ABSTRACT SPLINES IN KREIN SPACES

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ABSTRACT. We present generalizations to Krein spaces of the abstract interpolation and smoothing problems proposed by Atteia in Hilbert spaces: given $\mathcal{H}$, $\mathcal{K}$ Krein spaces and $\mathcal{E}$ a Hilbert space, (bounded) surjective operators $T\colon \mathcal{H} \to \mathcal{K}$ and $V\colon \mathcal{H} \to \mathcal{E}$, $\rho > 0$ and fixed $z_0 \in \mathcal{E}$, we study the existence of solutions of the problems $\operatorname{argmin}\{[\,Tx, Tx\,]_{\mathcal{K}} : Vx = z_0\}$ and

$$\operatorname{argmin}\left\{[\,Tx, Tx\,]_{\mathcal{K}} + \rho\|Vx - z_0\|^2_{\mathcal{E}} : x \in \mathcal{H}\right\}.$$

1. Introduction

Since I. J. Schoenberg introduced the spline functions [30], they have became an important notion in several branches of mathematics such us approximation theory, statistics, numerical analysis and partial differential equations, among others. Moreover, they have been useful to solve some practical issues in signal and image processing [18, 31, 32, 16], computer graphics [5, 23, 24], learning theory [9, 10] and other applications.

In the sixties, a Hilbert space formulation of spline functions, known as abstract splines, was introduced by M. Atteia [3] and developed by several authors, see for instance [2, 13, 22, 29]. Given Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{E}$, consider (bounded) surjective operators $T\colon \mathcal{H} \to \mathcal{K}$ and $V\colon \mathcal{H} \to \mathcal{E}$. The abstract interpolation problem in Hilbert spaces can be stated as follows: fixed $z_0 \in \mathcal{E}$, find $x_0 \in \mathcal{H}$ such that $Vx_0 = z_0$ and

$$\|Tx_0\|^2_{\mathcal{K}} = \min\{\|Tx\|^2_{\mathcal{K}} : Vx = z_0\}. \tag{1.1}$$

Observe that $x_0 \in V^{-1}(\{z_0\})$ is an abstract interpolating spline (i.e. x_0 satisfies Eq. (1.1)) if and only if Tx_0 realizes the distance between $TV^\dagger z_0$ and the subspace $T(N(V))$, where $V^\dagger$ stands for the Moore-Penrose inverse of V. So, the existence of x_0 depends on the existence of a suitable (contractive) projection of $TV^\dagger z_0$ onto $T(N(V))$. Then, if $T(N(V))$ is a closed subspace of $\mathcal{K}$, the existence of x_0 is guaranteed because the selfadjoint projection onto $T(N(V))$ is always contractive.

On the other hand, the abstract smoothing problem introduces a new parameter $\rho > 0$ in order to balance the amounts $\|Tx\|^2$ and $\|Vx - z_0\|^2$. Formally, given $\rho > 0$ and fixed $z_0 \in \mathcal{E}$, it consists in minimizing the function $F_\rho\colon \mathcal{H} \to \mathbb{R}^+$ defined by

$$F_\rho(x) = \|Tx\|^2_{\mathcal{K}} + \rho\|Vx - z_0\|^2_{\mathcal{E}}. \tag{1.2}$$

Key words and phrases. Krein spaces, abstract splines, oblique projections.

This problem can be reduced to a least squares problem. In fact,

$$F_\rho(x) = \|Lx - (0, z_0)\|_\rho^2,$$

where $\| \, \|_\rho$ is the norm associated to the inner product on $\mathcal{K} \times \mathcal{E}$ defined by $\langle (y,z), (y',z') \rangle_\rho = \langle y, y' \rangle_{\mathcal{K}} + \rho \langle z, z' \rangle_{\mathcal{E}}$ if $(y,z), (y',z') \in \mathcal{K} \times \mathcal{E}$, and L is an auxiliar operator from $\mathcal{H}$ into $\mathcal{K} \times \mathcal{E}$. Therefore, the abstract smoothing problem is also related to the existence of a selfadjoint (contractive) projection onto $R(L)$.

There is also a variational problem which mixes both abstract interpolation and smoothing problems. In the abstract mixed problem (as it is known) the "measurement operator" $V \colon \mathcal{H} \to \mathcal{E}$ splits up into two surjective operators. The technique used by A. Rozhenko [26] to solve this problem is similar to the one mentioned above to solve the abstract smoothing problem. So, the existence of "abstract mixed splines" also depends on the existence of a suitable contractive projection.

For a complete exposition on these subjects see the books by Atteia [4], A. Bezhaev and V. Vasilenko [6], and the survey by R. Champion et al. [12].

In this work, mainly motivated by the ideas exposed above, we present generalizations of the abstract interpolation, smoothing and mixed problems to Krein spaces. As we have mentioned before, the techniques used to solve these problems in the Hilbert space setting, involved contractive projections onto some subspaces. So, they (or their complementary subspaces) are asked to be closed. In order to reproduce this geometrical approach for Krein spaces, the hypothesis on the subspaces has to be modified. Recall that to guarantee the existence of a selfadjoint projection onto a subspace of a Krein space, it has to be regular. Moreover, if the projection has to be contractive then its nullspace has to be uniformly J-positive, where J stands for the fundamental symmetry of the Krein space (see [17]).

First, we study the indefinite abstract interpolation problem. Specifically, if $\mathcal{H}$ and $\mathcal{K}$ are two Krein spaces and $\mathcal{E}$ is a Hilbert space, given (bounded) surjective operators $T \colon \mathcal{H} \to \mathcal{K}$ and $V \colon \mathcal{H} \to \mathcal{E}$ and fixed $z_0 \in \mathcal{E}$, we are interested in characterizing (if there is any) those $x_0 \in \mathcal{H}$ such that $Vx_0 = z_0$ and

$$[\,Tx_0, Tx_0\,]_{\mathcal{K}} = \min\{[\,Tx, Tx\,]_{\mathcal{K}} : V(x) = z_0\}.$$

Using a similar argument as in the definite interpolation problem, it can be shown that the existence of x_0 depends on the existence of a suitable (contractive) projection of $TV^\dagger z_0$ onto the J-orthogonal companion of $T(N(V))$ in $\mathcal{K}$. Then, if $T(N(V))$ is a closed uniformly positive subspace of $\mathcal{K}$, the existence of x_0 is guaranteed.

On the other hand, in the indefinite abstract smoothing problem, we look for the minimizers of the function $F_\rho \colon \mathcal{H} \to \mathbb{R}$ defined by

$$F_\rho(x) = [\,Tx, Tx\,]_{\mathcal{K}} + \rho \|Vx - z_0\|_{\mathcal{E}}^2, \quad x \in \mathcal{H}. \tag{1.3}$$

This problem can be no longer restated as a least squares problem, but as an indefinite least squares problem. The technique used to describe its solutions is similar to the one used in the definite smoothing problem, but a particular orthogonal decomposition of the range of a given operator is needed.

The last problem we consider is the indefinite abstract mixed problem. If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces and $\mathcal{E}_1$ and $\mathcal{E}_2$ are Hilbert spaces, consider (bounded) surjective operators $T\colon \mathcal{H} \to \mathcal{K}$, $V_1\colon \mathcal{H} \to \mathcal{E}_1$ and $V_2\colon \mathcal{H} \to \mathcal{E}_2$. Given $\rho > 0$ and fixed $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, we look for those $x_0 \in \mathcal{H}$ such that $V_1 x_0 = z_1$ which are minimizers of the function

$$G_\rho(x) = [\,Tx, Tx\,]_{\mathcal{K}} + \rho \|V_2 x - z_2\|^2_{\mathcal{E}_2}, \quad x \in V_1^{-1}(\{z_1\}). \tag{1.4}$$

Spline functions in indefinite metric spaces have already been studied in [9] to solve numerical aspects related to learning theory problems. Although the problems presented there are different from those studied in this work, they are closely related. In [10] another version of the abstract indefinite smoothing problem is studied: given $z_0 \in \mathcal{E}$, instead of finding the minimum of the function F_ρ given in Eq. (1.3), the authors are interested in stabilizing it.

The paper is organized as follows: Section 2 contains the preliminaries. In Section 3 we study the indefinite abstract interpolation problem, we give necessary and sufficient conditions for the existence (and uniqueness) of solutions of this problem, and characterize them. Also, given a frame $\{f_n\}_{n\in\mathbb{N}}$ for the Hilbert space $\mathcal{E}$, we give conditions to obtain different frames for subspaces of $|\mathcal{H}|$ (the Hilbert space associated to $\mathcal{H}$) composed by interpolating splines corresponding to the family $\{f_n\}_{n\in\mathbb{N}}$.

Section 4 is devoted to the study of the indefinite abstract smoothing problem: after characterizing its set of solutions (for a fixed ρ), we show that it is related to the set of solutions of an indefinite interpolation problem for a certain $z_\rho \in \mathcal{E}$. Then, as it was studied by Atteia in Hilbert spaces, we analyze the convergence of the solutions of the indefinite smoothing problem to the solutions of the indefinite interpolation problem as ρ goes to infinity.

In section 5 the abstract mixed problem studied by A. Rozhenko and V. Vasilenko [26, 27, 28], is extended to Krein spaces.

2. Preliminaries

Along this work $\mathcal{E}$ denotes a complex (separable) Hilbert space. If $\mathcal{F}$ is another Hilbert space then $L(\mathcal{E}, \mathcal{F})$ is the algebra of bounded linear operators from $\mathcal{E}$ into $\mathcal{F}$, $L(\mathcal{E}) = L(\mathcal{E}, \mathcal{E})$ and denote by $\mathcal{Q}$ the set of (oblique) projections, i.e. $\mathcal{Q} = \{Q \in L(\mathcal{E}) : Q^2 = Q\}$. If $T \in L(\mathcal{E}, \mathcal{F})$ then $T^* \in L(\mathcal{F}, \mathcal{E})$ denotes the adjoint operator of T, $R(T)$ stands for its range and $N(T)$ for its nullspace. Also, if $T \in L(\mathcal{E}, \mathcal{F})$ has closed range, $T^\dagger$ denotes the Moore-Penrose inverse of T.

If $\mathcal{S}$ and $\mathcal{T}$ are two (closed) subspaces of $\mathcal{E}$, denote by $\mathcal{S} \dotplus \mathcal{T}$ the direct sum of $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S} \oplus \mathcal{T}$ the (direct) orthogonal sum of them and $\mathcal{S} \ominus \mathcal{T} := \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$. If $\mathcal{E} = \mathcal{S} \dotplus \mathcal{T}$, the oblique projection onto $\mathcal{S}$ along $\mathcal{T}$, $P_{\mathcal{S}//\mathcal{T}}$, is the unique $Q \in \mathcal{Q}$ with $R(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{S}$ and $N(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{T}$. In particular, $P_{\mathcal{S}} := P_{\mathcal{S}//\mathcal{S}^\perp}$ is the orthogonal projection onto $\mathcal{S}$.

2.1. **Krein spaces.** In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject (and the proofs of the results below) see the books by J. Bognár [7] and T. Ya. Azizov and I. S. Iokhvidov [19], the monographs by T. Ando [1] and by M. Dritschel and J. Rovnyak [15] and the paper by J. Rovnyak [25].

Given a Krein space $(\mathcal{H}, [\ ,\ ])$ with a *fundamental decomposition* $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$, the direct (orthogonal) sum of the Hilbert spaces $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ is denoted by $(|\mathcal{H}|, \langle\ ,\ \rangle)$. Sometimes we use the notation $[\ ,\ ]_{\mathcal{H}}$ instead of $[\ ,\ ]$ to emphasize the Krein space considered.

Observe that the indefinite metric of $\mathcal{H}$ and the inner product of $|\mathcal{H}|$ are related by means of a *fundamental symmetry*, i.e. a unitary selfadjoint operator $J \in L(\mathcal{H})$ which satisfies:

$$[\,x, y\,] = \langle\, Jx, y\,\rangle, \qquad x, y \in \mathcal{H}.$$

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces then $L(\mathcal{H}, \mathcal{K})$ stands for $L(|\mathcal{H}|, |\mathcal{K}|)$, and $L(\mathcal{H})$ for $L(|\mathcal{H}|)$. Given $T \in L(\mathcal{H}, \mathcal{K})$, the J-adjoint operator of T is defined by $T^{\#} = J_{\mathcal{H}} T^* J_{\mathcal{K}}$, where $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$ are the fundamental symmetries associated to $\mathcal{H}$ and $\mathcal{K}$, respectively. An operator $T \in L(\mathcal{H})$ is said to be J-selfadjoint if $T = T^{\#}$.

Given a subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$, the *J-orthogonal companion* to $\mathcal{S}$ is defined by

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} : [\,x, s\,] = 0, \text{ for every } s \in \mathcal{S}\}.$$

Notice that if $T \in L(\mathcal{H}, \mathcal{K})$ and $\mathcal{S}$ is a closed subspace of $\mathcal{K}$ then

$$T^{\#}(\mathcal{S})^{[\perp]_{\mathcal{H}}} = T^{-1}(\mathcal{S}^{[\perp]_{\mathcal{K}}}). \tag{2.1}$$

A subspace $\mathcal{S}$ of $\mathcal{H}$ is non degenerated if $\mathcal{S} \cap \mathcal{S}^{[\perp]} = \{0\}$. A vector $x \in \mathcal{H}$ is *J-positive* if $[\,x, x\,] > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is J-positive if every $x \in \mathcal{S}$, $x \neq 0$, is a J-positive vector. Moreover, it is said to be *uniformly J-positive* if there exists $\alpha > 0$ such that

$$[\,x, x\,] \geq \alpha \|x\|^2, \qquad \text{for every } x \in \mathcal{S},$$

where $\|\ \|$ stands for the norm of the associated Hilbert space $|\mathcal{H}|$. J-nonnegative, J-neutral, J-negative and J-nonpositive vectors (and subspaces) are defined analogously. Notice that if $\mathcal{S}$ is a J-definite subspace of $\mathcal{H}$ then it is non degenerated.

Definition. Let $\mathcal{H}$ be a Krein space with fundamental symmetry J. A subspace $\mathcal{S}$ of $\mathcal{H}$ is called *regular* if $\mathcal{S}$ is the range of a J-selfadjoint projection.

Proposition 2.1 ([19], Cor. 7.17)**.** *Let $\mathcal{H}$ be a Krein space with fundamental symmetry J and $\mathcal{S}$ a J-nonnegative closed subspace of $\mathcal{H}$. Then, $\mathcal{S}$ is regular if and only if $\mathcal{S}$ is uniformly J-positive.*

Corollary 2.2 ([7], Thm. 8.4)**.** *Let $\mathcal{H}$ be a Krein space with fundamental symmetry J and $\mathcal{S}$ a closed uniformly J-positive subspace of $\mathcal{H}$. If Q is the J-selfadjoint projection onto $\mathcal{S}$ then, given $x \in \mathcal{H}$,*

$$[\,x - Qx, x - Qx\,] = \min_{y \in \mathcal{S}} [\,x - y, x - y\,].$$

2.2. Angles between subspaces and reduced minimum modulus.

Definition. Let $\mathcal{S}$ and $\mathcal{T}$ be two closed subspaces of a Hilbert space $\mathcal{E}$. The cosine of the *Friedrichs angle* between $\mathcal{S}$ and $\mathcal{T}$ is defined by

$$c(\mathcal{S},\mathcal{T}) = \sup\{|\langle x, y\rangle| : x \in \mathcal{S} \ominus \mathcal{T}, \|x\| = 1,\ y \in \mathcal{T} \ominus \mathcal{S}, \|y\| = 1\}.$$

It is well known that

$$c(\mathcal{S},\mathcal{T}) < 1 \quad \Leftrightarrow \quad \mathcal{S}+\mathcal{T} \text{ is closed} \quad \Leftrightarrow \quad \mathcal{S}^{\perp}+\mathcal{T}^{\perp} \text{ is closed} \quad \Leftrightarrow \quad c(\mathcal{S}^{\perp},\mathcal{T}^{\perp}) < 1.$$

Furthermore, if $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ are the orthogonal projections onto $\mathcal{S}$ and $\mathcal{T}$, respectively, then $c(\mathcal{S},\mathcal{T}) < 1$ if and only if $(I-P_{\mathcal{S}})P_{\mathcal{T}}$ has closed range, or equivalently, $(I-P_{\mathcal{T}})P_{\mathcal{S}}$ has closed range. See [14] for further details.

Proposition 2.3 ([8, 20])**.** *Given a Hilbert space $\mathcal{H}$, let $A, B \in L(\mathcal{H})$ be closed range operators. Then, AB has closed range if and only if*

$$c(R(B), N(A)) < 1.$$

The next definition is due to T. Kato, see [21, Ch. IV, § 5] for a complete exposition on this subject.

Definition. The *reduced minimum modulus* $\gamma(T)$ of an operator $T \in L(\mathcal{E})$ is defined by

$$\gamma(T) = \inf\{\|Tx\| : \|x\| = 1;\ x \in N(T)^{\perp}\}.$$

It is well known that $\gamma(T) = \gamma(T^*) = \gamma(T^*T)^{1/2}$. Also, it can be shown that an operator $T \neq 0$ has closed range if and only if $\gamma(T) > 0$. In this case, $\gamma(T) = \|T^{\dagger}\|^{-1}$.

3. Indefinite abstract splines: definitions and basic results

Recently, some interpolation methods in Reproducing Kernel Hilbert Spaces (RKHS) have shown to be useful to deal with machine learning problems. Given a data set $X = \{x_1, \ldots, x_m\} \subseteq \mathcal{X}$ and labels $Y = \{y_1, \ldots, y_m\} \subset \mathbb{R}$, it is necessary to estimate the minimal norm function $f \in \mathcal{H}$ such that $f(x_i) = y_i$, where $\mathcal{H}$ is a RKHS with kernel

$$k\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}.$$

If $E\colon \mathcal{H} \to \mathbb{R}^m$ is the *evaluation map* given by $Ef = (f(x_1), \ldots, f(x_m))$, the above interpolation problem consists in finding $f \in \mathcal{H}$ such that

$$Ef = (y_1, \ldots, y_m) = y \quad \text{and} \quad \|f\|^2 = \min_{g \in E^{-1}(y)} \|g\|^2.$$

Notice that the adjoint operator $E^*\colon \mathbb{R}^m \to \mathcal{H}$ is given by

$$E^*\alpha = \sum_{i=1}^{m} \alpha_i k(x_i, x),$$

where $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$. Then, it follows that $K = EE^*$ is the Gram matrix associated to the kernel k, i.e. $K_{ij} = k(x_i, x_j)$.

Since $\mathcal{H} = R(E^*) \oplus N(E)$, it is easy to see that there is a solution to the above problem if and only if there exists $f \in R(E^*)$ such that $Ef = y$, or equivalently, there exists $\alpha \in \mathbb{R}^m$ such that $K\alpha = y$ (in this case the minimizing function is reconstructed as $f(x) = E^*\alpha = \sum_{i=1}^m \alpha_i k(x_i, x)$). So, the interpolation spline can be defined without using the norm of the RKHS but only its kernel.

In order to admit indefinite kernels to study machine learning problems, S. Canu et al. provided a definition of interpolating splines in a Reproducing Kernel Krein Space (RKKS), see [10, Definition 3.3]. Since the interpolation problem in RKHS mentioned above is a particular case of the abstract interpolation problem considered by M. Atteia (see Eq. (1.1)), it is natural to considerer a general indefinite version of the abstract interpolation problem.

Throughout this work, $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces with fundamental symmetries $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$, respectively, $\mathcal{E}$ is a Hilbert space and the operators $T \in L(\mathcal{H}, \mathcal{K})$ and $V \in L(\mathcal{H}, \mathcal{E})$ are surjective. Consider the following generalization of the abstract interpolation problem [3]:

Problem 1. Given $z_0 \in \mathcal{E}$, find $x_0 \in V^{-1}(\{z_0\})$ such that

$$[\,Tx_0, Tx_0\,]_{\mathcal{K}} = \min\{[\,Tx, Tx\,]_{\mathcal{K}} : \, Vx = z_0\}. \tag{3.1}$$

Definition. Any element $x_0 \in V^{-1}(\{z_0\})$ satisfying Eq. (3.1) is called an *indefinite abstract spline* or, more specifically, a (T, V)*-interpolant* to $z_0 \in \mathcal{E}$. The set of (T, V)-interpolants to z_0 is denoted by $sp(T, V, z_0)$.

Considering the Moore-Penrose inverse of V, the above problem can be restated as: Fixed $z_0 \in \mathcal{E}$, find $u_0 \in N(V)$ such that

$$[\,T(V^\dagger z_0 + u_0), T(V^\dagger z_0 + u_0)\,]_{\mathcal{K}} = \min_{u \in N(V)} [\,T(V^\dagger z_0 + u), T(V^\dagger z_0 + u)\,]_{\mathcal{K}}. \tag{3.2}$$

As it was mentioned in the introduction, the following lemma shows under which conditions indefinite abstract splines do exists.

Lemma 3.1. *Given $z_0 \in \mathcal{E}$, $x_0 \in V^{-1}(\{z_0\})$ is a (T, V)-interpolant to z_0 if and only if $T(N(V))$ is a $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$ and $Tx_0 \in T(N(V))^{[\perp]_{\mathcal{K}}}$.*

Proof. Suppose that $x_0 \in \mathcal{H}$ is a (T, V)-interpolant to z_0. Then, for every $u \in N(V)$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned}[\,Tx_0, Tx_0\,] &\le [\,T(x_0 + \alpha u), T(x_0 + \alpha u)\,] \\ &= [\,Tx_0, Tx_0\,] + 2\alpha\,\mathrm{Re}[\,Tx_0, Tu\,] + \alpha^2 [\,Tu, Tu\,].\end{aligned}$$

Therefore, $2\alpha\,\mathrm{Re}[\,Tx_0, Tu\,] + \alpha^2[\,Tu, Tu\,] \ge 0$ for every $\alpha \in \mathbb{R}$, and a standard argument shows that $\mathrm{Re}[\,Tx_0, Tu\,] = 0$. Analogously, if $\beta = i\alpha$, $\alpha \in \mathbb{R}$, it follows that $\mathrm{Im}[\,Tx_0, Tu\,] = 0$. Then, $[\,Tx_0, Tu\,] = 0$ and $[\,Tu, Tu\,] \ge 0$ for every $u \in N(V)$.

Conversely, suppose that $T(N(V))$ is a $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$ and there exists $x_0 \in V^{-1}(\{z_0\})$ such that $Tx_0 \,[\perp]_{\mathcal{K}}\, T(N(V))$. If $u_0 = x_0 - V^{\dagger}z_0 \in N(V)$ then, for every $u \in N(V)$,

$$\begin{aligned}[\,T(V^{\dagger}z_0+u), T(V^{\dagger}z_0+u)\,] &= [\,T(V^{\dagger}z_0+u_0), T(V^{\dagger}z_0+u_0)\,] \\ &\quad + [\,T(u-u_0), T(u-u_0)\,] \geq [\,Tx_0, Tx_0\,].\end{aligned}$$

Therefore, x_0 is a (T,V)-interpolant to z_0. □

As a consequence of Eq. (2.1), $sp(T,V,z_0)$ can be characterized as the intersection of a subspace and an affine manifold of $\mathcal{H}$.

Corollary 3.2. *Suppose that $T(N(V))$ is a $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$ and let $z_0 \in \mathcal{E}$. Then,*

$$sp(T,V,z_0) = (V^{\dagger}z_0 + N(V)) \cap T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}.$$

Proof. Given $z_0 \in \mathcal{E}$, suppose that $x_0 \in \mathcal{H}$ is a (T,V)-interpolant to z_0. Then, $u_0 = x_0 - V^{\dagger}z_0 \in N(V)$ and by the above lemma, $Tx_0 \in T(N(V))^{[\perp]_{\mathcal{K}}}$, or equivalently by Eq. (2.1), $x_0 \in T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}$. Therefore, $x_0 \in (V^{\dagger}z_0 + N(V)) \cap T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}$.

On the other hand, if $x = V^{\dagger}z_0 + u \in T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}$ with $u \in N(V)$, then $Tx \in T(N(V))^{[\perp]_{\mathcal{K}}}$ and $Vx = z_0$. So, applying Lemma 3.1, it follows that $x \in sp(T,V,z_0)$. □

The following lemma shows how regularity conditions on $T(N(V))$ determine relationships between the subspaces $N(T)$ and $T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}$.

Lemma 3.3.

(1) *If $T(N(V))$ is non degenerated, then $N(V) \cap T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}} = N(V) \cap N(T)$.*

(2) *If $T(N(V))$ is regular, then $\mathcal{H} = N(V) + T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}$.*

Proof. (i.) The inclusion $N(T) \cap N(V) \subseteq N(V) \cap T^{\#}T(N(V))^{[\perp]}$ is straightforward, see Eq. (2.1). On the other hand, if $x \in N(V) \cap T^{\#}T(N(V))^{[\perp]}$ then $Tx \in T(N(V)) \cap T(N(V))^{[\perp]} = \{0\}$. Thus, $x \in N(V) \cap N(T)$.
(ii.) If $T(N(V))$ is a regular subspace of $\mathcal{K}$ then $\mathcal{K} = T(N(V)) + T(N(V))^{[\perp]}$. Therefore,

$$\mathcal{H} = T^{-1}(T(N(V))) + T^{-1}(T(N(V))^{[\perp]}) = N(V) + T^{\#}T(N(V))^{[\perp]}$$

(see Eq. (2.1)). □

As mentioned above, if $T(N(V))$ is a regular subspace of $\mathcal{K}$ then $\mathcal{H} = N(V) + T^{\#}T(N(V))^{[\perp]}$. But this may not be a direct sum. Therefore, there is a family of closed subspaces of $T^{\#}T(N(V))^{[\perp]}$ which are complementary to $N(V)$. Along this work, if $T(N(V))$ is a regular subspace of $\mathcal{K}$ we will consider the following projection:

$$Q_0 = P_{N(V)//T^{\#}T(N(V))^{[\perp]} \ominus N(V)}. \tag{3.3}$$

Proposition 3.4. *Suppose that $T(N(V))$ is a closed subspace of $\mathcal{K}$. Then, the set $sp(T,V,z) \neq \varnothing$ for every $z \in \mathcal{E}$ if and only if $T(N(V))$ is uniformly $J_{\mathcal{K}}$-positive. In this case, $sp(T,V,z)$ is an affine manifold parallel to $N(V) \cap N(T)$.*

Proof. Suppose that $T(N(V))$ is a closed uniformly $J_{\mathcal{K}}$-positive subspace of $\mathcal{K}$. Then, by Proposition 2.1, $T(N(V))$ is a regular subspace of $\mathcal{K}$, and $Q_0 \in \mathcal{Q}$ (see Lemma 3.3). Fixed $z \in \mathcal{E}$, let $x = (I - Q_0)V^\dagger z \in \mathcal{H}$. Then, $Vx = z$ and $Tx \in T(N(V))^{[\perp]}$. So, by Lemma 3.1, $x \in sp(T,V,z)$, i.e. $sp(T,V,z) \neq \varnothing$ for every $z \in \mathcal{E}$.

Conversely, suppose that $sp(T,V,z) \neq \varnothing$ for every $z \in \mathcal{E}$. Then, as a consequence of Lemma 3.1, $T(N(V))$ is a $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$. Furthermore, for each $z \in \mathcal{E}$, there exists a vector $x_z \in \mathcal{H}$ such that $Vx_z = z$ and $Tx_z \in T(N(V))^{[\perp]_{\mathcal{K}}}$. Since $V^\dagger z = (V^\dagger z - x_z) + x_z$ and $V(V^\dagger z - x_z) = 0$ for every $z \in \mathcal{E}$, it is easy to see that $N(V)^\perp \subseteq N(V) + T^\# T(N(V))^{[\perp]}$. Therefore, $\mathcal{H} = N(V) + T^\# T(N(V))^{[\perp]}$ and $\mathcal{K} = T(N(V)) + T(N(V))^{[\perp]}$. So, $T(N(V))$ is a regular $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$, i.e. $T(N(V))$ is a uniformly $J_{\mathcal{K}}$-positive subspace of $\mathcal{K}$ (see Proposition 2.1).

Assuming that $T(N(V))$ is uniformly $J_{\mathcal{K}}$-positive, if $x_1, x_2 \in sp(T,V,z)$ then, by Lemma 3.3,

$$x_1 - x_2 \in N(V) \cap T^\# T(N(V))^{[\perp]} = N(V) \cap N(T). \qquad \square$$

Corollary 3.5. *Suppose that $T(N(V))$ is a closed uniformly $J_{\mathcal{K}}$-positive subspace of $\mathcal{K}$ and $N(T) \cap N(V) = \{0\}$. Then, given $z \in \mathcal{E}$, $sp(T,V,z)$ is a singleton. More precisely,*

$$sp(T,V,z) = \{P_{T^\# T(N(V))^{[\perp]}//N(V)} V^\dagger z\}.$$

In what follows, fixed $z_0 \in \mathcal{E}$, it is shown that $sp(T,V,z_0)$ can be parametrized by means of a family of projections onto $N(V)$.

Proposition 3.6. *Suppose that $T(N(V))$ is a closed uniformly $J_{\mathcal{K}}$-positive subspace of $\mathcal{K}$. Given $z_0 \in \mathcal{E}$, $x \in sp(T,V,z_0)$ if and only if there exists $Q \in \mathcal{Q}$ with $R(Q) = N(V)$ and $N(Q) \subseteq T^\# T(N(V))^{[\perp]_{\mathcal{H}}}$ such that $x = (I-Q)V^\dagger z_0$.*

To prove the above proposition, we need the following lemma.

Lemma 3.7. *Let $Q \in \mathcal{Q}$ and suppose that $T(N(V))$ is a regular subspace of $\mathcal{K}$. Then, $R(Q) = N(V)$ and $N(Q) \subseteq T^\# T(N(V))^{[\perp]_{\mathcal{H}}}$ if and only if $Q = Q_0 + Z$, where $Z \in L(\mathcal{H})$ is such that $N(V) \subseteq N(Z)$ and $R(Z) \subseteq N(V) \cap N(T)$.*

Proof. If $Q \in L(\mathcal{H})$ is a projection with $R(Q) = N(V)$ and $N(Q) \subseteq T^\# T(N(V))^{[\perp]}$, let $Z = Q - Q_0$. Since $R(Q) = R(Q_0) = N(V)$ it is trivial that $N(V) \subseteq N(Z)$. On the other hand, consider $y = Zx \in R(Z)$: $y = Qx - Q_0 x \in N(V)$ and $y = (I-Q_0)x - (I-Q)x \in T^\# T(N(V))^{[\perp]}$. Then $y \in N(V) \cap T^\# T(N(V))^{[\perp]} = N(V) \cap N(T)$.

Conversely, given $Z \in L(\mathcal{H})$ with $N(V) \subseteq N(Z)$ and $R(Z) \subseteq N(V) \cap N(T)$, consider $Q = Q_0 + Z$. Then, $Q^2 = Q$ because $Z^2 = 0$, $Q_0 Z = Z$ and $ZQ_0 = 0$.

It is easy to see that $R(Q) \subseteq N(V)$ and, if $x \in N(V)$ then $Qx = Q_0x = x$. Therefore, $R(Q) = N(V)$. Finally, observe that if $x \in N(Q)$ then $x = (I-Q)x = (I-Q_0)x - Zx \in T^{\#}T(N(V))^{[\perp]}$, because $N(Q_0) + R(Z) \subseteq T^{\#}T(N(V))^{[\perp]}$. □

Proof. (of Proposition 3.6) If $x = (I-Q)V^{\dagger}z_0$, where $Q \in \mathcal{Q}$ with $R(Q) = N(V)$ and $N(Q) \subseteq T^{\#}T(N(V))^{[\perp]}$, it is easy to see that $Vx = z_0$ and $Tx \in T(N(V))^{[\perp]}$. Then, by Lemma 3.1, $x \in sp(T, V, z_0)$.

Conversely, as a consequence of Proposition 3.4, $sp(T, V, z_0) = (I-Q_0)V^{\dagger}z_0 + N(V) \cap N(T)$ because $(I-Q_0)V^{\dagger}z_0 \in sp(T, V, z_0)$. Then, if $x \in sp(T, V, z_0)$ there exists $u \in N(V) \cap N(T)$ such that $x = (I-Q_0)V^{\dagger}z_0 + u$. So, consider $Z \in L(\mathcal{H})$ such that $Z(V^{\dagger}z_0) = -u$ and $Zy = 0$ if $y \perp V^{\dagger}z_0$. Then,

$$x = (I-Q_0)V^{\dagger}z_0 - ZV^{\dagger}z_0 = (I-(Q_0+Z))V^{\dagger}z_0,$$

$N(V) \subseteq N(Z)$ and $R(Z) \subseteq N(V) \cap N(T)$. Therefore, by the above lemma, $Q = Q_0 + Z \in \mathcal{Q}$ with $R(Q) = N(V)$ and $N(Q) \subset T^{\#}T(N(V))^{[\perp]}$. □

3.1. **Frames of indefinite abstract splines.** Recall that given a sequence $\{f_n\}_{n\in\mathbb{N}}$ in a Banach space X, it is called a *Schauder basis* of X if for every $x \in X$ there is a unique sequence of scalars $\{c_n\}_{n\in\mathbb{N}}$ so that $x = \sum_{n=1}^{\infty} c_n f_n$, where the series converges in the norm topology. A vector sequence $\{f_n\}_{n\in\mathbb{N}}$ in X is a *Riesz basis* if there exist constants $0 < A < B$ such that

$$A\sum_{n=1}^{m} |c_n|^2 \leq \left\| \sum_{n=1}^{m} c_n f_n \right\|^2 \leq B\sum_{n=1}^{m} |c_n|^2, \tag{3.4}$$

for all finite sequences $c_1, \dots, c_m$.

On the other hand, given a Hilbert space $\mathcal{E}$, a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$ is a *frame* for $\mathcal{E}$ if there exist constants $0 < A < B$ such that

$$A\|z\|^2 \leq \sum_{n=1}^{\infty} |\langle z, f_n \rangle|^2 \leq B\|z\|^2, \qquad \text{for every } z \in \mathcal{E}. \tag{3.5}$$

Observe that, if $\mathcal{E}$ is a Hilbert space, $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis of $\mathcal{E}$ if and only if $\{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{E}$ such that, if $\sum_{n=1}^{\infty} c_n f_n = 0$, then $c_n = 0$ for every $n \in \mathbb{N}$. See [11, 33] for further details on this subject.

In what follows, recall that $T \in L(\mathcal{H}, \mathcal{K})$ and $V \in L(\mathcal{H}, \mathcal{E})$ are surjective operators and suppose that $T(N(V))$ is a closed uniformly $J_{\mathcal{K}}$-positive subspace of $\mathcal{K}$.

Proposition 3.8. *Given a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$, suppose that there exists a frame $\{g_n\}_{n\in\mathbb{N}}$ for $\mathcal{W} = T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}$ such that $g_n \in sp(T, V, f_n)$ for every $n \in \mathbb{N}$. Then, $\{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{E}$.*

Proof. If $g_n \in sp(T, V, f_n)$ then, by Proposition 3.6, there exists $Q_n \in \mathcal{Q}$ with $R(Q_n) = N(V)$ and $N(Q_n) \subseteq \mathcal{W}$, such that $g_n = (I-Q_n)V^{\dagger}f_n$. Since $V(I-Q_n)V^{\dagger}$

$= I_{\mathcal{E}}$ for every $n \in \mathbb{N}$, it is easy to see that

$$\sum_{n=1}^{\infty} |\langle z, f_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle V^* z, (I - Q_n) V^\dagger f_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle P_{\mathcal{W}} V^* z, g_n \rangle|^2,$$

for every $z \in \mathcal{E}$, since $P_{\mathcal{W}}(I - Q_n) = (I - Q_n)$. Therefore, if $\{g_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{W}$ with frame bounds $0 < A < B$,

$$A\|P_{\mathcal{W}} V^* z\|^2 \leq \sum_{n=1}^{\infty} |\langle z, f_n \rangle|^2 \leq B\|P_{\mathcal{W}} V^* z\|^2 \leq B\|V\|^2\|z\|^2,$$

for every $z \in \mathcal{E}$. But $\|P_{\mathcal{W}} V^* z\|^2 \geq \gamma(P_{\mathcal{W}} V^*)^2 \|z\|^2 = \gamma(V P_{\mathcal{W}})^2\|z\|^2$. Since $c(\mathcal{W}, N(V)) < 1$ it follows by Proposition 2.3 that $VP_{\mathcal{W}}$ has closed range, so $\gamma(VP_{\mathcal{W}}) > 0$. Then, $\{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{E}$, with frame bounds $0 < A\gamma(VP_{\mathcal{W}})^2 < B\|V\|^2$. □

The next result shows that, given a frame $\{f_n\}_{n\in\mathbb{N}}$ for $\mathcal{E}$, it is possible to obtain frames of splines for any complement of $N(V)$ contained in $T^{\#}T(N(V))^{[\perp]}$.

Proposition 3.9. *Given a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$, consider $g_n = (I - Q)V^\dagger f_n \in sp(T, V, f_n)$, $n \in \mathbb{N}$, where $Q \in L(\mathcal{H})$ is any fixed projection such that $R(Q) = N(V)$ and $N(Q) \subseteq T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}$. Then,*

1. *$\{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{E}$ if and only if $\{g_n\}_{n\in\mathbb{N}}$ is a frame for $N(Q)$.*
2. *$\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis of $\mathcal{E}$ if and only if $\{g_n\}_{n\in\mathbb{N}}$ is a Riesz basis of $N(Q)$.*
3. *$\{f_n\}_{n\in\mathbb{N}}$ is a (Schauder) basis of $\mathcal{E}$ if and only if $\{g_n\}_{n\in\mathbb{N}}$ is a (Schauder) basis of $N(Q)$.*

Proof. Observe that, if $W = (I - Q)V^\dagger$, then $R(W) = R(I - Q) = N(Q)$ is closed. Then, $\gamma(W) > 0$.

(i.) Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{E}$. Notice that

$$\sum_{n=1}^{\infty} |\langle x, g_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle x, W f_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle W^* x, f_n \rangle|^2 \quad \text{for every } x \in \mathcal{H}.$$

So, if $0 < A < B$ are frame bounds for $\{f_n\}_{n\in\mathbb{N}}$ then

$$A\gamma(W)^2\|x\|^2 = A\gamma(W^*)^2\|x\|^2 \leq A\|W^* x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, g_n \rangle|^2$$
$$\leq B\|W^* x\|^2 \leq B\|W\|^2\|x\|^2,$$

for every $x \in N(W^*)^\perp = N(Q)$. Therefore, $\{g_n\}_{n\in\mathbb{N}}$ is a frame for $N(Q)$. The other implication is a consequence of Proposition 3.8.

(ii.) Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis of $\mathcal{E}$. Then it is also a frame for $\mathcal{E}$ and, by item *1*, the sequence $\{g_n\}_{n\in\mathbb{N}}$ is a frame for $N(Q)$. Furthermore, if there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} \alpha_k g_k = 0$, then applying V to both sides of the equation we obtain that $\sum_{k=1}^{\infty} \alpha_k f_k = 0$. So, $\alpha_k = 0$ for every

$k \in \mathbb{N}$ because $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis of $\mathcal{E}$. Therefore, $\{g_n\}_{n\in\mathbb{N}}$ is a Riesz basis of $N(Q)$. The other implication follows in the same way.

(iii.) It is analogous to the proof of [4, Ch. III, Proposition 1.1]. □

Given a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$, if $N(T)\cap N(V) = \{0\}$ it is easy to see that $\{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{E}$ if and only if $\{g_n\}_{n\in\mathbb{N}}$ is a frame for $T^{\#}T(N(V))^{[\perp]}$, where g_n is the (unique) (T,V)-interpolant to f_n (see Proposition 3.9). However, the following example shows that, if $N(T)\cap N(V) \neq \{0\}$, given a frame $\{f_n\}_{n\in\mathbb{N}}$ for $\mathcal{E}$ it is easy to construct $g_n \in sp(T,V,f_n)$ (for every $n\in\mathbb{N}$) such that $\{g_n\}_{n\in\mathbb{N}}$ is not a frame.

Example 1. Observe that if $\{f_n\}_{n\in\mathbb{N}}$ is a frame with frame bounds $0 < A < B$ then $\|f_n\|^2 \leq B$. Given $u \in N(T)\cap N(V)$ with $\|u\| = 1$, define

$$Z_n(x) = \begin{cases} n\alpha u & \text{if } x = \alpha V^\dagger f_n,\ \alpha\in\mathbb{C}; \\ 0 & \text{if } x \perp V^\dagger f_n. \end{cases}$$

Then, $Z_n \in L(\mathcal{H})$ and satisfies $N(V)\subseteq N(Z_n)$ and $R(Z_n)\subseteq N(T)\cap N(V)$. Furthermore, by Lemma 3.7, $Q_n = Q_0 + Z_n$ is a projection with $R(Q_n) = N(V)$ and $N(Q_n)\subseteq T^{\#}T(N(V))^{[\perp]}$. Therefore, $g_n = (I-Q_n)V^\dagger f_n \in sp(T,V,f_n)$ for every $n\in\mathbb{N}$.

But observe that $\{g_n\}_{n\in\mathbb{N}}$ can not be a frame because $\|g_n\|\to+\infty$ as $n\to\infty$. Indeed, it is easy to see that

$$\|g_n\| \geq \|Z_nV^\dagger f_n\| - \|(I-Q_0)V^\dagger f_n\| \geq n - \|I-Q_0\|\|V^\dagger\|B^{1/2} \to +\infty,$$

as $n\to\infty$.

4. Indefinite abstract smoothing splines

Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces with fundamental symmetries $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$, respectively, and consider a Hilbert space $\mathcal{E}$. Given surjective operators $T\in L(\mathcal{H},\mathcal{K})$ and $V\in L(\mathcal{H},\mathcal{E})$, consider the following generalization of the abstract smoothing problem [4]:

Problem 2. Given $\rho > 0$ and fixed $z_0\in\mathcal{E}$, find $x_0\in\mathcal{H}$ such that

$$[\,Tx_0, Tx_0\,]_{\mathcal{K}} + \rho\|Vx_0 - z_0\|^2_{\mathcal{E}} = \min_{x\in\mathcal{H}}\left([\,Tx,Tx\,]_{\mathcal{K}} + \rho\|Vx-z_0\|^2_{\mathcal{E}}\right). \tag{4.1}$$

Definition. Any element $x_0\in\mathcal{H}$ satisfying Eq. (4.1) is called a (T,V,ρ)-*smoothing spline* to $z_0\in\mathcal{E}$. The set of (T,V,ρ)-smoothing splines to z_0 is denoted by $sm(T,V,\rho,z_0)$.

To study this problem consider the indefinite metric defined on $\mathcal{K}\times\mathcal{E}$ by:

$$[\,(y,z),(y',z')\,]_\rho = [\,y,y'\,]_{\mathcal{K}} + \rho\,\langle\, z,z'\,\rangle_{\mathcal{E}}\,,\quad (y,z),(y',z')\in\mathcal{K}\times\mathcal{E}. \tag{4.2}$$

Notice that $\mathcal{K}\times\mathcal{E}$ is a Krein space with the indefinite metric defined above. In fact, considering the fundamental symmetry $J_{\mathcal{K}}$ of $\mathcal{K}$ and the inner product $\langle\ ,\ \rangle_\rho$ in

$\mathcal{K}\times\mathcal{E}$ given by $\langle\, (y,z),(y',z')\,\rangle_\rho = \langle\, y, z\,\rangle_{\mathcal{K}} + \rho\,\langle\, z, z'\,\rangle_{\mathcal{E}}$ where $(y,z),(y',z') \in \mathcal{K}\times\mathcal{E}$, the operator $J_\rho \in L(\mathcal{K}\times\mathcal{E})$ defined as

$$J_\rho(y,z) = (J_{\mathcal{K}}y, z), \qquad (y,z)\in\mathcal{K}\times\mathcal{E},$$

is a fundamental symmetry associated to $(\mathcal{K}\times\mathcal{E}, [\ ,\ ]_\rho)$. Also, considering the operator $L\colon \mathcal{H}\to\mathcal{K}\times\mathcal{E}$ defined by

$$Lx = (Tx, Vx), \qquad x\in\mathcal{H},$$

observe that Problem 2 can be restated as the following indefinite least squares problem: given $\rho>0$ and fixed $z_0\in\mathcal{E}$, find $x_0\in\mathcal{H}$ such that

$$[\, Lx_0 - (0,z_0), Lx_0 - (0,z_0)\,]_\rho = \min_{x\in\mathcal{H}} [\, Lx - (0,z_0), Lx - (0,z_0)\,]_\rho. \tag{4.3}$$

Using the formulation given above, the next results characterize the solutions of the indefinite abstract smoothing problem.

Lemma 4.1. *Given $z_0\in\mathcal{E}$, $x_0\in\mathcal{H}$ is a solution of Problem 2 if and only if $R(L)$ is J_ρ-nonnegative and x_0 is a solution of the equation:*

$$(T^\# T + \rho V^\# V)x = \rho V^\# z_0.$$

Proof. Following the same arguments as in Lemma 3.1, it is easy to see that $x_0\in\mathcal{H}$ satisfies Eq. (4.3) if and only if $R(L)$ is J_ρ-nonnegative and

$$[\, Lx_0 - (0,z_0), Lx\,]_\rho = 0, \qquad \text{for every } x\in\mathcal{H}.$$

or equivalently, $L^\#(Lx_0 - (0,z_0)) = 0$. Since $L^\# \in L(\mathcal{K}\times\mathcal{E},\mathcal{H})$ is given by $L^\#(y,z) = T^\# y + \rho V^\# z$, $(y,z)\in\mathcal{K}\times\mathcal{E}$, it follows that $(T^\# T + \rho V^\# V)x_0 = \rho V^\# z_0$. □

In order to obtain some alternative characterizations for the solutions of Problem 2, it is necessary to consider the particular case of a closed range operator L. The next lemma gives a condition between the operators V and T that guarantees that L has closed range. The proof is similar to the one given in [4, Ch. III, Lemma 2.1] for the Hilbert space case.

Lemma 4.2. *If $T(N(V))$ is a closed subspace of $\mathcal{K}$ then $R(L)$ is a closed subspace of $\mathcal{K}\times\mathcal{E}$.*

Proof. Given $(y,z)\in\mathcal{K}\times\mathcal{E}$, suppose that $\{x_n\}_{n\geq 1}\subseteq N(L)^\perp$ is such that $Lx_n \to (y,z)$. If $v_n = V^\dagger V x_n \in N(V)^\perp \subseteq N(L)^\perp$, then $v_n \to V^\dagger z\in\mathcal{H}$ and $u_n = x_n - v_n \in N(V)\cap N(L)^\perp$. Therefore, $Vv_n = Vx_n \to z$ and $Tu_n \to y - TV^\dagger z$.

Since $T(N(V))$ is a closed subspace of $\mathcal{K}$, the operator $W = T|_{N(V)}\colon N(V)\to\mathcal{K}$ has closed range and, for every $n\geq 1$, $u_n = W^\dagger T u_n$ because $u_n \in N(V)\cap N(L)^\perp = N(W)^\perp$. Thus, $x_n = v_n + u_n = v_n + W^\dagger T u_n \to V^\dagger z + W^\dagger(y - TV^\dagger z)$. Furthermore, if $x = V^\dagger z + W^\dagger(y - TV^\dagger z)$, it follows that $Tx = y$ and $Vx = z$ because $y - TV^\dagger z \in T(N(V))$. Therefore, $R(L)$ is a closed subspace of $\mathcal{K}\times\mathcal{E}$. □

As a consequence of Corollary 2.2, if there exists

$$\arg\min_{x\in\mathcal{H}}[\, Lx-(y,z), Lx-(y,z)\,]_\rho$$

for every $(y,z)\in\mathcal{K}\times\mathcal{E}$, then $R(L)$ is a regular subspace of $\mathcal{K}\times\mathcal{E}$. The following proposition shows that this assertion also holds considering the proper subspace of $\mathcal{K}\times\mathcal{E}$ obtained by embedding $\mathcal{E}$ into $\mathcal{K}\times\mathcal{E}$.

Proposition 4.3. *Problem 2 admits a solution for every $z\in\mathcal{E}$ if and only if $R(L)$ is a closed uniformly J_ρ-positive subspace of $\mathcal{K}\times\mathcal{E}$.*

Proof. Suppose that, Problem 2 admits a solution for every $z\in\mathcal{E}$. Applying Lemma 4.1, it follows that $R(L)$ is J_ρ-nonnegative. Given $(y,z)\in\mathcal{K}\times\mathcal{E}$, consider $w=u+T^\dagger y$, where $u\in\mathcal{H}$ satisfies $[\, Lu-(0,z-VT^\dagger y), Lu-(0,z-VT^\dagger y)\,]_\rho = \min_{x\in\mathcal{H}}[\, Lx-(0,z-VT^\dagger y), Lx-(0,z-VT^\dagger y)\,]_\rho$.

Then, for every $x\in\mathcal{H}$,

$$\begin{aligned}
[\, Lw-(y,z), Lw-(y,z)\,]_\rho &= \\
&= [\, Lu+(y,VT^\dagger y)-(y,z), Lu+(y,VT^\dagger y)-(y,z)\,]_\rho \\
&= [\, Lu-(0,z-VT^\dagger y), Lu-(0,z-VT^\dagger y)\,]_\rho \\
&\leq [\, L(x-T^\dagger y)-(0,z-VT^\dagger y), L(x-T^\dagger y)-(0,z-VT^\dagger y)\,]_\rho \\
&= [\, Lx-(y,z), Lx-(y,z)\,]_\rho.
\end{aligned}$$

Therefore, for every $(y,z)\in\mathcal{K}\times\mathcal{E}$, there exists $w\in\mathcal{H}$ such that

$$[\, Lw-(y,z), Lw-(y,z)\,]_\rho = \min_{x\in\mathcal{H}}[\, Lx-(y,z), Lx-(y,z)\,]_\rho.$$

Then, as in Lemma 3.1, it is easy to see that for every $(y,z)\in\mathcal{K}\times\mathcal{E}$ there exists $w\in\mathcal{H}$ such that $Lw-(y,z)\in R(L)^{[\perp]_\rho}$. So, $\mathcal{K}\times\mathcal{E}=R(L)+R(L)^{[\perp]_\rho}$, i.e. $R(L)$ is a regular subspace of $\mathcal{K}\times\mathcal{E}$. Thus, by Proposition 2.1, $R(L)$ is a closed uniformly J_ρ-positive subspace of $\mathcal{K}\times\mathcal{E}$.

The converse implication follows from Corollary 2.2, considering the J_ρ-selfadjoint projection $Q\in L(\mathcal{K}\times\mathcal{E})$ onto $R(L)$. □

4.1. **Every indefinite smoothing spline is an indefinite interpolating spline.** This subsection is devoted to show that $sm(T,V,\rho,z_0)=sp(T,V,z')$ for a suitable $z'\in\mathcal{E}$. In order to do so, a particular decomposition of $R(L)$ is needed. If $T(N(V))$ is a regular subspace of $\mathcal{K}$ and Q_0 is the projection considered in Eq. (3.3), consider the (bounded) operator $U\colon\mathcal{E}\to\mathcal{K}\times\mathcal{E}$ given by

$$Uz=(T(I-Q_0)V^\dagger z, z),\quad z\in\mathcal{E}.$$

Observe that $N(U)=\{0\}$ and $R(U)$ is closed (because it is isometrically isomorphic to the graph of the bounded operator $T(I-Q_0)V^\dagger$).

Lemma 4.4. *If $T(N(V))$ is a regular subspace of $\mathcal{K}$ then*

$$R(L) = (T(N(V)) \times \{0\}) \dot{+} R(U),$$

and this decomposition of $R(L)$ is orthogonal in the Krein space $(\mathcal{K} \times \mathcal{E}, [\ ,\ ]_\rho)$.

Proof. Since $R(Q_0) = N(V)$, observe that $R(L) = L(N(V)) + L(N(Q_0))$ and $L(N(V)) = T(N(V)) \times \{0\}$. In order to compute $L(N(Q_0))$, observe that $I - Q_0 = (I - Q_0)P_{N(V)^\perp} = (I - Q_0)V^\dagger V$ because $N(I - Q_0) = N(P_{N(V)^\perp}) = N(V)$. Therefore, if $x \in N(Q_0)$,

$$\begin{aligned} Lx = (Tx, Vx) &= (T(I - Q_0)x, Vx) = (T(I - Q_0)V^\dagger Vx, Vx) = \\ &= (T(I - Q_0)V^\dagger z, z) = Uz, \end{aligned}$$

where $z = Vx$. Since $V(N(Q_0)) = \mathcal{E}$, it follows that $L(N(Q_0)) = \{(T(I - Q_0)V^\dagger z, z) : z \in \mathcal{E}\} = R(U)$. Finally, since $T(N(Q_0)) \subseteq T(N(V))^{[\perp]}$, it follows that $L(N(V))\ [\perp]_\rho\ L(N(Q_0))$. □

The next theorem shows the existence of a vector $z' \in \mathcal{E}$ such that $sm(T, V, \rho, z_0) = sp(T, V, z')$. Also, along the proof, an expression of such z' is given in terms of the J_ρ-selfadjoint projection onto one of the subspaces of $R(L)$ presented in the above decomposition.

Theorem 4.5. *Suppose that $T(N(V))$ is a closed subspace of $\mathcal{K}$ and $R(L)$ is a uniformly J_ρ-positive subspace of $\mathcal{K} \times \mathcal{E}$. Then, given $z_0 \in \mathcal{E}$,*

$$sm(T, V, \rho, z_0) = sp(T, V, z'),$$

where z' is an adequate vector in $\mathcal{E}$.

Proof. If $z_0 = 0$ then $sm(T, V, \rho, 0) = N(L) = N(T) \cap N(V) = sp(T, V, 0)$. On the other hand, notice that $R(L)$ is closed (see Lemma 4.2). Then, by Proposition 2.1, $R(L)$ and $T(N(V))$ are regular subspaces of $\mathcal{K} \times \mathcal{E}$ and $\mathcal{K}$, respectively. So, the projection considered in Eq. (3.3) is bounded. Given $x \in \mathcal{H}$, it can be decomposed as

$$x = Q_0x + (I - Q_0)x = Q_0x + (I - Q_0)P_{N(V)^\perp}x = v + (I - Q_0)V^\dagger z,$$

where $v = Q_0x \in N(V)$ and $z = Vx \in \mathcal{E}$. Observe that, by Lemma 4.4,

$$[\,Lx - (0, z_0), Lx - (0, z_0)\,]_\rho = [\,(Tv, 0), (Tv, 0)\,]_\rho + [\,Uz - (0, z_0), Uz - (0, z_0)\,]_\rho.$$

So, $x_0 \in sm(T, V, \rho, z_0)$ if and only if $[\,TQ_0x_0, TQ_0x_0\,]_\mathcal{K} = \min_{u \in N(V)} [\,Tu, Tu\,]_\mathcal{K}$ and $z_1 = Vx_0$ satisfies

$$[\,Uz_1 - (0, z_0), Uz_1 - (0, z_0)\,]_\rho = \min_{z \in \mathcal{E}} [\,Uz - (0, z_0), Uz - (0, z_0)\,]_\rho.$$

Notice that $\min_{u \in N(V)} [\,Tu, Tu\,]_\mathcal{K}$ is attained at every $u \in N(T) \cap N(V)$, because $T(N(V))$ is uniformly $J_\mathcal{K}$-positive. Therefore, $Q_0x_0 \in N(T) \cap N(V)$.

On the other hand, since $R(U)$ is a regular subspace of $R(L)$ (see Lemma 4.4), $R(U)$ is a (closed) uniformly J_ρ-positive subspace of $\mathcal{K} \times \mathcal{E}$. Thus, by Corollary 2.2,

z_1 satisfies the above equation if and only if $Uz_1 = P(0, z_0)$, where P is the J_ρ-selfadjoint projection onto $R(U)$.

If $S\colon \mathcal{K} \times \mathcal{E} \to \mathcal{E}$ is defined as $S(y, z) = z$ then $SU = I_{\mathcal{E}}$ and $z_1 = SUz_1 = SP(0, z_0)$. So, $(I - Q_0)V^\dagger z_1 = (I - Q_0)V^\dagger SP(0, z_0)$.

Therefore, $x_0 \in sm(T, V, \rho, z_0)$ if and only if $x_0 \in (I - Q_0)V^\dagger SP(0, z_0) + N(T) \cap N(V)$, i.e.

$$sm(T, V, \rho, z_0) = sp(T, V, SP(0, z_0)). \qquad \square$$

4.2. **The smoothing splines converge to the interpolating spline.** In the following paragraph we show that, given $z_0 \in \mathcal{E}$, if $\{x_\rho\}_{\rho\geq 1}$ is a net in $\mathcal{H}$ such that $x_\rho \in sm(T, V, \rho, z_0)$, then it converges to an interpolating spline $x_0 \in sp(T, V, z_0)$ as $\rho \to \infty$. The proof of this result is analogous to [4, Ch. III, Proposition 2.2].

Proposition 4.6. *Given a fixed vector $z_0 \in \mathcal{E}$, suppose that $T(N(V))$ is a closed subspace of $\mathcal{K}$ and $R(L)$ is a uniformly J_ρ-positive subspace of $\mathcal{K} \times \mathcal{E}$. Let $x_\rho \in sm(T, V, \rho, z_0)$ for every $\rho \geq 1$. Then, there exists $x_0 \in sp(T, V, z_0)$ such that*

$$\lim_{\rho\to\infty} \|x_\rho - x_0\| = 0.$$

Proof. First, notice that if $x_\rho \in sm(T, V, \rho, z_0)$ then $\{[\,Tx_\rho, Tx_\rho\,]\}_{\rho\geq 1}$ is an increasing net in $\mathbb{R}$ with an upper bound, and $\|Vx_\rho - z_0\| \to 0$ as $\rho \to \infty$. Indeed, given $\rho_1, \rho_2 \geq 1$, notice that $[\,Tx_{\rho_i}, Tx_{\rho_i}\,] + \rho_i\|Vx_{\rho_i} - z_0\|^2 \leq [\,Tx_{\rho_j}, Tx_{\rho_j}\,] + \rho_i\|Vx_{\rho_j} - z_0\|^2$, if $i \neq j$. Then, if $\rho_1 < \rho_2$ it follows that $\|Vx_{\rho_1} - z_0\|^2 - \|Vx_{\rho_2} - z_0\|^2 \geq 0$ and

$$[\,Tx_{\rho_2}, Tx_{\rho_2}\,] - [\,Tx_{\rho_1}, Tx_{\rho_1}\,] \geq \rho_1(\|Vx_{\rho_1} - z_0\|^2 - \|Vx_{\rho_2} - z_0\|^2) \geq 0.$$

Furthermore, if $x \in sp(T, V, z_0)$ for every $\rho \geq 1$, $[\,Tx_\rho, Tx_\rho\,] + \rho\|Vx_\rho - z_0\|^2 \leq [\,Tx, Tx\,] + \rho\|Vx - z_0\|^2 = [\,Tx, Tx\,]$. So, $[\,Tx, Tx\,] - [\,Tx_\rho, Tx_\rho\,] \geq \rho\|Vx_\rho - z_0\|^2 \geq 0$ for every $\rho \geq 1$, and this inequality implies that

$$\lim_{\rho\to\infty} \|Vx_\rho - z_0\| = 0.$$

The next step is to prove that $\lim_{\rho\to\infty} \|x_\rho - x_0\| = 0$, where $x_0 = V^\dagger z_0 + u$ for some $u \in N(V)$. Let $y_\rho = P_{N(V)^\perp} x_\rho$ and observe that $y_\rho = V^\dagger V x_\rho \to V^\dagger z_0$ as $\rho \to \infty$.

If $u_\rho = x_\rho - y_\rho = P_{N(V)} x_\rho \in N(V)$, then $\{u_\rho\}_{\rho\geq 1}$ converges to some $u \in N(V)$. To prove this assertion, consider the closed range operator $W = T|_{N(V)} : N(V) \to \mathcal{K}$ (see Lemma 4.2). If Q is the $J_{\mathcal{K}}$-selfadjoint projection onto $T(N(V))$, let $W' = W^\dagger Q$. Then, W' satisfies $WW'W = W$, $W'WW' = W'$ and $N(W') = T(N(V))^{[\perp]}$. By Theorem 4.5, $x_\rho \in sp(T, V, z_\rho)$ for a suitable $z_\rho \in \mathcal{E}$; then, it follows that $Tx_\rho \in T(N(V))^{[\perp]}$ (see Lemma 3.1). Therefore, $W'Tx_\rho = 0$ for every $\rho \geq 1$, and

$$W'Tu_\rho = -W'Ty_\rho \to -W'TV^\dagger z_0 = u \in R(W') \subseteq N(V) \ \text{ as } \rho \to \infty. \qquad \square$$

5. The indefinite abstract mixed problem

Given Hilbert spaces $\mathcal{E}_1$ and $\mathcal{E}_2$, and Krein spaces $\mathcal{H}$ and $\mathcal{K}$ with fundamental symmetries $J_\mathcal{H}$ and $J_\mathcal{K}$ respectively, let $T \in L(\mathcal{H}, \mathcal{K})$, $V_1 \in L(\mathcal{H}, \mathcal{E}_1)$ and $V_2 \in L(\mathcal{H}, \mathcal{E}_2)$ be surjective operators. Then, consider the following problem:

Problem 3. Let $\rho > 0$. Fixed $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, find $x_0 \in \mathcal{H}$ such that $V_1 x_0 = z_1$ and

$$\left([\,Tx_0, Tx_0\,]_\mathcal{K} + \rho\|V_2 x_0 - z_2\|^2_{\mathcal{E}_2}\right) = \min_{V_1 x = z_1} \left([\,Tx, Tx\,]_\mathcal{K} + \rho\|V_2 x - z_2\|^2_{\mathcal{E}_2}\right).$$

This is a generalization to Krein spaces of the mixed problem in Hilbert spaces proposed by A. I. Rozhenko in [26] (see also [27, 28]).

It is clear that the indefinite abstract and smoothing problems are the partial cases of the indefinite abstract mixed problem corresponding to $\mathcal{E}_2 = \{0\}$, $V_2 = 0$ and $\mathcal{E}_1 = \{0\}$, $V_1 = 0$, respectively. Thus, it is expected that similar results to those given in the previous sections, can be stated with some additional restrictions. We prefer to introduce the indefinite abstract mixed problem after studying the other problems in order to motivate it.

As in the previous section, $\mathcal{K} \times \mathcal{E}_2$ is a Krein space with the indefinite metric defined in Eq. (4.2) and its fundamental symmetry $J_\rho \in L(\mathcal{K} \times \mathcal{E}_2)$ is given by $J_\rho(y, z) = (J_\mathcal{K} y, z)$, where $(y, z) \in \mathcal{K} \times \mathcal{E}_2$. Also, consider the operators $L \in L(\mathcal{H}, \mathcal{K} \times \mathcal{E}_2)$ given by

$$Lx = (Tx, V_2 x), \qquad x \in \mathcal{H},$$

and $L_1 = LP_{N(V_1)} \in L(\mathcal{H}, \mathcal{K} \times \mathcal{E}_2)$. Then, Problem 3 can be restated as: given $\rho > 0$ and fixed $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, find $x_0 \in \mathcal{H}$ such that

$$[\,L_1 x_0 - (w_1, w_2), L_1 x_0 - (w_1, w_2)\,]_\rho = \min_{x \in \mathcal{H}} [\,L_1 x - (w_1, w_2), L_1 x - (w_1, w_2)\,]_\rho, \tag{5.1}$$

where $w_1 = -TV_1^\dagger z_1$ and $w_2 = z_2 - V_2 V_1^\dagger z_1$.

Lemma 5.1. *Given $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, $x_0 \in \mathcal{H}$ is a solution of Problem 3 if and only if $R(L_1)$ is J_ρ-nonnegative and x_0 is a solution of the equation:*

$$P^{\#}_{N(V_1)}(T^{\#}T + \rho V_2^{\#} V_2) P_{N(V_1)} x_0 = P^{\#}_{N(V_1)}(T^{\#} w_1 + \rho V_2^{\#} w_2).$$

Proof. It is analogous to the proof of Lemma 4.1. Notice that, in this case, $L_1^{\#} \in L(\mathcal{K} \times \mathcal{E}_2, \mathcal{H})$ is given by $L_1^{\#}(y, z) = P^{\#}_{N(V_1)} L^{\#}(y, z) = P^{\#}_{N(V_1)}(T^{\#} y + \rho V_2^{\#} z)$, $(y, z) \in \mathcal{K} \times \mathcal{E}_2$. □

Proposition 5.2. *Problem 3 admits a solution for every $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$ if and only if $R(L_1)$ is a closed uniformly J_ρ-positive subspace of $\mathcal{K} \times \mathcal{E}_2$.*

Proof. Suppose that, Problem 3 admits a solution for every $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$. Given $(y, z) \in \mathcal{K} \times \mathcal{E}_2$, let $z_1 = -V_1 T^\dagger y$ and $z_2 = z - V_2 T^\dagger y$. Consider $x_0 = u + T^\dagger y$, where $u \in \mathcal{H}$ satisfies

$$[\, L_1 u - (w_1, w_2), L_1 u - (w_1, w_2)\,]_\rho = \min_{x\in\mathcal{H}} [\, L_1 x - (w_1, w_2), L_1 x - (w_1, w_2)\,]_\rho,$$

for this particular pair $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$.

Observe that $L_1 x_0 - (y, z) = L_1 u + (TP_{N(V_1)} T^\dagger y, V_2 P_{N(V_1)} T^\dagger y) - (y, z) = L_1 u - (-TV_1^\dagger z_1, z_2 - V_2 V_1^\dagger z_1) = L_1 u - (w_1, w_2)$. Then, for every $x \in \mathcal{H}$,

$$\begin{aligned}
[\, L_1 x_0 - (y, z), L_1 x_0 - (y, z)\,]_\rho &= [\, L_1 u - (w_1, w_2), L_1 u - (w_1, w_2)\,]_\rho \\
&\leq [\, L_1(x - T^\dagger y) - (w_1, w_2), L_1(x - T^\dagger y) - (w_1, w_2)\,]_\rho \\
&= [\, L_1 x - (y, z), L_1 x - (y, z)\,]_\rho,
\end{aligned}$$

because $L_1 x - (y, z) = L_1(x - T^\dagger y) - (w_1, w_2)$. Therefore, for every $(y, z) \in \mathcal{K} \times \mathcal{E}_2$, there exists $x_0 \in \mathcal{H}$ such that

$$[\, L_1 x_0 - (y, z), L_1 x_0 - (y, z)\,]_\rho = \min_{x\in\mathcal{H}} [\, L_1 x - (y, z), L_1 x - (y, z)\,]_\rho.$$

Following the same arguments as in the proof of Proposition 4.3, it is easy to see that the above condition holds if and only if $R(L_1)$ is a closed uniformly J_ρ-positive subspace of $\mathcal{K} \times \mathcal{E}_2$. □

5.1. **Parametrization of the set of solutions of the indefinite abstract mixed problem.** The following paragraphs follow analogous ideas to those presented in the previous section to show that every smoothing spline is an interpolating spline.

Consider the operator $V \in L(\mathcal{H}, \mathcal{E}_1 \times \mathcal{E}_2)$ given by $Vx = (V_1 x, V_2 x)$, $x \in \mathcal{H}$, and notice that $N(V) = N(V_1) \cap N(V_2)$ but V is not surjective. However, Lemma 3.3 also holds in this case. So, if $T(N(V))$ is a regular subspace of $\mathcal{K}$ then, denoting $\mathcal{W} = T^\# T(N(V))^{[\perp]} \ominus N(V)$, the projection $Q_0 = P_{N(V)//\mathcal{W}}$ is bounded. Before stating the main theorem, we need the following key lemma.

Lemma 5.3. *Suppose that $T(N(V))$ is a regular subspace of $\mathcal{K}$ and $N(V_1) + N(V_2)$ is closed in $\mathcal{H}$. Then,*

1. *$\mathcal{M}_1 = (I - Q_0)(N(V_1))$ and $\mathcal{M}_2 = V_2(N(V_1))$ are closed subspaces of $\mathcal{H}$ and $\mathcal{E}_2$, respectively.*
2. *$V_2|_{\mathcal{M}_1} \colon \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism.*
3. *$R(L_1) = (T(N(V)) \times \{0\}) \dotplus L(\mathcal{M}_1)$. Furthermore, $L(\mathcal{M}_1)$ is closed in $\mathcal{K} \times \mathcal{E}_2$ and the decomposition is orthogonal in the Krein space $(\mathcal{K} \times \mathcal{E}_2, [\ ,\ ]_\rho)$.*

Proof. (i.) First of all, notice that $\mathcal{M}_1 = R(I - Q_0) \cap N(V_1)$. Therefore, it is closed and $N(V_1) = N(V) \dotplus \mathcal{M}_1$ because $Q_0(N(V_1)) = N(V)$. On the other hand, by Proposition 2.3, $\mathcal{M}_2 = R(V_2 P_{N(V_1)})$ is closed if and only if $c(N(V_2), N(V_1)) < 1$, or equivalently, $N(V_1) + N(V_2)$ is closed. Therefore, $\mathcal{M}_2$ is closed.

(ii.) To show that $V_2|_{\mathcal{M}_1}\colon \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism observe that $V_2(\mathcal{M}_1) = V_2(\mathcal{M}_1 + N(V)) = V_2(N(V_1)) = \mathcal{M}_2$, so it only remains to prove that $V_2|_{\mathcal{M}_1}$ is injective. But, if $x \in \mathcal{M}_1$ and $V_2x = 0$ then $x \in N(V_2) \cap \mathcal{M}_1 = N(V) \cap R(I - Q_0) = \{0\}$.

(iii.) Observe that $R(L_1) = L(N(V_1)) = L(N(V)) + L(\mathcal{M}_1)$ because $N(V_1) = N(V) \dot{+} \mathcal{M}_1$. Furthermore, if $x \in N(V_1)$ then $Q_0x \in N(V)$ and $(I - Q_0)x \in \mathcal{M}_1$. So, $Lx = (TQ_0x, 0) + L(I - Q_0)x$. Therefore, $R(L_1) = L(N(V)) + L(\mathcal{M}_1) = (T(N(V)) \times \{0\}) + L(\mathcal{M}_1)$.

If $(y, 0) \in (T(N(V)) \times \{0\}) \cap L(\mathcal{M}_1)$, there exists $m \in \mathcal{M}_1$ such that $Tm = y$ and $V_2m = 0$. Then, $m = 0$ because $V_2|_{\mathcal{M}_1}$ is an isomorphism. So, $y = Tm = 0$ and $R(L_1) = L(N(V)) \dot{+} L(\mathcal{M}_1)$. As in Lemma 4.4, it is easy to see that this decomposition is orthogonal respect to the indefinite metric defined on $\mathcal{K} \times \mathcal{E}_2$.

It only remains to prove that $L(\mathcal{M}_1)$ is a closed subspace of $\mathcal{K} \times \mathcal{E}_2$. Given $(y, z) \in \overline{L(\mathcal{M}_1)}$ consider $\{m_k\}_{k\geq 1} \subseteq \mathcal{M}_1$ such that $Tm_k \to y$ and $V_2m_k \to z$ as $k \to \infty$. Notice that $m_k = (V_2|_{\mathcal{M}_1})^{-1}V_2m_k$, because $V_2|_{\mathcal{M}_1}\colon \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism. Therefore, $m_k \to (V_2|_{\mathcal{M}_1})^{-1}z \in \mathcal{M}_1$ and $(y, z) = L((V_2|_{\mathcal{M}_1})^{-1}z)$. □

Corollary 5.4. *If $T(N(V))$ is a regular subspace of $\mathcal{K}$ and $N(V_1) + N(V_2)$ is closed in $\mathcal{H}$ then $R(L_1)$ is closed in $\mathcal{K} \times \mathcal{E}_2$.*

The next theorem shows that that every mixed spline is an interpolating spline.

Theorem 5.5. *Suppose that $N(V_1) + N(V_2)$ is closed in $\mathcal{K}$, $T(N(V))$ is a closed subspace of $\mathcal{K}$ and $R(L_1)$ is a (closed) uniformly J_ρ-positive subspace of $\mathcal{K} \times \mathcal{E}_2$. Then, given $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, an element $x_0 \in \mathcal{H}$ is a solution of Problem 3 if and only if*

$$x_0 \in sp(T, V, (e_1, e_2)),$$

where (e_1, e_2) is a suitable vector in $\mathcal{E}_1 \times \mathcal{E}_2$.

Proof. Given $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, recall that if $x_0 \in \mathcal{H}$ is a solution of Problem3 then $V_1x_0 = z_1$, or equivalently, $P_{N(V_1)^\perp}x_0 = V_1^\dagger z_1$. Assuming that $T(N(V))$ is a regular subspace of $\mathcal{K}$, $V_1^\dagger z_1$ can be decomposed as $V_1^\dagger z_1 = u_1 + v_1$, where $u_1 = Q_0V_1^\dagger z_1 \in N(V)$ and $v_1 = (I - Q_0)V_1^\dagger z_1 \in \mathcal{W}$. Then, the pair (w_1, w_2) considered in Eq. (5.1) satisfies

$$-w_1 = TV_1^\dagger z_1 = Tu_1 + Tv_1 \in T(N(V)) \dot{+} T(N(V))^{[\perp]} \quad \text{and} \quad w_2 = z_2 - V_2v_1.$$

If $N(V_1) + N(V_2)$ is a closed subspace of $\mathcal{H}$, given $x \in \mathcal{H}$ there exist (unique) $u \in N(V)$ and $m \in \mathcal{M}_1$ such that $P_{N(V_1)}x = u + m$ (see Lemma 5.3). Thus, $x = u + m + P_{N(V)^\perp}x$ and

$$L_1x - (w_1, w_2) = (T(u + u_1), 0) + Lm - (-Tv_1, w_2).$$

Observe that $Lm-(-Tv_1,w_2) = L(m+v_1)-(0,z_2) \in (T(N(V))\times\{0\})^{[\perp]}$ because $m+v_1 \in N(Q_0)$. Then,

$$[\,L_1x-(w_1,w_2),L_1x-(w_1,w_2)\,]_\rho = \\ [\,T(u+u_1),T(u+u_1)\,]_\mathcal{K} + [\,Lm-(-Tv_1,w_2),Lm-(-Tv_1,w_2)\,]_\rho.$$

Therefore, x_0 is a solution to Problem 3 if and only if $P_{N(V_1)}x_0 = u_0+m_0$, with $u_0 \in N(V)$ and $m_0 \in \mathcal{M}_1$ satisfying $[\,T(u_0+u_1),T(u_0+u_1)\,]_\mathcal{K} = \min_{u\in N(V)}[\,T(u+u_1),T(u+u_1)\,]_\mathcal{K}$ and

$$[\,Lm_0-(-Tv_1,w_2),Lm_0-(-Tv_1,w_2)\,]_\rho = \\ \min_{m\in\mathcal{M}_1}[\,Lm-(-Tv_1,w_2),Lm-(-Tv_1,w_2)\,]_\rho.$$

Notice that, if $R(L_1)$ is a closed uniformly J_ρ-positive subspace of $\mathcal{K}\times\mathcal{E}_2$, then $T(N(V))$ is a closed uniformly $J_\mathcal{K}$-positive subspace of $\mathcal{K}$ and $\min_{u\in N(V)}[\,T(u+u_1),T(u+u_1)\,]_\mathcal{K}$ is attained at every $y \in -u_1+N(V)\cap N(T)$.

On the other hand, consider the bounded operator $U\colon \mathcal{M}_2 \to \mathcal{K}\times\mathcal{E}_2$ defined by

$$Uz = (T(V_2|_{\mathcal{M}_1})^{-1}z, z).$$

Observe that U has closed range, because it is isometrically isomorphic to the graph of the bounded operator $T(V_2|_{\mathcal{M}_1})^{-1}$, and

$$\min_{m\in\mathcal{M}_1}[\,Lm-(-Tv_1,w_2),Lm-(-Tv_1,w_2)\,]_\rho = \\ \min_{z\in\mathcal{M}_2}[\,Uz-(-Tv_1,w_2),Uz-(-Tv_1,w_2)\,]_\rho.$$

Thus, following the same argument as in Theorem 4.5 and observing that $R(U) = L(\mathcal{M}_1)$ is a closed uniformly J_ρ-positive subspace of $\mathcal{K}\times\mathcal{E}_2$, this last problem admits a (unique) solution given by $z_0 = V_2m_0 = SP(-Tv_1,w_2)$, where P is the J_ρ-selfadjoint projection onto $L(\mathcal{M}_1)$ and $S\colon \mathcal{K}\times\mathcal{E}_2 \to \mathcal{E}_2$ is defined by $S(y,z) = z$. So, $x_0 \in \mathcal{H}$ is a solution to Problem 3 if and only if

$$x_0 = V_1^\dagger z_1 + P_{N(V_1)}x_0 = u_1+v_1+u_0+m_0 \\ \in (v_1 + (V_2|_{\mathcal{M}_1})^{-1}SP(-Tv_1,w_2)) + N(T)\cap N(V).$$

Therefore, $x_0 \in \mathcal{H}$ solves Problem 3 if and only if $x_0 \in sp(T,V,(e_1,e_2))$, where $e_1 = z_1+V_1(V_2|_{\mathcal{M}_1})^{-1}SP(-Tv_1,w_2) \in \mathcal{E}_1$ and $e_2 = V_2V_1^\dagger z_1+SP(-Tv_1,w_2) \in \mathcal{E}_2$. □

Acknowledgements

The authors would like to acknowledge Prof. Gustavo Corach who brought into their attention the mixed problem for abstract splines in Hilbert spaces. Francisco Martínez Pería would also like to thanks Centre de Recerca Matemàtica staff for their hospitality while he held a Lluís Santaló visiting position in Barcelona, Spain, and to Fundación Carolina for supporting him during this visit with a Formación Permanente fellowship.

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