CALDERÓN-ZYGMUND CAPACITIES AND WOLFF POTENTIALS ON CANTOR SETS

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Abstract. We show that, for some Cantor sets in $\mathbb{R}^d$, the capacity γ_s associated to the s-dimensional Riesz kernel $x/|x|^{s+1}$ is comparable to the capacity $\dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}$ from non linear potential theory. It is an open problem to show that, when s is a positive and non integer, they are comparable for all compact sets in $\mathbb{R}^d$. We also discuss other open questions in the area.

1. Introduction

In the first part of this paper we show that, for some Cantor sets in $\mathbb{R}^d$, the capacity γ_s associated to the s-dimensional Riesz kernel $x/|x|^{s+1}$ is comparable to the capacity $\dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}$ from non linear potential theory. It is an open problem to show that, when s is a positive and non integer, they are comparable for all compact sets in $\mathbb{R}^d$. In the last part of the paper, we discuss other related open questions.

To state our results in detail we need to introduce some notation. For $0<s<d$, the $s-$dimensional Riesz kernel is defined by

$$K^s(x) = \frac{x}{|x|^{s+1}}, \quad x \in \mathbb{R}^d,\, x \neq 0.$$

Notice that this is a vectorial kernel. The $s-$dimensional Riesz transform (or $s-$Riesz transform) of a real Radon measure ν with compact support is

$$R^s\nu(x) = \int K^s(y-x)\,d\nu(y), \qquad x \notin \operatorname{supp}(\nu).$$

Although the preceding integral converges a.e. with respect to Lebesgue measure, the convergence may fail for $x \in \operatorname{supp}(\nu)$. This is the reason why one considers the truncated $s-$Riesz transform of ν, which is defined as

$$R^s_\varepsilon\nu(x) = \int_{|y-x|>\varepsilon} K^s(y-x)\,d\nu(y), \qquad x \in \mathbb{R}^d,\, \varepsilon > 0.$$

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These definitions also make sense if one consider distributions instead of measures. Given a compactly supported distribution T, set

$$R^s(T) = K^s * T$$

(in the principal value sense for $s = d$), and analogously

$$R^s_\varepsilon(T) = K^s_\varepsilon * T,$$

where $K^s_\varepsilon(x) = \chi_{|x|>\varepsilon}\, x/|x|^{s+1}$.

Given a positive Radon measure with compact support and a function $f \in L^1(\mu)$, we consider the operators $R^s_\mu(f) := R^s(f\,d\mu)$ and $R^s_{\mu,\varepsilon}(f) := R^s_\varepsilon(f\,d\mu)$. We say that R^s_μ is bounded on $L^2(\mu)$ if $R^s_{\mu,\varepsilon}$ is bounded on $L^2(\mu)$ uniformly in $\varepsilon > 0$, and we set

$$\|R^s_\mu\|_{L^2(\mu)\to L^2(\mu)} = \sup_{\varepsilon>0} \|R^s_{\mu,\varepsilon}\|_{L^2(\mu)\to L^2(\mu)}.$$

Given a compact set $E \subset \mathbb{R}^d$, the capacity γ_s of E is

$$\gamma_s(E) = \sup |\langle T, 1\rangle|, \tag{1.1}$$

where the supremum is taken over all distributions T supported on E such that $\|R^s(T)\|_{L^\infty(\mathbb{R}^d)} \leq 1$. Following [Vol03], we call γ_s the s-dimensional Calderón-Zygmund capacity. The case $s = d-1$ is particularly relevant: γ_{d-1} coincides with the capacity κ introduced by Paramonov [Par93] in order to study problems of $\mathcal{C}^1$ approximation by harmonic functions in $\mathbb{R}^d$ (the reader should notice that κ is called κ' in [Par93]). When $d = 2$ and $s = 1$, $z/|z|^{s+1}$ coincides with the complex conjugate of the Cauchy kernel $1/z$. Thus, if one allows T to be a complex distribution in the supremum above, then γ_1 is the analytic capacity.

If we restrict the supremum in (1.1) to distributions T given by positive Radon measures supported on E, we obtain the capacities $\gamma_{s,+}$. Clearly, we have $\gamma_s(E) \geq \gamma_{s,+}(E)$. On the other hand, the opposite inequality also holds (up to a multiplicative absolute constant c_s):

$$\gamma_s(E) \leq c_s\, \gamma_{s,+}(E).$$

This was first shown for $s = 1, d = 2$ by the author [Tol03], and it was extended to the case $s = d-1$ by Volberg [Vol03]. For other values of s, this can be proved by combining the techniques from [Vol03] with others from [MPV05].

Now we turn to non linear potential theory. Given $\alpha > 0$ and $1 < p < \infty$ with $0 < \alpha p < 2$, the capacity $\dot{C}_{\alpha,p}$ of $E \subset \mathbb{R}^d$ is defined as

$$\dot{C}_{\alpha,p}(E) = \sup_\mu \mu(E)^p,$$

where the supremum runs over all positive measures μ supported on E such that

$$I_\alpha(\mu)(x) = \int \frac{1}{|x-y|^{2-\alpha}}\, d\mu(x)$$

satisfies $\|I_\alpha(\mu)\|_{p'} \leq 1$, where as usual $p' = p/(p-1)$.

For our purposes, the characterization of $\dot C_{\alpha,p}$ in terms of Wolff potentials is more useful than its definition above. Consider

$$\dot W^{\mu}_{\alpha,p}(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^{2-\alpha p}}\right)^{p'-1} \frac{dr}{r}.$$

A well known theorem of Wolff asserts that

$$\dot C_{\alpha,p}(E) \approx \sup_{\mu} \mu(E), \tag{1.2}$$

where the supremum is taken over all measures μ supported on E such that $\dot W^{\mu}_{\alpha,p}(x) \leq 1$ for all $x \in E$ (see [AH96, Chapter 4], for instance). The notation $A \approx B$ means that there is an absolute constant $c > 0$, or depending on d and s at most, such that $c^{-1}A \leq B \leq cB$.

Mateu, Prat and Verdera showed in [MPV05] that if $0 < s < 1$, then

$$\gamma_s(E) \approx \dot C_{\frac{2}{3}(d-s),\frac{3}{2}}(E).$$

Notice that for the indices $\frac{2}{3}(d-s), \frac{3}{2}$, one has

$$\dot W^{\mu}_{\frac{2}{3}(d-s),\frac{3}{2}}(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^{s}}\right)^{2} \frac{dr}{r}.$$

When $s = 1$ and $d = 2$, from the characterization of $\gamma_{1,+}$ in terms of curvature of measures, one easily gets $\gamma_1(E) \gtrsim \dot C_{\frac{2}{3},\frac{3}{2}}(E)$. Using analogous arguments (involving a symmetrization of the kernel and the $T(1)$ theorem), in [ENV08] it has been shown that this also holds for all indices $0 < s < d$:

$$\gamma_s(E) \gtrsim \dot C_{\frac{2}{3}(d-s),\frac{3}{2}}(E),$$

for any compact set $E \subset \mathbb{R}^d$. The opposite inequality is false when s is integer (for instance, if E is contained in an s-plane and has positive s-dimensional Hausdorff measure, then $\gamma_s(E) > 0$, but $\dot C_{\frac{2}{3}(d-s),\frac{3}{2}}(E) = 0$). When $0 < s < d$ is non integer, it is an open problem to prove (or disprove) that

$$\gamma_s(E) \lesssim \dot C_{\frac{2}{3}(d-s),\frac{3}{2}}(E).$$

See Section 6 for more details and related questions.

In the present paper we show that the comparability $\gamma_s(E) \approx \dot C_{\frac{2}{3}(d-s),\frac{3}{2}}(E)$ holds for some Cantor sets $E \subset \mathbb{R}^d$, which are are defined as follows. Given a sequence $\lambda = (\lambda_n)_{n=1}^\infty$, $0 \leq \lambda_n < 1/2$, we construct E by the following algorithm. Consider the unit cube $Q^0 = [0,1]^d$. At the first step we take 2^d closed cubes inside Q^0, of side length $\ell_1 = \lambda_1$, with sides parallel to the coordinate axes, such that each cube contains a vertex of Q^0. At the second step 2 we apply the preceding procedure to each of the 2^d cubes produced at step 1, but now using the proportion factor λ_2. Then we obtain 2^{2d} cubes of side length $\ell_2 = \lambda_1\lambda_2$.

Proceeding inductively, we have at the $n-$th step 2^{nd} cubes Q_j^n, $1 \leq j \leq 2^{nd}$, of side length $\ell_n = \prod_{j=1}^n \lambda_j$. We consider

$$E_n = E(\lambda_1, \dots, \lambda_n) = \bigcup_{j=1}^{2^{nd}} Q_j^n,$$

and we define the Cantor set associated to $\lambda = (\lambda_n)_{n=1}^\infty$ as

$$E = E(\lambda) = \bigcap_{n=1}^{\infty} E_n.$$

For example, if $\lim_{n\to\infty} \ell_n/2^{-nd/s} = 1$, then the Hausdorff dimension of $E(\lambda)$ is s. If moreover $\ell_n = 2^{-nd/s}$ for each n, then $0 < \mathcal{H}^s(E(\lambda)) < \infty$, where $\mathcal{H}^s$ stands for the $s-$dimensional Hausdorff measure. In the planar case ($d = 2$), This class of Cantor sets first appeared in [Gar72] (as far as we know), and its study has played a very important role in the last advances concerning analytic capacity.

Our result reads as follows.

Theorem 1.1. *Assume that, for all n, $0 < \lambda_n \leq \tau_0 < \frac{1}{2}$. Denote $\theta_n = 2^{-nd}/\ell_n^s$. For any $N = 1, 2, \dots$ we have*

$$\gamma_s(E_N) \approx \dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}(E_N) \approx \left(\sum_{n=1}^N \theta_n^2\right)^{-1/2},$$

where the constants involved in the relationship $\approx$ depend on d, s and τ_0, but not on N.

Observe that if μ is for the probability measure on E_N given by $\mu = \frac{\mathcal{L}^d|E_N}{\mathcal{L}^d(E_N)}$, where $\mathcal{L}^d$ stands for the Lebesgue measure in $\mathbb{R}^d$, then $\theta_n = \mu(Q_j^n)/\ell_n^s$. So θ_n is the s-dimensional density of μ on a cube from the n-th generation.

Showing that $\dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}(E_N) \approx \left(\sum_{n=1}^N \theta_n^2\right)^{-1/2}$ is not difficult, using the characterization of $\dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}$ in terms of Wolff's potentials (see Section 2). The difficult part of the theorem consists in showing that

$$\gamma_s(E_N) \approx \left(\sum_{n=1}^N \theta_n^2\right)^{-1/2}. \tag{1.3}$$

The main step in proving this result consists in estimating the $L^2(\mu)$ norm of the s-dimensional Riesz transform R_μ^s.

Let us remark that (1.3) has been proved for analytic capacity ($s = 1$, $d = 2$) in [MTV03] (using previous results from Mattila [Mat96] and Eiderman [Èĭd98]). The arguments in [MTV03] (as well as the ones in [Mat96] and Eiderman [Èĭd98]) rely heavily on the relationship between the Cauchy transform and curvature of measures. See [Mel95] and [MV95] for more details on this relationship.

In the case $s = d - 1$, the comparability (1.3) was proved by Mateu and the author [MT04] under the additional assumption that $\lambda_n \geq 2^{-d/s}$ for all n, which is equivalent to saying that the sequence $\{\theta_n\}$ is non increasing. It is not difficult to show that the arguments in [MT04] extend to all indices $0 < s < d$. However, getting rid of the assumption $\lambda_n \geq 2^{-d/s}$ is much more delicate. This is what we carry out in this paper.

Let us also mention that in [GPT06] it was shown that the estimate (1.3) also holds if one replaces E_N by some bilipschitz image of itself, also under the assumption $\lambda_n \geq 2^{-d/s}$. On the other hand, recently in [ENV08] some examples of random Cantor sets where the comparability $\gamma_s \approx \dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}$ holds have been studied.

The plan of the paper is the following. In Section 2 we show that $\dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}(E_N) \approx \left(\sum_{n=1}^N \theta_n^2\right)^{-1/2}$. The proof of (1.3) is contained in Sections 3, 4, and 5. In the final Section 6 we discuss open problems in connection with Calderón-Zygmund capacities, Riesz transforms, and Wolff potentials.

Throughout all the paper, the letters c, C will stand for a absolute constants (which may depend on d and s) that may change at different occurrences. Constants with subscripts, such as C_1, will retain their values, in general.

2. Proof of $\dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}(E_N) \approx \left(\sum_{n=1}^N \theta_n^2\right)^{-1/2}$

The proof of this result is essentially contained in [AH96, Section 5.3]. However, for the reader's convenience we give a simple and almost self-contained proof.

Recall that μ stands for the probability measure on E_N defined by $\mu = \frac{\mathcal{L}^d|E_N}{\mathcal{L}^d(E_N)}$. Given $x \in E_N$, let $Q^n(x)$ denote the cube Q_j^n from the n-th generation in the construction of E_N that contains x, so that $\ell(Q^n(x)) = \ell_n$ is its side length. It is straightforward to check that for all $x \in E_N$,

$$\dot{W}^{\mu}_{\frac{2}{3}(d-s),\frac{3}{2}}(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^s}\right)^2 \frac{dr}{r} \approx \sum_{n\geq 0} \left(\frac{\mu(Q^n(x))}{\ell(Q^n(x))^s}\right)^2 = \sum_{n\geq 0} \theta_n^2.$$

Thus, if we consider the measure

$$\nu = \left(\sum_{n\geq 0} \theta_n^2\right)^{-1/2} \mu,$$

we have $\dot{W}^{\nu}_{\frac{2}{3}(d-s),\frac{3}{2}}(x) \lesssim 1$ for all $x \in E_N$. From (1.2) we infer that

$$\dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}(E_N) \gtrsim \nu(E_N) = \left(\sum_{n\geq 0} \theta_n^2\right)^{-1/2}.$$

To prove the converse inequality, we recall that given any Borel measure σ on $\mathbb{R}^d$, for any capacity $\dot{C}_{\alpha,p}$,

$$\dot{C}_{\alpha,p}\big(\{x\in\mathbb{R}^d:\, W^\sigma_{\alpha,p}(x)>\lambda\}\big)\leq c_{\alpha,p}\,\frac{\sigma(\mathbb{R}^d)}{\lambda^{p-1}},\quad \text{for all } \lambda>0.$$

See Proposition 6.3.12 of [AH96]. If we apply this estimate to $\dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}$, $\sigma=\mu$, and $\lambda\approx\sum_{n\geq0}\theta_n^2$, we get

$$\dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}(E_N)\leq \dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}\big(\{x\in\mathbb{R}^d:\, W^\sigma_{\frac{2}{3}(d-s),\frac{3}{2}}(x)>\lambda\}\big)\lesssim \frac{1}{\Big(\sum_{n\geq0}\theta_n^2\Big)^{1/2}}.$$

3. Preliminaries for the proof of $\gamma_s(E_N)\approx\Big(\sum_{n=1}^N\theta_n^2\Big)^{-1/2}$

To simplify notation, to denote the s-dimensional Riesz transform of μ we will write $R\mu$ instead of $R^s\mu$, and also $K(x)$ instead of $K^s(x)=x/|x|^{s+1}$. Moreover, $\|\cdot\|$ stands for the $L^2(\mu)$ norm.

Arguing as in [MT04, Lemma 4.2], it turns out that the estimate

$$\gamma_s(E_N)\approx\Big(\sum_{n=1}^N\theta_n^2\Big)^{-1/2} \tag{3.1}$$

follows from the next result.

Theorem 3.1. *Let μ be the preceding probability measure supported on E_N. We have*

$$\|R\mu\|^2\approx\sum_{j=0}^N\theta_j^2.$$

We will skip the arguments that show that (3.1) can be deduced from this theorem, which the interested reader can find in the aforementioned reference.

Sections 4 and 5 of this paper are devoted to the proof of Theorem 3.1. In the remaining part of the current section, we introduce some additional notation that we will use below, and we prove a technical estimate.

Denote $\widetilde{\Delta}=\big\{Q_j^n: n\geq0,\, 1\leq j\leq 2^{nd}\big\}$, where the Q_j^n's are the cubes which appear in the construction of the $E(\lambda)$. Let Δ_n be the family of cubes in Δ from the n-th generation. That is, $\Delta_n=\{Q_j^n\}_{j=1}^{2^{nd}}$. For a fixed $N\geq1$, we set $\Delta=\bigcup_{n=1}^N\Delta_n$ (so E_N is constructed using the cubes from Δ_N).

Given a cube $Q\subset\mathbb{R}^d$, we set

$$\theta(Q):=\frac{\mu(Q)}{\ell(Q)^s},$$

i.e. $\theta(Q)$ is the average s-dimensional density of μ over Q. Thus $\theta_n=\theta(Q)$ if $Q\in\Delta_n$.

Given a cube $Q \in \Delta$ and a function $f \in L^1_{loc}(\mu)$, we define

$$S_Q f(x) = \frac{1}{\mu(Q)} \int f\, d\mu \; \chi_Q(x).$$

Also, for $0 \leq j \leq N$, we set $S_j f = \sum_{Q\in\Delta_j} S_Q f$. If we denote by $\mathcal{F}(Q)$ the cubes from Δ which are sons of Q, we set

$$D_Q f(x) = \sum_{P\in\mathcal{F}(Q)} S_P f(x) - S_Q f(x),$$

and for $0 \leq j \leq N$ we denote $D_j f = \sum_{Q\in\Delta_j} D_Q f = S_{j+1} f - S_j f$.

Let $\Delta^0 = \Delta \setminus \Delta_N$. Notice that the functions $D_Q f$ and $D_P f$ are orthogonal for $P \neq Q$. If $\int f\, d\mu = 0$, then

$$S_N f = \sum_{j=0}^{N-1} D_j f = \sum_{Q\in\Delta^0} D_Q f,$$

and thus

$$\|f\|^2 \geq \|S_N f\|^2 = \sum_{Q\in\Delta^0} \|D_Q f\|^2.$$

In particular, if we take $f = R\mu$, by antisymmetry $\int R\mu\, d\mu = 0$, and thus

$$\|R\mu\|^2 \geq \|S_N(R\mu)\|^2 = \sum_{Q\in\Delta^0} \|D_Q(R\mu)\|^2. \tag{3.2}$$

Given cubes $Q, R \in \Delta$, we denote

$$p(Q) := \sum_{P\in\Delta: Q\subset P} \theta(P)\frac{\ell(Q)}{\ell(P)}, \qquad p(Q,R) := \sum_{P\in\Delta: Q\subset P\subset R} \theta(P)\frac{\ell(Q)}{\ell(P)}. \tag{3.3}$$

For $0 \leq j \leq N$, we denote $p_j := p(Q)$, for $Q \in \Delta_j$.

Lemma 3.2. *Let $Q \in \Delta$ and $x, x' \in Q$. Let $\widehat{Q}$ the parent of Q. Then we have*

$$\left|R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x) - R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x')\right| \leq C_1 \frac{\ell(Q)}{\ell(\widehat{Q})}\, p(\widehat{Q}).$$

Thus,

$$\left|R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x) - R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x')\right| \leq C_1\, p(\widehat{Q}) \leq C_2\, p(Q).$$

Proof. We have

$$\begin{aligned}\big|R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x) - R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x')\big| \\ &\leq \int_{\mathbb{R}^d\setminus Q} |K(x-y) - K(x'-y)|\,d\mu(y) \\ &\leq C|x-x'| \int_{\mathbb{R}^d\setminus Q} \frac{1}{|x-y|^{s+1}}\,d\mu(y) \\ &\leq C|x-x'| \sum_{P\in\Delta:\, Q\subsetneq P} \frac{\mu(P)}{\ell(P)^{s+1}} \leq C\,\frac{\ell(Q)}{\ell(\widehat{Q})}\,p(\widehat{Q}). \qquad \square\end{aligned}$$

4. Proof of $\|R\mu\|^2 \lesssim \sum_{j=0}^N \theta_j^2$.

Lemma 4.1. *If $Q\in\Delta^0$ and P is a son of Q, then*

$$|S_P(R\mu) - S_Q(R\mu)| \lesssim p(Q). \tag{4.1}$$

As a consequence,

$$\|D_Q(R\mu)\|^2 \lesssim p(Q)^2\,\mu(Q). \tag{4.2}$$

Proof. It is clear that (4.2) follows from (4.1). To prove (4.1), we use the antisymmetry of the kernel $K(x)$:

$$\begin{aligned}S_P(R\mu) - S_Q(R\mu) &= S_P(R(\chi_{\mathbb{R}^d\setminus P}\mu)) - S_Q(R(\chi_{\mathbb{R}^d\setminus Q}\mu)) \\ &= S_P(R(\chi_{Q\setminus P}\mu)) + S_P(R(\chi_{\mathbb{R}^d\setminus Q}\mu)) - S_Q(R(\chi_{\mathbb{R}^d\setminus Q}\mu)). \qquad (4.3)\end{aligned}$$

From Lemma 3.2 it follows that

$$|S_P(R(\chi_{\mathbb{R}^d\setminus Q}\mu)) - S_Q(R(\chi_{\mathbb{R}^d\setminus Q}\mu))| \lesssim p(Q).$$

To estimate $S_P(R(\chi_{Q\setminus P}\mu))$ we take into account that $\mathrm{dist}(Q\cap E_N\setminus P, P)\approx \ell(Q)$, and so for every $x\in P$,

$$|R(\chi_{Q\setminus P}\mu)(x)| \lesssim \frac{\mu(Q)}{\ell(Q)^s} = \theta(Q) \leq p(Q).$$

From the preceding estimates and (4.3), we get (4.1). $\square$

Lemma 4.2. *We have*

$$\|S_N(R\mu)\|^2 \lesssim \sum_{j=0}^{N-1}\theta_j^2 \qquad \text{and} \qquad \|R\mu\|^2 \lesssim \sum_{j=0}^{N}\theta_j^2.$$

Proof. By (3.2) and Lemma (4.1),

$$\|S_N(R\mu)\|^2 = \sum_{Q\in\Delta^0}\|D_Q(R\mu)\|^2 \lesssim \sum_{Q\in\Delta^0} p(Q)^2\mu(Q) = \sum_{j=0}^{N-1} p_j^2.$$

On the other hand, by Lemma 3.2, for each $Q\in\Delta_N$ and $x\in Q$,

$$|S_Q(R\mu) - R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x)| = |S_N(R\chi_{\mathbb{R}^d\setminus Q}\mu)) - R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x)| \lesssim p(Q).$$

Using also that

$$\|\chi_Q R(\chi_Q\mu)\| \leq \theta(Q)\,\mu(Q)^{1/2},$$

we obtain

$$\begin{aligned}\|R\mu\|^2 &= \sum_{Q\in\Delta_N} \|\chi_Q R(\mu)\|^2 \leq 2\sum_{Q\in\Delta_N}\Big(\|\chi_Q R(\chi_Q\mu)\|^2 + \|\chi_Q R(\chi_{\mathbb{R}^d\setminus Q}\mu)\|^2\Big)\\ &\leq 2\sum_{Q\in\Delta_N}\Big(\|\chi_Q R(\chi_Q\mu)\|^2 + \|R(\chi_{\mathbb{R}^d\setminus Q}\mu) - S_N(R\chi_{\mathbb{R}^d\setminus Q}\mu))\|^2 + \|S_N(R\mu)\|^2\Big)\\ &\lesssim \sum_{Q\in\Delta_N}\theta(Q)^2\mu(Q) + \sum_{Q\in\Delta_N} p(Q)^2\mu(Q) + \sum_{Q\in\Delta^0} p(Q)^2\mu(Q)\\ &\lesssim \sum_{Q\in\Delta} p(Q)^2\mu(Q) = \sum_{j=0}^N p_j^2.\end{aligned}$$

It only remains to show that $\sum_{j=0}^M p_j^2 \lesssim \sum_{j=0}^M \theta_j^2$ both for $M = N-1$ and $M = N$. This follows easily from the definition of p_j and Cauchy-Schwartz:

$$\begin{aligned}\sum_{j=0}^M p_j^2 &= \sum_{j=0}^M\left(\sum_{k=0}^j \theta_k\,\frac{\ell_j}{\ell_k}\right)^2 \leq \sum_{j=0}^M\left(\sum_{k=0}^j \theta_k^2\,\frac{\ell_j}{\ell_k}\right)\left(\sum_{k=0}^j \frac{\ell_j}{\ell_k}\right)\\ &\leq 2\sum_{j=0}^M\sum_{k=0}^j \theta_k^2\,\frac{\ell_j}{\ell_k} = 2\sum_{k=0}^M\theta_k^2\sum_{j=k}^M\frac{\ell_j}{\ell_k} \leq 4\sum_{k=0}^M\theta_k^2. \qquad (4.4)\end{aligned}$$

□

5. Proof of $\|R\mu\|^2 \gtrsim \sum_{j=0}^N \theta_j^2$

5.1. The main lemma. The main lemma to prove the estimate

$$\|R\mu\|^2 \gtrsim \sum_{j=0}^N \theta_j^2 \qquad (5.1)$$

is the following.

Lemma 5.1. *We have*

$$\sum_{Q\in\Delta^0}\|D_Q(R\mu)\|^2 \gtrsim \sum_{j=0}^{N-1}\theta_j^2. \qquad (5.2)$$

Let us see how one deduces (5.1) from the preceding inequality.

Proof of (5.1) using Lemma 5.1. From (3.2) and (5.2) we infer that

$$\|R\mu\|^2 \geq \|S_N(R\mu)\|^2 \geq C_3^{-1}\sum_{j=0}^{N-1}\theta_j^2. \qquad (5.3)$$

So we only have to show that $\|R\mu\|^2 \gtrsim \theta_N^2$.

Consider $Q \in \Delta_N$ and $x \in Q$. We split $R\mu(x)$ as follows:

$$\begin{aligned} R\mu(x) &= R(\chi_Q\mu)(x) + R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x) \\ &= R(\chi_Q\mu)(x) + S_N(R\mu)(x) + \big(R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x) - S_N(R\mu)(x)\big). \end{aligned}$$

So we get

$$\begin{aligned} \|R\mu\| \geq \Big\| \sum_{Q\in\Delta_N} \chi_Q R(\chi_Q\mu)\Big\| - \|S_N(R\mu)\| \\ - \Big\| \sum_{Q\in\Delta_N} \chi_Q R(\chi_{\mathbb{R}^d\setminus Q}\mu) - S_N(R\mu)\Big\|. \end{aligned} \tag{5.4}$$

It is easy to check that

$$\Big\| \sum_{Q\in\Delta_N} \chi_Q R(\chi_Q\mu)\Big\| \geq C_4^{-1}\theta_N.$$

To deal with $S_N(R\mu)$ we simply use the fact that

$$\|S_N(R\mu)\| \leq \|R\mu\|.$$

On the other hand, by Lemma 3.2, if $x \in Q \in \Delta_N$,

$$|R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x) - S_N(R\mu)(x)| = |R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x) - S_Q(R(\chi_{\mathbb{R}^d\setminus Q}\mu)(x)| \lesssim p_{N-1}.$$

By Cauchy-Schwartz, it follows easily that $p_{N-1} \leq C\big(\sum_{j=0}^{N-1}\theta_j^2\big)^{1/2}$. Then we deduce

$$\Big\| \sum_{Q\in\Delta_N} \chi_Q R(\chi_{\mathbb{R}^d\setminus Q}\mu) - S_N(R\mu)\Big\|^2 \leq C\sum_{j=0}^{N-1}\theta_j^2.$$

Then, by (5.4) and the estimates above, we get

$$\|R\mu\| \geq C_4^{-1}\theta_n - \|R\mu\| - C_5\big(\sum_{j=0}^{N-1}\theta_j^2\big)^{1/2}.$$

From (5.3), we infer that

$$C_4^{-1}\theta_n \leq 2\|R\mu\| + C_5\Big(\sum_{j=0}^{N-1}\theta_j^2\Big)^{1/2} \leq 2\|R\mu\| + C_3^{1/2}\,C_5\|R\mu\|,$$

and thus the lemma follows. $\square$

5.2. **The stopping scales and the intervals I_k.** To prove Lemma 5.1 we need to define some stopping scales on the squares from Δ. Let B be some big constant (say, $B > 100$) to be fixed below. We proceed by induction to define a subset $\text{Stop} := \{s_0, \dots, s_m\} \subset \{0, 1, \dots, N\}$. First we set $s_0 = 0$. If, for some $k \geq 0$, s_k has already been defined and $s_k < N - 1$, then s_{k+1} is the least integer $i > s_k$ which verifies at least one of the following conditions:

(a) $i = N$, or
(b) $\theta_i > B\,\theta_{s_k}$, or
(c) $\theta_i < B^{-1}\,\theta_{s_k}$.

We finish the construction of Stop when we find some $s_{k+1} = N$. Notice that we have

$$[0, N-1] \cap \mathbb{Z} = \bigcup_{k=0}^{m-1} [s_k, s_{k+1}) \cap \mathbb{Z} =: \bigcup_{k=0}^{m-1} I_k.$$

Moreover, the intervals I_k are pairwise disjoint.

If s_k satisfies the condition (a) above, then we say that I_k is terminal (in this case $k+1 = m$). If s_k satisfies (b) but not (a), then we say that I_k is an interval of increasing density, $I_k \in ID$. If (c) holds for s_k, but not (a) nor (b), then we say that I_k is an interval of decreasing density, $I_k \in DD$. We denote its length by $|I_k|$. Notice that it coincides with $\#I_k$.

For $0 \leq k \leq m$, we denote

$$T_k\mu = \sum_{j: s_k \leq j < s_{k+1}} D_j(R\mu).$$

In this way,

$$S_N(R\mu) = \sum_{k=0}^{m-1} T_k\mu,$$

and since the functions $D_j(R\mu)$ are pairwise orthogonal,

$$\|S_N(R\mu)\|^2 = \sum_{k=0}^{m-1} \|T_k\mu\|^2.$$

To simplify notation, given $A \subset \{0, \dots, N\}$, we denote

$$\sigma(A) := \sum_{j \in A} \theta_j^2.$$

So σ can be thought as a measure on $\{0, \dots, N\}$.

5.3. **Good and bad scales.** We say that $j \in \{0, N-1\}$ is a good scale, and we write $j \in \mathcal{G}$, if

$$p_j \leq 40\theta_j.$$

Otherwise, we say that j is a bad scale and we write $j \in \mathcal{B}$.

Lemma 5.2. *We have*

$$\sigma(\mathcal{B}) \leq \frac{1}{10}\,\sigma([0,N-1]).$$

Proof. As in (4.4) (replacing M by $N-1$),

$$\sum_{j=0}^{N-1} p_j^2 \leq 4\sum_{k=0}^{N-1}\theta_k^2 = 4\,\sigma([0,N-1]).$$

Thus,

$$\sigma(\mathcal{B}) \leq \frac{1}{40}\sum_{j=0}^{N-1} p_j^2 \leq \frac{1}{10}\,\sigma([0,N-1]). \qquad \square$$

5.4. **Good and bad intervals I_k.** We also say that an interval I_k is good if

$$\sigma(I_k\cap\mathcal{G}) \geq \frac{1}{10}\,\sigma(I_k).$$

Otherwise we say that it is bad.

Lemma 5.3.

$$\sigma([0,N-1]) \leq \frac{9}{8}\sum_{k:\,I_k\text{ good}} \sigma(I_k).$$

Proof. If I_k is bad, then

$$\sigma(I_k\cap\mathcal{B}) \geq \frac{9}{10}\,\sigma(I_k).$$

Thus,

$$\sum_{k:\,I_k\text{ bad}} \sigma(I_k) \leq \frac{10}{9}\sigma(\mathcal{B}) \leq \frac{10}{9}\,\frac{1}{10}\,\sigma([0,N-1]) = \frac{1}{9}\,\sigma([0,N-1]).$$

Therefore,

$$\begin{aligned}\sigma([0,N-1]) &= \sum_{k:\,I_k\text{ good}} \sigma(I_k) + \sum_{k:\,I_k\text{ bad}} \sigma(I_k)\\ &\leq \sum_{k:\,I_k\text{ good}} \sigma(I_k) + \frac{1}{9}\sigma([0,N-1]),\end{aligned}$$

and so

$$\sigma([0,N-1]) \leq \frac{9}{8}\sum_{k:\,I_k\text{ good}} \sigma(I_k). \qquad \square$$

5.5. **Long and and short intervals I_k.** Let N_L be some (big) integer to be fixed below. We say that an interval I_k is long if

$$|I_k| = s_{k+1}-s_k \geq N_L.$$

Otherwise we say that I_k is short.

5.6. **Estimates for long good intervals I_k. The key lemma.**

Lemma 5.4. *Let I_k be good, and set $j_0 = \min(I_k \cap \mathcal{G})$. Then,*

$$j_0 - s_k \leq \frac{B^4}{1+B^4}\,(s_{k+1} - s_k).$$

Proof. We denote $\ell = s_{k+1} - s_k$ and $\lambda = j_0 - s_k$. Then we have

$$\sigma(I_k \cap \mathcal{G}) \leq B^2 \theta_{s_k}^2 (\ell - \lambda),$$

and also

$$\sigma(I_k \cap \mathcal{B}) \geq B^{-2} \theta_{s_k}^2\, \lambda.$$

Since I_k is good, we have $\sigma(I_k \cap \mathcal{B}) \leq \sigma(I_k \cap \mathcal{G})$, and so we infer that

$$\lambda \leq B^4(\ell - \lambda),$$

and the lemma follows. □

Lemma 5.5. *Let $0 \leq k \leq N-1$. There exists some absolute constant C_6 such that if*

$$\frac{\ell_k}{\ell_{k-1}}\, p_{k-1} \leq C_6\big(\theta_k + \theta_{k+1} + \ldots \theta_{k+h}\big), \tag{5.5}$$

then

$$\sum_{j=k}^{k+h} \|D_j(R\mu)\|^2 \geq C_7^{-1} 2^{-hd}\big(\theta_k + \theta_{k+1} + \ldots \theta_{k+h}\big)^2.$$

Proof. Denote $f = \sum_{j=k}^{k+h} D_j(R\mu)$. Take $P \in \Delta_{k+h+1}$ and $Q \in \Delta_k$ containing P. Then, for $x \in P$ we have

$$f(x) = S_P(R\mu)(x) - S_Q(R\mu)(x).$$

By antisymmetry, as in (4.3), we get

$$f(x) = S_P(R(\chi_{Q\setminus P}\mu))(x) + S_P(R(\chi_{\mathbb{R}^d\setminus Q}\mu))(x) - S_Q(R(\chi_{\mathbb{R}^d\setminus Q}\mu))(x).$$

From Lemma 3.2 it follows that

$$|S_P(R(\chi_{\mathbb{R}^d\setminus Q}\mu))(x) - S_Q(R(\chi_{\mathbb{R}^d\setminus Q}\mu))(x)| \leq C_8\,\frac{\ell_k}{\ell_{k-1}}\, p_{k-1}.$$

On the other hand, if $P \in \Delta_{k+h+1}$ is a cube containing a corner of Q, then it is easy to check that

$$|m_P(R(\chi_{Q\setminus P}\mu))| \geq C^{-1}\big(\theta_k + \theta_{k+1} + \ldots \theta_{k+h}\big).$$

Therefore,

$$|f(x)| \geq C_9^{-1}\big(\theta_k + \theta_{k+1} + \ldots \theta_{k+h}\big) - C_8 \frac{\ell_k}{\ell_{k-1}}\, p_{k-1}.$$

As a consequence, if $C_6 \leq C_9^{-1} C_8^{-1}/2$, then

$$\begin{aligned}\|\chi_Q f\|^2 &\geq C^{-1}\big(\theta_k + \theta_{k+1} + \dots \theta_{k+h}\big)^2 \mu(P) \\ &= 2^{-(h+1)d} C^{-1}\big(\theta_k + \theta_{k+1} + \dots \theta_{k+h}\big)^2 \mu(Q).\end{aligned}$$

Summing over all the cubes $Q \in \Delta_k$, the lemma follows. □

Lemma 5.6. *[Key lemma] Let A, c_0 be positive constants, and $r, q \in [0, N-1] \cap \mathbb{Z}$ such that $q \leq r$, $\frac{\ell_q}{\ell_{q-1}}\, p_{q-1} \leq c_0 \theta_q$ and, for all j with $q \leq j \leq r$,*

$$A^{-1}\theta_q \leq \theta_j \leq A\theta_q.$$

There exists $N_1 = N_1(c_0, A)$ such that if $|q - r| > N_1$, then

$$\sum_{j=q}^{r} \|D_j(R\mu)\|^2 \geq C|q-r|\, \theta_q^2,$$

where C is some positive constant depending on c_0 and A.

Proof. Set $f = \sum_{j=q}^r D_j(R\mu)$. We have to show that $\|f\|^2 \geq C|q-r|\,\theta_q^2$.

Let M_0 some positive integer depending on c_0, A to be fixed below. We decompose f as follows

$$f = \sum_{j=q}^{q+t\,M_0-1} D_j(R\mu) + \sum_{j=q+t\,M_0}^{r} D_j(R\mu), \tag{5.6}$$

where t is the biggest integer such that $q + t\,M_0 - 1 \leq r$. Assuming N_1 big enough we will have $t \approx |q-r|$, with constants depending on M_0, and so on c_0, A.

We write the first sum on the right side of (5.6) as follows:

$$\sum_{j=q}^{q+t\,M_0-1} D_j(R\mu) = \sum_{h=0}^{t-1} \sum_{j=q+hM_0}^{q+(h+1)M_0-1} D_j(R\mu) =: \sum_{h=0}^{t-1} U_h(\mu).$$

By orthogonality, we have

$$\|f\|^2 \geq \sum_{h=0}^{t-1} \|U_h(\mu)\|^2.$$

We will show below that if the parameter $M_0 = M_0(c_0, A)$ is chosen big enough, then

$$\|U_h(\mu)\|^2 \geq C(c_0, A)\theta_q^2 \quad \text{for all } 0 \leq h \leq t-1, \tag{5.7}$$

and thus

$$\|f\|^2 \geq C(c_0, A)\, |q-r|\, \theta_q^2,$$

if $N_1 \geq 2M_0$, say.

To prove (5.7) we intend to apply Lemma 5.5. Recall that $\frac{\ell_q}{\ell_{q-1}}\,p_{q-1} \leq c_0\theta_q$, and since

$$p_{q+hM_0-1} = \sum_{i\leq q+hM_0-1} \frac{\ell_{q+hM_0-1}}{\ell_i}\,\theta_i$$
$$= \sum_{q-1<i\leq q+hM_0-1} \frac{\ell_{q+hM_0-1}}{\ell_i}\,\theta_i + \frac{\ell_{q+hM_0-1}}{\ell_{q-1}}\,p_{q-1},$$

we infer that

$$p_{q+hM_0-1} \leq 2A\theta_q + \frac{\ell_{q+hM_0-1}}{\ell_{q-1}}\,p_{q-1}.$$

Therefore,

$$\frac{\ell_{q+hM_0}}{\ell_{q+hM_0-1}}\,p_{q+hM_0-1} \leq 2A\theta_q + \frac{\ell_{q+hM_0}}{\ell_{q-1}}\,p_{q-1} \leq 2A\theta_q + \frac{\ell_q}{\ell_{q-1}}\,p_{q-1} \leq (2A + c_0)\theta_q.$$

On the other hand,

$$\sum_{j=q+hM_0}^{q+(h+1)M_0-1} \theta_j \geq M_0 A^{-1}\theta_q.$$

If M_0 is big enough then $2A + c_0 \leq C_6 M_0 A^{-1}$ and so the assumption (5.5) in Lemma 5.5 is satisfied. Thus

$$\|U_h\mu\|^2 \geq C_7^{-1} 2^{-M_0 d}\left(\sum_{j=q+hM_0}^{q+(h+1)M_0-1} \theta_j\right)^2 \geq C_7^{-1} 2^{-M_0 d} M_0^2 A^{-2}\theta_q^2,$$

and so our claim (5.7) follows. □

Lemma 5.7. *Suppose that the constant N_L is chosen big enough (depending on B). If I_k is long and good, then*

$$\sigma(I_k) \leq C(B)\|T_k\mu\|^2.$$

Recall that $T_k\mu = \sum_{j:s_k\leq j<s_{k+1}} D_j(R\mu)$.

Proof. Set $\ell = s_{k+1} - s_k$. Notice that

$$\sigma(I_k) \leq \ell\, B^2\theta_{s_k}.$$

Let $j_0 = \min(I_k \cap \mathcal{G})$. We suppose that $N_L \gg B^4$, so that by Lemma 5.4,

$$s_{k+1} - j_0 \geq \frac{1}{B^4}\,\ell \gg 1.$$

We split $T_k\mu$ as follows

$$T_k\mu = \sum_{j=s_k}^{j_0-1} D_j(R\mu) + \sum_{j=j_0}^{s_{k+1}-1} D_j(R\mu),$$

Now we apply Lemma 5.6, with $A = B$, $c_0 = 40$, and we we deduce that if N_L is big enough, then

$$\sum_{j=j_0}^{s_{k+1}-1} \|D_j(R\mu)\|^2 \geq C(B)^{-1}|s_{k+1} - j_0|\,\theta_{s_k}^2$$

By orthogonality,

$$\|T_k\mu\|^2 \geq \sum_{j=j_0}^{s_{k+1}-1} \|D_j(R\mu)\|^2,$$

and thus the lemma follows. □

5.7. **The intervals** J_h**.** By Lemmas 5.3 and 5.7, to finish our proof of $\sigma([0, N-1]) \lesssim \sum_j \|D_j(R\mu)\|^2$, it is enough to show that

$$\sum_{k:\, I_k \text{ short good}} \sigma(I_k) \lesssim \sum_j \|D_j(R\mu)\|^2. \tag{5.8}$$

To this end, we have to define some auxiliary intervals J_h.

We consider the following partial ordering in the family of intervals contained in $\mathbb{R}$: if I, J are disjoint intervals such that all $x \in I$, $y \in J$ satisfy $x < y$, then we write $I \prec J$.

An interval J_h, $h \geq 1$, is the union of two intervals I_k, I_{k+1}, so that I_k is of type ID and I_{k+1} is either of type DD or it is the terminal interval I_m. Then $\{J_h\}_{1\leq h\leq m_J}$ is the collection of all these intervals. We assume that $J_h \prec J_{h+1}$ for all h. Moreover, for convenience, if I_0 is of type DD, we set $J_0 = I_0$.

Remark 5.8. Of course, there may be intervals I_k which are not contained in any interval J_h. Suppose that, for some $0 \leq h \leq m_J$, there are intervals I_k such that

$$J_h \prec I_k \prec I_{k+1} \prec \ldots \prec I_{k+r} \prec J_{h+1}.$$

Then, from the definition of the intervals J_h, it turns out that either all the intervals $I_k, \ldots, I_{k+r}$ are of type ID, or all are of type DD, or there exists $1 \leq s \leq r$ such that $I_k, \ldots, I_{k+s-1}$ are of type DD, and $I_{k+s}, \ldots, I_{k+r}$ are of type ID.

Given an interval $I \subset [0, N]$, we denote

$$\theta^{\max}(I) = \max_{j\in I} \theta_j.$$

Lemma 5.9. *Let J_h, $1 \leq h \leq m_J$, be such that*

$$J_h \prec I_k \prec I_{k+1} \prec \ldots \prec I_{k+r} \prec J_{h+1}.$$

Then,

$$\sum_{\substack{k\leq i\leq k+r\\ I_i \text{ short}}} \sigma(I_i) \leq C(B, N_L)\big[\theta^{\max}(J_h)^2 + \theta^{\max}(J_{h+1})^2\big]. \tag{5.9}$$

Proof. Notice that any short interval I_k satisfies

$$\sigma(I_k) \leq B^2 \, N_L \theta_{s_k}^2. \tag{5.10}$$

If there is some $q \geq 1$ such that the intervals $I_k, \ldots, I_{k+q-1}$ are of type DD, then

$$\theta_{s_{k+q-1}} \leq B^{-1}\theta_{s_{k+q-2}} \leq \ldots \leq B^{1-q}\theta_{s_k} \leq B^{-q}\theta^{\max}(J_h).$$

Thus,

$$\sum_{\substack{k\leq i\leq k+q-1\\ I_i \text{ short}}} \sigma(I_i) \leq C(B, N_L)\theta^{\max}(J_h)^2.$$

Analogously, one deduces that

$$\sum_{\substack{k+q\leq i\leq k+r\\ I_i \text{ short}}} \sigma(I_i) \leq C(B, N_L)\theta^{\max}(J_{h+1})^2,$$

and the lemma follows. □

Lemma 5.10. *We have*

$$\sum_{k: I_k \text{ short}} \sigma(I_k) \leq C(B, N_L) \sum_h \theta^{\max}(J_h)^2. \tag{5.11}$$

Proof. This is a direct consequence of Lemma 5.9. □

5.8. **The standard intervals J_h.** By Lemma 5.10, in order to prove (5.8), it is enough to show that

$$\sum_h \theta^{\max}(J_h)^2 \lesssim \sum_j \|D_j(R\mu)\|^2.$$

To this end, we need to distinguish different types of intervals J_h. For $h \geq 1$, let $t_h \in J_h$ be the least integer such that

$$\theta_{t_h} > B^{-1/2}\,\theta^{\max}(J_h).$$

Notice that, if $J_h = I_k \cup I_{k+1}$, then $\theta^{\max}(J_h) \leq B\theta_{s_{k+1}}$. However we cannot assure that $\theta^{\max}(J_h) \leq B^2\theta_{s_k}$ because it may happen that $\theta_{s_{k+1}} \gg B\theta_{s_k}$. We say that J_h is **standard** if

$$\frac{\ell_{t_h}}{\ell_{t_h-1}}\, p_{t_h-1} \leq \, C_{10}\,\theta^{\max}(J_h), \tag{5.12}$$

where $C_{10} = C_6/2$ (with C_6 from (5.5). For convenience, if J_0 exists (and thus $J_0 = I_0 \in DD$) we also say that J_0 is standard.

Lemma 5.11. *Suppose that B has been chosen big enough. If J_h is standard, then*

$$\theta^{\max}(J_h)^2 \leq C(B) \sum_{j\in J_h} \|D_j(R\mu)\|^2.$$

Proof. In the special case $h = 0$ (with $J_0 = I_0$), it is immediate to check that $\|D_0(R\mu)\|^2 \geq C^{-1}\theta_0^2 \geq C^{-1}B^{-2}\theta^{\max}(J_0)^2$ (for instance, one can apply Lemma 5.5 with $p_{-1} = 0$), and thus the lemma holds.

For $h \geq 1$, let k be such that $J_h = I_k \cup I_{k+1}$. Notice that

$$p_{s_{k+1}-1} = \sum_{t_h \leq j < s_{k+1}} \frac{\ell_{s_{k+1}-1}}{\ell_j}\,\theta_j + \frac{\ell_{s_{k+1}-1}}{\ell_{t_h-1}} \sum_{j \leq t_h-1} \frac{\ell_{t_h-1}}{\ell_j}\,\theta_j =: P_1 + P_2. \tag{5.13}$$

In the sum P_1 we have $\theta_j \leq B\theta_{s_k}$, and so $P_1 \leq 2B\theta_{s_k}$. On the other hand,

$$P_2 = \frac{\ell_{s_{k+1}-1}}{\ell_{t_h-1}}\,p_{t_{h-1}} \leq \frac{\ell_{t_h}}{\ell_{t_h-1}}\,p_{t_h-1} \leq C_{10}\,\theta^{\max}(J_h).$$

Therefore,

$$p_{s_{k+1}-1} \leq C_{10}\,\theta^{\max}(J_h) + 2B\theta_{s_k}, \tag{5.14}$$

and thus $p_{s_{k+1}} \leq C(B)\,\theta_{s_{k+1}}$.

We distinguish several cases:

Case 1. Suppose first that the length $|I_{k+1}|$ is big. That is, $|I_{k+1}| = s_{k+2} - s_{k+1} > N_2$, where $N_2 = N_2(C_{10}, B)$ is some big integer. From (5.14) and Lemma 5.6, we infer that if N_2 is chosen big enough, then

$$\theta^{\max}(J_h)^2 \leq C(B) \sum_{j \in I_{k+1}} \|D_j(R\mu)\|^2,$$

and thus the lemma holds in this case.

Case 2. Assume that $|I_{k+1}| \leq N_2$. If moreover $\theta^{\max}(J_h) > C_{11}B\theta_{s_k}$, with $C_{11} = 4C_6^{-1}$ (i.e. C_{11} is big enough), from (5.14), recalling that $C_{10} = C_6/2$, we infer that

$$p_{s_{k+1}-1} \leq \left(\frac{C_6}{2} + \frac{2}{C_{11}}\right)\theta^{\max}(J_h) = C_6\,\theta^{\max}(J_h).$$

Then, by Lemma 5.5,

$$\sum_{j=s_{k+1}}^{s_{k+2}-1} \|D_j(R\mu)\|^2 \geq C_7^{-1}2^{-N_2 d}\Big(\sum_{j=s_{k+1}}^{s_{k+2}-1} \theta_j\Big)^2 \geq C_7^{-1}2^{-N_2 d}\,\theta^{\max}(J_h)^2,$$

and so the lemma also holds in this situation.

Case 3. Suppose now that $|I_{k+1}| \leq N_2$, that $\theta^{\max}(J_h) \leq C_{11}B\theta_{s_k}$, and, moreover, that $|s_{k+1} - t_h| > N_3$, where $N_3 = N_3(B)$ is some fixed big integer to be fixed below. Using the fact that $\frac{\ell_{t_h}}{\ell_{t_h-1}}\,p_{t_h-1} \leq C_{10}\,\theta^{\max}(J_h)$ and that $\theta_j \approx \theta_{s_k} \approx \theta^{\max}(J_h)$ for all $j \in [t_h, s_{k+2})$, with constants depending on B, if N_3 big enough, from Lemma 5.6 we get

$$\sum_{j=t_h}^{s_{k+2}-1} \|D_j(R\mu)\|^2 \geq C(B)^{-1} \sum_{j=t_h}^{s_{k+2}-1} \theta_j^2 \geq C(B)^{-1}\theta^{\max}(J_h)^2.$$

Case 4. Finally, suppose that $|I_{k+1}| \leq N_2$, and that $|s_{k+1} - t_h| \leq N_3$, with $N_3 = N_3(B)$. Since $\sum_{j=t_h}^{s_{k+2}-1} \theta_j \geq \theta^{\max}(J_h)$, from the condition (5.12), recalling that $C_{10} = C_6/2$, by Lemma 5.5, we infer that

$$\sum_{j=t_h}^{s_{k+2}-1} \|D_j(R\mu)\|^2 \geq C^{-1}\Big(\sum_{j=t_h}^{s_{k+2}-1} \theta_j\Big)^2 \geq C_7^{-1} 2^{-(N_2+N_3)d}\, \theta^{\max}(J_h)^2,$$

and so the lemma also holds under these assumptions. □

5.9. **The non standard intervals J_h.**

Lemma 5.12. *Suppose that B has been chosen big enough. We have*

$$\sum_{h: J_h \text{ non standard}} \theta^{\max}(J_h)^2 \leq C(B) \sum_{h: J_h \text{ standard}} \theta^{\max}(J_h)^2.$$

Proof. Denote by $\{J_n^{st}\}_n$ the subfamily of the standard intervals from $\{J_h\}_h$, ordered so that $J_n^{st} \prec J_{n+1}^{st}$ for all n. For a fixed n, denote by $\Lambda_1, \ldots, \Lambda_m$ the collection of all non standard intervals from the family $\{J_h\}$ such that either

$$J_n^{st} \prec \Lambda_1 \prec \Lambda_2 \prec \ldots \prec \Lambda_m \prec J_{n+1}^{st} \qquad \text{if } J_{n+1}^{st} \text{ exists},$$

or

$$J_n^{st} \prec \Lambda_1 \prec \Lambda_2 \prec \ldots \prec \Lambda_m \qquad \text{if } J_{n+1}^{st} \text{ does not exist}.$$

We will prove that

$$\theta^{\max}(\Lambda_i) \leq B^{-i/8s}\, \theta^{\max}(J_n^{st}) \quad \text{for } i \geq 1, \tag{5.15}$$

by induction on i. The lemma follows easily from this estimate.

To simplify notation, we set $\Lambda_0 = J_n^{st}$ and $\theta_i^{\max} = \theta^{\max}(\Lambda_i)$. Also, if $\Lambda_i = I_k \cup I_{k+1}$, we denote by Q_i is a cube from Δ_{s_k}, by $\widetilde{Q}_i$ a cube from Δ_{t_h-1} (see (5.12)), and by $Q_i^{\max}$ a cube from $\bigcup_{j\in I_{k+1}} \Delta_j$ such that $\theta_i^{\max} = \theta_j$. Moreover, we assume that

$$Q_i \supset \widetilde{Q}_i \supset Q_i^{\max} \supset Q_{i+1} \supset \widetilde{Q}_{i+1} \supset Q_{i+1}^{\max} \supset \ldots$$

First we prove (5.15) for $\boldsymbol{i = 1}$. Since Λ_1 is not standard,

$$\theta_1^{\max} \leq C_{10}^{-1} \frac{\ell(s(\widetilde{Q}_1))}{\ell(\widetilde{Q}_1)}\, p(\widetilde{Q}_1), \tag{5.16}$$

where $s(\widetilde{Q}_1)$ stands for a son of $\widetilde{Q}_1$. To estimate $p(\widetilde{Q}_1)$, we decompose it as follows:

$$p(\widetilde{Q}_1) \leq p(\widetilde{Q}_1, Q_1) + \frac{\ell(\widetilde{Q}_1)}{\ell(Q_1)}\, p(Q_1, Q_0^{\max}) + \frac{\ell(\widetilde{Q}_1)}{\ell(Q_0^{\max})}\, p(Q_0^{\max}).$$

Now observe that

$$p(\widetilde{Q}_1, Q_1) \leq 2B^{-1/2}\theta_1^{\max}, \tag{5.17}$$

since $\theta(P) \leq B^{-1/2}\theta_1^{\max}$ for $\widetilde{Q}_1 \subset P \subset Q_1$. Also,

$$p(Q_1, Q_0^{\max}) \leq 2\theta(Q_1) + 2\theta_0^{\max}, \tag{5.18}$$

because $\theta(P) \leq \theta(Q_1) + \theta_0^{\max}$ for $Q_1 \subset P \subset Q_0^{\max}$, taking into account Remark 5.8. And finally,

$$\begin{aligned} p(Q_0^{\max}) &\leq p(Q_0^{\max}, \widetilde{Q}_0) + \frac{\ell(Q_0^{\max})}{\ell(\widetilde{Q}_0)}\, p(\widetilde{Q}_0) \\ &\leq p(Q_0^{\max}, \widetilde{Q}_0) + \frac{\ell(s(\widetilde{Q}_0))}{\ell(\widetilde{Q}_0)}\, p(\widetilde{Q}_0) \leq 4\theta_0^{\max}, \end{aligned} \tag{5.19}$$

because $\theta(P) \leq \theta_0^{\max}$ for $Q_0^{\max} \subset P \subset \widetilde{Q}_0$ and moreover Λ_0 is standard. Thus we infer that

$$\begin{aligned} p(\widetilde{Q}_1) &\leq 2B^{-1/2}\theta_1^{\max} + \frac{2\ell(\widetilde{Q}_1)}{\ell(Q_1)} \big(\theta(Q_1) + \theta_0^{\max}\big) + \frac{4\ell(\widetilde{Q}_1)}{\ell(Q_0^{\max})}\, \theta_0^{\max} \\ &\leq 4B^{-1/2}\theta_1^{\max} + \frac{6\ell(\widetilde{Q}_1)}{\ell(Q_1)}\, \theta_0^{\max}, \end{aligned}$$

using that $\theta(Q_1) \leq B^{-1}\theta_1^{\max} \leq B^{-1/2}\theta_1^{\max}$ in the second inequality. If we plug this estimate into (5.16) we deduce

$$\theta_1^{\max} \leq 4C_{10}^{-1}B^{-1/2}\theta_1^{\max} + 6C_{10}^{-1}\frac{\ell(s(\widetilde{Q}_1))}{\ell(Q_1)}\, \theta_0^{\max}.$$

If we assume B big enough, so that $4C_{10}^{-1}B^{-1/2} \leq 1/2$ (recall that $C_{10} = C_6/2$ does not depend on B), we obtain

$$\theta_1^{\max} \leq 12C_{10}^{-1}\frac{\ell(s(\widetilde{Q}_1))}{\ell(Q_1)}\, \theta_0^{\max}.$$

On the other hand, since $\theta(s(\widetilde{Q}_1)) > B^{1/2}\theta(Q_1)$ (by the definition of $\widetilde{Q}_1$), we infer that

$$\ell(s(\widetilde{Q}_1))^s \leq B^{-1/2}\ell(Q_1)^s, \tag{5.20}$$

and so

$$\theta_1^{\max} \leq 12C_{10}^{-1}B^{-1/2s}\, \theta_0^{\max}.$$

If we suppose B big enough again, (5.15) follows in the particular case $i = 1$.

The proof of (5.15) for an arbitrary integer $\boldsymbol{i \geq 2}$ when we assume that it holds for $1, \ldots, i-1$ is analogous to the one for the case $i = 1$. For the sake of completeness we will show the detailed arguments. As in (5.16), we have

$$\theta_i^{\max} \leq C_{10}^{-1}\frac{\ell(s(\widetilde{Q}_i))}{\ell(\widetilde{Q}_i)}\, p(\widetilde{Q}_i), \tag{5.21}$$

because Λ_i is not standard. Now we split $p(\widetilde{Q}_i)$ as follows:

$$\begin{aligned} p(\widetilde{Q}_i) \le p(\widetilde{Q}_i, Q_i) &+ \frac{\ell(\widetilde{Q}_i)}{\ell(Q_i)}\, p(Q_i, Q_{i-1}^{\max}) \\ &+ \sum_{j=1}^{i-1} \frac{\ell(\widetilde{Q}_i)}{\ell(Q_j^{\max})}\, p(Q_j^{\max}, Q_{j-1}^{\max}) + \frac{\ell(\widetilde{Q}_i)}{\ell(Q_0^{\max})}\, p(Q_0^{\max}). \end{aligned}$$

We will estimate each of the terms in the preceding inequality separately. As in (5.17), we have

$$p(\widetilde{Q}_i, Q_i) \le 2B^{-1/2}\theta_i^{\max},$$

and as in (5.18),

$$p(Q_i, Q_{i-1}^{\max}) \le 2\theta(Q_i) + 2\theta_{i-1}^{\max} \le 2B^{-1/2}\theta_i^{\max} + 2\theta_{i-1}^{\max}.$$

By analogous arguments,

$$p(Q_j^{\max}, Q_{j-1}^{\max}) \le 2\theta_j^{\max} + 2\theta_{j-1}^{\max}.$$

On the other hand, the term $p(Q_0^{\max})$ has been estimated in (5.19). By the preceding inequalities and the induction hypothesis, we obtain

$$\begin{aligned} p(\widetilde{Q}_i) &\le 2B^{-1/2}\theta_i^{\max} + \frac{\ell(\widetilde{Q}_i)}{\ell(Q_i)} \left(2B^{-1/2}\theta_i^{\max} + 2\theta_{i-1}^{\max}\right) \\ &\qquad + \sum_{j=1}^{i-1} \frac{\ell(\widetilde{Q}_i)}{\ell(Q_j^{\max})} \left(2\theta_j^{\max} + 2\theta_{j-1}^{\max}\right) + \frac{4\ell(\widetilde{Q}_i)}{\ell(Q_0^{\max})}\theta_0^{\max} \\ &\le 4B^{-1/2}\theta_i^{\max} + 2\,\frac{\ell(\widetilde{Q}_i)}{\ell(Q_i)}\, B^{-(i-1)/8s}\, \theta_0^{\max} \\ &\qquad + 4\sum_{j=1}^{i-1} \frac{\ell(\widetilde{Q}_i)}{\ell(Q_j^{\max})}\, B^{-(j-1)/8s}\, \theta_0^{\max} + \frac{4\ell(\widetilde{Q}_i)}{\ell(Q_0^{\max})}\theta_0^{\max}. \end{aligned}$$

If we plug this inequality into (5.21) and we assume B big enough, we deduce that

$$\begin{aligned} \theta_i^{\max} \le C \Bigg[&\frac{\ell(s(\widetilde{Q}_i))}{\ell(Q_i)}\, B^{-(i-1)/8s}\, \theta_0^{\max} \\ &+ \sum_{j=1}^{i-1} \frac{\ell(s(\widetilde{Q}_i))}{\ell(Q_j^{\max})}\, B^{-(j-1)/8s}\, \theta_0^{\max} + \frac{\ell(s(\widetilde{Q}_i))}{\ell(Q_0^{\max})}\, \theta_0^{\max} \Bigg], \end{aligned} \tag{5.22}$$

with C independent of B. As in (5.20), we have

$$\frac{\ell(s(\widetilde{Q}_i))}{\ell(Q_i)} \le B^{-1/2s},$$

and for $0 \leq j \leq i-1$,

$$\frac{\ell(s(\widetilde{Q}_i))}{\ell(Q_j^{\max})} \leq \frac{\ell(s(\widetilde{Q}_i))}{\ell(Q_i)} \cdots \frac{\ell(s(\widetilde{Q}_{j+1}))}{\ell(Q_{j+1})} \leq B^{(j-i)/2s}.$$

From the latter estimates and (5.22) we obtain

$$\begin{aligned}\theta_i^{\max} &\leq C \Bigg[B^{-1/2s}\, B^{-(i-1)/8s}\, \theta_0^{\max} \\ &\qquad + \sum_{j=1}^{i-1} B^{(j-i)/2s}\, B^{-(j-1)/8s}\, \theta_0^{\max} + B^{-i/2s}\, \theta_0^{\max} \Bigg] \\ &\leq C\, B^{-1/4s}\, B^{-i/8s}\, \theta_0^{\max},\end{aligned}$$

and so (5.15) holds if we assume B big enough. □

5.10. **Proof of Lemma 5.1.** From Lemmas 5.3, 5.7, and 5.10, we get

$$\begin{aligned}\sum_{j=0}^{N-1} \theta_j^2 &= \sum_k \sigma(I_k) \leq C \sum_{k:\, I_k \text{ good}} \sigma(I_k) \\ &= C \sum_{k:\, I_k \text{ long good}} \sigma(I_k) + C \sum_{k:\, I_k \text{ short good}} \sigma(I_k) \\ &\leq C \sum_{j=0}^{N-1} \|D_j(R\mu)\|^2 + C \sum_h \theta^{\max}(J_h)^2.\end{aligned}$$

By Lemmas 5.12 and 5.11,

$$\sum_{h:\, J_h} \theta^{\max}(J_h)^2 \lesssim \sum_{h:\, J_h \text{ standard}} \theta^{\max}(J_h)^2 \lesssim \sum_{j=0}^{N-1} \|D_j(R\mu)\|^2.$$

We are done. □

6. Open problems

In this section we discuss some open problems in connection with Riesz transforms and Wolff potentials.

1) Riesz transforms and rectifiability.

Let $E \subset \mathbb{R}^d$ be a compact set with $0 < \mathcal{H}^n(E) < \infty$, for some integer $0 < n < d$, and set $\mu = \mathcal{H}^n_{|E}$. If R^n_μ is bounded in $L^2(\mu)$, is then E n-rectifiable? Recall that E is called n-rectifiable if there exist Lipschitz mappings $g_i : \mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mu\Big(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} g_i(\mathbb{R}^n)\Big) = 0.$$

When $n = 1$, David and Léger [Lég99] answered the question in the affirmative, using the relationship between curvature and the Cauchy kernel. By [Vol03], when $n = d-1$ this question is equivalent to the following: is it true that $\kappa(E) = 0$ if and only if E is purely $(d-1)$-unrectifiable? (E is called purely $(d-1)$-unrectifiable if it does not contain any n-rectifiable subset F with $\mathcal{H}^{d-1}(F) > 0$).

A partial result was obtained in [Tol08], where it was shown that the existence of the principal values $\lim_{\varepsilon\to 0} R^n_\varepsilon \mu(x)$ for μ-a.e. $x \in \mathbb{R}^d$ implies E to be n-rectifiable. Under the additional assumption

$$\theta^n_{\mu,*}(x) := \liminf_{r\to 0} \frac{\mu(B(x,r))}{r^n} > 0 \qquad \mu\text{-a.e. on } \mathbb{R}^d, \tag{6.1}$$

this had been proved previously by Mattila and Preiss in [MP95]. Unfortunately, it is not known if the $L^2(\mu)$ boundedness of the Riesz transform R^n_μ implies the existence of principal values, and so the results in [Tol08] and [MP95] do not help to solve the problem above.

Another related result is given in [MP95, Theorem 5.5], where it is proved that if (6.1) holds and all the operators

$$Tf(x) = \int K(x-y)f(y)\,d\mu(x),$$

with kernel of the form $K(x) = \varphi(|x|)x/|x|^{n+1}$ satisfying $|\nabla^j K(x)| \leq \frac{C(j)}{|x|^{n+j}}$ for $j \geq 0$ are bounded in $L^2(\mu)$, then E is n-rectifiable.

A variant of this problem, posed by David and Semmes, consists in taking E Ahlfors-David regular and n-dimensional. That is,

$$\mathcal{H}^n(E \cap B(x,r)) \approx r^n \quad \text{for all } x \in E,\ 0 < r \leq \operatorname{diam}(E).$$

Again, set $\mu = \mathcal{H}^n_{|E}$. If R^n_μ is bounded in $L^2(\mu)$, is then E uniformly n-rectifiable? For the definition of uniform rectifiability, see [DS91] and [DS93] (for the reader's convenience let us say that, roughly speaking, uniform rectifiability is the same as rectifiability plus some quantitative estimates). For $n = 1$ the answer is true again, because of curvature. The result is from Mattila, Melnikov and Verdera [MMV96]. For $n > 1$, in [DS91] and [DS93] some partial answers are given. In particular, it is shown that if all the operators T with kernel K as above are bounded in $L^2(\mu)$, then E is uniformly rectifiable.

2) Calderón-Zygmund capacities and Wolff potentials of non integer dimension.

This problem was already mentioned in the Introduction: is it true that when for $0 < s < d$ non integer we have

$$\gamma_s(E) \approx \dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}(E) \tag{6.2}$$

with constants independent of E? Recall that this was shown to be true when $0 < s < 1$ by Mateu, Prat and Verdera [MPV05], while for the other values of s it is proved in [ENV08] that the estimate $\gamma_s(E) \gtrsim \dot{C}_{\frac{2}{3}(d-s),\frac{3}{2}}(E)$ holds.

The main obstacle to prove the opposite inequality is the following. It is not known if, for $s \notin \mathbb{Z}$, there are sets E with $0 < \mathcal{H}^s(E) < \infty$ such that the Riesz transform R^s_μ, with $\mu = \mathcal{H}^s_{|E}$, is bounded in $L^2(\mu)$. If (6.2) holds, then such sets do not exist. This is the case for $0 < s < 1$, as shown by Prat [Pra04] using the curvature method, and for other $s \notin \mathbb{Z}$ by Vihtila [Vih96] under the additional assumption that $\theta^s_{\mu,*}(x) > 0$ for μ-a.e. $x \in \mathbb{R}^d$, where $\theta^s_{\mu,*}(x)$ is defined in (6.1).

On the other hand, in [RdVT] it has been proved that, for $0 < s < d$ and $\mu = \mathcal{H}^s_{|E}$, with $0 < \mathcal{H}^s(E) < \infty$, the existence of the principal values $\lim_{\varepsilon\to 0} R^s_\varepsilon\mu(x)$ for μ-a.e. $x \in \mathbb{R}^d$ forces s to be integer. Notice that if one combines the results on principal values from [Tol08] mentioned above with the ones from [RdVT], then one gets:

Theorem. *For $0 < s \leq d$, let $E \subset \mathbb{R}^d$ be a set satisfying $0 < \mathcal{H}^s(E) < \infty$. The principal value*

$$\lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{s+1}} d\mathcal{H}^s_{|E}(y)$$

exists for $\mathcal{H}^s$-almost every $x \in E$ if and only if s is integer and E is s-rectifiable.

It is interesting to compare the last theorem with well known results in geometric measure theory due essentially to Marstrand [Mar64] and Preiss [Pre87]:

For $0 < s \leq m$, let $E \subset \mathbb{R}^m$ be a set satisfying $0 < \mathcal{H}^s(E) < \infty$. The density $\theta^s_{\mathcal{H}^s|E}(x)$ exists for $\mathcal{H}^s$-almost every $x \in E$ if and only if s is integer and E is s-rectifiable.

3) L^2 boundedness of Riesz transforms and square functions.

Given a non-increasing radial $\mathcal{C}^\infty$ function ψ such that $\chi_{B(0,1/2)} \leq \psi \leq \chi_{B(0,2)}$, for each $j \in \mathbb{Z}$, we set $\psi_j(z) := \psi(2^j z)$ and $\varphi_j := \psi_j - \psi_{j+1}$, so that each function φ_j is non-negative and supported in the annulus $A(0, 2^{-j-2}, 2^{-j+1})$, and moreover we have $\sum_{j\in\mathbb{Z}} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^d \setminus \{0\}$. For each $j \in Z$ we denote $K^s_j(x) = \varphi_j(x)\, x/|x|^{s+1}$ and

$$R^s_j\mu(x) = \int K^s_j(x-y)\, d\mu(y). \tag{6.3}$$

Notice that, at a formal level, we have $R\mu = \sum_{j\in\mathbb{Z}} R_j\mu$, and so

$$\|R^s\mu\|^2_{L^2(\mu)} = \sum_{j\in\mathbb{Z}} \|R^s_j\mu\|^2_{L^2(\mu)} + \sum_{j\neq k} \langle R^s_j\mu,\, R^s_k\mu\rangle.$$

Consider the square function

$$Q^s\mu(x) = \Big(\sum_{j\in\mathbb{Z}} |R^s_j\mu(x)|^2\Big)^{1/2},$$

and set $Q^s_\mu(f) = Q^s(f\,d\mu)$. Notice that

$$\|Q^s_\mu(f)\|^2_{L^2(\mu)} = \sum_{j\in\mathbb{Z}} \|R^s_j(f\,d\mu)\|^2_{L^2(\mu)}.$$

One should view $Q^s_\mu(f)$ as a square function associated to the Riesz transform $R^s_\mu(f)$.

When s is integer and $E \subset \mathbb{R}^d$ uniformly rectifiable, with $\mu = \mathcal{H}^s_{|E}$, then Q^s_μ is bounded in $L^2(\mu)$. Moreover, the converse is also true: if E is Alhfors-David regular, the $L^2(\mu)$ boundedness of Q_μ implies that E is uniformly rectifiable (at least for an appropriate choice of the function ψ above), as shown in [Tol09]. In the non Ahlfors-David regular case it is also true that the boundedness of Q_μ implies the rectifiability of E [MV09a].

On the other hand, given $E \subset \mathbb{R}^d$ such that $0 < \mathcal{H}^s(E) < \infty$, $0 < s < d$, and $\mu = \mathcal{H}^s_{|E}$, if Q_μ is bounded in $L^2(\mu)$, then $s \in \mathbb{Z}$. This follows easily from the results of [RdVT], as shown in [MV09b]. Thus the following question arises naturally:

Let $0 < s < d$ and let μ be a Radon measure on $\mathbb{R}^d$ with no atoms. Is it true that R^s_μ is bounded in $L^2(\mu)$ if and only if Q^s_μ is bounded in $L^2(\mu)$?

As remarked above, solving this question would be a fundamental contribution for the solution of the problems explained above in 1) and 2).

4) Bilipschitz and affine invariance, and other problems.

Let μ be a Radon measure on $\mathbb{C}$ such that the Cauchy transform $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$. Recall that

$$\mathcal{C}_\mu f(z) = \int \frac{1}{z-\xi}\, f(\xi)\, d\mu(\xi).$$

In [Tol05] it has been shown that if $\varphi : \mathbb{C} \to \mathbb{C}$ is a bilipschitz map and $\sigma = \varphi\#\mu$ is the image measure of μ, then $\mathcal{C}_\sigma$ is bounded in $L^2(\sigma)$. The analogous problem for the $(d-1)$-dimensional Riesz transform R^{d-1}_μ in $\mathbb{R}^d$ is open, and it seems that before trying to solve it, one should understand better the relationship between the L^2 boundedness of the Riesz transforms and rectifiability [i.e. one should first solve the questions in 1)], since this is a basic ingredient in the proof of the analogous result for the Cauchy transform in [Tol05]. However, in the case $d > 2$, the problem is open even when φ is an affine map. For instance, let

$$\varphi(x_1, x_2, x_3, \dots, x_d) = (2x_1, x_2, x_3, \dots, x_d).$$

If R^{d-1}_μ is bounded in $L^2(\mu)$ and we set $\sigma = \varphi\#\sigma$, is then R^{d-1}_σ bounded in $L^2(\sigma)$?

A similar question in terms of the capacity κ is the following. Is it true that for any compact set $E \subset \mathbb{R}^d$, $\kappa(E) \approx \kappa(\varphi(E))$? Analogous questions can be posed for the other capacities γ_s and the Riesz transforms of codimension different from 1.

Let us discuss another problem whose solution may help to understand the relationship between the L^2 boundedness of Riesz transforms and geometry. Let $R^s_{(j)}$, $0 \leq j \leq d$, denote the scalar components of the (vectorial) Riesz transform R^s. Let μ be a Radon measure on $\mathbb{R}^d$ such that $\mu(B(x,r)) \leq C\, r^s$ for all $x \in \mathbb{R}^d$, $r > 0$. Suppose that $d-1$ components of R^s_μ, say $R^s_{(1),\mu}, \ldots, R^s_{(d-1),\mu}$ are bounded in $L^2(\mu)$. Is then R^s_μ bounded in $L^2(\mu)$? When $s = 1$ the answer is yes, because of the curvature method. However, for other values of s, the problem is open again.

An analogous question can be posed in terms of the capacities associated to these kernels. That is, for a compact set $E \subset \mathbb{R}^d$, let $\widetilde{\gamma}_s(E) = \sup |\nu(E)|$, where the supremum is taken over signed measures (or distributions) supported on E such that $\|R^s_{(j)}\nu\|_{L^\infty(\mathbb{R}^d)} \leq 1$ for $0 \leq j \leq d-1$ and $|\nu(B(x,r))| \leq r^s$ for all $x \in \mathbb{R}^d$, $r > 0$ (in case ν is a distribution the latter condition should be reformulated appropriately). Is $\widetilde{\gamma}_s(E) \approx \gamma_s(E)$? The answer is affirmative for $s = 1$ and negative for $0 < s < 1$ ([Pra09]), while it is unknown when $s > 1$.

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