GENERIC SUPER-EXPONENTIAL STABILITY OF INVARIANT TORI IN HAMILTONIAN SYSTEMS

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Abstract. In this article, we consider solutions starting close to some linearly stable invariant tori in an analytic Hamiltonian system and we prove results of stability for a super-exponentially long interval of time, under generic conditions. The proof combines classical Birkhoff normal forms and a new method to obtain generic Nekhoroshev estimates developed by the author and L. Niederman in another paper. We will mainly focus on the neighbourhood of elliptic fixed points, the other cases being completely similar.

1. Introduction

1.1. Introduction and results. In this paper, we are interested in the stability properties, in the sense of Lyapounov, of some linearly stable invariant tori in analytic Hamiltonian systems. Let us begin by the case of elliptic fixed points.

As the problem is local, it is enough to consider a Hamiltonian H, defined and analytic on an open neighbourhood of $0 \in \mathbb{R}^{2n}$, having the origin as a fixed point. Up to an irrelevant additive constant, expanding the Hamiltonian as a power series at the origin we can write

$$H(z) = H_2(z) + V(z)$$

where $z = (x, y)$ is sufficiently close to $0 \in \mathbb{R}^{2n}$, H_2 is the Hessian of H at 0 and $V(z) = O(||z||^3)$. Recall that the fixed point is said to be elliptic if the spectrum of the linearized system is purely imaginary. Due to the symplectic character of the equations, these are the only linearly stable fixed points and the spectrum has the form $\{\pm i\alpha_1, \dots, \pm i\alpha_n\}$ for some vector $\alpha = (\alpha_1, \dots, \alpha_n)$ which is called the normal (or characteristic) frequency. Now we assume that the components of α are all distinct so that we can make a symplectic linear change of variables that diagonalizes the quadratic part, hence

$$H(z) = \sum_{i=1}^{n} \frac{\alpha_i}{2}(z_i^2 + z_{n+i}^2) + V(z) = \alpha.\tilde{I} + V(z)$$

where $\tilde{I} = \tilde{I}(z)$ are the "formal actions", that is

$$\tilde{I}(z) = \frac{1}{2}(z_1^2 + z_{n+1}^2, \dots, z_n^2 + z_{2n}^2) \in \mathbb{R}^n.$$

Assuming the components of α are all of the same sign, it is easy to see that H (or $-H$) is a Lyapounov function so the fixed point is stable. But in the general case one has to study the influence of the higher order terms $V(z)$, and we will explain how it can be done using classical perturbation theory.

Let us first note that given a solution $z(t)$ of H, if $\tilde{I}(t) = \tilde{I}(z(t))$ then

$$|\tilde{I}(t)| = \sum_{i=1}^{n} |\tilde{I}_i(t)|$$

is (up to a factor one-half) the square of the euclidian distance of $z(t)$ to the origin so that stability can be proved if $|\tilde{I}(t) - \tilde{I}(0)|$ does not vary much for all times.

Now in order to study the dynamics on a small neighbourdhood of size ρ around the origin in $\mathbb{R}^{2n}$, it is more convenient to change coordinates by performing the standard scalings

$$z \longmapsto \rho z, \qquad H \longmapsto \rho^{-2} H$$

to have a Hamiltonian defined on a fixed neighbourhood of zero in $\mathbb{R}^{2n}$, and then by analyticity, to extend the resulting Hamiltonian as a holomorphic function on some complex neighbourhood of zero in $\mathbb{C}^{2n}$. So we will consider the following setting: we define

$$\mathcal{D}_s = \{z \in \mathbb{C}^{2n} \mid ||z|| < s\}$$

the euclidean ball in $\mathbb{C}^{2n}$ or radius s around the origin, and if $\mathcal{A}_s$ is the space of holomorphic functions on $\mathcal{D}_s$ which are real valued for real arguments, and $|.|_s$ its usual supremum norm, then we consider

$$(A) \qquad \begin{cases} H(z) = \alpha.\tilde{I} + f(z) \\ H \in \mathcal{A}_s, |f|_s < \rho. \end{cases}$$

Let us emphasize that the small parameter ρ, which was originally describing the size of the neighbourhood of 0, now describes the size of the "perturbation" f on a neighbourhood of *fixed size* s. Without loss of generality, we may assume $s > 3$.

Probably the main tool to investigate stability properties is the construction of normal forms using averaging methods, and in this case these are the so-called Birkhoff normal forms. For an integer $m \geq 1$, assuming α is non-resonant up to order $2m$, that is

$$k.\alpha \neq 0, \quad k \in \mathbb{Z}^n,\ 0 < |k| \leq 2m$$

there exists an analytic symplectic transformation Φ_m close to identity such that $H \circ \Phi_m$ is in Birkhoff normal form up to order $2m$, that is

$$H \circ \Phi_m(z) = h_m(\tilde{I}) + f_m(z)$$

where h_m is a polynomial of degree at most m in the $\tilde{I}$ variables and the remainder f_m is roughly of order ρ^m (see [Bir66], or [Dou88] for a more recent exposition). The polynomials h_m are uniquely defined once α is fixed and are usually called the Birkhoff invariants. The transformed Hamiltonian is therefore the sum of an integrable part h_m, for which the origin is trivially stable in the sense that $\tilde{I}(t)$ is constant for all times, and a smaller perturbation f_m. Moreover, if α is non-resonant up to any order, we can even define a formal symplectic transformation Φ_∞ and a formal power series $h_\infty = \sum_{k\geq 1} h^k$, with $h_m = \sum_{k=1}^m h^k$, such that

$$H \circ \Phi_\infty(z) = h_\infty(\tilde{I}). \tag{1}$$

However, in general the series h_∞ is divergent (this is a result of Siegel) and the convergence properties of the transformation Φ_∞ is even more subtle (see [PM03]). But Birkhoff normal forms at finite order are still very useful, not only because the "perturbation" f_m is made smaller but also because the "integrable" part h_m, for $m \geq 2$, is now non-linear and other classical techniques from perturbation theory can be used.

In the case $n = 1$, that is in dimension 2, a complete result of stability is given by an application of KAM theory. Assume that α is non-resonant up to order 4 so that our system reduce to

$$H(z) = \alpha.\tilde{I} + \beta\tilde{I}^2 + f_2(z)$$

where f_2 is a small perturbation. If the coefficient β is non-zero (the so-called twist condition), then a celebrated theorem of Moser ([Mos62], see also [SM71]) ensures stability for all times provided the size of the perturbation is small enough, that is given any solution $z(t)$ the variations $|\tilde{I}(t) - \tilde{I}(0)|$ is almost constant for all times. More geometrically, scaling back to the original variables in a neighbourhood of the origin, the theorem gives us the existence of invariant circles, bounding invariant discs, which accumulates the origin and this ensures stability. Such results have been widely considered in the litterature, and it is not possible to mention all the improvements. In fact KAM theory equally applies in higher dimensions, yielding the existence of invariant tori, the dimension of which ranges from 0 to n, in a small neighbourhood of the origin. However in this case because of the dimension it is not possible to deduce any perpetual stability results, and in fact it is believed that "generic" elliptic fixed points are unstable but this is totally unclear for the moment (see [DLC83] and [Dou88] for examples and [KMV04] for an announcement).

Therefore, for $n \geq 2$ stability results under general assumptions are concerned with finite but hopefully long interval of times, and this is the content of the paper. More precisely we will prove, under generic assumptions and provided ρ is sufficiently small, that for all initial conditions the variation $|\tilde{I}(t) - \tilde{I}(0)|$ is of order ρ for $t \in T(\rho)$, where $T(\rho)$ is an interval of time of order $\exp(\exp(\rho^{-1}))$. The interpretation in the original coordinates is that for a solution starting in a sufficiently small neighbourhood of the origin, then it stays in some larger neighbourhood for an interval of time which is super-exponentially long with respect to the distance to the origin. Let us first describes previously known results of exponential stability, where they were basically two strategy.

In a first approach, one assumes a Diophantine condition on α, that is there exist $\gamma > 0$ and $\tau \geq 0$ such that

$$|k.\alpha| \geq \gamma |k|^{-\tau}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$, but no conditions on Birkhoff invariants. From the point of view of perturbation theory, the linear part is considered as the integrable system. In particular α is completely non-resonant, hence one can perform any finite number m of Birkhoff normalizations, and since we have a control on the small divisors, one can precisely estimate the size of the remainder f_m (in terms of γ and τ). The usual trick is then to optimize the choice of m as a function of ρ in order to have an exponentially small remainder with respect to ρ^{-1}. Therefore the exponential stability is immediately read from the normal form, and this requires only an assumption on the linear part. The Diophantine condition is "generic" (it has full Lebesgue measure), however as we will see later, the threshold of the perturbation and the constants of stability are very sensitive to the Diophantine properties of α, in particular the small parameter γ.

The second approach is fundamentally different, and it does not rely on the arithmetic properties of α. Indeed, one just assume that α is non-resonant up to order 4 so that the Hamiltonian reduces to

$$H(z) = \alpha.\tilde{I} + \beta\tilde{I}.\tilde{I} + f_2(z)$$

where β is a symmetric matrix of size n and f_2 a small perturbation. But this time we consider the non-linear part $h_2(I) = \alpha.\tilde{I} + \beta\tilde{I}.\tilde{I}$ as the integrable system, and we assume it is convex, which is equivalent to β being sign definite. Under those assumptions, it was predicted and partially proved by Lochak ([Loc92] and [Loc95]) and completely proved independently by Niederman ([Nie98]) and Benettin, Fasso and Guzzo ([BFG98]) that exponential stability holds. Their proofs are based on the implementation of Nekhoroshev estimates in cartesian coordinates, but they are radically different: the first one uses Lochak method of periodic averagings and simultaneous Diophantine approximations while the second is based on Nekhoroshev original mechanism. The proof of Niederman was later clarified by Pöschel ([Pös99]). However, the method of Lochak was restricted to the convex case and it was not clear how to remove this hypothesis to have a result valid in a more general context.

In this paper, using the method of [BN09] we are able to replace the convexity condition by a generic assumption and to combine both approaches to obtain the following result.

Theorem 1.1. *Suppose H as in (A), with α non-resonant. Then under a generic condition (G) on h_∞, there exist constants a, a', c_1, c_2 and ρ_0 such that for $\rho \leq \rho_0$, every solution $z(t)$ of H with $|\tilde{I}(0)| < 1$ satisfies*

$$|\tilde{I}(t) - \tilde{I}(0)| < c_1\rho, \quad |t| < \exp\left(\rho^{-a'} \exp(c_2 a' \rho^{-a})\right).$$

Denoting $h_\infty = \sum_{k\geq 1} h^k$ and $h_m = \sum_{k=1}^m h^k$, let us first explain our generic condition (G) on the formal power series h_∞. In fact

$$(G) = \bigcup_{m\in\mathbb{N}^*} (G_m)$$

is countably many conditions, where (G_m) is a condition on h_m. The first condition (G_1) requires that $h_1(I) = \alpha.I$ with a (γ,τ)-Diophantine vector α. The other conditions (G_m), for $m \geq 2$, ask that each polynomial function h_m belongs to a special class of functions called $SDM_{\gamma'}^{\tau'}$ which was introduced in [BN09] (see the appendix A for a definition). We will show in this appendix that each condition (G_m) is of full Lebesgue measure in the finite dimensional space of polynomials of degree m with n variables, assuming τ and τ' are large enough. This is well-known for $m = 1$, it will be elementary for $m = 2$ (see theorem A.8) but it requires the quantitative Morse-Sard theory of Yomdim ([Yom83], [YC04], see theorem A.3) for $m > 2$. Let us point out that this would have not been possible if we had assumed h_m, for $m \geq 2$, to be steep in the sense of Nekhoroshev since polynomials are generically steep only if their degrees is sufficiently large with respect to the number of degrees of freedom (see [LM88]).

Our condition (G) on the formal series h_∞ is therefore of "full Lebesgue measure at any order". From an abstract point of view, this condition defines a prevalent set in the space of formal power series, where prevalence is an analog of the notion of full Lebesgue measure in the context of infinite dimensional vector spaces. This will be proved in appendix A, theorem A.6. Consequently, in the above statement we can choose the exponents

$$a = (1+\tau)^{-1},\ a' = 3^{-1}(2(n+1)\tau')^{-n}$$

and our threshold ρ_0 depends in particular on γ and γ'. Moreover our constants c_1 and c_2 also depend on γ but not on γ' and we shall be a little more precise later on.

As we already explained, the proof is based on a combination of Birkhoff normalizations up to an exponentially small remainder, which is well-known (a statement is recalled in proposition 2.1 below), and Nekhoroshev estimates for a generic integrable Hamiltonian near an elliptic fixed point (theorem 2.2 below). The latter result is new, and it will follow rather easily from the new approach of Nekhoroshev theory in a generic case taken in [BN09].

As a direct consequence of our Nekhoroshev estimates near an elliptic fixed point, we can derive an exponential stability result more general than those obtained in [BFG98] and [Nie98]. Like in those papers, we only require α to be non-resonant up to order 4 and after the scalings

$$z \longmapsto \rho z, \qquad H \longmapsto \rho^{-4} H, \qquad \alpha \longmapsto \rho^{-2}\alpha$$

we consider

$$(B) \qquad \begin{cases} H(z) = \alpha.\tilde{I} + \beta\tilde{I}.\tilde{I} + f(z) \\ H \in \mathcal{A}_s, |f|_s < \rho. \end{cases}$$

However, instead of assuming that β is sign definite, our result applies to Lebesgue almost all symmetric matrix β without any condition on α. Let $S_n(\mathbb{R})$ be the space of symmetric matrix of size n with real coefficients.

Theorem 1.2. *Suppose H as in (B). For Lebesgue almost all $\beta \in S_n(\mathbb{R})$, there exist a', b' and ρ_0 such that for $\rho \leq \rho_0$, every solution $z(t)$ of H with $|\tilde{I}(0)| < 1$ satisfies*

$$|\tilde{I}(t) - \tilde{I}(0)| < n(n^2+1)\rho^{-b'}, \quad |t| < \exp(\rho^{-a'}).$$

The above theorem is a direct consequence of theorem 2.2 below, provided that $h_2(\tilde{I}) = \alpha.\tilde{I} + \beta\tilde{I}.\tilde{I}$ belongs to $SDM_{\gamma'}^{\tau'}$, but we will see in the appendix A that this happens for almost all symmetric matrix β independently of α, and moreover the proof is elementary as it does not rely on Morse-Sard theory (see theorem A.8). Once again, let us also mention that this result is not possible in the steep case, as the quadratic part $h^2(\tilde{I}) = \beta\tilde{I}.\tilde{I}$ is steep only if β is sign definite. In the above statement one can choose

$$a' = b' = 3^{-1}(2(n+1)\tau')^{-n}$$

and the threshold ρ_0 depends on γ'.

The idea of combining both Birkhoff theory and Nekhoroshev theory to obtain super-exponential stability was discovered by Morbidelli and Giorgilli ([MG95]) in the context of Lagrangian Diophantine tori. Evidently, we can also state results in this context.

So consider a Hamiltonian system on a manifold which has an invariant Lagrangian Diophantine tori, that is an invariant manifold $\mathcal{T}$ which is diffeomorphic to the standard torus $\mathbb{T}^n$ and whose induced flow is conjugated to a linear flow on $\mathbb{T}^n$ with a Diophantine frequency. Since the torus is Lagrangian, one can locally reduce the situation to a Hamiltonian defined on $T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$, having $\mathbb{T}^n \times \{0\}$ as the invariant torus. Moreover, by invariance and transitivity of the torus, in the coordinates $(\theta, I) \in \mathbb{T}^n \times \mathbb{R}^n$ we can write

$$H(\theta, I) = \omega.I + F(\theta, I)$$

where ω is a (γ, τ)-Diophantine vector and $F(\theta, I) = O(|I|^2)$. After some scalings one is led to consider

$$(C) \qquad \begin{cases} H(\theta, I) = \omega.I + f(\theta, I) \\ H \in \mathcal{A}_s, |f|_s < \rho \end{cases}$$

where $\mathcal{A}_s$ is the space of holomorphic functions on the domain

$$\mathcal{D}_s = \{(\theta, I) \in (\mathbb{C}^n/\mathbb{Z}^n) \times \mathbb{C}^n \mid |\mathcal{I}(\theta)| < s, \ |I| < s\}$$

with $\mathcal{I}(\theta)$ the imaginary part of θ. Here one can also define polynomials h_m and a formal power series h_∞ and we can state the following result.

Theorem 1.3. *Suppose H as in (C). Then under a generic condition on h_∞, there exist constants a, a', c_1, c_2 and ρ_0 such that for $\rho \leq \rho_0$, every solution $(\theta(t), I(t))$ of H with $|I(0)| < 1$ satisfies*

$$|I(t) - I(0)| < c_1\rho, \quad |t| < \exp\left(\rho^{-a'} \exp(c_2 a' \rho^{-a})\right).$$

The assumption on h_∞ and the values of the constants a and a' are the same as in theorem 1.1, as the proof is completely analogous. In fact, it is is even simpler since we are using action-angles coordinates and therefore we can immediately use Nekhoroshev estimates obtained in [Nie07b] without any modifications.

However, it is important to note that one cannot obtain a statement similar to theorem 1.2, simply because in this case a non-resonant condition up to a finite order does not allow to build the corresponding Birkhoff normal form.

If we compare this result with [MG95], note that our assumption is generic and we don't require any convexity, but of course the price to pay is that one has to consider the full set of Birkhoff invariants, but this is almost inevitable as in the multi-dimensional case no generic property can be detected by looking only at the jet of order two of a given function.

As a final result, one could also obtain similar estimates for the general case of a linearly stable lower-dimensional torus, under the common assumptions of isotropy and reducibility (which were automatic for a fixed point or a Lagrangian torus). In that context, it is enough to consider a Hamiltonian defined in $\mathbb{T}^k \times \mathbb{R}^k \times \mathbb{R}^{2l}$ (by isotropy) of the form

$$H(\theta, I, z) = \omega.I + \frac{1}{2}Bz.z + F(\theta, I, z)$$

where B is a symmetric matrix (constant by reducibility) such that $J_{2l}B$ has a purely imaginary spectrum (J_{2l} being the canonical symplectic structure of $\mathbb{R}^{2l}$), and finally $F(\theta, I, z) = O(|I|^2, ||z||^3)$. In those coordinates, the invariant torus is simply given by $I = z = 0$, and this generalizes both the case of an elliptic fixed point (when the action-angle variables (θ, I) are absent) and of a Lagrangian invariant torus (when the cartesian coordinates z are absent). If the spectrum $\{\pm i\alpha_1, \dots, \pm i\alpha_n\}$ of $J_{2l}B$ is simple, one can assume further that

$$H(\theta, I, z) = \omega.I + \alpha.\tilde{I} + F(\theta, I, z)$$

where $\tilde{I}$ are the "formal actions" associated to the z variables. Therefore after some appropriate scalings we can consider

$$(D) \qquad \begin{cases} H(\theta, I, z) = \omega.I + \alpha.\tilde{I} + F(\theta, I, z) \\ H \in \mathcal{A}_s, |f|_s < \rho \end{cases}$$

where $\mathcal{A}_s$ is the space of holomorphic functions on the domain

$$\mathcal{D}_s = \{(\theta, I, z) \in (\mathbb{C}^k/\mathbb{Z}^k) \times \mathbb{C}^k \times \mathbb{C}^{2l} \mid |\mathcal{I}(\theta)| < s,\ |I| < s,\ ||z|| < s\}.$$

Under a suitable Diophantine condition on the vector $(\omega, \alpha) \in \mathbb{R}^{k+l}$, one can define polynomials h_m and a formal series h_∞ depending on $J = (I, \tilde{I})$, and Birkhoff exponential estimates in this more difficult situation have been obtained in [JV97]. Regarding Nekhoroshev estimates for a generic integrable Hamiltonian which depends both on actions and formal actions, they can be obtained by slight modifications of the method of [BN09] as it will done here. Therefore we can state the following result.

Theorem 1.4. *Suppose H as in (D). Then under a generic condition on h_∞, there exist constants a, a', c_1, c_2 and ρ_0 such that for $\rho \leq \rho_0$, every solution $(\theta(t), I(t), z(t))$ of H with $|J(0)| < 1$ satisfies*

$$|J(t) - J(0)| < c_1\rho, \quad |t| < \exp\left(\rho^{-a'} \exp(c_2 a' \rho^{-a})\right).$$

Once again, the condition on h_∞ and the values of the exponents are the same.

1.2. **Comments and prospects.** Let us add that one could easily give similar estimates in the discrete case, that is for exact symplectic diffeomorphisms near an elliptic fixed point, an invariant Lagrangian torus or an invariant linearly stable isotropic reducible torus. Even if one has the possibility to re-write the proof in that context, the easiest way is to use suspension arguments, as it is done qualitatively in [Dou88] or quantitavely in [KP94] (see also [PT97] for a different approach) and deduce stability results in the discrete case from the corresponding results in the continuous case.

To conclude, let us mention that important examples of invariant tori satisfying our assumptions (linearly stable, reducible, isotropic) are those given by KAM theory. However, for Lagrangian tori the latter not only gives individual tori but a whole Cantor family (see [Pös01] for a modern exposition). In this context, Popov proved exponential stability estimates for this family of KAM tori, if the Hamiltonian is analytic or Gevrey ([Pop00] and [Pop04]). His proof relies on a KAM theorem with Gevrey smoothness on the parameters (in the sense of Whitney) and some kind of simultaneous Birkhoff normal form over the Cantor set of tori. We believe that our method should be useful in trying to extend those results to obtain super-exponential stability under generic conditions. But clearly this is a more difficult problem, and the first step is to obtain Nekhoroshev estimates in Gevrey regularity for a generic integrable Hamiltonian, the quasi-convex case having been settled in [MS02].

Plan of the paper

The next section is devoted to the proof of theorem 1.1, and an appendix is devoted to our genericity assumptions. In section 2.1, we give a statement of a Birkhoff normal form up to an exponentially small remainder and in 2.2, we will explain how the Nekhoroshev estimates obtained in [BN09] generalizes in the neighbourhood of elliptic fixed points. Eventually, in 2.3, we will show how a simple combination of Birkhoff estimates and Nekhoroshev estimates implies our theorem 1.1, provided our assumption on h_∞ is satisfied.

Finally, let us add that in order to avoid meaningless expressions, we will only keep track of the small parameters ρ, γ and γ' and replace the other constants by a $\cdot$ when it is convenient.

2. Proof of the main theorem

In the sequel, we recall that we will use the "formal" actions

$$\tilde{I} = \tilde{I}(z) = \frac{1}{2}(z_1^2 + z_{n+1}^2, \dots, z_n^2 + z_{2n}^2) \in \mathbb{R}^n$$

but one has to remember that these are nothing but notations for expressions in $z \in \mathbb{R}^{2n}$. Moreover, we will also need to use complex coordinates for the normal forms, and abusing notations we will denote them by $z \in \mathbb{C}^{2n}$, but of course the solutions we will consider will be real.

2.1. **Birkhoff estimates.**

Here we consider the Hamiltonian as in (A), and we assume that α is (γ, τ)-Diophantine. In this context, the following result is classical.

Proposition 2.1. *Under the previous assumptions, if $\rho \lessdot \gamma$, there exist an integer $m = m(\rho)$ and an analytic transformation*

$$\Phi_m : \mathcal{D}_{3s/4} \longrightarrow D_s$$

such that

$$H \circ \Phi_m(z) = h_m(\tilde{I}) + f_m(z)$$

is in Birkhoff normal form with a remainder f_m satisfying the estimate

$$|f_m|_{3s/4} \lessdot \rho \exp\left(-(\gamma\rho^{-1})^a\right)$$

with the exponent $a = (1+\tau)^{-1}$. Moreover, $|\Phi_m - Id|_{3s/4} \lessdot \gamma^{-1}\rho$ and the image of Φ_m contains the domain $\mathcal{D}_{s/2}$.

For a proof, we refer to [GDF+89] and [DG96]. The analogous result for invariant Lagrangian tori can be found in [PW94] or [Pös93], and in the more general case of an isotropic and reducible linearly stable invariant tori in [JV97].

Even though the technical estimates are complicated, the idea is in fact rather simple: if the size of the perturbation satisfy $\rho \lessdot \gamma$, after m steps of classical Birkhoff normalizations one arrives at

$$H \circ \Phi_m = h_m + f_m$$

and the remainder f_m is of order $\rho(\gamma^{-1}\rho)^m (m!)^{\tau+1}$, so that we may choose $m = m(\rho)$ of order $(\gamma\rho^{-1})^{(\tau+1)^{-1}}$ to obtain the exponential estimate by Stirling formula.

Note that letting ρ goes to zero, the degree of the polynomial h_m goes to infinity, and this explains why in the proof of theorem 1.1 we require a condition on the whole formal power series h_∞.

2.2. **Nekhoroshev estimates.** Here we consider the Hamiltonian

$$\text{(E)} \qquad \begin{cases} H(z) = h(\tilde{I}) + f(z) \\ H \in \mathcal{A}_s,\ h \in SDM^{\tau'}_{\gamma'},\ |f|_s < \varepsilon \end{cases}$$

and we assume that the derivative up to order 3 of h are uniformly bounded by some constant $M > 1$ on the real part of the domain. The definition of the set $SDM^{\tau'}_{\gamma'}(B)$ is recalled in appendix A.

Theorem 2.2. *Let H as in (E), with $\tau' \geq 2$ and $\gamma' \leq 1$. Then there exists ε_0 such that if $\varepsilon \leq \varepsilon_0$, for every solution $z(t)$ with $|\tilde{I}(0)| < 1$, we have*

$$|\tilde{I}(t) - \tilde{I}(0)| < n(n^2+1)\varepsilon^{b'}, \quad |t| < \exp(\varepsilon^{-a'})$$

with the exponents $a' = b' = 3^{-1}(2(n+1)\tau')^{-n}$.

Theorem 1.2 is now an immediate consequence of this result and theorem A.8 (see the appendix). The above statement is the exact analog of the main statement of [BN09], the only difference being the extra-factor n in the bound of the variations of the actions, coming from the fact that for this part it will be more convenient to use the supremum norm $|.|_\infty$ for vectors.

However, the main difference is that here we are using cartesian coordinates and not action-angles coordinates (*i.e.* symplectic polar coordinates), and we *cannot* use the latter since they become singular at the origin so we *cannot* apply directly the main result of [BN09] (and a fortiori [Nie07b]). In fact, this is not a serious issue when applying KAM theory in this context (see [Pös82]) but this becomes problematic in Nekhoroshev theory (see [Loc92] or [Loc95] for detailed explanations). This result was only conjectured by Nekhoroshev in [Nek77] and it took a long time before it can be solved in the convex case ([Nie98],[BFG98]). Here we are able to solve this problem in the generic case. The reason is that even though we cannot apply the result of [BN09], we can use exactly the same approach, since the method of averagings along unperturbed periodic flows is in some sense intrinsic, *i.e.* independent of the choice of coordinates, a fact that was first used implicitly in [Nie98] and made completely clear in [Pös99].

The proof of such estimates usually requires an analytic part, which boils down to the construction of suitable normal forms, and a geometric part. The geometric part of [BN09] goes exactly the same way, so in the sequel we will restrict ourself to indicate the very slight modifications in the construction of the normal forms.

So consider linearly independent periodic vectors $(\omega_1, \dots, \omega_n)$, with periods $(T_1, \dots, T_n)$, that is the vectors $T_1\omega_1, \dots, T_n\omega_n$ belong to $\mathbb{Z}^n$. Now we define the domains

$$\mathcal{D}_{r_j,s_j}(\omega_j) = \{z \in \mathcal{D}_{s_j} \mid |\nabla h(\tilde{I}) - \omega_j|_\infty < r_j\}$$

for $j \in \{1, \dots, n\}$, with two sequences $(r_1, \dots, r_n)$ and $(s_1, \dots, s_n)$ (recall that $\mathcal{D}_{s_j}$ is the complex ball of radius s_j). We will denote by l_i the linear Hamiltonian with frequency ω_i, that is $l_i(\tilde{I}) = \omega_i.\tilde{I}$.

The supremum norm of a function f defined on $\mathcal{D}_{r_j,s_j}(\omega_j)$ will be simply denoted by

$$|f|_{r_j,s_j} = |f|_{\mathcal{D}_{r_j,s_j}(\omega_j)}$$

and for a Hamiltonian vector field X_f, we will write

$$|X_f|_{r_j,s_j} = \max_{1 \leq i \leq 2n} |\partial_{z_i} f|_{r_j,s_j}.$$

To obtain normal forms on these domains we will make the following assumptions (A_j), for $j \in \{1, \dots, n\}$, where (A_1) is

$$(A_1) \qquad \begin{cases} mT_1\varepsilon \cdot\!< r_1, \ mT_1 r_1 \cdot\!< s_1, \ 0 < r_1 <\!\cdot\, s_1 \\ \mathcal{D}_{r_1,s_1}(\omega_1) \neq \emptyset, \ s_1 \cdot\!< s \end{cases}$$

and for $j \in \{2, \dots, n\}$, (A_j) is

$$(A_j) \qquad \begin{cases} mT_j\varepsilon \cdot\!< r_j, \ mT_j r_j \cdot\!< s_j, \ 0 < r_j <\!\cdot\, s_j \\ \mathcal{D}_{r_j,s_j}(\omega_j) \neq \emptyset, \ \mathcal{D}_{r_j,s_j}(\omega_j) \subseteq \mathcal{D}_{2r_{j-1}/3,2s_{j-1}/3}(\omega_{j-1}). \end{cases}$$

With these assumptions, one can proof the following proposition.

Proposition 2.3. *Consider $H = h + f$ on the domain $\mathcal{D}_{r_1,s_1}(\omega_1)$, with $|X_f|_{r_1,s_1} < \varepsilon$, and let $j \in \{1, \dots, n\}$. If (A_i) is satisfied for any $i \in \{1, \dots, j\}$, then there exists an analytic symplectic transformation*

$$\Psi_j \colon \mathcal{D}_{2r_j/3,2s_j/3}(\omega_j) \to \mathcal{D}_{r_1,s_1}(\omega_1)$$

such that

$$H \circ \Psi_j = h + g_j + f_j$$

with $\{g_j, l_i\} = 0$ for $i \in \{1, \dots, j\}$ and the estimates

$$|X_{g_j}|_{2r_j/3,2s_j/3} <\!\cdot\, \varepsilon, \quad |X_{f_j}|_{2r_j/3,2s_j/3} <\!\cdot\, e^{-m}\varepsilon.$$

Moreover, we have $\Psi_j = \Phi_1 \circ \dots \circ \Phi_j$ with

$$\Phi_i \colon \mathcal{D}_{2r_i/3,2s_i/3}(\omega_i) \to \mathcal{D}_{r_i,s_i}(\omega_i)$$

such that $|\Phi_i - Id|_{2r_i/3,2s_i/3} \cdot\!< r_i$.

The proof is analogous to the corresponding one in [BN09], appendix A, for which we refer for more details, and in fact it is a bit simpler since one does not have to use "weighted" norms for vector fields. It relies on a finite composition of averagings along the periodic flows generated by l_j, $j \in \{1, \dots, n\}$. The case $j = 1$ is due to Pöschel ([Pös99]) and for $j > 1$, the proof goes by induction using our assumption (A_j), $j \in \{1, \dots, n\}$.

Once we have this normal form, the rest of the proof in [BN09] goes exactly the same way, every solution $z(t)$ of H with $|\tilde{I}(0)|_\infty < 1$ satisfy

$$|\tilde{I}(t) - \tilde{I}(0)|_\infty < (n^2+1)\varepsilon^{b'}, \quad |t| < \exp(\varepsilon^{-a'})$$

provided that $\varepsilon \leq \varepsilon_0$, with ε_0 depending on n, s, M, γ' and τ' and with the exponents

$$a' = b' = 3^{-1}(2(n+1)\tau')^{-n}.$$

In particular, every solution $z(t)$ of H with $|\tilde{I}(0)| < 1$ satisfy

$$|\tilde{I}(t) - \tilde{I}(0)| < n(n^2+1)\varepsilon^{b'}, \quad |t| < \exp(\varepsilon^{-a'}).$$

2.3. **Proof of theorem 1.1.** Now we can finally prove theorem 1.1, by using successively Birkhoff estimates and Nekhoroshev estimates.

Proof of theorem 1.1. Let H as in (A), first assume that $\rho < \rho_1$ with $\rho_1 =\cdot\, \gamma$ so that we can apply proposition 2.1 using our assumption (G_1): there exist an integer $m = m(\rho)$ and an analytic transformation

$$\Phi_m \colon \mathcal{D}_{3s/4} \longrightarrow \mathcal{D}_s$$

such that

$$H \circ \Phi_m(z) = h_m(\tilde{I}) + f_m(z)$$

is in Birkhoff normal form with a remainder f_m satisfying the estimate

$$|f_m|_{3s/4} <\cdot\, \rho \exp\left(-(\gamma\rho^{-1})^{a^{-1}}\right)$$

with $a = \tau + 1$. So let $H_m = H \circ \Phi_m$, and define

$$\varepsilon = \rho \exp\left(-(\gamma\rho^{-1})^{a^{-1}}\right).$$

By our assumption (G_m), for $m \geq 2$, the Hamiltonian H_m, which is defined on the domain $\mathcal{D}_{3s/4}$, satisfy (E). Now assume that $\varepsilon <\cdot\, \varepsilon_0$ which gives another threshold $\rho < \rho_2$, with ρ_2 also depending on γ', and our final threshold is $\rho_0 = \min\{\rho_1, \rho_2\}$. We can apply theorem 2.2 so that every solution $z_m(t)$ of H_m with $|\tilde{I}_m(0)| < 1$ satisfy

$$|\tilde{I}_m(t) - \tilde{I}_m(0)| <\cdot\, \varepsilon^{b'}, \quad |t| < \exp(\varepsilon^{-a'})$$

with

$$a' = b' = 3^{-1}(2(n+1)\tau')^{-n}.$$

Recalling the definition of ε this gives

$$|\tilde{I}_m(t) - \tilde{I}_m(0)| <\cdot\, \rho^{b'} \exp\left(-b'(\gamma\rho^{-1})^{a}\right), \quad |t| < \exp\left(\rho^{-a'} \exp\left(a'(\gamma\rho^{-1})^{a}\right)\right).$$

However, one has

$$\rho^{b'} \exp\left(-b'(\gamma\rho^{-1})^{a}\right) < \gamma^{-1}\rho$$

and as Φ_m satisfies $|\Phi_m - Id|_{3s/4} \lessdot \gamma^{-1}\rho$ and its image contains the domain $\mathcal{D}_{s/2}$, a standard argument gives

$$|I(t) - I(0)| \lessdot \gamma^{-1}\rho, \quad |t| < \exp\left(\rho^{-a'} \exp\left(a'(\gamma\rho^{-1})^{a}\right)\right)$$

for any solution $z(t)$ of H with $|\tilde{I}(0)| < 1$. □

Appendix A. Generic assumptions

In this appendix, we will show that our assumption (G) is generic, in the sense that it defines a prevalent set in the infinite dimensional space of formal power series.

But first we recall the definition of SDM functions. Let $G(n,k)$ be the set of all vector subspaces of $\mathbb{R}^n$ of dimension k. We equip $\mathbb{R}^n$ with the euclidian scalar product and given an integer $L \in \mathbb{N}^*$, we define $G^L(n,k)$ as the subset of $G(n,k)$ consisting in subspaces which orthogonal can be generated by integer vectors with components bounded by L. In the sequel, B will be an arbitrary open ball of $\mathbb{R}^n$.

Definition A.1. *A smooth function $h\colon B \to \mathbb{R}$ is said to be SDM if there exist $\gamma' > 0$ and $\tau' \geq 0$ such that for any $L \in \mathbb{N}^*$, any $k \in \{1, \dots, n\}$ and any $\Lambda \in G^L(n,k)$, there exists $(e_1, \dots, e_k)$ (resp. $(f_1, \dots, f_{n-k})$) an orthonormal basis of Λ (resp. of $\Lambda^\perp$) such that the function h_Λ defined on B by*

$$h_\Lambda(\alpha,\beta) = h\left(\alpha_1 e_1 + \cdots + \alpha_k e_k + \beta_1 f_1 + \cdots + \beta_{n-k} f_{n-k}\right)$$

satisfies the following: for any $(\alpha,\beta) \in B$,

$$|\partial_\alpha h_\Lambda(\alpha,\beta)| \leq \gamma' L^{-\tau'} \Longrightarrow |\partial_{\alpha\alpha} h_\Lambda(\alpha,\beta).\eta| > \gamma' L^{-\tau'} |\eta|$$

for any $\eta \in \mathbb{R}^k$.

This definition is basically a quantitative transversality condition, it is inspired by the steepness condition of Nekhoroshev and the quantitative Morse-Sard theory of Yomdin (see [BN09] for more explanations). It depends on a choice of coordinates adapted to the orthogonal decomposition $\Lambda + \Lambda^\perp$, so for $\Lambda \in G^L(n,k)$ and $(\alpha,\beta) \in B$, $\partial_\alpha h_\Lambda(\alpha,\beta)$ is a vector in $\mathbb{R}^k$ and $\partial_{\alpha\alpha} h_\Lambda(\alpha,\beta)$ is a symmetric matrix of size k with real coefficients.

Remark A.2. *Note also that the definition can be stated, for any $(\alpha,\beta) \in B$, as the following alternative: either we have $|\partial_\alpha h_\Lambda(\alpha,\beta)| > \gamma L^{-\tau}$ or $|\partial_{\alpha\alpha} h_\Lambda(\alpha,\beta).\eta| > \gamma L^{-\tau}|\eta|$ for any $\eta \in \mathbb{R}^k$. Hence for a given function it is sufficient to verify that $|\partial_{\alpha\alpha} h_\Lambda(\alpha,\beta).\eta| > \gamma L^{-\tau}|\eta|$ for any $\eta \in \mathbb{R}^k$, and we will use this fact later (in theorem A.8).*

The set of SDM functions on B with respect to $\gamma' > 0$ and $\tau' \geq 0$ will be denoted by $SDM_{\gamma'}^{\tau'}(B)$, and we will also use the notation

$$SDM^{\tau'}(B) = \bigcup_{\gamma'>0} SDM_{\gamma'}^{\tau}(B).$$

The following theorem was proved in [BN09], and it relies on non-trivial results from quantitative Morse-Sard theory ([Yom83],[YC04]).

Proposition A.3 ([BN09])**.** *Let $\tau > 2(n^2+1)$, and $h \in C^{2n+2}(B)$. Then for Lebesgue almost all $\xi \in \mathbb{R}^n$, the function h_ξ, defined by $h_\xi(I) = h(I) - \xi.I$ for $I \in B$, belongs to $SDM^{\tau'}(B)$.*

Now let us recall the definition of a prevalent set ([HSY92], see also [OY05]).

Definition A.4. *Let E be a completely metrizable topological vector space. A Borel subset $S \subseteq E$ is said to be shy if there exists Borel measure μ on E, with $0 < \mu(C) < \infty$ for some compact set $C \subseteq E$, and such that $\mu(x+S) = 0$ for all $x \in E$.*

An arbitrary set is called shy if it contains a shy Borel subset, and finally the complement of a shy set is called prevalent.

For a finite dimensional vector space E, by an easy application of Fubini theorem, prevalence is equivalent to full Lebesgue measure. The following "genericity" properties can be checked ([OY05]): a prevalent set is dense, a set containing a prevalent set is also prevalent, and prevalent sets are stable under translation and countable intersection. Furthermore, we have an easy but useful criterion for a set to be prevalent.

Proposition A.5 ([HSY92])**.** *Let A be a Borel subset of E. Suppose there exists a finite-dimensional subspace F of E such that, if we denote λ_F the Lebesgue measure on F, the set $x + A$ has full λ_F-measure for all $x \in E$. Then A is prevalent.*

It is an obvious consequence of proposition A.3 and proposition A.5 that $SDM^{\tau'}(B)$ is prevalent in $C^{2n+2}(B)$ for $\tau' > 2(n^2+1)$.

Now let $\mathcal{P}_\infty = \mathbb{R}[[X_1,\dots,X_n]]$ be the space of all formal power series in n variables with real coefficients. It is naturally a Fréchet space, for example as the projective limit of the finite dimensional spaces $\mathcal{P}_m$ consisting of polynomials in n variables of degree less or equal than m. We define the subset

$$\mathcal{P}_\infty^{\tau'} = \{h_\infty \in \mathcal{P}_\infty \mid h_m \in SDM^{\tau'}(B),\ \forall m \geq 2\}$$

where $h_m = \sum_{k=1}^m h^k$ if $h_\infty = \sum_{k\geq 1} h^k$, and we identify the polynomial h_m with the associated function defined on B. Let us also define

$$\mathcal{P}_\infty^{\tau} = \{h_\infty \in \mathcal{P}_\infty \mid h_1(X) = \alpha.X,\ \alpha \in \mathcal{D}^\tau\}$$

where $\mathcal{D}^\tau$ is the set of Diophantine vector of $\mathbb{R}^n$ with exponent $\tau > 0$, and finally

$$\mathcal{P}_\infty^{\tau,\tau'} = \mathcal{P}_\infty^{\tau} \cap \mathcal{P}_\infty^{\tau'}.$$

The set $\mathcal{P}_\infty^{\tau,\tau'}$ is the set of formal power series for which condition (G) holds.

Theorem A.6. *For $\tau > n-1$ and $\tau' > 2(n^2+1)$, the set $\mathcal{P}_\infty^{\tau,\tau'}$ is prevalent in $\mathcal{P}_\infty$.*

Proof. As the intersection of two prevalent set is prevalent, it is enough to prove that both set $\mathcal{P}_\infty^\tau$ for $\tau > n-1$ and $\mathcal{P}_\infty^{\tau'}$ for $\tau' > 2(n^2+1)$ are prevalent.

For the set $\mathcal{P}_\infty^\tau$, it is an easy consequence of the fact that $\mathcal{D}^\tau$ is of full Lebesgue measure in $\mathbb{R}^n$ for $\tau > n-1$ and proposition A.5 with $F = \mathcal{P}_1$ the space of linear forms. For the set $\mathcal{P}_\infty^{\tau'}$, first note that we can write

$$\mathcal{P}_\infty^{\tau'} = \bigcap_{m \geq 2} \mathcal{P}_{\infty,m}^{\tau'}$$

where for an integer $m \geq 2$,

$$\mathcal{P}_{\infty,m}^{\tau'} = \{h_\infty \in \mathcal{P}_\infty \mid h_m \in SDM_{\tau'}(B)\}.$$

As a countable intersection of prevalent sets is prevalent, it is enough to prove that for each $m \geq 2$, the set $\mathcal{P}_{\infty,m}^{\tau'}$ is prevalent in $\mathcal{P}_\infty$. But once again this is just a consequence of proposition A.3 and proposition A.5 with $F = \mathcal{P}_1$ the space of linear forms. □

The set of polynomials h_m for which condition (G_m) is satisfied is given by

$$\mathcal{P}_m^{\tau'} = \{h_m \in \mathcal{P}_m \mid h_m \in SDM_{\tau'}(B)\}$$

and the proof of the above theorem immediately gives the following result.

Theorem A.7. *or $\tau' > 2(n^2+1)$, the set $\mathcal{P}_m^{\tau'}$ is of full Lebesgue measure in $\mathcal{P}_m$.*

In the special case $m = 2$, we can state a refined result which is due to Niederman ([Nie07a]).

Theorem A.8. *For Lebesgue almost all $\beta \in S_n(\mathbb{R})$, the function $h(I) = \alpha.I + \beta I.I$ belongs to $\in SDM^{\tau'}(B)$ provided $\tau' > n^2+1$.*

Note that in the above theorem, there is no condition on α, and contrary to proposition A.3, the proof does not rely on Morse-Sard theory as it uses the following elementary lemma. We will denote by λ the one-dimensional Lebesgue measure and by I_k the identity matrix of size k.

Lemma A.9. *Let $k \in \{1, \dots, n\}$, $\beta_k \in S_k(\mathbb{R})$ and $\kappa > 0$. Then there exists a subset $\mathcal{C}_\kappa \subseteq \mathbb{R}$ such that*

$$\lambda(\mathcal{C}_\kappa) \leq 2k\kappa$$

and for any $\xi \notin \mathcal{C}_\kappa$, the matrix $\beta_{k,\xi} = \beta_k - \xi I_k$ satisfy

$$|\beta_{k,\xi}.\eta| > \kappa|\eta|$$

for any $\eta \in \mathbb{R}^k$.

Of course, the set $\mathcal{C}_\kappa$ depends on the matrix β_k.

Proof. Let $\{\lambda_1, \dots, \lambda_k\}$ be the eigenvalues of β_k, then in an orthonormal basis of eigenvectors for β_k, the matrix $\beta_{k,\xi}$ is also diagonalized, with eigenvalues

$\{\lambda_1-\xi,\dots,\lambda_k-\xi\}$. Then one has $|\beta_{k,\xi}.\eta|>\kappa|\eta|$ for any $\eta\in\mathbb{R}^k$ provided that for all $i\in\{1,\dots,k\}$, $|\lambda_i-\xi|>\kappa$, that is if ξ does not belong to

$$\mathcal{C}_\kappa=\bigcup_{i=1}^{k}[\lambda_i-\kappa,\lambda_i+\kappa].$$

The measure estimate $\lambda(\mathcal{C}_\kappa)\leq 2k\kappa$ is trivial. □

With this lemma, the proof is now similar to proposition A.3.

Proof of theorem A.8. Let $h(I)=\alpha.I+\beta I.I$, and given $\Lambda\in G^L(n,k)$, we denote by $\beta_\Lambda\in S_k(\mathbb{R})$ the matrix which represents the quadratic form $\beta I.I$ restricted on Λ. Since the second derivative of h along any subspace is constant, then coming back to the definition A.1 and using the remark A.2, $h\in SDM_{\gamma'}^{\tau'}$ if

$$|\beta_\Lambda.\eta|>\gamma' L^{-\tau'}|\eta| \tag{2}$$

for any $\Lambda\in G^L(n,k)$ and $\eta\in\mathbb{R}^n$. Let $A_{\gamma'}^{\tau'}$ be the subset of $S_n(\mathbb{R})$ whose elements contradicts the condition (2) and $A^{\tau'}=\cap_{\gamma'>0}A_{\gamma'}^{\tau'}$, what we need to show is that $A^{\tau'}$ has zero Lebesgue measure in $S_n(\mathbb{R})$ provided $\tau'>n^2+1$.

So we apply lemma A.9 to $\beta_\Lambda\in S_k(\mathbb{R})$ with $\kappa=\gamma' L^{-\tau'}$ to have a subset $\mathcal{C}_{\gamma',\tau',L,\Lambda}\subseteq\mathbb{R}$ such that

$$\lambda(\mathcal{C}_{\gamma',\tau',L,\Lambda})\leq 2k\gamma' L^{-\tau'} \tag{3}$$

and for any $\xi\notin\mathcal{C}_{\gamma',\tau',L,\Lambda}$, the matrix $\beta_{\Lambda,\xi}=\beta_\Lambda-\xi I_k$ satisfy

$$|\beta_{\Lambda,\xi}.\eta|>\gamma' L^{-\tau'}|\eta|$$

for any $\eta\in\mathbb{R}^n$. If we define

$$\mathcal{C}_{\gamma',\tau'}=\bigcup_{L\in\mathbb{N}^*}\bigcup_{k\in\{1\dots,n\}}\bigcup_{\Lambda\in G^L(n,k)}\mathcal{C}_{\gamma',\tau',L}$$

then

$$\mathcal{C}_{\gamma',\tau'}=\{\xi\in\mathbb{R}\mid\beta_\xi\in A_{\gamma'}^{\tau'}\}$$

and so

$$\mathcal{C}_{\tau'}=\bigcap_{\gamma'>0}\mathcal{C}_{\gamma',\tau'}=\{\xi\in\mathbb{R}\mid\beta_\xi\in A^{\tau'}\}.$$

It remains to prove that the Lebesgue measure of $\mathcal{C}_{\tau'}$ is zero, since by Fubini theorem, this will imply that the Lebesgue measure of $A^{\tau'}$ is zero. By our estimate (3) we have

$$\begin{aligned}\lambda(\mathcal{C}_{\gamma',\tau'}) &\leq \sum_{L\in\mathbb{N}^*}\sum_{k=1}^{n}|G^L(n,k)|2k\gamma' L^{-\tau'}\\ &\leq \sum_{L\in\mathbb{N}^*}\sum_{k=1}^{n}L^{n^2}2k\gamma' L^{-\tau'}\\ &\leq 2\left(\sum_{k=1}^{n}k\right)\left(\sum_{L\in\mathbb{N}^*}L^{n^2-\tau'}\right)\gamma'\end{aligned}$$

and since $\tau' > n^2+1$, the above series is convergent. To conclude, note that

$$\lambda(\mathcal{C}_{\tau'}) = \inf_{\gamma'>0}\lambda(\mathcal{C}_{\gamma',\tau'}) = 0. \qquad \square$$

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