mathrm{T}_\natural(f\sigma) > 2^{k+m}\,,\ \mathrm{U}_\tau(\sigma f) < \eta 2^k\}) \\ \lesssim \mathfrak{T}^p\|f\|^p_{L^p(\sigma)} + \eta 2^{kp}\omega(\{\mathrm{T}_\natural(f\sigma) > 2^k\})\,.\end{aligned} \tag{3.5}$$

Apply this inequality for a choice of k for which the left-hand side is close to its supremum. For sufficiently small η, this proves (3.1).

Apply the Whitney decomposition (2.11)—(2.14) for the set Ω_k as in (2.10), which we can do as (3.4) holds. For $Q \in \mathcal{Q}_k$, let

$$E_k(Q) := Q \cap \{\mathrm{T}_\natural(f\sigma) > 2^{k+m}\,,\ \mathrm{U}_\tau(\sigma f) < \eta 2^k\}\,.$$

Apply the Maximum Principle (2.16). We deduce that for $x \in E_k(Q)$ we have

$$\mathrm{T}_\natural(f\mathbf{1}_{(Q^{(\varsigma)})}\sigma) \ge 2^{k+m} - C2^k - C\eta 2^k\,.$$

For $0 < \eta < 1$ sufficiently small, m sufficiently large and $x \in E_k(Q)$ we will have

$$2^{k+1} \le \mathrm{T}_\natural(f\mathbf{1}_{(Q^{(\varsigma)})}\sigma)\,.$$

And so we have

$$2^k\omega(E_k(Q)) \lesssim \int_{E_k(Q)} \mathrm{T}_\natural(f\mathbf{1}_{(Q^{(\varsigma)})}\sigma)\,\omega(dx)\,.$$

The sets $E_k(Q)$ are disjoint by (2.11). And so we should sum this last inequality. But we do so subject to one more division of the $Q \in \mathcal{Q}_k$. Let

$$\mathcal{Q}_k^{\text{small}} := \{Q \in \mathcal{Q}_k \,:\, \omega(E_k(Q)) \le \eta\omega(Q)\},$$

and $\mathcal{Q}_k^{\text{big}} := \mathcal{Q}_k \backslash \mathcal{Q}_k^{\text{small}}$. Now,

$$\begin{aligned} 2^{kp} \sum_{Q \in \mathcal{Q}_k^{\text{small}}} \omega(E_k(Q)) &\le 2^{kp}\eta \sum_{Q \in \mathcal{Q}_k} \omega(Q) \\ &\le 2^{kp}\eta\omega(\{\mathrm{T}_\natural(f\sigma) > 2^k\}) \,. \end{aligned}$$

This is the first half of (3.5).

We also use the testing condition in (1.21) to estimate

$$\begin{aligned} 2^{kp} \sum_{Q \in \mathcal{Q}_k^{\text{big}}} \omega(E_k(Q)) &\lesssim \eta^{-p} \sum_{Q \in \mathcal{Q}} \omega(E_k(Q)) \Big[\mathbb{E}^w_{Q^{(\zeta)}} \mathrm{T}(f\mathbf{1}_{Q^{(\zeta)}}\sigma)\Big]^p \\ &\lesssim \mathfrak{T}^p \sum_{Q \in \mathcal{Q}} \omega(E_k(Q)) \big[\omega(Q^{(\zeta)})^{-1} \cdot \|f\mathbf{1}_{Q^{(\zeta)}}\|_{L^p(\sigma)} \omega(Q^{(\zeta)})^{1/p'}\big]^p \\ &\lesssim \mathfrak{T}^p \sum_{Q \in \mathcal{Q}} \|f\mathbf{1}_{Q^{(\zeta)}}\|^p_{L^p(\sigma)} \\ &\lesssim \mathfrak{T}^p \|f\|^p_{L^p(\sigma)} \,. \end{aligned}$$

The last line follows from the finite overlap condition (2.13). This completes the proof of (3.4).

4. First Steps in the Proof of the Strong Type Inequality

The conditions (1.24) easily follow from the strong type inequality (1.23). For the first condition, just note that $U_\tau f \le \mathrm{T}_\natural f$; the second follows from Hölder's inequality; and the last condition obviously follows from (1.23). Thus, the content of the Theorem is that the testing conditions (1.24) imply (1.23).

We will show that

$$\|\mathrm{T}_\natural(f\sigma)\|_{L^p(\omega)} \lesssim \big\{\mathfrak{T} + \mathfrak{T}^* + \mathfrak{U}\big\} \|f\|_{L^p(\sigma)} \,, \tag{4.1}$$

$$\mathfrak{T}^* := \sup_{\|g\|_\infty \le 1} \sup_Q \sigma(Q)^{-1/p} \|\mathrm{T}_\natural(g\mathbf{1}_Q)\|_{L^p(\omega)} \,, \tag{4.2}$$

$\mathfrak{T}$ is defined in (2.8), and $\mathfrak{U}$ is defined in (3.2). This proof requires some initial steps before the main steps can be taken. The reader can consult Figure 4.1 for a schematic tree of the proof of this estimate.

4.1. **Initial Decomposition of $\|\mathrm{T}_\natural(f\sigma)\|_{L^p(w)}$.** Take f to be a bounded function which is a finite sum of indicators of dyadic cubes. We use the notation (2.10), and apply the decomposition of the sets Ω_k into collections of cubes $\mathcal{Q}_k$ as in

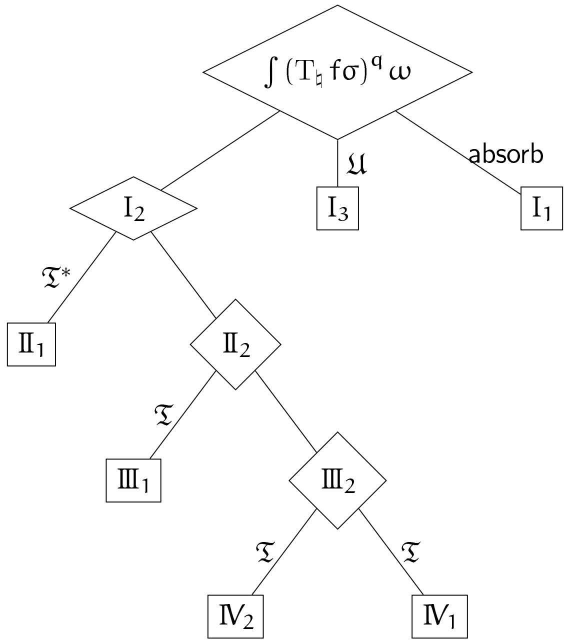

FIGURE 4.1. The schematic tree of the proof of the strong type inequality. For the purposes of this figure, we suppress the dependence of the estimates on K, and replace $S(K)$ in the root of the diagram by $\int (\mathrm{T}_\natural\, f\sigma)^p \omega(dx)$. Terms in diamonds are further decomposed and those in rectangles are final estimates. On edges leading into rectangles, we indicate how that term is controlled. By 'absorb' in the edge leading into I_1, we mean that I_1 can be absorbed into the left hand side. The term I_3 is the sole term controlled by $\mathfrak{U}$. The term II_1 is associated with the 'good function' of the Calderón-Zygmund decompositions, and is controlled by the dual testing condition $\mathfrak{T}^*$. The term II_2 and its descendent's are associated with the 'bad function' and are controlled by the testing condition $\mathfrak{T}$.

(2.11)—(2.15). In particular, we take $\rho = \zeta + 1$. We modify the notation of $E_k(Q)$ from the previous section to

$$E_k(Q) \coloneqq Q \cap \{\Omega_{k+m-1} - \Omega_{k+m}\} , \qquad Q \in \mathcal{Q}_k . \tag{4.3}$$

For an illustration, see Figure 4.2. Here, m will be a fixed constant depending upon dimension, ζ and τ. We emphasize that a given dyadic cube Q can be in many collections $\mathcal{Q}_k$, which fact will enter into a late stage of the proof. Also note that these sets are disjoint in Q, thus we have the estimate following from (2.3),

$$\sum_k \sum_{Q\in\mathcal{Q}_k} \omega(E_k(Q)) \big|\mathbb{E}^w_Q \phi\big|^p \lesssim \|\phi\|^p_{L^p(w)} . \tag{4.4}$$

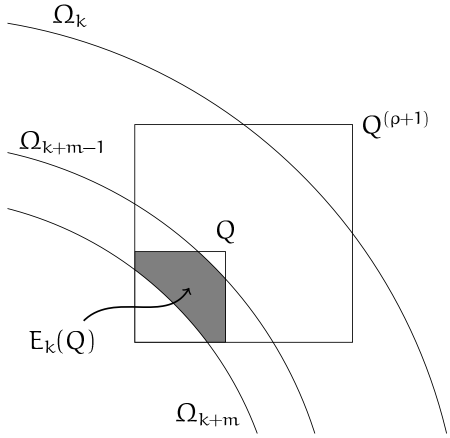

FIGURE 4.2. The set $E_k(Q)$.

Our testing conditions do not trivially imply that $\|\mathrm{T}_\natural f\sigma\|_{L^p(\omega)}$ is finite, which difficulty we circumvent with these definition. For an integer $K > 1$, we take

$$\mathbb{Z}' := \{k \in \mathbb{Z} \,:\, \omega(\Omega_{k+m-1} - \Omega_{k+m}) > \tfrac{1}{2m}\omega(\Omega_k)\}\,, \quad \mathbb{Z}'(K) = \mathbb{Z}' \cap [-K, K]\,. \tag{4.5}$$

Note that we have $\|\mathrm{T}_\natural(f\sigma)\|_{L^p(w)}^p \lesssim \sum_{k\in\mathbb{Z}'} 2^{kp}\omega(\Omega_{k+m-1} - \Omega_{k+m})$, and we will estimate

$$S(K) := \sum_{k\in\mathbb{Z}'(K)} \sum_{Q\in\mathcal{Q}_k} 2^{kp}\omega(E_k(Q))\,. \tag{4.6}$$

The point here is that the this sum is finite: The testing conditions imply that $\mathrm{T}_\natural$ satisfies the two weight weak type inequality (1.20), whence $S(K)$ is finite.

4.2. **Linearization and Maximum Principle.** We now make a choice of linearization, as in Definition 2.4, adapted to the collections $\mathcal{Q}_k$. For a choice of integer m that depends upon the implied constants occurring in Lemma 2.16, we can choose L, as in (2.5), so that these conditions hold for all k and all $Q \in \mathcal{Q}_k$, and $x \in E_k(Q)$:

$$\rho(x) \le \ell(Q) + \zeta + 1\,, \qquad x \in E_k(Q)\,, \tag{4.7}$$

$$\mathrm{T}_\natural(f\sigma)(x) \le C\,\mathrm{L}(f\sigma)(x) + \sup_{\substack{\zeta+2<r\le 2\zeta+2\\ Q^{(r)}\in\mathcal{U}_\tau}} |\langle f\sigma, h'_{Q^{(r)}}\rangle h_{Q^{(r)}}(x)| \tag{4.8}$$

Is certainly possible to achieve all of these conditions, as the sets $E_k(Q)$ are disjoint as $Q \in \mathcal{Q}_k$ and $k \in \mathbb{Z}$ vary. Note that we can then write

$$\mathrm{L}(f\sigma)(x) = \mathrm{L}(\mathbf{1}_{Q^{(\zeta+1)}}f\sigma)(x)\,, \qquad x \in E_k(Q)\,,\ Q \in \mathcal{Q}_k\,.$$

The sum $S(K)$ is estimated as follows. For a constant $0 < \eta < 1$, which we will take to be of order c^p for absolute constant c, estimate

$$S(K) \le I_1(K) + I_2(K) + I_3(K) \tag{4.9}$$

$$I_j(K) := \sum_{k\in\mathbb{Z}'(K)} \sum_{Q\in\mathcal{Q}_k^j} 2^{kp}\omega(E_k(Q)), \qquad j = 1, 2, 3$$

$$\mathcal{Q}_k^1 := \{Q \in \mathcal{Q}_k \ : \ \omega(E_k(Q)) < \eta\omega(Q)\} \tag{4.10}$$

$$\mathcal{Q}_k^2 := \Big\{Q \in \mathcal{Q}_k - Q_k^1 \ : \ \inf_{x\in E_k(Q)} \mathrm{L}(f\sigma)(x) > c2^k\Big\} \tag{4.11}$$

$$\mathcal{Q}_k^3 := \mathcal{Q}_k - \mathcal{Q}_k^1 - \mathcal{Q}_k^2 .$$

Observe that for appropriate m in (4.3), if $\mathcal{U}_\tau = \emptyset$ we have (2.18) in force, in which case $Q_k^3 = \emptyset$ for all k. In (4.11), we take $0 < c < 1$ to be a small constant depending upon the implied constant in (2.17).

The estimates we will prove are, with implied constants depending only on the parameter τ associated with the operator T and dimension.

$$\begin{aligned} I_1 &\lesssim \eta S(K), \\ I_2(K) &\lesssim \eta^{-p-1}\big\{\mathfrak{T} + \mathfrak{T}^*\big\}^p \|f\|_{L^p(\sigma)}^p, \\ I_3(K) &\lesssim \mathfrak{U}^p \|f\|_{L^p(\sigma)}^p . \end{aligned} \tag{4.12}$$

With these estimates proved, we have from (4.6) and (4.9)

$$S(K) \lesssim \eta S(K) + \eta^{-p-1}\big\{\mathfrak{T} + \mathfrak{T}^*\big\}^p \|f\|_{L^p(\sigma)}^p + \mathfrak{U}^p \|f\|_{L^p(\sigma)}^p .$$

For $0 < \eta < 1$ sufficiently small, the first term can be absorbed into the left hand side of the inequality. Letting $K \to \infty$ proves (4.1).

5. Two Easy Estimates.

Proof: Estimate for I_1. The proof of (4.12) is straight forward. By definitions, especially (4.5), we have

$$\begin{aligned} I_1(K) &\le \eta \sum_{k\in\mathbb{Z}'(K)} 2^{kp} \sum_{Q\in\mathcal{Q}_k} \omega(Q) \\ &\le 2\eta m \sum_{k\in\mathbb{Z}'(K)} 2^{kp}\omega(\Omega_{k+m-1} - \Omega_{k+m}) \le 2\eta m S(K) . \end{aligned}$$ □

It is for the proof above that we introduced the set $\mathbb{Z}'$, the parameter K being introduced to get the *a priori* finiteness of $S(K)$. Having exploited both of these points, we will suppress their appearance in the remaining arguments.

Proof: Estimate for I_3. For $Q \in \mathcal{Q}_k^3$, we necessarily have the estimate below from (2.17).

$$\sup_{\substack{\zeta+2\le r\le 2\zeta+2 \\ Q^{(r)}\in\mathcal{U}_\tau}} |\langle f\sigma, h'_{Q^{(r)}}\rangle h_{Q^{(r)}}(x)| \ge c2^k$$

And this means that the sum below can be controlled in terms of the maximal function U_τ.

$$\begin{aligned} I_3 &\lesssim \sum_{k=-\infty}^{\infty} \sum_{Q\in\mathcal{Q}_k} \omega(E_k(Q)) \times \sup_{\substack{\zeta+2\le r\le 2\zeta+2 \\ Q^{(r)}\in\mathcal{U}_\tau}} \lvert\langle f\sigma, h'_{Q^{(r)}}\rangle h_{Q^{(r)}}(x)\rvert \\ &\lesssim \mathfrak{U}^p \lVert f\rVert_{L^p(\sigma)}^p \,. \end{aligned}$$

Here, one dominates the supremum over r by a sum of at most ζ terms. □

6. The Estimate for I_2.

6.1. **The Calderón-Zygmund Decompositions.** We utilize Calderón-Zygmund decompositions adapted to measure σ. In addition, we will use the decomposition at *all heights simultaneously.* The purpose of this section is to describe this decomposition and apply it to the term I_2.

Let $\mathcal{G}_t$ be the maximal dyadic cubes Q such that

$$2^t \le \mathbb{E}^\sigma_Q \lvert f\rvert \le 2^{t+1}\,. \tag{6.1}$$

Then, we have the following variant of (2.2), which we will refer to below.

$$\sum_{t=-\infty}^{\infty} 2^{tp} \sum_{G\in\mathcal{G}_t} \sigma(G) \lesssim \lVert f\rVert_{L^p(\sigma)}^p \,. \tag{6.2}$$

It is the cubes in $\mathcal{G}_t$ that we use to organize our proof. Let $\mathcal{G} = \bigcup_t \mathcal{G}_t$. Let $\tau\colon \mathcal{G} \mapsto \mathbb{Z}$ be given by $\tau(\mathcal{G}_t) := t$, for all t. This map is well-defined as each cube is a member of a unique $\mathcal{G}_t$. For any cube Q, let $\Gamma(Q)$ denote the minimal cube $G\in\mathcal{G}$ such that $Q\subset G$. ($\Gamma(Q)$ is the 'father' of Q in $\mathcal{G}$.) We have the following nested property.

$$G\in\mathcal{G}_k\,,\ G'\in\mathcal{G}_l\,,\ \ G\subsetneq G' \quad \text{implies} \quad k\ge l\,. \qquad \text{(nested property)}$$

That is, the containment is strict.

We construct the Calderón-Zygmund decompositions. We set

$$\mathcal{C}(G) := \{G'\in\mathcal{G}_{\tau(G)+1} \ :\ G'\subset G\}\,. \tag{6.3}$$

These are the 'children' of G in the collection $\mathcal{G}$. For $G\in\mathcal{G}$, we set $f\mathbf{1}_G = g_G + b_G$ where

$$g_G := \begin{cases} \mathbb{E}^\sigma_{G'} f & x\in G'\,,\ G'\in\mathcal{C}(G) \\ f(x) & x\in G - \bigcup\{G' \ :\ G'\in\mathcal{P}(G)\} \\ 0 & x\notin G\,. \end{cases} \tag{6.4}$$

This choice then specifies b_G. It is simple to see that $\lVert g_G\rVert_\infty \le 2^{\tau(G)+2}$, thus g_G is the 'good function at height $2^{\tau(G)}$' and b_G is the 'bad function.' The cancellation property that the bad function has is that $\mathbb{E}^\sigma_G b_G = 0$. See Figure 6.1.

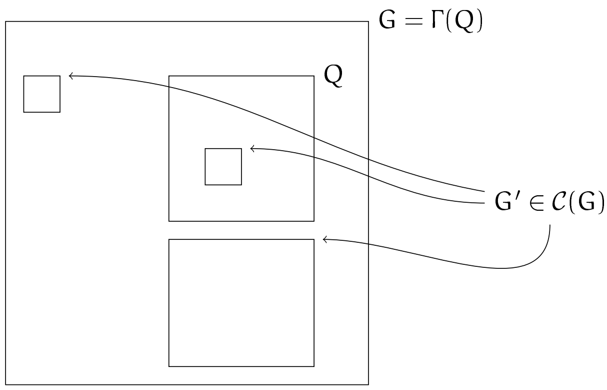

FIGURE 6.1. A given cube Q is contained in $G \in \Gamma(Q)$. Cubes $G' \in \mathcal{C}(G)$ are contained inside G. If no cube G' is contained in Q, then $|f|\mathbf{1}_Q \le 2^{\tau(G)+2}$ ω-a.e.

Let

$$\mathcal{H}(G) := \{(Q,k) \;:\; Q \in \mathcal{Q}_k^2\,,\, \Gamma(Q) = G\}\,, \qquad G \in \mathcal{G}\,.$$

We now refine the sum for I_2 according to the collections $\mathcal{H}$ and the Calderón-Zygmund Decomposition. Due to the definition of $\mathcal{Q}_k^2$, we can estimate

$$I_2 \lesssim \beta^{-p}\{II_1 + II_2\}$$

$$II_1 := \sum_{G\in\mathcal{G}} \sum_{(Q,k)\in\mathcal{H}(G)} \omega(E_k(Q)) \Big|\omega(Q)^{-1} \int_{E_k(Q)} \mathrm{L}(g_G\sigma)(x)\,\omega(dx)\Big|^p \tag{6.5}$$

$$II_2 := \sum_{G\in\mathcal{G}} \sum_{(Q,k)\in\mathcal{H}(G)} \omega(E_k(Q)) \Big|\omega(Q)^{-1} \int_{E_k(Q)} \mathrm{L}(b_G\sigma)(x)\,\omega(dx)\Big|^p \tag{6.6}$$

6.2. **The Estimate for the Good Functions.** The term II_1 in (6.5) involve the good functions. We have

$$II_1 \lesssim (\mathfrak{T}^*)^p \|f\|_{L^p(\sigma)}^p\,.$$

Proof. We estimate from (6.5), using the condition in (4.2), and the fact that the good functions are bounded, namely $\|g_G\|_\infty \le 2^{t+2}$ for $G \in \mathcal{G}_t$. But first, the sets $E_k(Q)$ are disjoint so

$$\begin{aligned} II_1 &\le \sum_{G\in\mathcal{G}} \sum_{(Q,k)\in\mathcal{H}(G)} \omega(E_k(Q)) \big|\mathbb{E}_Q^w \mathrm{L}(g_G\sigma)\big|^p \\ &\lesssim \sum_{G\in\mathcal{G}} \int_G \big|\mathrm{L}(g_G\sigma)\big|^p\,\omega(dx) && \text{(by (4.4))} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=-\infty}^{\infty} 2^{p(t+2)} \sum_{G\in\mathcal{G}_t} \int_G |\mathrm{L}(2^{-t-2} g_G \sigma)|^p \,\omega(dx) && \text{(recall (6.4))} \\
&\le (\mathfrak{T}^*)^p \sum_{t=-\infty}^{\infty} 2^{p(t+1)} \sum_{G\in\mathcal{G}_t} \sigma(G) && \text{(by (4.2))} \\
&\lesssim (\mathfrak{T}^*)^p \|f\|_{L^p(\sigma)}^p \,. && \text{(by (6.2))}
\end{aligned}
$$

Here, we can apply the testing condition (4.2) since $\|2^{-t-2} g_G\|_\infty \le 1$. □

7. The Analysis of II_2

We turn our attention to the term II_2 defined in (6.6), which is the term arising from the 'bad functions.' Its analysis will occupy the remainder of the proof. We did not explicitly define the 'bad' functions before, so let us do so now. They are

$$
b_G := \begin{cases} \beta_{G'} & x \in G' \,,\ G' \in \mathcal{C}(G) \qquad \text{(See (6.3))} \\ 0 & x \notin G \,. \end{cases} \tag{7.1}
$$

$$
\beta_{G'} := \mathbf{1}_{G'}\big(f(x) - \mathbb{E}^{\sigma}_{G'} f\big) \,, \qquad G' \in \mathcal{G} \,. \tag{7.2}
$$

In particular, $\beta_{G'}$ is supported on G' and has σ-mean zero.

The operator L is selected in (4.7)—(4.8). We are considering the expressions below, where $G \in \mathcal{G}$, and $(Q,k) \in \mathcal{H}(G)$:

$$
\int_{E_k(Q)} \mathrm{L}(b_G\sigma)(x)\,\omega(dx) = \sum_{G'\in\mathcal{C}(G)} \int_{E_k(Q)} \mathrm{L}(\beta_{G'}\sigma)(x)\,\omega(dx) \tag{7.3}
$$

These conditions mean:

- $G \in \mathcal{G}$: The average value of f on G, with respect to σ-measure is about $2^{\tau(G)}$, and G is a maximal dyadic cube with this property, see (6.1).
- $(Q,k) \in \mathcal{H}(G)$: G is the minimal cube in $\mathcal{G}$ containing Q, and $Q \in \mathcal{Q}^2_k$.
- Recall that $G' \in \mathcal{G}_{\tau(Q)+1}$, $G' \subsetneq G$ and $\Gamma(Q) = G$.

There is a cancellation that takes place here: Recall the role of the integer ζ in (1.2). We have

$$
\operatorname{supp}(\mathrm{L}(\beta_{G'}\sigma)) \subset (G')^{(\zeta)} \,. \tag{7.4}
$$

If $x \notin (G')^{(\zeta)}$, and P is any dyadic cube that contains x and intersects $(G')^{(\zeta)}$, let P' be the subcube with $\ell(P') = 2^{-\zeta}\ell(P)$ that contains G'. It follows that and that h'_P is constant on P' and $\int_{P'} \beta_{G'}\ \sigma(dy) = 0$. Hence, $\langle \beta_{G'}, h'_Q\rangle = 0$. This proves the assertion above.

The cubes $G' \in \mathcal{C}(G)$ are disjoint, but this does not apply to the cubes $(G')^{(\zeta)}$ to address this point we make the following construction. Take $\overline{\mathcal{C}}(Q)$ to be the maximal cubes among the collection

$$
\{(G')^{(\zeta)} \ :\ G' \in \mathcal{C}(Q)\,,\, Q \cap (G')^{(\zeta)} \neq \emptyset\}
$$

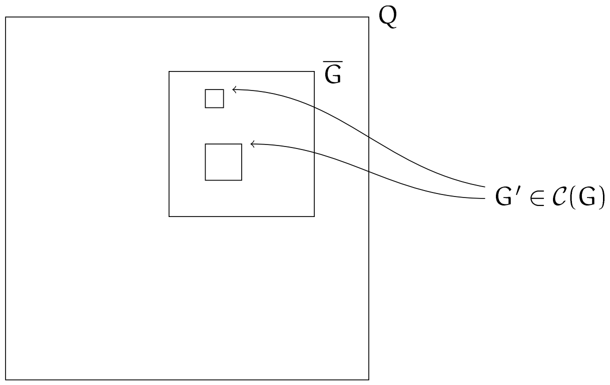

FIGURE 7.1. The largest cube is $Q \in \mathcal{Q}_k$, with $\Gamma(Q) = G$. The cube Q contains two members of $G' \in \mathcal{C}(G)$, and the maximal cube of the form $(G')^{(\zeta)}$ is denoted as $\overline{G}$.

If any G' satisfies $Q \subset (G')^{(\zeta)}$, then this collection consists of a single tile, a complication we will have to track in the analysis of below. See Figure 7.1 for an illustration. Take $\overline{\Gamma}_Q : \mathcal{C}(Q) \mapsto \overline{\mathcal{C}}(Q)$ to be the map that assigns to $G' \in \mathcal{C}(Q)$ the minimal element of $\overline{\mathcal{C}}(Q)$ that contains it, if such a cube exists. (No such cube exists for G' if $Q \cap (G')^{(\zeta)} = \emptyset$.) Set

$$\beta_{\overline{G}} := \sum_{\substack{G' \in \mathcal{C}(G) \\ \overline{\Gamma}_Q(G) = \overline{G}}} \beta_{G'} \,. \tag{7.5}$$

We argue that we have the estimate

$$|\beta_{\overline{G}}| \lesssim \mathrm{M}_\sigma f \times \sum_{\substack{G' \in \mathcal{C}(G) \\ \overline{\Gamma}_Q(G) = \overline{G}}} \mathbf{1}_{G'} \le \mathrm{M}_\sigma f \times \mathbf{1}_Q \,. \tag{7.6}$$

Indeed, the collection of cubes $\{G' \in \mathcal{C}(G) \,:\, \overline{\Gamma}_Q(G') = \overline{G}\}$ are pairwise disjoint, a property inherited from $\mathcal{C}(Q)$, and the estimate above follows from (7.2).

By (7.4), we have the equality

$$\mathrm{L}(\beta_{\Gamma(Q)}\sigma)\mathbf{1}_Q = \mathbf{1}_Q \sum_{\overline{G} \in \overline{\mathcal{C}}(Q)} \mathrm{L}(\beta_{\overline{G}}\sigma) \,.$$

Therefore, we can continue the equality in (7.3) to conclude that for $(Q,k) \in \mathcal{H}(G)$ we have

$$\begin{aligned} \int_{E_k(Q)} \mathrm{L}(b_G\sigma)(x)\,\omega(dx) &= \sum_{\overline{G} \in \overline{\mathcal{C}}(Q)} \int_{E_k(Q)\cap\overline{G}} \mathrm{L}(\overline{\beta}_{\overline{G}}\sigma)(x)\,\omega(dx) \\ &= \sum_{\overline{G} \in \overline{\mathcal{C}}(Q)} \int \overline{\beta}_{\overline{G}}\, \mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap\overline{G}}\omega)(y)\,\sigma(dy) \end{aligned}$$

$$= \mathit{III}_1(Q) + \mathit{III}_2(Q)$$

$$\mathit{III}_1(Q) := \sum_{\overline{G} \in \overline{\mathcal{C}}(Q)} \int_{\overline{G} \setminus \Omega_{k+m}} \overline{\beta}_{\overline{G}} \, \mathrm{L}^*(\mathbf{1}_{E_k(Q) \cap \overline{G}} \omega)(y) \, \sigma(dy) \, .$$

The definition of $\mathit{III}_2(Q)$ is similar with the integration being done over $\overline{G} \cap \Omega_{k+m}$.

With these definitions, we set

$$\mathit{III}_v := \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{Q}_k^2} \omega(E_k(Q)) \big[\omega(Q)^{-1} \mathit{III}_v(Q)\big]^p \, , \qquad v = 1, 2 \, . \tag{7.7}$$

We have $\mathit{II}_2 \lesssim \mathit{III}_1 + \mathit{III}_2$. And we turn to the proof of

$$\mathit{III}_1 \lesssim \mathfrak{T}^p \|f\|_{L^p(\sigma)}^p \, .$$

Proof: Estimate for III_1. Use Hölder's inequality and the testing condition (2.8) in its dual linearized form (2.7).

$$\begin{aligned}
|\mathit{III}_1(Q)| &\le \sum_{\overline{G} \in \overline{\mathcal{C}}(Q)} \|\mathbf{1}_{\overline{G} \setminus \Omega_{k+m}} \overline{\beta}_{\overline{G}}\|_{L^p(\sigma)} \|\mathrm{L}^*(\mathbf{1}_{E_k(Q) \cap \overline{G}} \omega)\|_{L^{p'}(\sigma)} \\
&\le \mathfrak{T} \sum_{\overline{G} \in \overline{\mathcal{C}}(Q)} \|\mathbf{1}_{\overline{G} \setminus \Omega_{k+m}} \overline{\beta}_{\overline{G}}\|_{L^p(\sigma)} \omega(Q \cap \overline{G})^{1/p'} \\
&\le \mathfrak{T} \omega(Q)^{1/p'} \|\mathbf{1}_{Q \setminus \Omega_{k+m}} \mathrm{M}_\sigma f\|_{L^p(\sigma)} \, .
\end{aligned}$$

Here, we have used Hölder's inequality, in the variable $\overline{G} \in \overline{\mathcal{C}}(Q)$, as well as (7.6).

We can now appeal to the definition of III_1, (7.7), to see that

$$\begin{aligned}
\mathit{III}_1 &\le \mathfrak{T}^p \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{Q}_k^2} \omega(E_k(Q)) \big[\omega(Q)^{-1/p} \|\mathbf{1}_{Q \setminus \Omega_{k+m}} \mathrm{M}_\sigma f\|_{L^p(\sigma)}\big]^p \\
&\le \mathfrak{T}^p \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{Q}_k^2} \|\mathbf{1}_{Q \setminus \Omega_{k+m}} \mathrm{M}_\sigma f\|_{L^p(\sigma)}^p \\
&\le \mathfrak{T}^p \sum_{k=-\infty}^{\infty} \|\mathbf{1}_{\Omega_k \setminus \Omega_{k+m}} \mathrm{M}_\sigma f\|_{L^p(\sigma)}^p \\
&\le \mathfrak{T}^p \|f\|_{L^p(\sigma)}^p \, .
\end{aligned}$$

The proof of this estimate is complete. □

8. The Analysis of III_2

We analyze the term III_2. To be specific, one has

$$\mathit{III}_2(Q) := \sum_{\overline{G} \in \overline{\mathcal{C}}(Q)} \int_{\overline{G} \cap \Omega_{k+m}} \overline{\beta}_{\overline{G}} \, \mathrm{L}^*(\mathbf{1}_{E_k(Q) \cap \overline{G}} \omega) \sigma(dy)$$

We turn to the collection $\mathcal{Q}_{k+m}$, namely the Whitney decomposition of Ω_{k+m}.

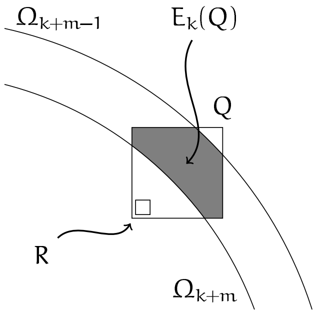

FIGURE 8.1. A cube $Q \in \mathcal{Q}_k$, sets Ω_{k+m-1}, Ω_{k+m} and $E_k(Q)$ are indicated. A set $R \in \mathcal{Q}_{k+m}$ is indicated. This particular cube is also contained inside of Q, which is not general the case.

Note that as $E_k(Q) \cap \Omega_{k+m} = \emptyset$, the function $\mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap G'}\omega)$ is constant on each cube of the form $R \in \Omega_{k+m}$. Indeed, the cube $R^{(\rho)} = R^{(\zeta+1)}$ does not intersect $E_k(Q)$, so this follows from Lemma 2.9. (This conclusion is our rationale for linking the Whitney decompositions to the structure of the operators we consider. See Figure 8.1 for an illustration.) Thus,

$$\int_{R\cap\overline{G}\cap Q} \overline{\beta}_{\overline{G}} \cdot \mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap\overline{G}}\omega)\sigma(dy)$$
$$= \int_{R\cap\overline{G}\cap Q} \mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap\overline{G}}\omega)\ \sigma(dy) \times \mathbb{E}^{\sigma}_{R\cap\overline{G}\cap Q}\overline{\beta}_{\overline{G}}\,, \quad R \in \mathcal{Q}_{k+m}\,.$$

There are two points to note in the last display: We can apply the testing condition to the integral, and we have the σ-averages of $\overline{\beta}_{\overline{G}}$ above. We define

$$\mathcal{R}(R,\overline{G}) := \{R \in \mathcal{Q}_{k+m} \ :\ R \cap \overline{G} \cap Q \neq \emptyset\}\,, \qquad \overline{G} \in \overline{\mathcal{C}}(Q)\,. \tag{8.1}$$

By the nested property of the $\mathcal{Q}_k$, if $R \cap Q \neq \emptyset$, we must have $R \subset Q$. We have by (7.1) and (7.5)

$$\mathbb{E}^{\sigma}_{R\cap\overline{G}\cap Q}|\beta_{\overline{G}}| \le \mathbb{E}^{\sigma}_{R\cap\overline{G}\cap Q}\big|f - \mathbb{E}^{\sigma}_{\overline{G}}f\big|$$
$$\le \mathbb{E}^{\sigma}_{R\cap\overline{G}\cap Q}|f| + \Bigg|\mathbb{E}^{\sigma}_{R\cap\overline{G}\cap Q} \sum_{\substack{G'\in\mathcal{C}(G)\\ \overline{\Gamma}_Q(G)=\overline{G}}} \beta_{G'}\Bigg| \qquad \text{(by (7.5))}$$
$$\le 2\mathbb{E}^{\sigma}_{R\cap\overline{G}\cap Q}|f| + 2^{\tau(\Gamma(Q))+1} \qquad \text{(by (6.1))} \tag{8.2}$$

For the last line, recall the definition of $\beta_{G'}$ from (7.2). By selection of G' we have $|\mathbb{E}^{\sigma}_{G'}f| \le 2^{\tau(\Gamma(G))+1}$ for all G'. Hence, the last line follows.

This estimate reveal that there is a distinguished subset of $\mathcal{R}(Q,\overline{G})$, it is

$$\mathcal{R}_1(Q,\overline{G}) := \{R \in \mathcal{R}(Q,\overline{G}) \ :\ \mathbb{E}^{\sigma}_{R\cap\overline{G}\cap Q}|f| \le 2^{\tau(\Gamma(Q))+4}\}\,. \tag{8.3}$$

Let $\mathcal{R}_2(Q,\overline{G}) := \mathcal{R}(Q,\overline{G})\backslash\mathcal{R}_1(Q,\overline{G})$. By the construction of the cubes $\mathcal{G}$, we have the inclusion

$$R \subset G' \cap Q\,, \qquad R \in \mathcal{R}_2(Q,\overline{G})\,,\ G' \in \mathcal{C}(Q),\ G' \subset \overline{G}\,. \tag{8.4}$$

Now, we estimate

$$\begin{aligned}&\mathit{III}_2(Q) \lesssim \mathit{IV}_1(Q) + \mathit{IV}_2(Q)\,,\\ &\mathit{IV}_v(Q) := \sum_{\overline{G}\in\overline{\mathcal{C}}(Q)} \sum_{R\in\mathcal{R}_v(Q,\overline{G})} \int_{R\cap\overline{G}\cap Q} |\mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap\overline{G}}\omega)|\,\sigma(dy)\times\mathbb{E}^\sigma_R|\beta_{\overline{G}}|\,,\ v=1,2.\end{aligned} \tag{8.5}$$

We define IV_1, IV_2 as in (7.7).

We claim

$$\mathit{IV}_1 \lesssim [\![\sigma,w]\!]_{\mathrm{T},p}^{p'}\|f\|^p_{L^p(\sigma)}$$

Proof: Estimate for IV_1. In this case, we are nicely set up to appeal to the maximal function inequality (2.2). For $Q\in\mathcal{Q}^2_k$, we estimate as follows.

$$\begin{aligned}\mathit{IV}_1(Q) &\lesssim 2^{\tau(\Gamma(Q))}\sum_{\overline{G}\in\overline{\mathcal{C}}(Q)} \sum_{R\in\mathcal{R}_1(Q,\overline{G})} \int_{R\cap\overline{G}\cap Q} |\mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap\overline{G}}\omega)|\,\sigma(dy)\\
&\lesssim 2^{\tau(\Gamma(Q))}\sum_{\overline{G}\in\overline{\mathcal{C}}(Q)} \int_{\overline{G}\cap Q} |\mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap\overline{G}}\omega)|\,\sigma(dy)\\
&\lesssim 2^{\tau(\Gamma(Q))}\sum_{\overline{G}\in\overline{\mathcal{C}}(Q)} \sigma(Q\cap\overline{G})^{1/p}\|\mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap\overline{G}}\omega)\|_{L^{p'}(\sigma)} && \text{(Hölder's inequality)}\\
&\lesssim \mathfrak{T}2^{\tau(\Gamma(Q))}\sum_{\overline{G}\in\overline{\mathcal{C}}(Q)} \sigma(Q\cap\overline{G})^{1/p}\omega(Q\cap\overline{G})^{1/p'} && \text{(See (4.2), (2.7).)}\\
&\lesssim \mathfrak{T}2^{\tau(\Gamma(Q))}\sigma(Q)^{1/p}\cdot\omega(Q)^{1/p'} && \text{(Hölder's inequality)}\\
&\lesssim \eta^{-1}\mathfrak{T}2^{\tau(\Gamma(Q))}\sigma(Q)^{1/p}\cdot\omega(E_k(Q))^{1/p'}\,.\end{aligned}$$

In the last line, we have used the fact that $Q\notin\mathcal{Q}^1_k$ to pass from $\omega(Q)$ to $\omega(E_k(Q))$. See (4.10). Let us also note, as follows from definitions, that $Q\subset\Gamma(Q)\subset\{\mathrm{M}_\sigma f > 2^{\tau(\Gamma(Q))}\}$, which fact we will use below.

We estimate as follows.

$$\begin{aligned}\mathit{IV}_1 &\lesssim \sum_{k=-\infty}^{\infty}\sum_{Q\in\mathcal{Q}^2_k}\omega(E_k(Q))\big[\omega(Q)^{-1}\mathit{IV}_1(Q)\big]^p\\
&\lesssim \eta^{-1}\mathfrak{T}^p\sum_{k=-\infty}^{\infty}\sum_{Q\in\mathcal{Q}^2_k}\omega(E_k(Q))\big[2^{\tau(\Gamma(Q))}\omega(Q)^{-1/p}\sigma(Q)^{1/p}\big]^p\\
&\lesssim \eta^{-1}\mathfrak{T}^p\sum_{G\in\mathcal{G}}2^{p\tau(G)}\sigma(\{\mathrm{M}_\sigma f > 2^{\tau(G)}\})\end{aligned}$$

$$\lesssim \eta^{-1}\mathfrak{T}^p \|f\|_{L^p(\sigma)}^p .$$

The last line follows from (6.2). □

9. Analysis of IV_2

We turn to the estimate for IV_2, showing that

$$\mathit{IV}_2 \lesssim [\![\sigma, w]\!]_{\mathrm{T},p}^{p'} \|f\|_{L^p(\sigma)}^p \tag{9.1}$$

This is case in which combinatorics of our decomposition are essential to obtain an estimate on this sum. This proof also completes the proof of our Theorem.

We begin with an estimate for $\mathit{IV}_2(Q)$, taking these steps: (a) invoke (8.4) to simplify the range of integration that appears in (8.5); (b) use the definition of $\mathcal{R}_2(Q,\overline{G})$ and (8.2) to insert the term $\mathbb{E}_R^\sigma|f|$; and (c) insert $\sigma(Q)^{\pm 1/p}$ into the sum below and apply Hölder's inequality in the summing indicies.

$$\begin{aligned}
\mathit{IV}_2(Q) &= \sum_{\overline{G}\in\overline{\mathcal{C}}(Q)} \sum_{R\in\mathcal{R}_2(Q,\overline{G})} \int_R |\mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap\overline{G}}\omega)| \,\sigma(dy) \times \mathbb{E}_R^\sigma|f| \\
&\le A(Q)\cdot B(Q)\,, \\
A(Q)^p &:= \sum_{\overline{G}\in\overline{\mathcal{C}}(Q)} \sum_{R\in\mathcal{R}_2(Q,\overline{G})} \sigma(R)\big[\mathbb{E}_R^\sigma|f|\big]^p\,, \\
B(Q)^{p'} &:= \sum_{\overline{G}\in\overline{\mathcal{C}}(Q)} \sum_{R\in\mathcal{R}_2(Q,\overline{G})} \sigma(R)^{-p'/p}\left[\int_R |\mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap\overline{G}}\omega)| \,\sigma(dy)\right]^{p'} \\
&\le \sum_{\overline{G}\in\overline{\mathcal{C}}(Q)} \sum_{R\in\mathcal{R}_2(Q,\overline{G})} \int_R |\mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap\overline{G}}\omega)|^{p'} \,\sigma(dy) \\
&\le \sum_{\overline{G}\in\overline{\mathcal{C}}(Q)} \int_{Q\cap\overline{G}} |\mathrm{L}^*(\mathbf{1}_{E_k(Q)\cap\overline{G}}\omega)|^{p'} \,\sigma(dy) \\
&\le \mathfrak{T}^{p'} \sum_{\overline{G}\in\overline{\mathcal{C}}(Q)} \omega(Q\cap\overline{G}) \qquad \text{(See (4.2), (2.7).)} \\
&\le \mathfrak{T}^{p'}\omega(Q)\,.
\end{aligned}$$

Recall that the rectangles R above are disjoint, since they are members of $\mathcal{Q}_{k+m}$, and contained in Q, by (8.4).

Therefore, we have

$$\mathit{IV}_2 = \sum_{k=-\infty}^{\infty} \sum_{Q\in\mathcal{Q}_k^2} \omega(E_k(Q))\big[\omega(Q)^{-1}\mathit{IV}_2(Q)\big]^p$$

$$\lesssim \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{Q}_k^2} \omega(E_k(Q)) \big[\omega(Q)^{-1} A(Q) \cdot B(Q)\big]^p$$

$$\lesssim \mathfrak{T}^p \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{Q}_k^2} \omega(E_k(Q)) \cdot \omega(Q)^{-1} \times \sum_{\overline{G} \in \overline{\mathcal{C}}(Q)} \sum_{R \in \mathcal{R}_2(Q, \overline{G})} \sigma(R) \big[\mathbb{E}_R^\sigma |f|\big]^p$$

$$\lesssim \mathfrak{T}^p \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{Q}_k^2} \sum_{\overline{G} \in \overline{\mathcal{C}}(Q)} \sum_{R \in \mathcal{R}_2(Q, \overline{G})} \sigma(R) \big[\mathbb{E}_R^\sigma |f|\big]^p . \tag{9.2}$$

It is our intention to dominate this last sum (9.2) by $\|\mathrm{M}_\sigma f\|_{L^p(\sigma)}^p$. To accomplish this, there are two points that should be demonstrated. (1) Any given cube R can arise in the sum above at most a bounded number of times. (2) Two given cubes $R_1 \subsetneq R_2$ that arise in the sum above satisfy $\mathbb{E}_{R_1}^\sigma |f| > 4\mathbb{E}_{R_2}^\sigma |f|$. Both points are true, and require some combinatorial arguments to verify.

Definition 9.3. Say that a cube R is of *type IV_2* if it arises in the sum (9.2). More specifically, we say that (k, Q, G', R) is a *type IV_2 quadruple* if

1. $k \in \mathbb{Z}$,
2. $Q \in \mathcal{Q}_k^2$,
3. $G' \in \mathcal{C}(Q)$ and $R \subset G'$, (See (8.4).)
4. $R \in \mathcal{R}_2(Q, \overline{\Gamma}(G'))$. In particular $R \in \mathcal{Q}_{k+m}$ and $\mathbb{E}_R^\sigma |f| > 2^{\tau(G')+3}$. (See (8.1) and (8.3).)

Let us set $\mathcal{IV}_2$ the collection of type IV_2 quadruples.

Proposition 9.4. *A given cube R can occur as a type IV_2 cube in only a bounded number of ways. Namely, if (k_t, Q_t, R, G'_t), for $1 \le t < T$ are distinct type IV_2 quadruples, with R fixed, then we have $T \le m + \eta^{-1}$.*

Proof. Let us note that we have $k_t \neq k_s$ for $1 \le s \neq t \le T$. Assuming that $k_s = k_t$ we would have $R \subset Q_{k_s} \cap Q_{k_t}$ by (8.4), so that by (2.11), we would have $Q = Q_{k_s} = Q_{k_t}$. Likewise, we have $R \subset G'_{k_s} \cap G'_{k_t}$, with $G'_{k_s}, G'_{k_t} \in \mathcal{C}(Q)$, hence $G'_{k_s} = \overline{G}'_{k_t}$, and the data are equal.

Now, let us assume that $k_1 < k_2 < \cdots < k_T$. We have $R \in \mathcal{Q}_{k_t+m}$. We see that of necessity, that $R \in \mathcal{Q}_k$ for $k_1 + m \le k \le k_T + m$. That is, for $m \le t \le T$ *both* R and Q_{k_t} are in $\mathcal{Q}_{k_t}$. (We can assume $T > m$.) Then the disjoint cover condition (2.11) implies that $R = Q_{k_t}$.

An important difficulty that arises here is that a given cube Q can be a member of an unbounded number of $\mathcal{Q}_k$. But the additional assumption is that we have $R \in \mathcal{Q}_{k_t}^2$ for $m \le t \le T$ implies that $\omega(E_{k_t}(Q)) \ge \eta\omega(Q)$. These last sets are disjoint and contained in Q, so that we see our claim is proved. □

This last Proposition is an important step towards our goal. Still, the cubes R of type IV_2, even if distinct, can still overlap. We address this point in the next two propositions.

Proposition 9.5. *For each $G_0 \in \mathcal{G}$, we have*

$$\sum_{\substack{(k,Q,R,G')\in \mathcal{IV}_2 \\ \Gamma(Q)=G_0,\, k=\varepsilon mod\ m}} \mathbf{1}_R \le m\,, \qquad \varepsilon = 0,1,\dots,m-1\,.$$

That is, there are bounded overlaps of the type IV_2 cubes if we hold constant both the principle cube G_0, and the parity of $k \mod m$.

Proof. Let $(k_s, Q_s, R_s, G'_s) \in \mathcal{IV}_2$ for $s = 1, 2$ be two distinct type IV_2 quadruples with $R_1 \subset R_2$ and $\Gamma(Q_1) = \Gamma(Q_2) = G_0$. Then, we must have $k_1 \ge k_2$ by the nested property and the fact that $R_s \in \mathcal{Q}_{k_s+m}$. The case of $k_1 = k_2$ would imply that the quadruples are the same, and so seeking contradiction we must have $k_1 > k_2 + m$. Then, we have

$$R_1 \subset Q_1 \subset R_2 \subset Q_2\,.$$

But, by definition we have

$$\mathbb{E}^\sigma_{R_2}|f| > 8\mathbb{E}^\sigma_{Q_2}|f| > 4\mathbb{E}^\sigma_{\Gamma(Q_2)}|f|\,.$$

That is, we cannot have $\Gamma(Q_1) = \Gamma(Q_2)$. We have our contradiction. □

Proposition 9.6. *If we have two type IV_2 quadruples $(k_s, Q_s, R_s, G'_s) \in \mathcal{IV}_2$ for $s = 1, 2$, with $R_1 \subsetneq R_2$ and $k_1 = k_2 \mod m$ then we have*

$$\mathbb{E}^\sigma_{R_1}|f| \ge 4\mathbb{E}^\sigma_{R_2}|f|\,.$$

Proof. The hypotheses, with the nested property (2.15) give us that $k_1 \ge k_2 + m$. By the previous Proposition, we must have $G'_1 \ne G'_2$, hence $R_1 \subsetneq G'_1 \subsetneq R_2$. And from this we see that

$$\mathbb{E}^\sigma_{R_1}|f| \ge 8\mathbb{E}^\sigma_{G'_1}|f| \ge 8\mathbb{E}_{\Gamma(R_2)|f|} \ge 4\mathbb{E}^\sigma_{R_2}|f|\,.$$

And so our claim is proved. □

We can complete the estimate for IV_2. Let

$$\mathcal{T} := \{R \,:\, \exists k, Q, G \ni (k, Q, R, G) \in \mathcal{IV}_2\}\,.$$

Note that Proposition 9.6 implies that we have

$$\sum_{R\in\mathcal{T}} \sigma(Q)\big[\mathbb{E}^\sigma_R|f|\big]^p \lesssim \|\mathrm{M}_\sigma f\|^p_{L^p(\sigma)}\,.$$

And so we can estimate using Proposition 9.4,

$$(9.2) \lesssim \beta^{-1}\sum_{R\in\mathcal{T}} \sigma(Q)\big[\mathbb{E}^\sigma_R|f|\big]^p \lesssim \|f\|_{L^p(\sigma)}$$

This completes the proof of (9.1).

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