DIRECTIONAL DISCREPANCY IN TWO DIMENSIONS.

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ABSTRACT. In the present paper, we study the *geometric discrepancy* with respect to families of rotated rectangles. The well-known extremal cases are the axis-parallel rectangles (logarithmic discrepancy) and rectangles rotated in all possible directions (polynomial discrepancy). We study several intermediate situations: lacunary sequences of directions, lacunary sets of finite order, and sets with small Minkowski dimension. In each of these cases, extensions of a lemma due to Davenport allow us to construct appropriate rotations of the integer lattice which yield small discrepancy.

1. INTRODUCTION

In the present paper we address the following two-dimensional question in the theory of irregularities of distribution. Let $\Omega \subset [0, \pi/2]$ be a set of directions. We consider the collection of rectangles pointing in the directions of Ω:

$$\mathcal{A}_\Omega = \{\text{rectangles } R : \text{ one side of } R \text{ makes angle } \phi \in \Omega \text{ with the } x\text{-axis}\}. \tag{1.1}$$

Taking a set of N points in the unit square, $\mathcal{P}_N \subset [0,1]^2$, we measure its discrepancy with respect to $\mathcal{A}_\Omega$:

$$D_\Omega(\mathcal{P}_N) = \sup_{R \in \mathcal{A}_\Omega,\, R \subset [0,1]^2} |D_\Omega(\mathcal{P}_N, R)| = \sup_{R \in \mathcal{A}_\Omega,\, R \subset [0,1]^2} \Big| \#\mathcal{P}_N \cap R - N \cdot |R| \Big|. \tag{1.2}$$

We are interested in the behavior of the quantity

$$D_\Omega(N) = \inf_{\mathcal{P}_N \subset [0,1]^2} D_\Omega(\mathcal{P}_N). \tag{1.3}$$

as N goes to infinity, depending on the properties of Ω. It is also of interest to consider suitable (e.g., L^2) averages in place of the supremum in (1.2).

The motivation for this question comes from several classical results:

- In the case $\Omega = \{0\}$, i.e. $\mathcal{A}_\Omega$ is the set of axis-parallel rectangles we have

$$D_\Omega \approx \log N. \tag{1.4}$$

  Here, and throughout the paper, we use the notation $A \lesssim B$ meaning that there exists an absolute constant C, independent of N, such that $A \le CB$, and write $A \approx B$ if $A \lesssim B \lesssim A$. The lower bound in the estimate above

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is a celebrated theorem of W. Schmidt [14], while the upper bound goes back to a century-old result due to Lerch [11]. The inequalities above continue to hold when Ω is finite (This result is essentially contained in [7]).

- When $\Omega = [0, \pi/2]$, i.e. $\mathcal{A}_\Omega$ consists of rectangles rotated in all possible directions, we have

$$N^{\frac{1}{4}} \lesssim D_\Omega(N) \lesssim N^{\frac{1}{4}} \log^{\frac{1}{2}} N. \tag{1.5}$$

Here both inequalities are due to J. Beck ([2], [3]).

We see that the behavior of $D_\Omega(N)$ in these two extreme situations differs drastically. We would like to know what happens in the intermediate cases, how the geometry of Ω effects the discrepancy, and where is the threshold between the logarithmic and polynomial estimates.

In this work we look at particular examples: Ω being 1) a lacunary sequence of directions; 2) a lacunary set of finite order (for the definition of such sets and a brief discussion of their role in analysis see §2.3); or 3) a set with small upper Minkowski dimension, and prove the following theorem:

Theorem 1.6.
1) Let Ω be a lacunary sequence. Then we have

$$D_\Omega(N) \lesssim \log^3 N. \tag{1.7}$$

2) Let Ω be a lacunary set of order $M > 1$. Then we have

$$D_\Omega(N) \lesssim \log^{2M+1} N. \tag{1.8}$$

3) Assume Ω has upper Minkowski dimension $0 \le d < 1$. In this case,

$$D_\Omega(N) \lesssim N^{\frac{\tau}{2(\tau+1)}+\varepsilon}, \tag{1.9}$$

for any $\varepsilon > 0$, where $\tau = \frac{2}{(1-d)^2} - 2$.

We should point out that, in view of (1.5), the last part yields a new non-trivial estimate only if d is small enough.

In addition, we complement this theorem with the following L^2-averaging estimates. Denote $\mathcal{A}'_\Omega = \{R \in \mathcal{A}_\Omega : R \subset [0,1]^2\}$, or, alternatively, one may define $\mathcal{A}'_\Omega = \{R \in \mathcal{A}_\Omega : \operatorname{diam}(R) \le 1\}$ with $[0,1]^2$ viewed as a torus. We have

Theorem 1.10. *Let μ be any probability measure on $\mathcal{A}'_\Omega$. Then*
1) If Ω is a lacunary sequence, there exists $\mathcal{P} \subset [0,1]^2$, $\#\mathcal{P} = N$ such that

$$\left(\int_{\mathcal{A}'_\Omega} |D_\Omega(\mathcal{P}, R)|^2 d\mu(R) \right)^{\frac{1}{2}} \lesssim \log^{\frac{5}{2}} N. \tag{1.11}$$

2) If Ω is a lacunary set of order $M > 1$, there exists $\mathcal{P} \subset [0,1]^2$, $\#\mathcal{P} = N$ such that

$$\left(\int_{\mathcal{A}'_\Omega} |D_\Omega(\mathcal{P}, R)|^2 d\mu(R) \right)^{\frac{1}{2}} \lesssim \log^{2M+\frac{1}{2}} N. \tag{1.12}$$

3) If Ω has upper Minkowski dimension $0 \le d < 1$, there exists $\mathcal{P} \subset [0,1]^2$, $\#\mathcal{P} = N$ such that

$$\left(\int_{\mathcal{A}'_\Omega} |D_\Omega(\mathcal{P}, R)|^2 d\mu(R) \right)^{\frac{1}{2}} \lesssim N^{\frac{\tau}{2(\tau+1)}+\varepsilon}, \tag{1.13}$$

for any $\varepsilon > 0$, where $\tau = \frac{2}{(1-d)^2} - 2$ satisfies $\tau < 1$.

Comparing the first two parts of the above theorem to those of Theorem 1.6, we see a manifestation of the well-known discrepancy theory principle that the L^∞ (extremal) and L^2 (average) discrepancies differ by a factor of $\sqrt{\log N}$. This effect can be best seen if one compares (1.4) to the famous Roth's L^2 lower bound [12] of the order $\log^{1/2} N$ (which is sharp, [8]). In addition, the lower bound in (1.5) is known to be sharp in the L^2 sense [4].

In addition, we also address a 'sibling' problem: studying the discrepancy with respect to collections $\mathcal{B}_{\Omega,k}$ of convex polygons in $[0,1]^2$ with at most k sides whose normals point in the directions defined by Ω (cf. [5], [7] for earlier results) and prove inequalities analogous to Theorems 1.6 and 1.10 (see Theorems 4.14 and 5.10 in the text).

The paper is organized as follows. The core of the paper is §2 – here we obtain new diophantine inequalities which enable us to construct well-distributed sets. Section 3 describes how such inequalities can be translated into upper discrepancy estimates for one-dimensional sequences. In §4, we deduce our main Theorem 1.6, and §5 deals with bounds for the L^2 discrepancy in these settings. In the text, $\log n$ stands for $\max\{1, \log_2 n\}$.

2. Cassels-Davenport diophantine approximation arguments

In the case $\Omega = \{0\}$, one of the standard ways of constructing an example of a point-set satisfying the upper bound of (1.4) involves rotating the lattice $N^{-\frac{1}{2}}\mathbb{Z}^2$ by an angle α so that the slope $\tan\alpha$ is a *badly approximable* number, that is, for all $p \in \mathbb{Z}$, all $q \in \mathbb{N}$ we have

$$\left| \tan\alpha - \frac{p}{q} \right| \gtrsim \frac{1}{q^2}. \tag{2.1}$$

When Ω is an arbitrary finite set, the construction relies on the following result of Davenport [9] (which we state here in a particular case, relevant to our problem)

Lemma 2.2. *Let* $\Omega = \{\theta_1, \theta_2, \ldots, \theta_k\} \subset [0, \pi/2]$*. Then there exists* $\alpha \in [0, \pi/2]$ *so that*

$$\tan(\alpha - \theta_1), \ldots, \tan(\alpha - \theta_k)$$

are all badly approximable.

This allows us to find a rotation, which has a badly approximable slope with respect to all chosen directions θ_j. Davenport has, in fact, proven this fact for more general functions in place of the tangent. However, the argument is essentially due to Cassels [6] who proved a similar result earlier with $\tan(\alpha - \theta_k)$ replaced by $\alpha - \theta_k$.

Thus, analogs of the lemma above for infinite sets Ω may provide us with examples of low-discrepancy point distributions with respect to rotated rectangles. However, claiming "badly approximable" in the conclusion is, perhaps, too optimistic. Instead, we shall obtain results, in which inequalities similar to (2.1) have the right-hand side somewhat smaller than $1/q^2$. This, in turn, will lead to larger discrepancy bounds.

2.1. **General approach.** We first outline a general approach to the proof of statements akin to Lemma 2.2 extending the ideas of Cassels and Davenport. Assume that for a certain choice of parameters $R(n)$, $|I_n|$, $c(n)$, depending on the set Ω, a proposition of the following type holds:

Proposition 2.3. *Let* $\Omega \subset [0, \pi/2]$*. There exists a sequence of nested intervals* $I_0 \supset I_1 \supset \ldots \supset I_n \supset \ldots$ *in* $[0, \pi/2]$ *with* $|I_n| \to 0$ *such that for all* $\alpha \in I_n$ *and all* $p, q \in \mathbb{Z}$ *with* $R(n) \le q < R(n+1)$ *we have, for all* $\theta \in \Omega$*:*

$$\left| \tan(\alpha - \theta) - \frac{p}{q} \right| > \frac{c(n)}{q^2}. \tag{2.4}$$

This would of course imply that:

Lemma 2.5. *There exist* $\alpha \in [0, \pi/2]$ *and* $C > 0$ *such that for all* $\theta \in \Omega$*, all* $p \in \mathbb{Z}$*,* $q \in \mathbb{N}$ *we have*

$$\left| \tan(\alpha - \theta) - \frac{p}{q} \right| > \frac{C}{q^2 f(q)}, \tag{2.6}$$

where the function $f(q)$ *is determined by the relation between* $c(n)$ *and* $R(n)$*.*

To prove (2.4), one proceeds inductively. At the n^{th} step, the set Ω is covered by at most N_n intervals of length δ_n: the dependence between N_n and δ_n is governed by the geometry of the set Ω:

- $N = \text{const}$, if Ω is finite;
- $N \lesssim \log \frac{1}{\delta}$, if Ω is lacunary;
- $N \lesssim \log^M \frac{1}{\delta}$, if Ω is lacunary of order M;
- $N \le C_\varepsilon \left(\frac{1}{\delta}\right)^{d+\varepsilon}$, if Ω has upper Minkowski dimension d.

Next, one has to choose parameters $R(n)$, $|I_n|$, $c(n)$, δ_n, N_n so that they satisfy two inequalities, for an appropriately chosen constant C (We initially restrict our range of α to, say, $[\alpha_0, \pi/2-\alpha_0]$, so that, for all $\theta \in \Omega$, $\alpha-\theta \in [-\pi/2+\alpha_0, \pi/2-\alpha_0]$, where the derivative of tangent is bounded above by some $C > 0$):

$$\frac{2c(n)}{R^2(n)} + C(|I_{n-1}| + \delta_n) < \frac{1}{R^2(n+1)} \quad \text{and} \tag{2.7}$$

$$|I_{n-1}| - N_n \left(\frac{2c(n)}{R^2(n)} + \delta_n \right) \geq (N_n+1)|I_n|. \tag{2.8}$$

Indeed, assuming that I_{n-1} is constructed, fix one of the chosen intervals Ω_n^k of length δ_n. Suppose that the inequality (2.4) doesn't hold for two sets of numbers $\alpha', \alpha'' \in I_n$, $\theta', \theta'' \in \Omega_n^k$, $p', p'' \in \mathbb{Z}$, $R(n) \leq q', q'' < R(n+1)$, then by (2.7)

$$\begin{aligned}\left|\frac{p'}{q'} - \frac{p''}{q''}\right| &\leq \left|\frac{p'}{q'} - \tan(\alpha'-\theta')\right| + \left|\frac{p''}{q''} - \tan(\alpha''-\theta'')\right| + |\tan(\alpha'-\theta') - \tan(\alpha''-\theta'')| \\ &\leq \frac{2c(n)}{R^2(n)} + C(|\alpha'-\alpha''| + |\theta'-\theta''|) \leq \frac{2c(n)}{R^2(n)} + C(|I_{n-1}| + \delta_n) < \frac{1}{R^2(n+1)},\end{aligned}$$

which shows that $p'/q' = p''/q''$ (for otherwise they would have to differ by at least $\frac{1}{R^2(n+1)}$), i.e., there is at most one fraction p_k/q_k with $R(n) \leq q', q'' < R(n+1)$ for each Ω_n^k for which (2.4) is violated.

This implies that the inequality is true for α away from

$$S_n = \bigcup_{k=1}^{N_n} \left\{ \tan^{-1} \left\{ \left[\frac{p_k}{q_k} - \frac{c(n)}{R^2(n)}, \frac{p_k}{q_k} + \frac{c(n)}{R^2(n)} \right] \right\} + \Omega_n^k \right\}.$$

Obviously, $|S_n| \leq N_n \left(\frac{2c(n)}{R^2(n)} + \delta_n \right)$ and $I_{n-1} \setminus S_n$ consists of at most $N_n + 1$ intervals. Thus, the validity of (2.8) proves that $I_{n-1} \setminus S_n$ contains at least one interval of length $|I_n|$.

In particular, for a finite set Ω, to prove Davenport's lemma (Lemma 2.2), one can choose the parameters $R(n) = R^n$, $c(n) = c$ (for some $R, c > 0$), $\delta_n = 0$, $N_n = \#\Omega$. The task of proving similar lemmata for sets Ω of different types is therefore reduced to the proper choice of these parameters. The details are taken up in subsequent subsections.

2.2. **Lacunary sequences.** We recall that a sequence $\Omega = \{\omega_n\}_{n=1}^{\infty}$ is called lacunary if $\omega_{n+1}/\omega_n < A$ for some $A < 1$. For simplicity, we shall consider the set $\Omega = \{2^{-k}\}_{k=1}^{\infty}$, however the argument easily extends to more general lacunary sequences. The main geometrical feature of this set for our purposes is the fact that it can be covered by $\log_2(1/\delta)$ intervals of length δ. We prove

Lemma 2.9. *There exist $\alpha \in [0, \pi/2]$ and $C > 0$ such that for all $k \in \mathbb{N}$, all $p \in \mathbb{Z}$, $q \in \mathbb{N}$ we have*

$$\left|\tan\left(\alpha - 2^{-k}\right) - \frac{p}{q}\right| > \frac{C}{q^2 \log^2 q}. \tag{2.10}$$

The result of the lemma will follow from the following proposition similar to Proposition 2.3:

Proposition 2.11. *There exists a sequence of nested intervals $I_{n_0} \supset I_{n_0+1} \supset \ldots \supset I_n \supset \ldots$ with*

$$|I_n| = \delta(n+2)^{-(n+2)}\big(\log(n+2)\big)^{-(n+2)},$$

such that for all $\alpha \in I_n$ and all $p, q \in \mathbb{Z}$ with $n^{\frac{n}{2}}\big(\log n\big)^{\frac{n}{2}} \leq q < (n+1)^{\frac{n+1}{2}}\big(\log(n+1)\big)^{\frac{n+1}{2}}$ we have

$$\left|\tan\left(\alpha - 2^{-k}\right) - \frac{p}{q}\right| > \frac{c(n)}{q^2}, \tag{2.12}$$

where $c(n) = \frac{c}{(n+1)^2\log^2(n+1)}$ (for some absolute constants c, $\delta > 0$ and $n_0 \in \mathbb{N}$.)

Indeed, the proposition implies that there exists α such that for all $k \in \mathbb{N}$ we have

$$\left|\tan\left(\alpha - 2^{-k}\right) - \frac{p}{q}\right| > \frac{c'}{q^2 \log^2 q} \tag{2.13}$$

for $q \geq q_0 = (n_0)^{\frac{n_0+1}{2}}\big(\log(n_0+1)\big)^{\frac{n_0+1}{2}}$ and for some $c' > 0$.

Now consider $q \leq q_0$. Choose integer r, $1 \leq r \leq q_0$ so that $qr \geq q_0$. Then, if $q \geq 2$,

$$\begin{aligned}\left|\tan\left(\alpha - 2^{-k}\right) - \frac{p}{q}\right| &= \left|\tan\left(\alpha - 2^{-k}\right) - \frac{pr}{qr}\right| > \frac{c'}{(qr)^2\log^2(qr)} \\ &> \frac{c'}{q_0^2(1+\log q_0)^2}\frac{1}{q^2\log^2 q} = \frac{c''}{q^2\log^2 q}\end{aligned}$$

for some constant $c'' > 0$. The case $q = 1$ (without the log) is easy.

Proof of Proposition 2.11. We restrict the range of α to $[0, \pi/3]$ so that $\alpha - 2^{-k} \in [-1, \pi/3] \subset [-\pi/3, \pi/3]$, so that the derivatives of $\tan\left(\alpha - 2^{-k}\right)$ satisfy

$$1 \leq \frac{1}{\cos^2\left(\alpha - 2^{-k}\right)} \leq 4.$$

We arbitrarily choose an initial interval $I_{n_0-1} \subset [-\pi/3, \pi/3]$ with length

$$|I_{n_0-1}| = \varepsilon(n_0+1)^{-(n_0+1)}\big(\log(n_0+1)\big)^{-(n_0+1)},$$

where ε is a small constant, and proceed to construct the sequence inductively.

At the n^{th} step we cover Ω by at most $N_n = 2(n+1)\log(n+1)$ intervals of length $\delta_n = 2^{-N_n} = (n+1)^{-2(n+1)}$. We now show that with this choice of parameters $(c(n) = \frac{c}{(n+1)^2\log^2(n+1)}, R(n) = n^{\frac{n}{2}}\big(\log n\big)^{\frac{n}{2}}, |I_n| = \varepsilon(n+2)^{-(n+2)}\big(\log(n+2)\big)^{-(n+2)})$ the inequalities (2.7) and (2.8) hold for n large enough.

Indeed, one easily verifies (2.7):

$$\frac{2c(n)}{n^n(\log n)^n} + 4(|I_{n-1}| + \delta_n) < \frac{1}{(n+1)^{n+1}(\log(n+1))^{n+1}} = \frac{1}{R^2(n+1)}, \tag{2.14}$$

for c, ε small. Inequality (2.8) is slightly more subtle, as in this case both sides have roughly the same order of magnitude in n, so a little extra care should be given to constants. It is easy to see that, if $c \ll \varepsilon$ and n is large, the left-hand side satisfies

$$|I_{n-1}| - N_n\left(\frac{2c(n)}{R^2(n)} + \delta_n\right) > 0.99|I_{n-1}|, \tag{2.15}$$

(we have $N_n\frac{2c(n)}{R^2(n)} \approx |I_{n-1}|$ and $N_n\delta_n \ll |I_{n-1}|$ for n large)
On the other hand, for the right-hand side

$$\begin{aligned}
(N_n+1)|I_n| &\le \varepsilon(2(n+1)\log(n+1)+1) \times (n+2)^{-(n+2)}\big(\log(n+2)\big)^{-(n+2)} \\
&\le \varepsilon\cdot 2.5\cdot(n+2)^{-(n+1)}\big(\log(n+2)\big)^{-(n+1)} \quad \text{for } n \text{ large} \\
&\le \varepsilon\cdot 2.5\cdot(n+1)^{-(n+1)}\big(\log(n+1)\big)^{-(n+1)}\left(1+\frac{1}{n+1}\right)^{-(n+1)} \\
&\le \varepsilon\cdot\frac{2.5}{2.7}\cdot(n+1)^{-(n+1)}\big(\log(n+1)\big)^{-(n+1)} \quad \text{for } n \text{ large} \\
&< 0.99\cdot\varepsilon\cdot(n+1)^{-(n+1)}(\log(n+1))^{-(n+1)} = 0.99|I_{n-1}|,
\end{aligned}$$

where the second inequality from the bottom holds because $e > 2.7$. Thus, (2.8) holds and the proof is finished.

2.3. **Lacunary sets of finite order.** We now turn our attention to lacunary sets of finite order. They are defined inductively

Definition 2.16. *Lacunary set of order one is a lacunary sequence. We call a set Ω lacunary of order M if it can be covered by the union of a lacunary set Ω' of order $M-1$ with lacunary sequences converging to every point of Ω'.*

These sets play an important role in analysis. In particular, recently M. Bateman [1] proved that the directional maximal function

$$\mathcal{M}_\Omega f(x) = \sup_{R\in\mathcal{A}_\Omega:\, x\in R}\frac{1}{|R|}\int_R |f(x)|\,dx, \tag{2.17}$$

where $\mathcal{A}_\Omega$ is as defined in (1.1), is bounded on $L^p(\mathbb{R}^2)$, $1<p<\infty$, if and only if Ω is covered by a finite union of lacunary sets of finite order. This condition is also equivalent to the fact that Ω does not "admit Kakeya sets" (for details see [1], [15]).

One can check that a lacunary set of order M can be covered by $\mathcal{O}(\log^M(1/\delta))$ intervals of length δ. A simple example of a lacunary set of order M is a set

$$\Omega = \{2^{-j_1} + 2^{-j_2} + \ldots + 2^{-j_M}\}_{j_1,\ldots,j_M \in \mathbb{N}}. \tag{2.18}$$

In our setting, we have the following statement about such sets:

Lemma 2.19. *Let $\Omega \subset [0, \pi/2]$ be a lacunary set of order $M \geq 1$. Then there exist $\alpha \in [0, \pi/2]$ and $C > 0$ such that for all $\theta \in \Omega$, all $p \in \mathbb{Z}$, $q \in \mathbb{N}$ we have*

$$\left|\tan(\alpha - \theta) - \frac{p}{q}\right| > \frac{C}{q^2 \log^{2M} q}. \tag{2.20}$$

This lemma is a generalization of Lemma 2.9. For simplicity we deal with Ω as in (2.18) in which case $N(\delta) = \log_2^M(M/\delta)$. We follow the general approach of §2.1 and verify that inequalities (2.7) and (2.8) hold for the following choice of parameters

$$\begin{aligned}
R(n) &= (Mn)^{\frac{Mn}{2}} \big(\log n\big)^{\frac{Mn}{2}}, \\
|I_n| &= \varepsilon (M(n+2))^{-M(n+2)} \big(\log(n+2)\big)^{-M(n+2)}, \\
c(n) &= \frac{c}{(M(n+1))^{2M} \log^{2M}(n+1)}, \\
N_n &= (2M)^M (n+1)^M \log^M(n+1), \\
\delta_n &= M 2^{-N_n^{1/M}} = M(n+1)^{-2M(n+1)}.
\end{aligned}$$

The proof is verbatim the same as that of Proposition 2.11.

2.4. **Sets of fractional Minkowski dimension.** We now turn to an analogous lemma for the case when the set of directions has non-negative upper Minkowski dimension. Recall that the upper Minkowski dimension of a set $\Omega \subset \mathbb{R}$ is defined as the infimum of exponents d such that for any $0 < \delta \ll 1$ the set E can be covered by $\mathcal{O}(\delta^{-d})$ intervals of length δ.

Lemma 2.21. *Let $\Omega \subset (0, \pi/2)$ be a set of upper Minkowski dimension $d < 1$. Then, for each $\varepsilon > 0$, there exists $\alpha \in \mathbb{R}$ and a constant $c > 0$ such that for all $\gamma \in \Omega$, all $p \in \mathbb{Z}$, $q \in \mathbb{Z}_+$ we have*

$$\left|\tan(\alpha - \gamma) - \frac{p}{q}\right| > c\, q^{-\frac{2}{(1-d)^2} - \varepsilon}. \tag{2.22}$$

The proof is again based on the approach described in §2.1. Fix $t \in (d, 1)$ and denote $a = \frac{1}{1-t}$. We shall construct a system of nested intervals I_n with length $|I_n| = \varepsilon_1 2^{-2a^{n+2}}$ such that for $p \in \mathbb{Z}$, $R(n) = 2^{a^n} \leq q < 2^{a^{n+1}} = R(n+1)$ we have, for all $\alpha \in I_n$,

$$\left|\tan(\alpha - \theta) - \frac{p}{q}\right| > \frac{c(n)}{q^2},$$

where $c(n) = c 2^{-2a^n(a^2-1)}$. The lemma follows from this construction, since $c(n) \gtrsim q^{-2(a^2-1)}$ for this range of q's.

Initially, restrict the attention to α in $(\alpha_0, \pi/2 - \alpha_0)$, $\alpha_0 > 0$, so that $\alpha - \theta$ stays away from $\pm\pi/2$ and the derivative of $\tan(\alpha - \theta)$ is bounded above by some $C > 0$ in absolute value.

Assume I_{n-1} is constructed and consider $2^{a^n} \le q < 2^{a^{n+1}}$. Now fix a number s so that $d < s < t$. We cover Ω by at most $N_n = C_s\delta_n^{-s}$ intervals of length $\delta_n = \varepsilon_2 2^{-2a^{n+2}}$. Inequality (2.7) is obviously satisfied

$$\frac{2c(n)}{2^{2a^n}} + C(|I_{n-1}| + \delta) < 2^{-2a^{n+1}} = \frac{1}{R^2(n+1)}, \tag{2.23}$$

if the constants $c, \varepsilon_1, \varepsilon_2$ are small enough.

$$\begin{aligned} N_n \cdot \left(\frac{2c(n)}{2^{2a^n}} + \delta_n\right) &\le C_s\delta_n^{-s}\left(\frac{2c(n)}{2^{2a^n}} + \delta_n\right) \\ &\le C_s\delta_n^{-t}\left(\frac{2c(n)}{2^{2a^n}} + \delta_n\right) \\ &= C_s\left(2c\delta_n^{-t}2^{-2a^{n+2}} + \delta_n^{1-t}\right) \\ &= C_s\left(2c\varepsilon_2^{-t}2^{2a^{n+2}t}2^{-2a^{n+2}} + \varepsilon_2^{1-t}2^{-2a^{n+1}}\right) \\ &= C_s2^{-2a^{n+1}}\left(2c\varepsilon_2^{-t} + \varepsilon_2^{1-t}\right) \\ &< \frac{1}{2}\varepsilon_1 2^{-2a^{n+1}} = \frac{1}{2}|I_{n-1}|, \end{aligned}$$

if ε_2 and c are small (notice that $a(1-t) = 1$). Then $|I_{n-1}| - N_n \cdot \left(\frac{2c(n)}{2^{2a^n}} + \delta_n\right) \ge \frac{1}{2}|I_{n-1}|$ and

$$(C_s\delta^{-s} + 1)|I_n| \lesssim 2^{2a^{n+2}s}2^{-2a^{n+2}} = 2^{-2a^{n+2}(1-s)} = 2^{-2a^{n+1}\left(\frac{1-s}{1-t}\right)} \approx |I_{n-1}|^{\frac{1-s}{1-t}}. \tag{2.24}$$

Since $\frac{1-s}{1-t} > 1$, we conclude that $(C_s\delta^{-s} + 1)|I_n| < \frac{1}{2}|I_{n-1}|$ for n large enough. Thus (2.8) holds and the proof is finished.

3. One-dimensional discrepancy estimates

Denote by $\|\theta\|$ the distance from θ to the nearest integer. We say that a real number θ is of type $< \psi$ for some non-decreasing function ψ on $\mathbb{R}_+$ if for all natural q we have $q\|q\theta\| > 1/\psi(q)$, in other words for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$ we have

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{q^2 \cdot \psi(q)}. \tag{3.1}$$

In particular, our results in the previous section imply that the numbers $\tan(\alpha - \gamma)$ are of type $< \psi$ with

- $\psi(q) = C\,\log^2 q$ in the lacunary case,
- $\psi(q) = C\,\log^{2M} q$ in the "lacunary of order M" case,
- $\psi(q) = C\,q^{\frac{2}{(1-d)^2} - 2 + \varepsilon}$ in the case of upper Minkowski dimension d.

For a sequence $\omega = \{\omega_n\}_{n=1}^{\infty} \subset [0,1]$ its discrepancy is defined as

$$D_N(\omega) = \sup_{x\in[0,1]} \Big| \#\{\{\omega_1,\dots,\omega_N\} \cap [0,x)\} - Nx \Big| \tag{3.2}$$

The Erdös-Turan inequality (in a simplified form) says that, for any sequence $\omega \subset [0,1]$

$$D_N(\omega) \lesssim \frac{N}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h \omega_n} \right| \tag{3.3}$$

for all natural numbers m. It is particularly convenient to apply it to the sequence of the form $\{n\theta\}$, since in this case

$$\left| \sum_{n=1}^{N} e^{2\pi i h n\theta} \right| \le \frac{2}{|e^{2\pi i h\theta} - 1|} = \frac{1}{|\sin(\pi h\theta)|} = \frac{1}{\sin(\pi \|h\theta\|)} \le \frac{1}{2\|h\theta\|},$$

since $\sin(\pi x) \ge 2x$ for $x \in [0, 1/2]$. Thus, we obtain

$$D_N(\{n\theta\}) \lesssim \frac{N}{m} + \sum_{h=1}^{m} \frac{1}{h\|h\theta\|}. \tag{3.4}$$

If the number θ is of type $< \psi$, then the last sum above can be estimated as follows (see e.g., Exercise 3.12, page 131, [10])

$$\sum_{h=1}^{m} \frac{1}{h\|h\theta\|} \lesssim \log^2 m + \psi(m) + \sum_{h=1}^{m} \frac{\psi(h)}{h}. \tag{3.5}$$

Remark. The proof of the estimate above is somewhat delicate; a more straightforward summation by parts argument (Lemma 3.3, page 123, [10]) would have given

$$\sum_{h=1}^{m} \frac{1}{h\|h\theta\|} \lesssim \psi(2m)\log m + \sum_{h=1}^{m} \frac{\psi(2h)\log h}{h}. \tag{3.6}$$

However, in the case of lacunary directions, this inequality would have given us a weaker bound. It is interesting to note that in the case $\psi = const$, i.e. θ is badly approximable, both estimates, (3.5) and (3.6), only yield $\log^2 N$ as opposed to the sharp $\log^1 N$.

- The case $\psi(q) = C\, \log^2 q$. We have

$$\sum_{h=1}^{m} \frac{1}{h\|h\theta\|} \lesssim \log^2 m + \sum_{h=1}^{m} \frac{\log^2 h}{h} \approx \log^3 m,$$

  while (3.6) would only have given $\log^4 m$. Thus, for the discrepancy, inequality (3.4) with $m \approx N$ yields

$$D_N(\{n\theta\}) \lesssim \log^3 N. \tag{3.7}$$

- More generally, in the case $\psi(q) = C\,\log^{2M} q$, we obtain

$$D_N(\{n\theta\}) \lesssim \log^{2M+1} N. \tag{3.8}$$

- The case $\psi(q)=C\,q^{\frac{2}{(1-d)^2}-2+\varepsilon}$. Denote $\tau=\frac{2}{(1-d)^2}-2+\varepsilon$. From (3.5) we get

$$\sum_{h=1}^{m}\frac{1}{h\|h\theta\|}\lesssim m^\tau+\sum_{h=1}^{m}h^{\tau-1}\approx m^\tau.$$

Inequality (3.4) with $m\approx N^{\frac{1}{\tau+1}}$ shows that the discrepancy satisfies

$$D_N(\{n\theta\})\lesssim N^{\frac{\tau}{\tau+1}}. \tag{3.9}$$

4. Discrepancy with respect to rotated rectangles

In the present section we demonstrate how one can translate the one-dimensional discrepancy estimates into the estimates for $D_\Omega(N)$. These ideas are classical and go back to Roth [12]. The exposition of this and the next sections essentially follows the papers of Beck and Chen [5] and Chen and Travaglini [7].

The examples providing the upper bounds will be obtained using a rotation of the lattice $(N^{-1/2}\mathbb{Z})^2$. However, for technical reasons, it will be easier to rotate the unit square and the rectangles instead and leave the lattice intact. In addition, we shall consider a rescaled version of the problem.

Assume Ω is as described in parts 1,2, or 3 of Theorem 1.6. Let α be the angle provided by Lemma 2.9, 2.19, or 2.21, respectively. Denote by V the square $[0,N^{1/2})$ rotated counterclockwise by α, and by $\mathcal{A}_{\Omega,\alpha}$ the family of all rectangles $R\subset V$ which have a side that is either parallel to a side of V or makes angle $\theta-\alpha$ with the x-axis for some $\theta\in\Omega$. (Strictly speaking, we should have applied Lemma 2.9, 2.19, or 2.21 to the set $\Omega\cup\{0\}\cup(\Omega+\pi/2)\cup\{\pi/2\}$. It is easy to see that this change does not alter the proof.) For $R\subset V$, consider the quantity $D(R)=\#\{\mathbb{Z}^2\cap R\}-|R|$. We have the following lemma:

Lemma 4.1.

1) Let Ω be a lacunary sequence. For any $R\in\mathcal{A}_{\Omega,\alpha}$ we have

$$D(R)\lesssim\log^3 N. \tag{4.2}$$

2) Let Ω be a lacunary set of order M. For any $R\in\mathcal{A}_{\Omega,\alpha}$ we have

$$D(R)\lesssim\log^{2M+1} N. \tag{4.3}$$

3) Assume Ω has upper Minkowski dimension $0<d<1$. In this case, for each $R\in\mathcal{A}_{\Omega,\alpha}$,

$$D_\Omega(N)\lesssim N^{\frac{\tau}{2(\tau+1)}+\varepsilon}, \tag{4.4}$$

for any $\varepsilon>0$, where $\tau=\frac{2}{(1-d)^2}-2$.

We first show that the lemma above implies our main theorem.

Proof of Theorem 1.6. Denote by $\mathcal{P}_\alpha$ the intersection of the lattice $(N^{-1/2}\mathbb{Z})^2$ rotated by α and $[0,1]^2$. The only obstacle to proving the theorem is the fact that $\mathcal{P}_\alpha$ does not necessarily contain precisely N points. Let $\mathcal{P}'_\alpha$ be a set of N points obtained from $\mathcal{P}_\alpha$ by arbitrarily adding or removing $|\#\mathcal{P}_\alpha - N|$ points. Let $F(N)$ stand for the right-hand side of the inequality we are proving ((1.7), (1.8), or (1.9)). "Unscaling" the estimates of Lemma 4.1 and taking $R = [0,1]^2$, we obtain

$$|\#\mathcal{P}_\alpha - N| \lesssim F(N).$$

Then, for any $R \in \mathcal{A}_\Omega$ we have, again using Lemma 4.1

$$\begin{aligned} \Big|\#\mathcal{P}'_\alpha \cap R - N|R|\Big| &\leq \Big|\#\mathcal{P}_\alpha \cap R - N|R|\Big| + \Big|\#\mathcal{P}_\alpha \cap R - \#\mathcal{P}'_\alpha \cap R\Big| \\ &\lesssim F(N) + |\#\mathcal{P}_\alpha - N| \lesssim F(N), \end{aligned}$$

which finishes the proof. □

Remark. In view, of inequality (1.5), for any Ω we have the bound $D_\Omega(N) \lesssim N^{1/4}\log^{1/2} N$. Thus, the bound arising from (1.9) is meaningful only if $\frac{1}{2} - \frac{1}{2(1+\tau)} < \frac{1}{4}$, i.e. $\tau < 1$. So, in the context of rotated rectangles, this estimate is interesting only if the set of rotations has low Minkowski dimension:

$$d < 1 - \left(\frac{2}{3}\right)^{\frac{1}{2}} \approx 0.1835\ldots. \tag{4.5}$$

We now prove Lemma 4.1. For each point $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, consider a square of area one centered around it

$$S(\mathbf{n}) = \left[n_1 - \frac{1}{2}, n_1 + \frac{1}{2}\right) \times \left[n_2 - \frac{1}{2}, n_2 + \frac{1}{2}\right).$$

Obviously, we can write:

$$D(R) = \sum_{\mathbf{n}\in\mathbb{Z}^2} D(R \cap S(\mathbf{n})).$$

Denote the sides of R by T_1, T_2, T_3, T_4. Set

$$\mathcal{N}^- = \{\mathbf{n} : \ S(\mathbf{n}) \cap T_i = \emptyset, \ \text{for all } i = 1,2,3,4\},$$

i.e. the set of those $\mathbf{n}$ for which the corresponding square lies entirely within or entirely outside R – for such squares $D(R \cap S(\mathbf{n})) = 0$.

Also, take

$$\mathcal{N}^+ = \{\mathbf{n} : \ S(\mathbf{n}) \cap T_i \neq \emptyset, S(\mathbf{n}) \cap T_{i+1} \neq \emptyset, \ \text{for some } i = 1,2,3,4\},$$

(the addition is mod 4) to be those $\mathbf{n}$ for which $S(\mathbf{n})$ contains a corner of R. We have $\#\mathcal{N}^+ \leq 4$ and $|D(R \cap S(\mathbf{n}))| \leq 1$, thus $\sum_{\mathbf{n}\in\mathcal{N}^+} D(R \cap S(\mathbf{n})) \leq 4$.

Finally, for $i = 1,2,3,4$, set

$$\mathcal{N}^i = \{\mathbf{n} : \ S(\mathbf{n}) \cap T_i \neq \emptyset, \text{ but } \mathbf{n} \notin \mathcal{N}^+\},$$

to be the centers of those squares which intersect the side T_i but do not contain any corners. The collections $\mathcal{N}^i$ are not necessarily disjoint, e.g., when R is a thin rectangle. However, we have the following useful fact:

Proposition 4.6. *Let R be a convex polygon with sides $T_1, \ldots, T_m$. Denote by T_j^* the halfplane with boundary T_j which contains R. Assume the square $S(\mathbf{n})$ intersects R but does not contain any vertices of R. Let $T_{j_1}, \ldots, T_{j_k}$ be the sides of R that intersect $S(\mathbf{n})$. Then*

$$D(R \cap S(\mathbf{n})) = \sum_{i=1}^{k} D(T_{j_i}^* \cap S(\mathbf{n})). \tag{4.7}$$

We use the fact that discrepancy is an additive measure and that $D(S(\mathbf{n})) = 0$. Then

$$0 = D(S(\mathbf{n})) = \sum_{i=1}^{k} \Big(D(S(\mathbf{n})) - D(T_{j_i}^* \cap S(\mathbf{n})) \Big) + D(R \cap S(\mathbf{n})). \qquad \square$$

Since $\mathbb{Z}^2 = \mathcal{N}^- \cup \mathcal{N}^+ \cup \mathcal{N}^1 \cup \ldots \cup \mathcal{N}^4$, it remains to estimate the terms $\sum_{\mathbf{n} \in \mathcal{N}^j} D(T_j^* \cap S(\mathbf{n}))$. Assume that the j^{th} side of R lies on the line $\tan\phi = \frac{y_2 - a_2}{y_1 - a_1}$, i.e.

$$y_2 = a_2 + (y_1 - a_1)\tan\phi$$

for some constants a_1, a_2 and $\phi = \alpha - \theta$ or $\phi = \alpha - \theta + \pi/2$. Let $I_j = \{n_1 \in \mathbf{Z} : (n_1, n_2) \in \mathcal{N}^j$ for some $n_2 \in \mathbb{Z}\}$ be the projection of the $\mathcal{N}^j$ onto the x-axis. Fix $n \in I_j$ and let $h \in \mathbb{Z}$ be the smallest number such that $(n, h) \in \mathcal{N}^j$. Then it is easy to see that (here we assume that R is below T_j, the other case is analogous)

$$\sum_{\mathbf{n} \in \mathcal{N}^j,\, n_2 = n} \#\{\mathbb{Z}^2 \cap T_j^* \cap S(\mathbf{n})\} = [y_2(n_1) - h + 1], \tag{4.8}$$

and the area of the trapezoid is

$$\sum_{\mathbf{n} \in \mathcal{N}^j,\, n_2 = n} |T_j^* \cap S(\mathbf{n})| = y_2(n_1) - h + \frac{1}{2}. \tag{4.9}$$

(This relation may fail when n is an endpoint of I_j, but this gives us a bounded error.) Thus, the discrepancy can be described by the "sawtooth" function, $\psi(x) = x - [x] - \frac{1}{2} = \{x\} - \frac{1}{2}$,

$$\sum_{\mathbf{n} \in \mathcal{N}^j} D(T_j^* \cap S(\mathbf{n})) = \pm \sum_{n \in I_j} \psi(c - n\tan\phi). \tag{4.10}$$

The "sawtooth" function arises naturally in one dimensional discrepancy. If we define, for a sequence ω,

$$D_N(\omega, x) = \Big| \#\big\{ \{\omega_1, \ldots, \omega_N\} \cap [0, x) \big\} - Nx \Big|,$$

one can easily check that

$$D_N(\omega, x) = \sum_{n=1}^{N} \Big(\psi(\omega_n - x) - \psi(\omega_n) \Big). \tag{4.11}$$

Since $x \in [0,1]$ is arbitrary, it is possible to show that for all $x \in [0,1]$

$$\left| \sum_{n=1} \psi(\omega_n - x) \right| \lesssim D_N(\omega). \tag{4.12}$$

Indeed, one can find a point $x \in [0,1]$ with $D_N(\omega, x) = \sum_{n=1}^N \psi(\omega_n)$ (see, e.g., the proof of Erdös-Turan in [10]), thus $\left|\sum_{n=1}^N \psi(\omega_n)\right| \le D_N(\omega)$, but then for any $x \in [0,1]$, $|\sum_{n=1} \psi(\omega_n - x)| \le 2D_N(\omega)$. Thus, (4.10) and (4.12) imply

$$\left| \sum_{\mathbf{n} \in \mathcal{N}^j} D(R \cap S(\mathbf{n})) \right| \lesssim D_{|I_j|}(\omega). \tag{4.13}$$

Obviously, $|I_j| \lesssim N^{\frac{1}{2}}$. This fact, together with inequality (4.13) and the results of the previous section, proves the lemma. $\square$

To conclude this section, we formulate analogous results on the discrepancy with respect to convex polygons. We omit the proofs as they are verbatim the same as the proof of the main theorem.

Let Ω be a set of directions. Denote by $\mathcal{B}_{\Omega,k}$ the collection of all convex polygons in $[0,1]^2$ with at most k sides whose normals belong to $\pm\Omega$ and set

$$D_{\Omega,k}(N) = \inf_{\mathcal{P}_N:\,\#\mathcal{P}_N = N} \sup_{B \in \mathcal{B}_{\Omega,k}} \Big| \#\mathcal{P}_N \cap B - N \cdot |B| \Big|.$$

The following theorem holds (notice that the implied constants depend on k):

Theorem 4.14.
1) Let Ω be a finite union of lacunary sets of order at most $M \ge 1$. Then we have

$$D_{\Omega,k}(N) \lesssim_k \log^{2M+1} N. \tag{4.15}$$

2) Assume Ω has upper Minkowski dimension $0 < d < 1$. In this case,

$$D_{\Omega,k}(N) \lesssim_k N^{\frac{\tau}{2(\tau+1)} + \varepsilon}, \tag{4.16}$$

for any $\varepsilon > 0$, where $\tau = \frac{2}{(1-d)^2} - 2$.

5. An upper bound for the L^2 discrepancy

We now prove Theorem 1.10. In this case, the point set with low L^2 discrepancy is given by a suitably *shifted* rotation of the lattice $(N^{-1/2}\mathbb{Z})^2$; the idea of using random shifts to obtain distributions with low average discrepancy was first introduced by Roth [13]. As in the previous section we consider a rescaled and rotated version of the problem, that is we set V to be the square$[0, N^{1/2}]^2$ rotated counterclockwise by the angle α given by the Lemma 2.9, 2.19, or 2.21. Assume $\mathcal{A}_{\Omega,\alpha}$ is the family of all rectangles $R \subset V$ which have a side that is either parallel to a side of V or makes angle $\theta - \alpha$ with the x-axis for some $\theta \in \Omega$ and fix a rectangle $R \in \mathcal{A}_{\Omega,\alpha}$.

For any $\omega \in [0,1]^2$ define the shift of the integer lattice $\mathbb{Z}^2_\omega = \omega + \mathbb{Z}^2$. Consider the quantity $D_\omega(R) = D(\mathbb{Z}^2_\omega, R) = \#\{\mathbb{Z}^2_\omega \cap R\} - |R|$. We estimate the mean square of the shifted discrepancies in the following lemma:

Lemma 5.1.
1) Let Ω be a lacunary set of order $M \geq 1$. For any $R \in \mathcal{A}_{\Omega,\alpha}$, we have

$$\int_{[0,1]^2} \mid D(\mathbb{Z}^2_\omega, R) \mid^2 d\omega \lesssim \log^{4M+1} N \tag{5.2}$$

2) Let Ω be a set of upper Minkowski dimension $d < 1$. For any $R \in \mathcal{A}_{\Omega,\alpha}$, we have

$$\int_{[0,1]^2} \mid D(\mathbb{Z}^2_\omega, R) \mid^2 d\omega \lesssim N^{\frac{\tau}{\tau+1}+\varepsilon}, \tag{5.3}$$

for any $\varepsilon > 0$, where $\tau = \frac{2}{(1-d)^2} - 2$ and $\tau < 1$.

The lemma relies on the following important calculation which goes back to Davenport [8] (see also Beck and Chen [5]). Recall that $\| x \| = \min_{n \in Z} \mid x - n \mid$ denotes the distance from x to the nearest integer. We have

Lemma 5.4. *Let I be a finite interval of consecutive integers.*
1) Assume $\tan\phi$ satisfies $\nu\|\nu\tan\phi\| > \frac{c}{\log^{2M}\nu}$, for all $\nu \in \mathbb{N}$. Then

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n\in I} e^{-2\pi i \nu n \tan\phi} \right|^2 \lesssim \log^{4M+1} |I|. \tag{5.5}$$

2) Assume $\tan\phi$ satisfies $\nu\|\nu\tan\phi\| > c\nu^{-\tau+\varepsilon}$, for all $\varepsilon > 0$, where $0 \leq \tau < 1$. Then

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n\in I} e^{-2\pi i \nu n \tan\phi} \right|^2 \lesssim |I|^{\frac{2\tau}{\tau+1}+\varepsilon'}, \quad \text{where } \varepsilon' = \mathcal{O}(\varepsilon). \tag{5.6}$$

Proof. We will use a simple fact that

$$\left| \sum_{n\in I} e^{-2\pi i \nu n \tan\phi} \right| \lesssim \min\{|I|, \| \nu\tan\phi \|^{-1}\}.$$

We deal with part one first:

$$\begin{aligned} S &= \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n\in I} e^{-2\pi i \nu n \tan\phi} \right|^2 \\ &\lesssim \sum_{h=1}^{\infty} 2^{-2h} \sum_{2^{h-1} \leq \nu < 2^h} \min\{|I|^2, \| \nu\tan\phi \|^{-2}\}. \end{aligned}$$

Notice that our assumption on $\tan\phi$ implies that if $2^{h-1} \leq \nu < 2^h$, then $\| \nu\tan\phi \| > \frac{c}{2^h h^{2M}}$. On the other hand, for any pair $h, p \in \mathbb{N}$, there are at most two values of ν satisfying $2^{h-1} \leq \nu < 2^h$ and $p\frac{c}{2^h h^{2M}} \leq \| \nu \tan\phi \| < (p+1)\frac{c}{2^h h^{2M}}$. Indeed, otherwise the difference $(\nu_1 - \nu_2)$ of two of them would contradict the assumption. We have

$$\begin{aligned}
S &\lesssim \sum_{h=1}^{\infty}\sum_{p=1}^{\infty} \min\{2^{-2h}|I|^2, p^{-2}h^{4M}\} \\
&= \sum_{2^h \leq |I|}\sum_{p=1}^{\infty} \min\{2^{-2h}|I|^2, p^{-2}h^{4M}\} + \sum_{2^h > |I|}\sum_{p=1}^{\infty} \min\{2^{-2h}|I|^2, p^{-2}h^{4M}\} \\
&\lesssim \sum_{2^h \leq |I|}\sum_{p=1}^{\infty} p^{-2}h^{4M} + \sum_{2^h > |I|}\left(2^{-2h}|I|^2 2^h |I|^{-1} h^{2M} + \sum_{p > 2^h h^{2M}|I|^{-1}} h^{4M}p^{-2}\right) \\
&\lesssim \sum_{2^h \leq |I|} h^{4M} + \sum_{2^h > |I|} 2^{-h}|I| h^{2M} \\
&\lesssim \log^{4M+1} | I | .
\end{aligned}$$

Part 2 is proved in a similar fashion. The choice of ϕ yields that, for $2^{h-1} \leq \nu < 2^h$, we have $\| \nu \tan\phi \| > c2^{h(-1-\tau-\varepsilon)}$. And as before, for any pair $h, p \in \mathbb{N}$, no more than two values of ν satisfy $2^{h-1} \leq \nu < 2^h$ and $pc2^{h(-1-\tau-\varepsilon)} \leq \| \nu \tan\phi \| < (p+1)c2^{h(-1-\tau-\varepsilon)}$. Thus

$$\begin{aligned}
S &\lesssim \sum_{h=1}^{\infty}\sum_{p=1}^{\infty} \min\{2^{-2h}|I|^2, p^{-2}2^{2h(\tau+\varepsilon)}\} \\
&= \sum_{2^{h(1+\tau)} \leq |I|}\sum_{p=1}^{\infty} \min\{2^{-2h}|I|^2, p^{-2}2^{2h(\tau+\varepsilon)}\} \\
&\quad + \sum_{2^{h(1+\tau)} > |I|}\sum_{p=1}^{\infty} \min\{2^{-2h}|I|^2, p^{-2}2^{2h(\tau+\varepsilon)}\} \\
&\lesssim \sum_{2^{h(1+\tau)} \leq |I|}\sum_{p=1}^{\infty} p^{-2}2^{2h(\tau+\varepsilon)} \\
&\quad + \sum_{2^{h(1+\tau)} > |I|}\left(2^{-2h}|I|^2 2^{h(1+\tau+\varepsilon)} \, | I |^{-1} + \sum_{p > 2^{h(1+\tau+\varepsilon)}|I|^{-1}} p^{-2}2^{2h(\tau+\varepsilon)}\right) \\
&\lesssim \sum_{2^{h(1+\tau)} \leq |I|} 2^{2h(\tau+\varepsilon)} + \sum_{2^{h(1+\tau)} > |I|} 2^{h(-1+\tau+\varepsilon)}|I| \\
&\lesssim | I |^{\frac{2\tau}{1+\tau}+\varepsilon'},
\end{aligned}$$

where $\tau < 1$ is required for the second sum in the penultimate line above to converge for any choice of $\varepsilon > 0$.

We turn to the proof of Lemma 5.1. For any $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, $\omega = (\omega_1, \omega_2) \in [0,1]^2$, define

$$S(\mathbf{n}, \omega) = [n_1 + \omega_1 - 1/2, n_1 + \omega_1 + 1/2) \times [n_2 + \omega_2 - 1/2, n_2 + \omega_2 + 1/2).$$

Also define $\mathcal{N}^+ = \{\mathbf{n} : \exists \omega' \in [0,1]^2 \text{ such that } S(\mathbf{n}, \omega') \text{ contains a vertex of } R\ \}$, and

$$\begin{aligned} \mathcal{N} = \{\mathbf{n} :& \exists \omega' \in [0,1]^2 \text{ such that } S(\mathbf{n}, \omega') \cap R \neq \emptyset, \text{ and} \\ & \forall \omega \in [0,1]^2, \quad S(\mathbf{n}, \omega) \text{ contains no vertex of } R\ \}. \end{aligned}$$

Let $\widetilde{\mathcal{N}} = \mathcal{N}^+ \cup \mathcal{N}^-$. Then one can see that $D_\omega(R) = \sum_{\mathbf{n} \in \widetilde{\mathcal{N}}} D_\omega(R \cap S(\mathbf{n}, \omega))$. Obviously, $\#\mathcal{N}^+ = \mathcal{O}(1)$ and it remains to deal with $\mathcal{N}$. Write $\mathcal{N} = \mathcal{N}^1 \cup \ldots \cup \mathcal{N}^4$ in a natural way. Using Proposition 4.6, we can rewrite the discrepancy

$$\sum_{\mathbf{n} \in \mathcal{N}} D_\omega(R \cap S(\mathbf{n}, \omega)) = \sum_{j=1}^{4} \sum_{\mathbf{n} \in \mathcal{N}^j} D_\omega(S(\mathbf{n}, \omega) \cap T_j^*) \tag{5.7}$$

where T_j^* is the halfplane defined by the j^{th} side of R (see Proposition 4.6).

For each $j = 1, \ldots, 4$, define $I_j = \{n_1 \in \mathbf{Z} : \exists n_2 \, \text{such that} \, (n_1, n_2) \in \mathcal{N}^j\}$. Applying the argument, similar to the one preceding (4.10), we express the discrepancy arising from the j^{th} side in terms of the "sawtooth" function $\psi(x)$, up to a bounded error:

$$\sum_{\mathbf{n} \in \mathcal{N}^j} D_\omega(S(\mathbf{n}, \omega) \cap T_j^*) = \pm \sum_{n_1 \in I_j} \psi(a_2 - \omega_2 + (n_1 - a_1 + \omega_1) \tan \phi) \tag{5.8}$$

The "sawtooth" function $\psi(x)$ has the Fourier expansion $-\sum_{\nu \neq 0} \frac{e^{(2\pi i \nu x)}}{2\pi i \nu}$. Hence, using Parseval's theorem, one easily obtains

$$\int_{[0,1]^2} \left| \sum_{\mathbf{n} \in \mathcal{N}^j} D_\omega(S(\mathbf{n}, \omega) \cap T_j^*) \right|^2 d\omega \lesssim \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n \in I_j} e^{-2\pi i \nu n \tan \phi} \right|^2. \tag{5.9}$$

By applying Lemma 5.4 and the fact that, for each j, we have $| I_j | = \mathcal{O}(N^{1/2})$, we finish the proof of Lemma 5.1.

We are now ready to prove Theorem 1.10. Let μ be any probability measure on $\mathcal{A}'_\Omega$ and consider the induced probability measure μ' on the set $\mathcal{A}_{\Omega,\alpha}$ of rectangles $R \subset V$ (see the beginning of this section). Since, by Lemma 5.1,

$$\int_{[0,1]^2} | D(\mathbb{Z}^2_\omega, R) |^2 \, d\omega \lesssim F(N),$$

(where $F(N)$ denotes the right-hand side of (5.2) or (5.3), respectively), it follows that there exist $\omega_0 \in [0,1]^2$ such that

$$\int_{\mathcal{A}_{\Omega,\alpha}} | D(\mathbb{Z}^2_{\omega_0}, R) |^2 \, d\mu'(R) \lesssim F(N).$$

The only obstacle to finishing the proof is the fact that $\mathbb{Z}^2_{\omega_0} \cap V$ does not necessarily contain precisely N points. However, this can be handled as explained in the proof of Theorem 1.6.

Remark 1. Part 2 of Lemma 5.1 required that $\tau < 1$, which yields the same restriction $d < 1 - (2/3)^{1/2} \approx 0.1835\dots$ that arises in the L^∞ case, (4.5), for a different reason.

Remark 2. Often, when considering L^2 averages, it is more convenient, instead of imposing the condition $R \subset [0,1]^2$, to deal with all rectangles $R \in \mathcal{A}_\Omega$ with $\mathrm{diam}(R) \leq 1$, while treating $[0,1]^2$ as a torus. In this case, the proof of Theorem 1.10 presented above undergoes only minor changes: modulo V, any rectangle R with $\mathrm{diam}(R) \leq N^{1/2}$ can be represented as at most 4 polygons contained in V, having at most 6 sides each.

Remark 3. It is easy to see that the same argument also applies to convex polygons with a bounded number of sides. Thus we also have the following theorem.

Let, as before, $\mathcal{B}_{\Omega,k}$ denote the collection of all convex polygons in $[0,1]^2$ with at most k sides whose normals belong to $\pm\Omega$ and set, for $\mathcal{P} \subset [0,1]^2$ with $\#\mathcal{P} = N$ and for $B \in \mathcal{B}_{\Omega,k}$,

$$D_{\Omega,k}(\mathcal{P}, B) = \Big| \#\mathcal{P} \cap B - N \cdot |B| \Big|.$$

Theorem 5.10. *Let σ be any probability measure on $\mathcal{B}_{\Omega,k}$*

1) Let Ω be a finite union of lacunary sets of order at most $M \geq 1$. Then there exists $\mathcal{P} \subset [0,1]^2$ with $\#\mathcal{P} = N$ such that

$$\left(\int_{\mathcal{B}_{\Omega,k}} |D_{\Omega,k}(\mathcal{P}, B)|^2 \, d\sigma(B) \right)^{\frac{1}{2}} \lesssim_k \log^{2M+\frac{1}{2}} N. \tag{5.11}$$

2) Assume Ω has upper Minkowski dimension $0 \leq d < 1$. In this case, there exists $\mathcal{P} \subset [0,1]^2$ with $\#\mathcal{P} = N$ such that

$$\left(\int_{\mathcal{B}_{\Omega,k}} |D_{\Omega,k}(\mathcal{P}, B)|^2 \, d\sigma(B) \right)^{\frac{1}{2}} \lesssim_k N^{\frac{\tau}{2(\tau+1)}+\varepsilon}, \tag{5.12}$$

for any $\varepsilon > 0$, where $\tau = \frac{2}{(1-d)^2} - 2$ satisfies $\tau < 1$.

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