INVERSION OF ANALYTIC CHARACTERISTIC FUNCTIONS AND INFINITE CONVOLUTIONS OF EXPONENTIAL AND LAPLACE DENSITIES

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ABSTRACT. This paper shows that certain quotients of entire functions are characteristic functions. Under some conditions, we provide expressions for the densities of such characteristic functions which turn out to be generalized Dirichlet series which in turn can be expressed as an infinite linear combination of exponential or Laplace densities. We apply these results to several examples.

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1. INTRODUCTION

There are many cases in the literature where a characteristic function $\varphi(t)$ of a probability distribution can be written as $\varphi(t) = 1/g(it)$ for $t \in \mathbb{R}$, where $g(z)$ is an entire function of the complex variable z. A couple of important examples are the square of a Kolmogorov law and the Lévy area.

The characteristic function of the square of a Kolmogorov law is given by $\varphi_1(t) = \sqrt{2it}/\sin(\sqrt{2it})$, thus we are setting $g_1(z) = \sin(\sqrt{2z})/\sqrt{2z}$. This example was studied by Dugué [6], who determined that its distribution function is the Jacobi theta function

$$F_1(x) = \vartheta_4(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-\pi^2 k^2 x/2}\,, \qquad x > 0\,.$$

The characteristic function of the second example, given by Lévy in [14], is $\varphi_2(t) = \mathrm{sech}(t)$ and thus $g_2(z) = \cos(z)$. The density was computed by Lévy and given by

$$f_2(x) = \frac{1}{2}\mathrm{sech}\left(\frac{\pi}{2}x\right) = \frac{e^{-\pi|x|/2}}{1+e^{-\pi|x|}} = \sum_{k=1}^{\infty}(-1)^{k+1}e^{-\frac{(2k-1)\pi}{2}|x|}\,, \qquad x \neq 0\,,$$

where the last equality holds due to Newton's binomial series Theorem.

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These two examples share that regardless of the behaviour of the entire function, its inverse is a characteristic function. In fact, it is well known that for an entire function $g(z)$ of order $\rho < 2$ which has only real roots and $g(0) = 1$, the inverse $\varphi(t) = 1/g(it)$ is a characteristic function (see Lukacs [15, pp. 88 and 212] for an equivalent formulation). The result follows from Hadamard factorisation Theorem (see, for example, Levin [13, p. 26]) which states that an entire function of order $\rho < 2$ can be written as

$$g(z) = e^{cz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n} ,$$

where $\{a_n,\, n \geq 1\}$ are the zeros of $g(z)$ (c must be real so that the characteristic function assertion is true). Note that such factorisation of $g(z)$ induces a factorisation of $\varphi(t)$, and each factor is of type

$$\frac{1}{1 - \frac{it}{a_n}} e^{-it/a_n}$$

which is the characteristic function of a translated positive or negative exponential law. If $\rho < 1$, then we can leave aside the exponential part inside the infinite product for a canonical representation and $c = 0$. In any case $\varphi(t)$ is factorised as a convergent product of characteristic functions, and hence it is a characteristic function due to Lévy's continuity Theorem.

The two mentioned examples differ in the density function. Notice that the density deduced from $F_1(x)$ can be considered as an infinite linear combination of exponential densities, while $f_2(x)$ seems to be a mix of Laplace densities. The difference between both cases lies in the entire function $g(z)$. On one hand, $g_1(z)$ has order $1/2$ and all its zeros are simple and positive; as opposite to the second example, where $g_2(z)$ has order 1 and its zeros are simple but symmetric with respect to the origin.

The aim of this paper is to generalise the previous result. It will be shown that when $g(z)$ and $h(z)$ are both entire functions of order $\rho,\ \rho' \in (0, 2)$ respectively, satisfying $g(0) = h(0) = 1$ and with a further condition over their zeros then $\varphi(t) = h(it)/g(it)$ is a characteristic function. The main part of the paper is devoted to prove that when $\rho,\ \rho' \in (0, 1)$, the zeroes of $h(z)$ and $g(z)$ are simple and positive, and some additional hyphotesis then the law corresponding to $\varphi(t)$ has a density that can be written as a sum of exponential type densities

$$f(x) = -\sum_{n=1}^{\infty} \frac{h(a_n)}{g'(a_n)}\, e^{-a_n x},\ x > 0.$$

This series is a generalized Dirichlet series (see Mandelbrojt [16]) which has very good properties; in particular, we prove that it is uniformly convergent on every compact subset of $(0, \infty)$; the cumulative distribution function turns out to be also a generalized Dirichlet series, and since such series converge very fast, it is easy to simulate a random variable with such law.

When the zeroes are simple and symmetric, positive and negative, the existence of a density can be also proved for $\rho = 1$ and $h(z) \equiv 1$. Then the density can be written as a series of Laplace type densities. This case is important because it covers some elements of the double Wiener chaos (see Janson [8]) as the double Itô-Wiener integrals where the kernel has symmetric zeroes with multiplicity 2. In particular the Lévy area distribution.

We study some examples. In addition to the square of a Kolmogorov law and the Lévy area, we consider the law of the first hitting time of a Bessel process, whose characteristic function is expressed as a quotient of Bessel functions (Kent [12] and Borodin and Salminen [3]). We also show how this technique can be used to invert some Laplace transforms. In our last example we study a particular case of the Heston model used in mathematical finance and we prove that the general theory developed in the first part can be applied to it; such study was the starting point of this paper.

To summarize, the paper is two fold. On one side, it shows a way to construct a rich family of characteristic function. On the other side, given a characteristic function that can be identified to belongs to that family, our results give a procedure to invert that characteristic function.

2. Construction of characteristic functions

The first part of the paper is devoted to identify a rich family of characteristic functions which can be constructed by quotients of entire functions.

From now on we will consider that a *strictly increasing sequence of positive numbers*, $\{a_n n \geq 1\}$, is a sequence satisfying $0 < a_1 < a_2 < \cdots$ and such that $\lim_{n\to\infty} a_n = \infty$.

Proposition 2.1. *Consider two entire functions $g(z)$ and $h(z)$ of order lying in $(0,1)$ such that $g(0) = h(0) = 1$, with simple positive roots given by the strictly increasing sequences $\{a_n,\, n \geq 1\}$ and $\{b_n,\, n \geq 1\}$ respectively. Assume that $a_n < b_n$ for all n. Then $\varphi(t) = h(it)/g(it)$ is a characteristic function of a probability measure in $[0,\infty)$.*

Proof. By Hadamard's factorisation Theorem (see, for example, Levin [13, p. 26]),

$$g(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) \qquad \text{and} \qquad h(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{b_n}\right) .$$

We conclude from the convergence of both products that

$$\varphi(t) = \frac{h(it)}{g(it)} = \prod_{n=1}^{\infty}\frac{1 - it/b_n}{1 - it/a_n} = \prod_{n=1}^{\infty}\left(\frac{a_n}{b_n} + \left(1 - \frac{a_n}{b_n}\right)\left(1 - \frac{it}{a_n}\right)^{-1}\right) . \tag{1}$$

Since $0 < a_n/b_n < 1$ it follows that each factor of the above product is the characteristic function of the probability measure

$$\frac{a_n}{b_n}\,\delta_0 + \left(1 - \frac{a_n}{b_n}\right)\mathcal{E}\mathrm{xp}(a_n) ,$$

where δ_0 is a Dirac measure at 0 and $\mathcal{E}\mathrm{xp}(a_n)$ is an exponential law with parameter a_n. The result is a consequence of Lévy's continuity Theorem. ■

Remarks 2.2.

1) The condition $g(0) = h(0) = 1$ is just a way to ease the notation, in fact the same result would be true if we let $g(0) = h(0) \neq 0$.

2) By means of standard manipulations, Proposition 2.1 holds true for $h \equiv 1$. In this case each factor of $\varphi(t)$ is the characteristic function of the probability law $\mathcal{E}\mathrm{xp}(a_n)$.

3) It is easy to show that the restriction of simple roots in Proposition 2.1 can be relaxed if we let a_n and b_n have the same multiplicity for all n.

Proposition 2.3. *Consider two even entire functions $g(z)$ and $h(z)$ of order lying in $(0, 2)$ such that $g(0) = h(0) = 1$. Let $\{a_n, \, n \geq 1\}$ and $\{b_n, \, n \geq 1\}$ be two strictly increasing sequence of positive numbers; and let $\{\pm a_n, \, n \geq 1\}$ and $\{\pm b_n, \, n \geq 1\}$ be the simple roots of $g(z)$ and $h(z)$ respectively; assume that $a_n < b_n$ for all n. Then $\varphi(t) = h(it)/g(it)$ is a characteristic function of a probability measure in $\mathbb{R}$.*

Proof. By Hadamard's factorisation Theorem

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right) \qquad \text{and} \qquad h(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{b_n^2}\right)$$

due to the symmetry of the entire functions. The rest of the proof follows the same argument as the derivation of Proposition 2.1. In this setting each factor of $\varphi(t) = h(it)/g(it)$ is the characteristic function of of the probability measure

$$\frac{a_n^2}{b_n^2}\, \delta_0 + \left(1 - \frac{a_n^2}{b_n^2}\right) \mathcal{L}\mathrm{aplace}(a_n) \, ,$$

where $\mathcal{L}\mathrm{aplace}(a_n)$ is a Laplace law with parameter a_n. ■

As pointed out in Remark 2.2 there is a straight forward generalisation of Proposition 2.3 when $h \equiv 1$.

The writting of the statements of Propositions 2.1 and 2.3 are related to the main results of the following sections. The reader will notice that it is possible to state more general results using the same ideas and hence with similar derivations. We may state Proposition 2.3 without requesting that the functions were symmetric but with their zeros given by the sequences $\{a_n, \, n \geq 1\}$ and $\{b_n, \, n \geq 1\}$ described in the proposition. In such case

$$\varphi(t) = \frac{h(it)}{g(it)} = e^{c+dit} \prod_{n=1}^{\infty} \left(\frac{a_n}{b_n} + \left(1 - \frac{a_n}{b_n}\right) \left(1 - \frac{it}{a_n}\right)^{-1} \right) e^{it\left(\frac{1}{b_n} - \frac{1}{a_n}\right)}$$

and each factor is the characteristic function of the probability measure

$$\left(\frac{a_n}{b_n}\, \delta_0 + \left(1 - \frac{a_n}{b_n}\right) \mathcal{E}\mathrm{xp}(a_n) \right) \delta_{\left(\frac{1}{b_n} - \frac{1}{a_n}\right)} \, .$$

For example, we can set $\varphi(t) = \mathrm{Ai}(uit)/\mathrm{Ai}(vit)$, where $0 < u < v < \infty$ and $\mathrm{Ai}(z)$ is the Airy's function. The Airy function is an entire function of order $3/2$ and its zeros are real and negative (see Katori and Tanemura [11]). In this case, although we can state that $\varphi(t)$ is a characteristic function we will not be able to write explicitly its density function.

2.1. **Finite convolution of exponential and Laplace densities.** The next two lemmas will prove useful for determining the density of the characteristic functions in Proposition 2.1 and 2.3; we will consider a finite product approximation of $\varphi(t)$ and determine its density so we can take the limit afterwards. This section and the next one will develop the first steps of the procedure.

Lemma 2.5 is well known in the literature (see, for example, problem 12 of chapter 1 of Feller [7]). For the sake of completeness we will reproduce the derivation here, and hence it will be written in a way that best suits our objective. The Euler-Jacobi formulae is an auxiliary proposition that makes the proof easier and short.

Let us recall some standard notations. Given two probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, P_1 and P_2, we denote by $P_1 \star P_2$ its convolution

$$P_1 \star P_2(B) := \int_{\mathbb{R}} P_1(B-y)P_2(dy) \ ,$$

where $B \in \mathcal{B}(\mathbb{R})$ and $B - y = \{x - y,\, x \in B\}$. The characteristic function of $P_1 \star P_2$ is the product of the characteristic functions of P_1 and P_2. Moreover, if P_1 and P_2 are absolutely continuous with density f_1 and f_2 respectively, then the density of $P_1 \star P_2$ is given by the convolution of f_1 and f_2

$$f_1 \star f_2(x) := \int_{-\infty}^{+\infty} f_1(y) f_2(x-y)\, dy.$$

The convolution of $P_1 \star \cdots \star P_n$ (resp. $f_1 \star \cdots \star f_n$) is denoted by $\star_{j=1}^n P_j$ (resp. $\star_{j=1}^n f_j$).

Proposition 2.4 (1-dimensional Euler-Jacobi formulae). *Let Q be a polynomial from $\mathbb{C}$ to $\mathbb{C}$ of degree n with simple zeros $w_1, w_2, \ldots, w_n$. Then for any complex polynomial R such that deg$(R) <$ deg$(Q') =$ deg$(Q) - 1$,*

$$\sum_{k=1}^n \frac{R(w_k)}{Q'(w_k)} = 0 \ .$$

Lemma 2.5. *Fix $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$ and define the couple $A(n) := \prod_{i=1}^n \lambda_i$ and $B(k,n) := \prod_{\substack{i=1 \\ i \neq k}}^n (\lambda_k - \lambda_i)$. Then $\star_{j=1}^n \mathcal{E}\mathrm{xp}(\lambda_j)$ has density given by*

$$f_n(x) = (-1)^{n+1} A(n) \sum_{i=1}^n \frac{e^{-\lambda_i x}}{B(i,n)} \qquad x \geq 0 \ .$$

Proof. Consider $Q(z) = \prod_{j=1}^n (z - \lambda_j)$ and $R(z) \equiv 1$ in Proposition 2.4 to notice that $\sum_{i=1}^n B(i,n)^{-1} = 0$. Now we proceed by induction. Obviously $f_1(x) = \lambda_1 e^{-\lambda_1 x}$. Assuming that the formula holds for n it follows that the density of $\star_{j=1}^{n+1} \mathcal{E}\mathrm{xp}(\lambda_j) =$

$\star_{j=1}^{n}\mathcal{E}\text{xp}(\lambda_j)\star\mathcal{E}\text{xp}(\lambda_{n+1})$ is given by

$$\begin{aligned}\int_0^x & f_n(y)\lambda_{n+1}e^{-\lambda_{n+1}(x-y)}dy \\ &= (-1)^{n+1}A(n+1)e^{-\lambda_{n+1}x}\sum_{i=1}^{n}\frac{1}{B(i,n)}\frac{(e^{(\lambda_{n+1}-\lambda_i)x}-1)}{(\lambda_{n+1}-\lambda_i)} \\ &= (-1)^{n+1}A(n+1)\left(\sum_{i=1}^{n}\frac{-e^{-\lambda_i x}}{B(i,n+1)}+\frac{-e^{-\lambda_{n+1}x}}{B(n+1,n+1)}\right) \\ &= f_{n+1}(x)\end{aligned}$$

as required. ■

The proof of the following result uses an interesting property given by Bondesson (see [2]) in the context of generalized gamma convolutions that, in our setup, states that if Y is a non–negative random variable with moment generating function $M_Y(u)$, for u in a neighborhood of 0, and T is a centered normal random variable with variance 2, independent of Y, then the random variable $X:=\sqrt{Y}\,T$ has moment generating function $M_X(u)=M_Y(u^2)$. The proof is completed in the line

$$M_X(u)=\mathbb{E}\big[e^{u\sqrt{Y}\,T}\big]=\mathbb{E}\Big[\mathbb{E}\big[e^{u\sqrt{Y}\,T}/Y\big]\Big]=\mathbb{E}\big[e^{u^2\,Y}\big]=M_Y(u^2)\ .$$

In fact, next result is a consequence of the previous lemma and it will be useful to derive the density of the characteristic function of Proposition 2.3.

Lemma 2.6. *Fix $0<\lambda_1<\lambda_2<\ldots<\lambda_n$ and define $E(k,n)=\prod_{\substack{i=1\\ i\neq k}}^{n}(\lambda_k^2-\lambda_i^2)$. Then $\star_{j=1}^{n}\mathcal{L}\text{aplace}(\lambda_j)$ has density given by*

$$f_n(x)=\frac{(-1)^{n+1}}{2}A^2(n)\sum_{i=1}^{n}\frac{e^{-\lambda_i|x|}}{\lambda_i E(i,n)}\qquad x\in\mathbb{R}\ .$$

Proof. Lets consider the characteristic function

$$\tilde{\varphi}(t)=\prod_{j=1}^{n}\left(1-\frac{it}{\lambda_j^2}\right)^{-1}\ ,$$

which corresponds to a random variable, Y, that is the sum of n independent exponential random variables with parameters $\{\lambda_j^2,\ 1\leq j\leq n\}$. By the previous lemma, the density of Y is

$$f_Y(y)=(-1)^{n+1}A^2(n)\sum_{i=1}^{n}\frac{e^{-\lambda_i^2 y}}{E(i,n)}\qquad y\geq 0\ .$$

Let T be a centered normal random variable with variance 2, hence the random variable $X:=\sqrt{Y}T$ has characteristic function

$$\varphi(t)=\prod_{j=1}^{n}\left(1+\frac{t^2}{\lambda_j^2}\right)^{-1}\ ,$$

which corresponds to the sum of n independent Laplace random variables with parameters $\{\lambda_j,\ 1 \le j \le n\}$. In order to find the density of X, consider the pair (X,T) and compute its marginal density by means of a change of variables to the pair (Y,T). The Jacobian determinant is $\frac{2|X|}{T^2}$, which means that the change of variables is an almost everywhere diffeomorphism of the plane. Fix $x \in \mathbb{R}$, then

$$\begin{aligned}
f_X(x) &= \mathbf{1}_{x\ge 0}\int_0^\infty f_{(X,T)}(x,t)dt + \mathbf{1}_{x<0}\int_{-\infty}^0 f_{(X,T)}(x,t)dt \\
&= \mathbf{1}_{x\ge 0}\int_0^\infty f_{(Y,T)}\left(\frac{x^2}{t^2},t\right)\frac{2|x|}{t^2}dt + \mathbf{1}_{x<0}\int_{-\infty}^0 f_{(Y,T)}\left(\frac{x^2}{t^2},t\right)\frac{2|x|}{t^2}dt \\
&= \int_0^\infty f_Y\left(\frac{x^2}{t^2}\right) f_T(t)\frac{2|x|}{t^2}dt \\
&= \int_0^\infty \sum_{i=1}^n \frac{(-1)^{n+1}A^2(n)}{E(i,n)} e^{-\lambda_i^2x^2/t^2}e^{-t^2/4}\frac{1}{2\sqrt{\pi}}\frac{2|x|}{t^2}dt \\
&= \sum_{i=1}^n \frac{(-1)^{n+1}A^2(n)|x|}{\sqrt{\pi}E(i,n)}\int_0^\infty \frac{e^{-\lambda_i^2x^2/t^2}e^{-t^2/4}}{t^2}dt \\
&= \frac{(-1)^{n+1}}{2}A^2(n)\sum_{i=1}^n \frac{e^{-\lambda_i|x|}}{\lambda_i E(i,n)}
\end{aligned}$$

as required. ■

2.2. **Finite density approximation of the characteristic function.** We derive here the density function for a finite approximation of the characteristic functions of Proposition 2.1 and 2.3.

Lemma 2.7. *Let $0 < a_1 < a_2 < \cdots < a_n$ and $0 < b_1 < b_2 < \cdots < b_n$ such that $a_i < b_i$ for $1 \le i \le n$. Write either*

(a) $g_n(z) = \prod_{i=1}^n (1 - z/a_i)$ and $h_n(z) = \prod_{i=1}^n (1 - z/b_i)$, or
(b) $g_n(z) = \prod_{i=1}^n (1 - z^2/a_i^2)$ and $h_n(z) = \prod_{i=1}^n (1 - z^2/b_i^2)$.

Let $\varphi_n(t)$ be the characteristic function $\varphi_n(t) = h_n(it)/g_n(it)$. Then the corresponding law of $\varphi_n(t)$ is

$$\left(\prod_{i=1}^n \frac{a_i}{b_i}\right)\delta_0 + \mu_n\ ,$$

where μ_n is a finite measure in $(0,\infty)$ – case (a) – or in $\mathbb{R}\setminus\{0\}$ – case (b) –, with density given by

$$\text{(a)}\quad \frac{d\mu_n}{dx} = -\sum_{i=1}^n \frac{h_n(a_i)}{g_n'(a_i)}e^{-a_jx}\quad x>0 \quad\text{or (b)}\quad \frac{d\mu_n}{dx} = -\sum_{i=1}^n \frac{h_n(a_i)}{g_n'(a_i)}e^{-a_j|x|}\quad x\neq 0\ .$$

Proof. We will prove the result for the case (a); the case (b) is similar. Since the convolution of two finite measures is absolutely continuous when one of them is, we abuse of the notation and write

$$\left(\frac{a_j}{b_j}\,\delta_0 + \left(1-\frac{a_j}{b_j}\right)\mathcal{E}\mathrm{xp}(a_j)\right)(x)$$

to denote the probability *density* function of the measure which is a mixture of a delta measure and an exponential distribution. Let us denote by I_n^i the set of all subsets of $\{1,2,\ldots,n\}$ of cardinal $1 \le i \le n$. Then

$$\begin{aligned}
&\star_{j=1}^{n}\left(\frac{a_j}{b_j}\,\delta_0 + \left(1-\frac{a_j}{b_j}\right)\mathcal{E}\mathrm{xp}(a_j)\right)(x) = \\
&= \left(\prod_{i=1}^{n}\frac{a_i}{b_i}\right)\delta_0(x) + \sum_{i=1}^{n}\sum_{J\in I_n^{n-i}}\left[\prod_{k\in J}\frac{a_k}{b_k}\right]\left[\prod_{k\in J^c}\left(1-\frac{a_k}{b_k}\right)\right] \\
&\qquad\qquad \left[\sum_{k\in J^c}\frac{a_k}{\prod_{r\in J^c\setminus\{k\}}\left(1-\frac{a_k}{a_r}\right)}e^{-a_k x}\right] \\
&= \left(\prod_{i=1}^{n}\frac{a_i}{b_i}\right)\delta_0(x) + \\
&\qquad \sum_{k=1}^{n}\frac{e^{-a_k x}a_k}{\prod_{\substack{r=1\\ r\neq k}}^{n}\left(1-\frac{a_k}{a_r}\right)}\sum_{i=1}^{n}\sum_{\substack{J\in I_n^{n-i}\\ k\notin J}}\left[\prod_{r\in J}\frac{a_r}{b_r}\left(1-\frac{a_k}{a_r}\right)\right]\left[\prod_{r\in J^c}\left(1-\frac{a_r}{b_r}\right)\right] \\
&= \left(\prod_{i=1}^{n}\frac{a_i}{b_i}\right)\delta_0(x) + \sum_{k=1}^{n}e^{-a_k x}\frac{a_k}{\prod_{\substack{r=1\\ r\neq k}}^{n}\left(1-\frac{a_k}{a_r}\right)}\prod_{r=1}^{n}\left(\left(1-\frac{a_r}{b_r}\right)+\frac{a_r}{b_r}\left(1-\frac{a_k}{a_r}\right)\right) \\
&= \left(\prod_{i=1}^{n}\frac{a_i}{b_i}\right)\delta_0(x) + \sum_{k=1}^{n}e^{-a_k x}\frac{\prod_{r=1}^{n}\left(1-\frac{a_k}{b_r}\right)}{\frac{1}{a_k}\prod_{\substack{r=1\\ r\neq k}}^{n}\left(1-\frac{a_k}{a_r}\right)},
\end{aligned}$$

where we have used Lemma 2.5. ■

Remark 2.8. If $h \equiv 1$ in Lemma 2.7 then the law of $\varphi_n(t)$ is just μ_n.

Lemma 2.7 shows that the density has an atom at the origin. This is a handicap towards applying the conventional limit theorems to the expressions of the densities therein. Section 4.2 will give further assumptions and criteria to overcome this difficulty. Finally, just notice that the key point to this end is to determine the behaviour of

$$\lim_{n\to\infty}\frac{d\mu_n}{dx}\,.$$

3. FUNDAMENTAL LEMMAS

The following lemmas are essential for the purposes of the paper, they give the technical results to allow us to use the standard convergence results on the expressions of Lemma 2.7. Before that, lets remark a key property of an entire function: the order of an entire function is greater or equal to the exponent of convergence of its zeros (see Titchmarsh [19, p. 251]). That means, let f be an entire function of order ρ and $\{a_n,\, n \geq 1\}$ its zeros, then

$$\rho \geq \inf\{\alpha > 0 \,:\, \sum_{n=1}^{\infty} |a_n|^{-\alpha} < \infty\}\,.$$

In particular, $\sum_{n=1}^{\infty} |a_n|^{-\beta} < \infty$ for $\beta > \rho$. The equality holds if f has a representation as a canonical product, in fact Proposition 2.1 and 2.3 give such representation for h and g.

Lemma 3.1. *Let $g(z)$ be an entire function of order $\rho \in (0,1)$ such that $g(0) = 1$; and $\{a_n,\, n \geq 1\}$ be a strictly increasing sequence of positive roots. Let $h(z)$ be another function that either*
(a) $h(z) \equiv 1$ or
(b) $h(z)$ is an entire function of order $\rho' \in (0,1)$ such that $h(a_n) \neq 0$ for all n.
Then, for every $x > 0$, the series $\sum_{n\geq 1} \frac{h(a_n)}{g'(a_n)}\, e^{-a_n x}$ is convergent.

Proof. We will prove the lemma for the case (b); the case (a) is similar and indeed easier. Consider the closed contour $D(R)$ in Figure 1, where $R > 0$, $\theta_0 \in (0, \pi/2)$ and $R \neq a_j$ for all j. Fix $x > 0$. The function $\frac{h(z)}{g(z)}\, e^{-xz}$ is analytic in the region bounded

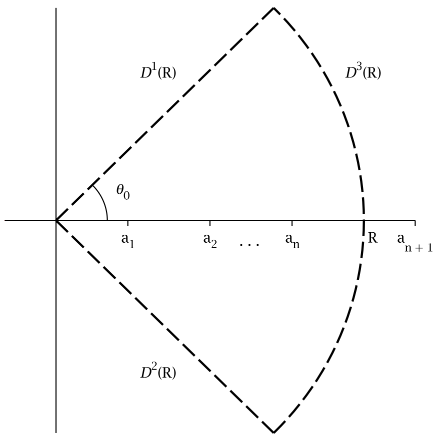

Figure 1. Dashed line is the contour $D(R)$ of integration.

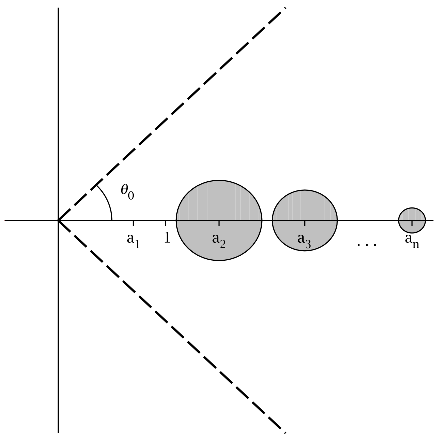

Figure 2. The function $|g(z)|$ is bounded below in the region excluded from the gray circles.

by $D(R)$, except at the zeros a_j that lie in the interior of $D(R)$. The Residue Theorem allows us to write

$$\frac{1}{2\pi i}\oint_{D(R)}\frac{h(z)}{g(z)}\,e^{-xz}\,dz=\sum_{a_j<R}\text{Res}(a_j)=\sum_{a_j<R}\frac{h(a_j)}{g'(a_j)}\,e^{-a_jx}\,.$$

Our objective is to prove that there is an increasing sequence, $\{R_n,\, n\geq 1\}$, towards infinity such that the limit

$$\lim_n \oint_{D(R_n)}\frac{h(z)}{g(z)}\,e^{-xz}\,dz \tag{2}$$

exists and is finite. The key point of the proof is that an entire function of order $\rho<1$ can be bounded below out small circles around its zeros. Since the zeros of $g(z)$ are located on the positive real axis, convenient bounds of $1/|g(z)|$ can be obtained. Specifically, the proof is based on:

1. For every $\varepsilon>0$, $g(z)=\mathcal{O}(e^{\epsilon|z|})$ and $h(z)=\mathcal{O}(e^{\epsilon|z|})$. This is due to $\sum_n |a_n|^{-1}<\infty$, and hence we can apply issue 15 from Titchmarsh [19, p. 286].

2. As a consequence of the previous statement, for every $\varepsilon>0$ there is an increasing sequence $\{R_n, n\geq 1\}$ such that $\lim_n R_n=\infty$ and $|g(R_ne^{i\theta})|>\exp\{-\varepsilon R_n\}$ uniformly on $\theta\in[0,2\pi]$. See Titchmarsh [19, p. 276, it. 8.75].

3. For every $\varepsilon>0$, there is $\theta_0\in(0,\pi/2)$ and $r_0>0$ such that

$$|g(re^{\pm i\theta_0})|>\exp\{-\varepsilon r\}\ ,\ \text{for } r\geq r_0.$$

This is deduced from Titchmarsh [19, p. 273, it. 8.71]. Take straight lines through the origin with angles θ_0 and $-\theta_0$ that do not intersect with any of the circles with centres a_j and radius $1/a_j$ (for $a_j>1$), see Figure 2. Titchmarsh proves that for all $\varepsilon'>0$ exists r_0' such that if $r\geq r_0'$ we have $|g(z)|>\exp\{-r^{\rho+\varepsilon'}\}$ in the region excluded from these discs. Take ε' satisfying $\rho+\varepsilon'<1$ and thus for sufficiently large r we have $r^{\rho+\varepsilon'}<\varepsilon r$. Choose r_0 a bit larger than r_0' to obtain the claimed inequality.

Now we are ready to complete the proof. Fix $\varepsilon>0$ such that $2\varepsilon<x\cos\theta_0$, where θ_0 is the angle depending on ε such that property **3** is fulfilled. Denote by $D^1(R_n)$ and $D^2(R_n)$ the lower and upper straight segment of $D(R_n)$ while $D^3(R_n)$ stands for the arch. All paths are considered with the corresponding orientation. Then we chop the integral into three parts :

$$\oint_{D(R_n)}\frac{h(z)}{g(z)}\,e^{-xz}\,dz=\int_{D^1(R_n)}+\int_{D^2(R_n)}+\int_{D^3(R_n)}.$$

We first consider the integral over $D^3(R_n)$. Due to points **1** and **2**, we can bound the module of this integral in the following way:

$$\left|\int_{D^3(R_n)} \frac{h(z)}{g(z)}\, e^{-xz}\, dz\right| \leq R_n \int_{-\theta_0}^{\theta_0} \left|\frac{h(R_n e^{i\theta})}{g(R_n e^{i\theta})}\right| e^{-xR_n\cos\theta}\, d\theta$$
$$\leq R_n e^{-xR_n\cos\theta_0} \int_{-\theta_0}^{\theta_0} \left|\frac{h(R_n e^{i\theta})}{g(R_n e^{i\theta})}\right| d\theta$$
$$\leq K\theta_0 R_n e^{-R_n(x\cos\theta_0 - 2\varepsilon)},$$

for R_n large enough. This goes to zero as $n \to \infty$.

The integral over $D^1(R_n)$ can be parametrised as

$$\int_{D^1(R_n)} \frac{h(z)}{g(z)}\, e^{-xz}\, dz = e^{-i\theta_0} \int_0^{R_n} \frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\, e^{-xre^{-i\theta_0}} dr\ .$$

We claim that

$$\int_0^\infty \left|\frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\, e^{-xre^{-i\theta_0}}\right| dr < \infty. \tag{3}$$

According to observations **1** and **3**, there is $r_0 > 0$ depending on ε such that

$$\int_{r_0}^\infty \left|\frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\, e^{-xre^{-i\theta_0}}\right| dr = \int_{r_0}^\infty \left|\frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\right| e^{-xr\cos\theta_0}\, dr$$
$$\leq K \int_{r_0}^\infty e^{-r(x\cos\theta_0 - 2\varepsilon)}\, dr < \infty.$$

Since $g(re^{-i\theta_0}) \neq 0$ for $r \geq 0$, it turns out that the function $\left|\frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\right| e^{-xr\cos\theta_0}$ is continuous on $[0, r_0]$ and hence

$$\int_0^{r_0} \left|\frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\, e^{-xre^{-i\theta_0}}\right| dr < \infty.$$

Adding up the two upper bounds we end up with (3). For $D^2(R_n)$ the computations are equivalent and (2) exists and is finite. ■

Lemma 3.2. *Under assumptions of Lemma 3.1, the series*

$$\sum_{n\geq 1} \frac{h(a_n)}{g'(a_n)}\, e^{-a_n x} \tag{4}$$

converges absolutely for $x > 0$, and the convergence is uniform on any compact subset of $(0, \infty)$.

Proof. The above expression is a generalised Dirichlet series. As a consequence, if (4) is convergent for some x_0, then does so for all $x > x_0$ and the convergence is uniform on every compact subset of the half-line (see Mandelbrojt [16, p. 9]). Therefore the second part of the lemma follows from Lemma 3.1. Each general Dirichlet series has associated an abscissa σ_c of convergence and an abscissa σ_a of absolute convergence. In general these two values are not equal but its distance is bounded (see Mandelbrojt [16, p. 11]) by

$$0 \leq \sigma_a - \sigma_c \leq \limsup_{n\to\infty} \frac{\ln n}{a_n} .$$

From the previous observation we know that $\sigma_c \leq 0$; we claim that the above limit is zero to derive the first part of the result. The claim is proved through Jensen's formula which gives the relationship $n(r) = \mathcal{O}(r^{\rho+\varepsilon})$ for the entire function $g(z)$, where $\varepsilon > 0$ and $n(r)$ stands for the number of zeros with norm less or equal than r (see Titchamrsh [19, p. 249]). Choose ε such that $\rho+\varepsilon < 1$, then $n \leq K a_n^{\rho+\varepsilon}$ for some positive constant K. Finally

$$\ln n \leq n - 1 \leq K a_n^{\rho+\varepsilon} - 1 ,$$

and hence $\sigma_c = \sigma_a$. ■

Remark 3.3. Straight forward manipulations lead to generalize Lemmas 3.1 and 3.2 for the case where the simple roots of $g(z)$ are $\{\pm a_n,\, n \geq 1\}$, but still has order lying in $(0, 1)$.

The next result extends somehow case (a) of Lemma 3.1 but with a penalization of extra hypotheses. As it will be shown in the examples, this particular case is also of great interest.

Lemma 3.4. *Consider an even entire function* $g(z)$ *of order* 1 *such that* $g(0) = 1$ *and* $g(z) = \mathcal{O}(e^{A|z|})$ *for some positive constant A. Let* $\{a_n,\, n \geq 1\}$ *be a strictly increasing sequence of positive numbers; let* $\{\pm a_n,\, n \geq 1\}$ *be the simple roots of* $g(z)$*; assume that exists the limit*

$$\lim_{n\to\infty} \frac{n}{a_n} = \delta > 0 . \tag{5}$$

Then, for every $x > 0$*, the series* $\sum_{n\geq 1} \frac{1}{g'(a_n)}\, e^{-a_n x}$ *is absolute convergent and the convergence is uniform on any compact subset of* $(0, \infty)$.

Proof. It is clear from (5) that $\sigma_a = \sigma_c$, thus if we prove the plain convergence of the series in $(0, \infty)$ the result will follow. The proof is very similar to that of Lemma 3.1 but the fact that the order of $g(z)$ is 1 demands some modifications. We will consider as before the closed contour $D(R)$ in Figure 1, where $R > 0$, $\theta_0 \in (0, \pi/2)$ and $R \neq a_j$ for all j. Our objective is to prove that there is an increasing sequence, $\{R_n,\, n \geq 1\}$, towards infinity such that

$$\lim_n \oint_{D(R_n)} \frac{1}{g(z)}\, e^{-xz} \tag{6}$$

exists and is finite for a fix $x > 0$. Split the above integral as in Lemma 3.1 into $D(R_n) = D^1(R_n) \cup D^2(R_n) \cup D^3(R_n)$.

The function $g(z)$ admits an expression as a canonical product given by

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right) .$$

Define the function

$$\widetilde{g}(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n^2}\right)$$

of order $1/2$ and notice $\widetilde{g}(z^2) = g(z)$. Fortunately, the function $g(z)$ inherits the good properties of $\widetilde{g}(z)$. To start with, since $\widetilde{g}(z)$ has order $1/2$, for every $\varepsilon > 0$ there is a sequence $\widetilde{R}_n \nearrow \infty$ such that for all $\theta \in [0, 2\pi]$

$$|\widetilde{g}(\widetilde{R}_n e^{i\theta})| > M_{\widetilde{g}}(\widetilde{R}_n)^{-\epsilon},$$

where $M_{\widetilde{g}}(r) = \max_{\{|z|=r\}} |\widetilde{g}(z)|$ (see Titchmarsh [19, p. 275, it. 8.74]). Since $g(z)$ is even, we deduce that there is a sequence $R_n := \widetilde{R}_n^{1/2} \nearrow \infty$ such that for all $\theta \in [0, 2\pi]$

$$|g(R_n e^{i\theta})| > M_g(R_n)^{-\epsilon} .$$

Given that $g(z) = \mathcal{O}(e^{A\,|z|})$, for R_n large enough,

$$\left|\frac{1}{g(R_n e^{i\theta})}\right| \le C e^{\varepsilon A R_n} \quad \text{uniformly on } \theta \in [0, 2\pi].$$

Now consider ε such that $A\varepsilon < x\cos\theta_0$, and the bound

$$\begin{aligned} \left| \int_{D^3(R_n)} \frac{1}{g(z)}\, e^{-xz}\, dz \right| &\le R_n \int_{-\theta_0}^{\theta_0} \left|\frac{1}{g(R_n e^{i\theta})}\right| e^{-xR_n\cos\theta}\, d\theta \\ &\le R_n e^{-xR_n\cos\theta_0} \int_{-\theta_0}^{\theta_0} \left|\frac{1}{g(R_n e^{i\theta})}\right| d\theta \\ &\le K\theta_0 R_n e^{-R_n(x\cos\theta_0 - A\varepsilon)}, \end{aligned}$$

for R_n large enough. This goes to zero as $n \to \infty$.

In order to bound the integral over $D^1(R_n)$ consider the following Remark in Levin [13, p. 82, eqn. $(2')$]:

$$\ln|\widetilde{g}(\widetilde{r}e^{i\widetilde{\theta}})| \approx \widetilde{r}^{1/2}\pi\delta\sin(\widetilde{\theta}/2) + \frac{\mathcal{O}(r^{1/2})}{\sin(\widetilde{\theta}/2)}$$

for $\widetilde{\theta} \in (0, 2\pi)$ and as $\widetilde{r} \to \infty$. Notice that $\widetilde{g}(\widetilde{r}e^{i\widetilde{\theta}}) = g(\widetilde{r}^{1/2}e^{i\widetilde{\theta}/2}) := g(re^{i\theta}) = g(re^{i(\theta+\pi)})$, hence the above asymptotic equation is translated into $g(z)$ as

$$\ln|g(re^{i\theta})| \approx r\pi\delta|\sin(\theta)| + \frac{\mathcal{O}(r)}{|\sin(\theta)|}$$

for $\theta \in (0, 2\pi) \setminus \{\pi\}$. Therefore

$$\left| \int_{D^1(R_n)} \frac{1}{g(z)}\, e^{-xz}\, dz \right| \leq \int_0^\infty \left| \frac{1}{g(re^{-i\theta_0})} \right| e^{-xr\cos\theta_0}\, dr$$

$$\sim \int_0^\infty e^{-r\left(x\cos\theta_0 + \pi\delta|\sin(\theta_0)| + \frac{\mathcal{O}(r)}{r|\sin(\theta_0)|} \right)}\, dr\ ,$$

where $\sim$ means that both converge or diverge together. Clearly last integral is finite. The same derivation is valid for the integral over $D^2(R_n)$ and (6) holds. ■

4. Construction of density functions

This section will derive the density function associated to the characteristic functions of Proposition 2.1 and Proposition 2.3 under the restrictions of Lemma 3.4.

Lemma 2.7 shows the density function for a finite product approximation of the characteristic function. From Lévy's continuity theorem, this means a convergence in distribution of the associated laws. We first show a pointwise convergence of the distribution function to finally obtain the convergence of densities. In order to avoid the Dirac's delta measure at 0, instead of using the distribution function $F(x)$ we work with $\overline{F}(x) := 1 - F(x)$.

Theorem 4.1. *Under assumptions of Proposition 2.1, the probability measure on* $[0, \infty)$ *corresponding to the characteristic function* $\varphi(t) = h(it)/g(it)$ *is absolutely continuous on* $(0, \infty)$ *with (perhaps defective) density given by*

$$f(x) = -\sum_{n=1}^{\infty} \frac{h(a_n)}{g'(a_n)}\, e^{-a_n x}, \qquad x > 0\ .$$

Proof. Recall $h_n(z)$, $g_n(z)$ and $\varphi_n(t)$ from Lemma 2.7 case (a). Denote by $F_n(x)$ the distribution function corresponding to the characteristic function $\varphi_n(t)$. It is clear that $\varphi_n(t) \to \varphi(t)$ pointwise, where $\varphi(t)$ is defined in (1). Since $a_n \neq 0$ for all n, it follows that $\varphi(t)$ is continuous in 0. Therefore there exists a distribution function $F(x)$, such that

$$\lim_{n\to\infty} \overline{F}_n(x) = \overline{F}(x)$$

for all x where $F(x)$ is continuous. From the expression of the density of $F_n(x)$ we deduce that

$$\overline{F}_n(x) = -\sum_{i=1}^{n} \frac{h_n(a_i)}{a_i g_n'(a_i)} e^{-a_i x}, \qquad x > 0\ .$$

Our objective is to prove that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{h_n(a_i)}{a_i g_n'(a_i)} e^{-a_i x} = \sum_{i=1}^{\infty} \frac{h(a_i)}{a_i g'(a_i)} e^{-a_i x}, \qquad x > 0\ . \tag{7}$$

Fix $x > 0$, set $\Lambda = \{a_i,\, i \geq 1\}$ and denote the counting measure by Γ. The proof of (7) is done using the dominated convergence theorem. First notice the convergence

$$\lim_{n\to\infty} \frac{h_n(y)}{yg_n'(y)} e^{-yx} = \frac{h(y)}{yg'(y)} e^{-yx}, \qquad \forall\, y \in \Lambda\,.$$

Secondly, we observe that for $n > j$

$$\frac{h_{n+1}(a_j)}{g_{n+1}'(a_j)} = \frac{h_n(a_j)}{g_n'(a_j)} \frac{1 - a_j/b_{n+1}}{1 - a_j/a_{n+1}}\,,$$

and since $a_{n+1} < b_{n+1}$ we conclude that

$$\left|\frac{h_n(a_j)}{g_n'(a_j)}\right| \leq \left|\frac{h_{n+1}(a_j)}{g_{n+1}'(a_j)}\right| \leq \left|\frac{h(a_j)}{g'(a_j)}\right|\,. \tag{8}$$

Finally, let $\Phi(y)$ be the function

$$\Phi(y) := \left|\frac{h(y)}{yg'(y)}\right| \mathbf{1}_{[0,1)}(y) e^{-yx} + \left|\frac{h(y)}{g'(y)}\right| \mathbf{1}_{[1,\infty)}(y) e^{-yx}, \qquad y > 0\,.$$

Then

$$\left|\frac{h_n(y)}{yg_n'(y)} e^{-yx}\right| \leq \left|\frac{h_n(y)}{yg_n'(y)}\right| \mathbf{1}_{[0,1)}(y) e^{-yx} + \left|\frac{h_n(y)}{g_n'(y)}\right| \mathbf{1}_{[1,\infty)}(y) e^{-yx} \leq \Phi(y), \qquad y > 0\,.$$

By Remark 3.2

$$\int_\Lambda |\Phi| d\mu = \sum_{i:\, a_i < 1} \left|\frac{h(a_i)}{a_i g'(a_i)}\right| e^{-a_i x} + \sum_{i:\, a_i \geq 1} \left|\frac{h(a_i)}{g'(a_i)}\right| e^{-a_i x} < \infty\,.$$

Hence we can apply the dominated convergence theorem to prove (7). It follows

$$\overline{F}(x) = -\sum_{i=1}^{\infty} \frac{h(a_i)}{a_i g'(a_i)} e^{-a_i x}, \qquad x > 0\,.$$

Also from Remark 3.2, we check that the set of continuity of $F(x)$ is $(0, \infty)$. Moreover, from the uniform convergence on compact sets of (4),

$$F'(x) = f(x) = -\sum_{n=1}^{\infty} \frac{h(a_n)}{g'(a_n)} e^{-a_n x}, \quad x > 0$$

as required. ■

Remark 4.2. Standard manipulations show that Theorem 4.1 is also true for $h \equiv 1$.

Remark 4.3. Notice that inequality (8) also holds true for Lemma 2.7 case (b). Thus, Theorem 4.1 for such $g(z)$ and $h(z)$ with orders lying in $(0, 1)$ still holds and concludes that the limiting density is

$$f(x) = -\sum_{n=1}^{\infty} \frac{h(a_n)}{g'(a_n)} e^{-a_n |x|}, \quad x \neq 0\,.$$

Moreover, the same proof of Theorem 4.1 will be valid for functions $g(z)$ and $h(z)$ of order lying in $[1, 2)$, provided that $\sigma_a \leq 0$ for the series (4).

The next result is an extension of the preceding theorem about the existence of a density function. One would like to generalise such result for characteristic functions $\varphi(t) = h(it)/g(it)$ where $g(z)$ and $h(z)$ have zeros $\{\pm a_n,\, n \geq 1\}$ and $\{\pm b_n,\, n \geq 1\}$ and orders lying in $[1, 2)$. Entire functions of order greater or equal to one are much more difficult to treat than the ones that have order less than one. Therefore we will restrict to the setup of Lemma 3.4. As pointed out in the above remark, we only need the absolute convergence of the general Dirichlet series to follow the same proof of Theorem 4.1 and end up with the density

$$f(x) = -\sum_{n=1}^{\infty} \frac{1}{g'(a_n)}\, e^{-a_n|x|}, \quad x \neq 0\ . \tag{9}$$

However, we would like to give a different proof, Theorem 4.4, which shows that both problems, the exponential and Laplace convolutions, are the head and tail of the same coin. It is worthwhile to remark that the random variables corresponding to that case are in the homogeneous second Wiener chaos (see Janson [8, chap. 6]), so our result is a step in the study of the densities of such interesting space of random variables; in particular, this case includes the Lévy area.

Theorem 4.4. *Under assumptions of Lemma 3.4, the probability measure on $\mathbb{R}$ corresponding to the characteristic function $\varphi(t) = 1/g(it)$ is absolutely continuous on $\mathbb{R} \setminus \{0\}$ with (perhaps defective) density given by (9).*

Proof. Recall the expressions of $g(z)$ and $\widetilde{g}(z)$ defined in the proof of Lemma 3.4. Notice that

$$2a_j \widetilde{g}'(a_j^2) = g'(a_j)\ . \tag{10}$$

Due to Proposition 2.1 and Theorem 4.1, $\varphi(t) = 1/\widetilde{g}(it)$ is a characteristic function of a non–negative random variable, denoted by Y, with density

$$f_Y(x) = \sum_{j=1}^{\infty} \frac{1}{\widetilde{g}'(a_j^2)} e^{-a_j^2 x}, \quad x > 0\ .$$

Fix $x \neq 0$ and proceed heuristically as in Lemma 2.6 using the mentioned Bondesson's argument and (10) to end up with the required expression (9). To make the argument accurate we need to apply Fubini's theorem since $f_Y(x)$ is an infinite series. It turns that the absolute convergence of (4) allows us to use Fubini's result. ∎

4.1. **Infinite convolution of exponential and Laplace densities.** For the sake of completeness we will rewrite Remark 4.2 and Theorem 4.4 in a way that will extend Lemmas 2.5 and 2.6. Moreover, we prove that in this setup the resulting density is continuous in $[0, \infty)$ or $\mathbb{R}$.

Following Wintner, for a sequence of densities, $\{f_n,\ n \geq 1\}$, we will say that $\star_{n=1}^{\infty} f_n$ is a convergent infinite convolution if the product

$$\prod_{n=1}^{\infty} \psi_n(t) \ , \qquad \text{where} \qquad \psi_n(t) = \int_{-\infty}^{\infty} e^{itx} f_n(x) dx \ ,$$

is uniformly convergent in every fixed finite t–interval.

Proposition 4.5. *Let $\{\lambda_n\, n \geq 1\}$ be a strictly increasing sequence of positive numbers such that for some $\rho \in (0,1)$, $\sum_{n\geq 1} \lambda_n^{-\rho} < \infty$. Then $\star_{n=1}^{\infty} \mathcal{E}\mathrm{xp}(\lambda_n)$ converges to a continuous density on $[0,\infty)$ which can be written as*

$$\star_{n=1}^{\infty} \mathcal{E}\mathrm{xp}(\lambda_n)(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{(-1)^{n+1} A(n)}{B(i,n)} e^{\lambda_i x} = \sum_{i=1}^{\infty} \left[\prod_{\substack{k=1 \\ k\neq i}}^{\infty} \left(1 - \frac{\lambda_i}{\lambda_k}\right) \right]^{-1} e^{-\lambda_i x} \ ,$$

where the first equality holds for $x \geq 0$ and the second just for $x > 0$.

Proof. We will first point out that the infinite convolution is convergent, to this end we will use a very useful result from Wintner [20] that ensures the convergence of the infinite convolution if

$$\sum_{n=1}^{\infty} M_n < \infty \qquad \text{where} \qquad M_n = \mathbb{E}[|\mathcal{E}\mathrm{xp}(\lambda_n)|] = \int_{-\infty}^{\infty} |x| \lambda_n e^{-\lambda_n x} \mathbf{1}_{[0,\infty)} dx \ .$$

Clearly $M_n = \lambda_n^{-1}$ and the condition to guarantee the convergence of the infinite convolution is fulfilled. Moreover, if one term of the infinite convolution has continuous density and bounded variation, then so is the infinite convolution; and the continuous function $\star_{n=1}^{m} \mathcal{E}\mathrm{xp}(\lambda_n)$ tends, as $m \to \infty$, to the infinite convolution uniformly in every bounded range (see Wintner [21]). Just need to notice that $\mathcal{E}\mathrm{xp}(\lambda_1) \star \mathcal{E}\mathrm{xp}(\lambda_2)$ is continuous and of finite variation. This will show the first equality of the proposition. For the last equality let

$$g(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\lambda_k}\right) \ ,$$

notice that $g(z)$ is an entire function of order less that 1 and apply Theorem 4.1. ∎

Proposition 4.5 introduces an extra result in Remark 4.2. It says that the density function in this case can be expressed as a series which might be divergent at the origin but its right limit must exists and is equal to the infinite convolution at this point. This will be made explicit in Proposition 4.7 of next section.

Proposition 4.6. *Let $\{\lambda_n\, n \geq 1\}$ be a strictly increasing sequence of positive numbers such that $\sum_{n\geq 1} \lambda_n^{-2} < \infty$. Then $\star_{n=1}^{\infty} \mathcal{L}\mathrm{aplace}(\lambda_n)$ converges to a continuous density on $\mathbb{R}$ which can be written as*

$$\star_{n=1}^{\infty} \mathcal{L}\mathrm{aplace}(\lambda_n)(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{(-1)^{n+1} A^2(n)}{E(i,n)} e^{\lambda_i |x|} \ , \qquad x \in \mathbb{R} \ .$$

Proof. As done in Proposition 4.5, we start by showing that the infinite convolution is convergent. Jessen and Wintner show in [9] that if

$$\sum_{n=1}^{\infty} M_n^1 \qquad \text{and} \qquad \sum_{n=1}^{\infty} M_n^2$$

are convergent, where M_n^1 and M_n^2 are the first and second moment of $\mathcal{L}\text{aplace}(\lambda_n)$ respectively, then so is the infinite convolution. One can check that $M_n^1 = 0$ and $M_n^2 = \lambda^{-2}$ to obtain the convergence of the infinite convolution. Therefore we can apply Wintner [21] to obtain the proposition since a Laplace density is continuous and of finite variation. ■

Again, this results states that the infinite convolution in Theorem 4.4 is continuous in $\mathbb{R}$, although the infinite series respresentation might not be convergent at 0.

4.2. **Existence of densities.** In this section we give three different conditions that guarantee that the probability law corresponding to $h(it)/g(it)$ is absolutely continuous.

Proposition 4.7. *Under the hypotheses of Theorem 4.1, if $h(z) \equiv 1$, then the probability measure corresponding to $1/g(it)$ is absolutely continuous and its density is continuous.*

This results follows from Proposition 4.5.

In some cases it is known that the law corresponding to $h(it)/g(it)$ has no atom at zero. Since that law is concentrated on $[0, \infty)$, this suffices for the existence of a density.

Proposition 4.8. *Assume the hypothesis of Theorem 4.1 and denote by μ the probability measure corresponding to $h(it)/g(it)$. If $\mu(\{0\}) = 0$, then μ is absolutely continuous.*

The following lemma gives a classical condition for the existence of a continuous density.

Lemma 4.9. *Let $g(z)$ and $h(z)$ be two entire functions of order $\rho \in (0,1)$, both with non-zero positive simple real zeros, and the zeros of $g(z)$ different from the zeros of $h(z)$. Denote by $n_g(r)$ (respectively $n_h(r)$) the number of the zeros of $g(z)$ with module less than r. Assume the existence of the limits:*

$$\delta = \lim_{r\to\infty} \frac{n_g(r)}{r^\rho} > 0 \qquad \textit{and} \qquad \delta' = \lim_{r\to\infty} \frac{n_h(r)}{r^\rho} > 0,$$

with $\delta' < \delta$. Then

$$\int_{-\infty}^{\infty} \left| \frac{h(it)}{g(it)} \right| dt < \infty \,. \tag{11}$$

Proof. Levin ([13, p. 82, eqn. (2')]) proves that if $\theta \in (0, 2\pi)$ and $r \to \infty$ then

$$\log |g(re^{i\theta})| \approx \frac{\pi\delta r^\rho \cos\big(\rho(\theta-\pi)\big)}{\sin(\pi\rho)} + \frac{\mathcal{O}(r^\rho)}{\sin(\theta/2)} \,. \tag{12}$$

The analogous holds true for $h(re^{i\theta})$ with δ' instead of δ.

Split integral (11) into two parts as

$$\int_{-\infty}^{\infty} \left|\frac{h(it)}{g(it)}\right| dt = \int_0^\infty \left|\frac{h(te^{i\pi/2})}{g(te^{i\pi/2})}\right| dt + \int_0^\infty \left|\frac{h(te^{i3\pi/2})}{g(te^{i3\pi/2})}\right| dt. \tag{13}$$

Due to (12) there is $r_0 > 0$ and $C > 0$ such that for $r > r_0$ we get

$$\begin{aligned}\left|\frac{h(te^{i\pi/2})}{g(te^{i\pi/2})}\right| &= \exp\{-\cos(\rho\pi/2)\pi(\delta-\delta')\csc(\pi\rho)r^\rho + \mathcal{O}(r^\rho)\} \\ &= \exp\{-r^\rho\big(\cos(\rho\pi/2)\pi(\delta-\delta')\csc(\pi\rho) + \mathcal{O}(r^\rho)/r^\rho\big)\} \le e^{-Cr^\rho}.\end{aligned}$$

Hence

$$\int_{r_0}^\infty \left|\frac{h(te^{i\pi/2})}{g(te^{i\pi/2})}\right| dt \le \int_{r_0}^\infty e^{-Cr^\rho}\, dr < \infty$$

since the last expression can be reduced to a convergent gamma integral. For the integral over $[0, r_0]$ there is a straight forward bound using the same sort of derivations used in Lemma 3.1. The other integral in the right hand side of (13) is bounded in a similar way. ■

With the notations of this lemma,

Proposition 4.10. *Under the hypothesis of Theorem 4.1, and assume that exist the limits*

$$\delta = \lim_{r\to\infty} \frac{n_g(r)}{r^\rho} > 0 \qquad \textit{and} \qquad \delta' = \lim_{r\to\infty} \frac{n_h(r)}{r^\rho} > 0,$$

with $\delta' < \delta$. Then the probability measure corresponding to $h(it)/g(it)$ has a continuous density.

5. EXAMPLES

We will now see different situations where we can apply the results obtained in the previous sections. Most results are known, but here we get all them using the same technique.

5.1. **Lévy area.**

Let $\varphi(t) = \mathrm{sech}(tT)$ be the characteristic function for the Lévy area (see Lévy [14]), where the time component of the process varies in $[0, T]$. Here we set $h \equiv 1$ and $g(z) = \cos(zT)$, where $g(z)$ has order 1. It is clear that $\{\pm(2k-1)\pi/2T,\ k \ge 1\}$ and $\{\pm(-1)^k/T,\ k \ge 1\}$ are the poles and the residues of $1/g(z)$ respectively, thus $\varphi(t)$ fulfills Proposition 2.3. Moreover, $g(z)$ is even and of exponential type, thus we apply Theorem 4.4 to end up with

$$f(x) = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{T} e^{-\frac{(2k-1)\pi}{2T}|x|} \qquad x \ne 0 \,.$$

Proposition 4.6 ensures that the density function is continuous in $\mathbb{R}$.

5.2. **The first hitting time of a Bessel process.** The second example is a characteristic function obtained from

$$k(z) = z^{-\nu/2} 2^{\nu/2} \Gamma(\nu+1) J_\nu(\sqrt{2z}) \, ,$$

where $J_\nu(z)$ is the Bessel function of first kind and order $\nu > -1$. Consider $0 < u < v < \infty$ and let $h(z) = k(u^2 z)$ and $g(z) = k(v^2 z)$. The law of the corresponding characteristic function $\varphi(t) = h(it)/g(it)$ describes the first hitting time of the point v by a Bessel process of order ν that starts at u, see Kent [12], and can be expressed as

$$\varphi(t) = \left(\frac{v}{u}\right)^\nu \frac{J_\nu(u\sqrt{2it})}{J_\nu(v\sqrt{2it})} \, .$$

From the Taylor expansion of

$$\left(\frac{2}{z}\right)^\nu J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(n+\nu+1)4^n} z^{2n} \, ,$$

we can deduce the order of $k(z)$, since the order of an entire function is

$$\rho = \limsup_{n\to\infty} \frac{n\ln(n)}{\ln(1/|c_n|)} \, ,$$

where c_n are the coefficients of the Taylor expansion, see Levin [13, p. 6]. Due to Stirling formula the above limit for $h(z)$ and $g(z)$ is $1/2$. Denote by $\{j_{\nu,k},\ k \geq 1\}$the positive zeros in order of magnitude of the Bessel function $J_\nu(z)$ and by $a_k = j_{\nu,k}^2/(2v^2)$ the zeros of $g(z)$. Finally we can apply Theorem 4.1 to end up with the density

$$f(x) = \sum_{k=1}^{\infty} \frac{j_{v,k} v^{\nu-2} J_v(j_{\nu,k} u/v)}{u^\nu J_{\nu+1}(j_{\nu,k})} e^{-\frac{j_{\nu,k}^2}{2v^2}x} \qquad x > 0 \, , \tag{14}$$

where we have used

$$\frac{d}{dz} J_\nu(z) = -J_{\nu+1}(z) + \frac{\nu}{z} J_\nu(z) \, .$$

Equation (14) is also derived in Borodin and Salminen [3, p. 387]. Notice that the distribution function is absolutely continuous in $[0,\infty)$ since the probability distribution gives no mass to $\{0\}$ due to the continuous paths of the Bessel process.

5.2.1. *Exit time from a n–dimensional sphere by a Brownian motion.* Let T_n denote the random variable of the total time spent by a n–dimensional Brownian motion starting at 0 inside the sphere $S^{n-1}(r)$ of radius $r > 0$ and $n \geq 3$. Let P_n denote the first exit time for a n–dimensional Brownian motion starting at 0 from the sphere $S^{n-1}(r)$ for $n \geq 1$. Ciesielski and Taylor in [4] show the remarkable equality of the distribution functions of T_n and P_{n-2} for $n \geq 3$. Their paper derives the distribution function of T_n using methods developed by Kac in [10]. They first compute the solution for $n = 3$ and then make a guess for the general framework. Finally they compare the result with the distribution of P_n which was computed by Lévy.

We can use Theorem 4.1 to derive the density function of T_n since Ciesielski and Taylor give its characteristic function, in fact they establish the following result

$$\mathbb{E}[e^{zT_n}] = \frac{(r\sqrt{2z})^{\nu-1}}{2^{\nu-1}\Gamma(\nu)J_{\nu-1}(r\sqrt{2z})} = \prod_{i=1}^{\infty}\left(1-\frac{2r^2z}{j_{\nu-1,i}^2}\right)^{-1} ,$$

where $\nu = (n-2)/2$. This is a particular case of the previous example, where we let $h(z) = 1$ and $g(z) = k(r^2z)$. Use the same arguments as before to derive the density function

$$f(x) = \frac{1}{2^{\nu-1}\Gamma(\nu)r^2}\sum_{k=1}^{\infty}\frac{j_{\nu-1,k}^{\nu}}{J_\nu(j_{\nu-1,k})}e^{-\frac{j_{\nu-1,k}^2}{2r^2}x} \qquad x > 0 .$$

Notice that T_n is absolutely continuous due to Proposition 4.7, and we can set $f(0) = 0$ to obtain a continuous density.

5.2.2. *The area under a squared Bessel bridge.* The characteristic function of T_n has an easy representation when $\nu = n + 1/2$ for $n \in \mathbb{N}$. Let $\nu = 3/2$ and $r = 1$ to end up with the expression

$$\varphi(t) = \frac{\sqrt{-2it}}{\sinh(\sqrt{-2it})} .$$

Revuz and Yor [18, p. 465] describe the characteristic function of the area under a squared Bessel process starting at any point and arriving at zero. It turns out that the above characteristic function corresponds to a Bessel process starting and arriving at zero of order 2. Such functions appear recursively in the literature and many authors have study them, for instance [1] and [17].

The factorization of the characteristic function is particularly easy, and one can check the identity

$$\frac{1}{g(z)} = \prod_{k=1}^{\infty}\left(1-\frac{2z}{\pi^2k^2}\right)^{-1} ,$$

where the residues of $1/g(z)$ are $\{(-1)^k\pi^2k^2,\ k \geq 1\}$. Finally, the density function can be written as

$$f(x) = \begin{cases} \sum_{k=1}^{\infty}(-1)^{k+1}\pi^2k^2e^{-\pi^2k^2x/2} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases} ,$$

as stated in [1]. This characteristic function also corresponds (with a change of parameters) to the square of a Kolmogorov law. The corresponding distribution function is

$$\vartheta_4 = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-\pi^2k^2x/2} \qquad \text{for } x > 0 ,$$

which was obtained by Dugué [6].

5.3. **Inverse Laplace transform.** Theta functions and related expressions have proved useful for manipulations of functionals of Brownian motion. For instance, Borodin and Salminen use the inverse Laplace transform of

$$\varphi(z) = \frac{v \sinh(u\sqrt{2z})}{u \sinh(v\sqrt{2z})} \quad 0 < u < v \ ,$$

which turns out to be

$$\mathcal{L}^{-1}(\varphi)(y) = \frac{v}{u} \sum_{k=-\infty}^{\infty} \frac{v-u+2kv}{\sqrt{2\pi} y^{3/2}} e^{-\frac{(v-u+2kv)^2}{2y}} \quad y > 0 \ . \tag{15}$$

Let us consider $\varphi(t) = h(it)/g(it)$, where $g(z) = \frac{\sinh(v\sqrt{2z})}{v\sqrt{2z}}$ and similar for $h(z)$. Standard manipulations lead to the computation of the sequences $\{-\frac{k^2\pi^2}{2v^2},\ k \geq 1\}$ and $\{(-1)^{k+1}\frac{k\pi}{u}\sin\left(\frac{u}{v}k\pi\right),\ k \geq 1\}$, which are the poles and the residues of $h(z)/g(z)$ respectively. As pointed out in the previous example, both functions $h(z)$ and $g(z)$ are entire functions of order $1/2$ and it is easy to check that $\varphi(t)$ satisfies Proposition 2.1. Notice that the poles are negative and hence we have to apply a generalization of Theorem 4.1 for the negative case, this is straight forward and would end up with the expression

$$f(y) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k\pi}{u} \sin\left(\frac{u}{v}k\pi\right) e^{\frac{k^2\pi^2}{2v^2}y} \quad y < 0 \ .$$

Since we consider the Laplace transform we need to change the sign of y in the above expression and consider it in the range $(0,\infty)$. After doing that, it turns out that the above series is equal to (15) due to Poisson summation formulae. Moreover, we are able to say that the density function $f(y)$ is continuous in $[0,\infty)$ due to Proposition 4.10, just need to notice that $n_g(r) = \left[\frac{v\sqrt{2r}}{\pi}\right]$, where $[x]$ stands for the integer part of x and similar for $n_h(r)$.

5.4. **Heston density function.** The authors proved in [5] that the density function of the Heston model is $\mathcal{C}^\infty$ and can be expressed as an infinite convolution of Bessel type densities. We give here another expression in a particular case. In fact the search of such an expression for the general case was the starting point for the present paper, thus we think is worthwile to write it down although is just a partial result.

The heston model for the log-spot is driven by the follwing system of stochastic differential equations

$$\left.\begin{aligned} dX_t &= -\frac{1}{2}V_t\,dt + \sqrt{V_t}\,dZ_t \\ dV_t &= a(b-V_t)\,dt + c\sqrt{V_t}\,dW(t) \end{aligned}\right\}$$

where a, b and c are real positive constants. The processes W and Z are two standard correlated Brownian motions such that $\langle Z, W\rangle_t = \rho t$ for some $\rho \in [-1,1]$. For the particular case of interest we set $2ab = c^2$ and consider that the volatility process V starts at 0. Then the complex moment generating function of log–spot is

$$\mathbb{E}[e^{zX_t}] = \frac{e^{z(x_0-\rho ct/2)}e^a}{\cosh(P(z)) + \frac{a-\rho cz}{P(z)}\sinh(P(z))} = \frac{e^{z(x_0-\rho ct/2)}}{g(z)} ,$$

where x_0 is the initial point for the process X and $P(z) = \sqrt{(a - c\rho z)^2 + c^2(z - z^2)}$. The term $e^{z(x_0-\rho ct/2)}$ is just a decentralisation term, while the function $g(z)$ is an entire function of order $1/2$ with all real zeros (see [5]). The function $g(z)$ has negative roots for $\rho = -1$, while for $\rho = 1$ and $a \geq c$ all roots are positive. In any case we can apply Theorem 4.1 to obtain an expression of the density function as a series of exponential type densities. Obviously, the roots of $g(z)$ must be computed numerical.

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