EXPONENTIALLY SMALL SPLITTING OF SEPARATRICES IN THE PERTURBED MCMILLAN MAP

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Abstract. The McMillan map is a one-parameter family of integrable symplectic maps of the plane, for which the origin is a hyperbolic fixed point with a homoclinic loop, with small Lyapunov exponent when the parameter is small. We consider a perturbation of the McMillan map for which we show that the loop breaks in two invariant curves which are exponentially close one to the other and which intersect transversely along two primary homoclinic orbits. We compute the asymptotic expansion of several quantities related to the splitting, namely the Lazutkin invariant and the area of the lobe between two consecutive primary homoclinic points. Complex matching techniques are in the core of this work. The coefficients involved in the expansion have a resurgent origin, as shown in [MSS08].

1. Preliminaries and Main Results

1.1. **Introduction.** This article and its companion [MSS08] are devoted to the study of the exponentially small splitting of separatrices in a particular family of maps of the plane: a two-parameter family of analytic symplectic maps, which contains a one-parameter subfamily composed of integrable maps known as the McMillan map. The McMillan map was introduced in [McM71] in connection with the modelization of particle accelerator dynamics; it has a hyperbolic fixed point at the origin, for which there is a homoclinic loop. We prove that, generically, for the perturbed McMillan map (i.e. for our two-parameter family) the homoclinic connection is destroyed: it splits in two invariant curves (stable and unstable manifolds of the hyperbolic fixed point) which intersect transversely. We obtain an asymptotic formula for the area of the lobe delimited by the two curves between two consecutive intersection points and for the Lazutkin invariant, a quantity related to the angle of intersection, introduced in [GLT91] and commonly used in the literature about splitting. Our results generalize and improve those of [DRR98].

In the problem considered, the two parameters play very different roles. One of them, which we will call ε, is a regular parameter. It measures the size of the perturbation (the integrable McMillan map corresponds to $\varepsilon = 0$), and all the quantities and geometric objects under consideration will depend analytically on it; this parameter will not be assumed to be small. The other parameter, h, is precisely the Lyapunov exponent of the origin for the McMillan map. Hence, when this parameter tends to zero, the origin is a weakly hyperbolic fixed point; as a consequence, a well-known result in [FS90] shows that the splitting of the curves must be exponentially small with respect to h.

The problem of exponentially small splitting has been addressed by several authors (e.g. [SMH91, DS97, DGJS97, Tre97, LMS03, OSS03, DG04]), because of its relevance for the non-integrability of Hamiltonian systems (see [Yoc06] for its relation with Poincaré's mistake in his 1889 memoir) and for the Arnold diffusion mechanism in the case of at least three degrees of freedom. The problem was studied in detail mainly for flows, but there are relatively few works dealing with symplectic maps. The famous Lazutkin paper of 1984 (see [Laz03] for the English translation) was the first work concerning the exponentially small separatrix splitting for a one-parameter family of maps, namely the standard map. Although important ideas where already present in that work, the complete proof of the results did not appear till fifteen years later, in [Gel99]. Some asymptotic computations related with the problem of the exponentially small splitting of the standard map were done in [HM93, Sur94] and for the Hénon map in [TTJ98].

The two-parameter family of maps considered in the present paper is essentially the same as in the article [DRR98]. That article provided a rigorous asymptotic formula for the separatrix splitting in the case where the regular perturbation parameter ε is small enough with respect to the singular parameter h, validating the prediction of the Melnikov formula adapted for maps given by [DRR96] (the possibility of taking $\varepsilon = O(h^p)$ with $p > 0$ is an advantage of the presence of two parameters which has no analogue in a one-parameter family like the standard map). We shall remove the smallness assumption on ε, thus reaching a situation which displays the same complexity as the standard map. We shall see that in the non-perturbative case the Melnikov formula does not predict the correct size of the splitting, whereas it does when ε and h are small but independent. Furthermore, the formula we obtain provides the full asymptotic expansion in h of the first exponentially small term in the splitting.

We now give a brief description of our method and its innovative features. Our study splits in two parts, corresponding to "outer" and "inner" domains; we found it convenient to devote a separate article [MSS08] to the inner part.

As in [Laz03, Gel99, DRR98], the detection of the exponentially small splitting relies on considering suitable parametrizations of the invariant curves. These parametrizations will be analytic in a complex strip whose size is limited by the singularities of the unperturbed homoclinic orbit. When the perturbation parameter ε is small with respect to h, the manifolds are well approximated by the unperturbed homoclinic even off the real line, as in [DRR98]. However, when ε is of order one, we need to deal with different approximations of the parametrizations of the invariant curves in different zones of the complex plane; the leading terms in the asymptotic expansion will be found as solutions of the so-called "inner equation". This equation needs its own study, using Borel resummation techniques and resurgence theory, and this is done in [MSS08]. (A study of an inner equation of the same kind but for the Hénon map can be found in [GS01].) "Complex matching" techniques are then needed to conclude.

In order to have access to the whole asymptotic expansion with respect to h in the first exponentially small term of the formula of the splitting, we need to study not only the "first inner equation" but all the "inner equations" involved in the problem, related to higher order powers in h. This entails the use of resurgence theory in equations with parameters in [MSS08] and matching procedures at any order in the present article.

One of the main differences between our work and the previous ones is the fact that we do not use "complex flow box variables" to obtain a good "splitting function" which measures the distance between both manifolds. Instead, we provide a formula for the difference of the parametrizations of the manifolds directly in the original variables of the problem—see formula (25) below. The key idea, that was already used in [Sau01] in the case of flows, is to exploit a linear difference equation which is satisfied by this difference and for which a basis of solutions can be described precisely enough; the difference has two be a linear combination of the basis solutions with h-periodic coefficients and one can then resort to a classical lemma about periodic functions of a complex variable (Lemma 3.3) to obtain exponentially small bounds on the real line from larger bounds in a complex strip.

1.2. **The unperturbed and perturbed McMillan maps.** The McMillan map is defined by the formulas

$$F_{h,0}\colon (x,y) \mapsto (x^*,y^*) \qquad \left| \begin{array}{l} x^* = y \\ y^* = -x + \dfrac{2(\cosh h)y}{1+y^2}, \end{array} \right.$$

where $h > 0$ is a parameter. It is a symplectic transformation of $\mathbb{R}^2$ (for the standard structure $\mathrm{d}x \wedge \mathrm{d}y$), which is integrable in the sense that it admits the following polynomial first integral:

$$H^0(x,y) = x^2 - 2(\cosh h)xy + y^2 + x^2y^2.$$

The origin is a hyperbolic fixed point, with characteristic exponents $\pm h$. Its stable and unstable manifolds coincide: the level curve $\{\, H^0 = 0 \,\}$ is formed of two homoclinic loops, one of which lies in the first quadrant and is explicitly given by $\mathcal{W}^0 = \{z^0(t),\ t \in \mathbb{R}\}$, with $z^0(t) = \big(\xi^0(t-h/2), \xi^0(t+h/2)\big)$ and

$$\xi^0(t) = \xi^0(t,h) = \frac{\gamma}{\cosh t}, \qquad \gamma = \sinh h, \tag{1}$$

in such a way that $F_{h,0}\big(z^0(t)\big) = z^0(t+h)$. Unless is convenient for clarity, we will not write explicitly the dependence of ξ^0 on h. We shall refer to $\mathcal{W}^0$ as "the unperturbed separatrix"; the other loop is obtained by symmetry with respect to the origin—see Figure 1.

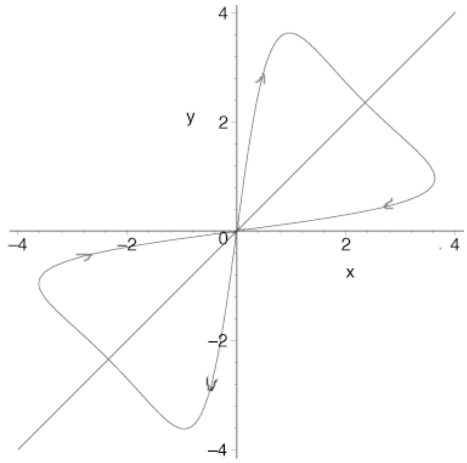

FIGURE 1. Unperturbed separatrix ($\varepsilon = 0$, $h = 2$)

Observe that for small h the homoclinic loops are small: $\|z^0(t)\|$ is $O(h)$ uniformly in t. See [Sur89] and [DRR98] for more on the McMillan map.

From now on, we shall use the notations

$$f(y) = \frac{2y}{1+y^2}, \qquad \mu = \cosh h. \tag{2}$$

The perturbation of the McMillan map that we consider is

$$F_{h,\varepsilon}\colon (x,y) \mapsto (x^*,y^*) \qquad \left| \begin{array}{l} x^* = y \\ y^* = -x + \mu f(y) + \varepsilon \tilde{V}'(y), \end{array} \right. \tag{3}$$

where the "perturbative potential"

$$\tilde{V}(y) = \sum_{k\geq 2} V_k y^{2k}$$

is an even analytic function, which is defined in a neighborhood of 0 and supposed to be $O(y^4)$, and $\varepsilon \in \mathbb{R}$ is a new parameter (not necessarily small). The maps $F_{h,\varepsilon}$ are defined in a neighborhood of the origin and symplectic. The only difference with [DRR98] is that we do not assume $\tilde{V}$ to be entire.

Since $\tilde{V}'(y) = O(y^3)$, the origin is still a hyperbolic fixed point with characteristic exponents $\pm h$; its stable and unstable manifolds are curves which have no reason to coincide any longer. The aim of this paper is precisely to show that, generically, for nonzero ε and small h the stable and unstable curves intersect transversely, and to measure the way they depart one from the other; the homoclinic loops are broken, this is the so-called "separatrix splitting" phenomenon—see Figure 2. As is well-known, the existence of a transversal homoclinic intersection has dramatic dynamical consequences, even though the phenomenon is exponentially small.

We shall focus on the part $\mathcal{W}^s_{h,\varepsilon}$, resp. $\mathcal{W}^u_{h,\varepsilon}$, of the stable curve, resp. unstable curve, which lies in the first quadrant. Anyway, since the function $\mu f + \varepsilon \tilde{V}'$ is odd, the dynamics of $F_{h,\varepsilon}$ is symmetric with respect to the origin. The analysis will be simplified by another kind of symmetry: the map $F_{h,\varepsilon}$ and its inverse $F^{-1}_{h,\varepsilon}$ are conjugate by the involution $R\colon (x,y) \mapsto (y,x)$ (the map is "reversible"); this implies that

$$\mathcal{W}^s_{h,\varepsilon} = R(\mathcal{W}^u_{h,\varepsilon}). \tag{4}$$

Moreover, at least for small $|\varepsilon|$, both curves intersect the symmetry line $\Delta_R = \{x = y\}$ because they are close to the unperturbed separatrix $\mathcal{W}^0$ and, by (4), a point of $\mathcal{W}^u_{h,\varepsilon} \cap \Delta_R$ is necessarily a homoclinic point (i.e. it also belongs to $\mathcal{W}^s_{h,\varepsilon}$).

1.3. **Main Theorem, geometrical version.** The article [DRR98] shows that, when $\tilde{V}$ is entire and $\varepsilon = o\big(h^6/|\ln h|\big)$, there are generically two primary homoclinic orbits in the first quadrant for small h (one of which has a point on Δ_R), and it yields an estimate of the lobe area enclosed by $\mathcal{W}^u_{h,\varepsilon}$ and $\mathcal{W}^s_{h,\varepsilon}$ between two successive intersection points (this area is invariant under the dynamics of $F_{h,\varepsilon}$). We shall see that the same result holds generically in our case with independent parameters ε and h (we shall assume h small but remove the smallness assumption on $|\varepsilon|$).

We shall estimate the *algebraic lobe area* A (with the same convention for its sign as in [DRR98]—see below) and another quantity: the *Lazutkin homoclinic invariant* ω [GLT91], the definition of which we now recall.

It is known that there must exist a "natural parametrization" for $\mathcal{W}^u_{h,\varepsilon}$, i.e. this curve can be injectively parametrized by a solution $t \mapsto z^u(t)$ of

$$F_{h,\varepsilon}\big(z^u(t)\big) = z^u(t+h), \qquad z^u(t) \xrightarrow[t\to-\infty]{} (0,0) \tag{5}$$

(see e.g. [DRR98], p. 328, or [GLT91], and also Proposition 1.4 below). We shall see that there exists t_* such that $z^u(t_*) \in \Delta_R$. We can assume that this occurs for $t_* = 0$ (by shifting the parametrization if necessary: $t \mapsto z^u(t + t_*)$ is also solution of (5)). Using reversibility and defining

$$z^s(t) = R\big(z^u(-t)\big), \tag{6}$$

we then get a natural parametrization of $\mathcal{W}^s_{h,\varepsilon}$ and $z^s(0) = z^u(0)$ is a homoclinic point. In this situation, the Lazutkin homoclinic invariant is

$$\omega = \det(\dot{z}^s(0), \dot{z}^u(0)). \tag{7}$$

This is an intrinsic quantity, related to the splitting angle.

Here is the convention for the definition of the algebraic lobe area A: if the intersection of the curves is transversal, i.e. if $\omega \neq 0$, the preservation of orientation by $F_{h,\varepsilon}$ implies that there must exist another homoclinic point between

$z^u(0) = z^s(0)$ and its image $z^u(h) = z^s(h)$; we say that there are only two primary homoclinic orbits if there is only one such other point, say $z^s(t^*) = z^u(t^{**})$ with $0 < t^*, t^{**} < h$; we then have[1]

$$z^s(t^*) = z^u(h - t^*) \tag{8}$$

and we call A the area enclosed by the simple loop made of the path $t \in [0, t^*] \mapsto z^s(t)$ followed by $t \in [t^*, h] \mapsto z^u(h-t)$, counted positively if and only if this loop is traveled anticlockwise (as on Figure 2).

Theorem 1.1 (Main Theorem, geometrical version). *Let* ε_0 *be positive* $< 1/|2V_2|$ *and*

$$\widehat{V}(\zeta) = \sum_{k\geq 2} V_k \frac{\zeta^{2k-1}}{(2k-1)!}. \tag{9}$$

There exist constants $h_0, c > 0$ *and real analytic functions* $B_k^+(\varepsilon)$, $k \in \mathbb{N}$, *holomorphic for complex* ε *of modulus* $< 1/|2V_2|$, *such that*

$$B_0^+(\varepsilon) = 4\pi^2 \widehat{V}(2\pi) + O(\varepsilon), \tag{10}$$

satisfying the following:
– If $0 < h < h_0$ *and* $-\varepsilon_0 < \varepsilon < \varepsilon_0$, *then* $\mathcal{W}^s_{h,\varepsilon}$ *and* $\mathcal{W}^u_{h,\varepsilon}$ *have an intersection point on the half-line* $\{x = y > 0\}$ *at which the Lazutkin homoclinic invariant admits the following asymptotic expansion with respect to* h*:*

$$\omega \sim \frac{4\pi\varepsilon}{\alpha^2 h^2} e^{-\frac{\pi^2}{h}} \sum_{k\geq 0} h^{2k} B_k^+(\varepsilon), \tag{11}$$

where α *is the positive constant defined by*

$$\alpha^2 = 1 - \frac{2\varepsilon V_2}{\cosh h}. \tag{12}$$

– If moreover

$$0 < h^2 < c|B_0^+(\varepsilon)|, \tag{13}$$

then the aforementioned intersection is transversal, there are only two primary homoclinic orbits in the first quadrant and the lobe area admits the following asymptotic expansion with respect to h*:*

$$A \sim \frac{2\varepsilon}{\pi\alpha^2} e^{-\frac{\pi^2}{h}} \sum_{k\geq 0} h^{2k} B_k^+(\varepsilon). \tag{14}$$

Theorem 1.1 will be proved in Section 1.5, as a consequence of Theorem 1.5 below.

[1] because (6), (5) and the property of z^s analogous to (5) deduced from reversibility imply that $z^u(h-t^*) = R\big(z^s(t^*-h)\big) = R\circ F^{-1}_{h,\varepsilon}\big(z^s(t^*)\big) = R\circ F^{-1}_{h,\varepsilon}\big(z^u(t^{**})\big) = R\big(z^u(t^{**}-h)\big) = z^s(h-t^{**})$ and, since $0 < h-t^*, h-t^{**} < h$, the uniqueness of the other primary homoclinic orbit imply (8).

Remark 1.2. In fact, we shall see in Section 1.5 that condition (13) can be replaced by a more technical but also more general one: there are constants $c_0, c_1, \dots$ such that the result holds as soon as there exists an integer N_0 such that

$$0 < h^{2N_0+2} < c_{N_0} \left| B_0^+(\varepsilon) + h^2 B_1^+(\varepsilon) + \dots + h^{2N_0} B_{N_0}^+(\varepsilon) \right| \tag{15}$$

(still with $0 < h < h_0$ and $-\varepsilon_0 < \varepsilon < \varepsilon_0$). Thus in principle, by an appropriate choice of N_0, one may increase the range of validity of the result. In particular, if condition (13) fails because $B_0^+(\varepsilon)$ happens to be zero, one can still try condition (15) with $N_0 = 1$, and so on. However, notice that we have little information on the numbers $B_k^+(\varepsilon)$ (apart from the value of B_0^+ at $\varepsilon = 0$—see Remark 1.3).

In (11) and (14), the symbol "$\sim$" means that the series in the right hand sides are asymptotic to the left hand sides in the classical sense, i.e. truncating the series at order N provides an expression for the left hand side with an error that is of the order of the first neglected term within the range $h \in (0, h_0)$ uniformly with respect to $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, with the restriction (13) or (15) in the case of A. However, the series in the right hand sides need not be convergent. In fact, numerical studies in [GS08] indicate that these series are Gevrey-1, i.e. that there exist constants $C, M > 0$ such that the coefficient $B_k^+(\varepsilon)$ of h^{2k} is bounded by $CM^k(2k)!$.

The function $\widehat{V}$ defined by (9) is an entire function (because of the Cauchy estimates for the Taylor coefficients of $\tilde{V}$ at the origin); it is the *Borel transform* of $\tilde{V}$ with respect to $1/y$ (see [MSS08] for more on the Borel transform).

Remark 1.3. Suppose $\widehat{V}(2\pi) \neq 0$ (which is true for generic $\tilde{V}$). Then there exists $\varepsilon_1 < \varepsilon_0$ such that $B_0^+(\varepsilon) \neq 0$ for $|\varepsilon| \leq \varepsilon_1$; thus condition (13) is fulfilled for $-\varepsilon_1 < \varepsilon < \varepsilon_1$ and $0 < h < h_1$ with a value of h_1 independent of ε. This is thus an improvement of the range of validity of the result obtained in [DRR98]: the Melnikov approximation

$$\omega \sim 16\pi^3 \widehat{V}(2\pi) \frac{\varepsilon}{h^2} e^{-\frac{\pi^2}{h}} \big[1 + O(h^2) + O(\varepsilon)\big],$$
$$A \sim 8\pi \widehat{V}(2\pi) \varepsilon\, e^{-\frac{\pi^2}{h}} \big[1 + O(h^2) + O(\varepsilon)\big]$$

is valid for ε and h small and independent—one can relax the assumption $\varepsilon = o\big(h^6/|\ln h|\big)$. But our result is at the same time an extension to the case when ε is not small; then the Melnikov approximation is no longer correct: one must use the coefficient $\alpha^{-2} B_0^+(\varepsilon)\varepsilon$ instead of $4\pi^2 \widehat{V}(2\pi)\varepsilon$.

Another improvement is the fact that Theorem 1.1 provides the full asymptotic expansion, involving the new coefficients $B_k^+(\varepsilon)$, $k \geq 1$, for the Lazutkin invariant and the lobe area.

Furthermore, the appearance of the Borel transform $\widehat{V}$ in the Melnikov approximation will receive a very natural explanation in our proof; this proof indeed relies on the Borel-Laplace summation process, which is at the basis of resurgence theory, and it attributes to the coefficients $B_k^+(\varepsilon)$ a resurgent origin. The reader is referred to Section 2.7 and [MSS08].

1.4. **Rephrasing in terms of solutions of a second-order difference equation. Analytic version of the theorem.** To study the stable and unstable curves, we shall use natural parametrizations as alluded above, i.e.

$$\mathcal{W}^u_{h,\varepsilon} = \{z^u(t)\}, \qquad \mathcal{W}^s_{h,\varepsilon} = \{z^s(t)\},$$

with z^u and z^s particular solutions of the system of first-order difference equations

$$z(t+h) = F_{h,\varepsilon}\big(z(t)\big). \tag{16}$$

The property $x^* = y$ in (3) implies that $t \mapsto z(t)$ is solution of (16) if and only if it can be written

$$z(t) = \big(\xi(t-h/2), \xi(t+h/2)\big)$$

with $t \mapsto \xi(t)$ solution of the second-order difference equation

$$\xi(t+h) + \xi(t-h) = \mu f(\xi(t)) + \varepsilon \tilde{V}'(\xi(t)). \tag{17}$$

For instance, for the McMillan map ($\varepsilon = 0$), the function ξ^0 defined in (1) satisfies

$$\xi^0(t+h) + \xi^0(t-h) = \mu f(\xi^0(t)). \tag{18}$$

Finding a parametrization z^u of $\mathcal{W}^u_{h,\varepsilon}$ which satisfies (5) is thus equivalent to finding a solution of (17) which satisfies

$$\lim_{t\to-\infty} \xi^u(t) = 0 \text{ and } \xi^u(t) > 0 \text{ for } -t \text{ large enough}, \tag{19}$$

and writing $z^u(t) = (\xi^u(t-h/2), \xi^u(t+h/2))$ (the positivity condition in (19) is meant to distinguish the part of the unstable curve which starts in the first quadrant; the symmetry of this curve with respect to the origin is reflected in the fact that $-\xi^u$ is solution of (17) if ξ^u is).

Proposition 1.4. *For any $h > 0$ and $\varepsilon \in \mathbb{R}$, there exists a solution $\tilde{\xi}^u$ of equation* (17) *which satisfies the boundary condition* (19) *and which is real-analytic and $2\pi i$-periodic in a half-plane $\{\operatorname{Re} t < -T^*\}$, with a constant $T^* > 0$ (which depends on h and ε). Moreover, such a solution $\tilde{\xi}^u(t)$ is unique up to a translation $\tilde{\xi}^u(t) \to \tilde{\xi}^u(t-\tau)$ with arbitrary $\tau \in \mathbb{R}$ (which may depend on h and ε).*

Proof. With the change of variable $\zeta = e^t$, this corresponds to searching a solution $\zeta \mapsto Z(\zeta)$ of the equation $Z(e^h\zeta) = F_{h,\varepsilon}\big(Z(\zeta)\big)$, the components of which are holomorphic real-analytic near $\zeta = 0$ and positive for small $\zeta > 0$, with $Z(0) = 0$ (indeed, the $2\pi i$-periodicity, the holomorphy in a half-plane and (19) imply the existence of a convergent Fourier expansion $\sum_{n\geq 1} e^{nt} Z_n$). It is easy to see that this problem has a solution which is unique up to rescaling $Z(\zeta) \to Z(c\zeta)$ with arbitrary $c > 0$ (this is the analytic version of the stable manifold theorem for $F_{h,\varepsilon}^{-1}$; it is sufficient to look at the equations obtained by expanding a solution in the form $Z(\zeta) = \sum_{n\geq 1} \zeta^n Z_n$, one finds Z_1 proportional to $(1, e^h)$ with an arbitrary positive proportionality factor, the other terms are determined inductively and easy to bound). □

From now on, we denote by $\tilde{\xi}^u(t)$ one of the solutions given by Proposition 1.4. We shall see that it has an analytic continuation to any real interval $(-\infty, T]$, provided h is small enough,[2] and choose $\tau \in \mathbb{R}$ (depending on h and ε) so that

$$\xi^u(t) = \tilde{\xi}^u(t - \tau) \tag{20}$$

satisfies the condition

$$\xi^u(-h/2) = \xi^u(h/2). \tag{21}$$

Equation (21) corresponds to the condition $z^u(0) \in \Delta_R$ which was introduced at the beginning of Section 1.3.

The reversibility property of $F_{h,\varepsilon}$ is reflected in the fact that $t \mapsto \xi(-t)$ is solution of (17) whenever $t \mapsto \xi(t)$ is. Once ξ^u is found, the formula $\xi^s(t) = \xi^u(-t)$ defines a solution ξ^s of (17) which satisfies the boundary conditions (21) and

$$\lim_{t\to+\infty} \xi^s(t) = 0 \text{ and } \xi^s(t) > 0 \text{ for } t \text{ large enough}, \tag{22}$$

hence $z^s(t) = (\xi^s(t - h/2), \xi^s(t + h/2))$ is a natural parametrization of $\mathcal{W}^s_{h,\varepsilon}$ which intersects $\mathcal{W}^u_{h,\varepsilon}$ at $t = 0$. The splitting problem is thus reduced to studying the difference

$$D(t) = \xi^u(t) - \xi^s(t) = \xi^u(t) - \xi^u(-t).$$

Theorem 1.5 (Main Theorem, analytical version). *Let $\varepsilon_0 < 1/|2V_2|$ and $T > 0$. There exist $h_0, C_0 > 0$ such that, for any h and $\varepsilon \in \mathbb{R}$ with $0 < h < h_0$ and $|\varepsilon| < \varepsilon_0$, there exists a unique $\tau \in \mathbb{R}$ such that $\xi^u(t) = \tilde{\xi}^u(t - \tau)$ extends analytically to $(-\infty, T]$, satisfies (21) and $|\xi^u(t) - \xi^0(t)| \leq C_0|\varepsilon|h^3$ for all $t \in (-\infty, T]$.*

Moreover, there exists a sequence $(\xi^{N,\text{out}})_{N\geq 0}$ of even real-analytic functions defined on $\mathbb{R}$, with $\xi^{0,\text{out}} = \xi^0$, and constants $C_N > 0$ such that, for any $N \geq 0$,

$$\Big|\frac{d^j}{dt^j}(\xi^u - \alpha^{-1}\xi^{N,\text{out}})(t)\Big| \leq C_N|\varepsilon|h^{2N+3}, \qquad t \in (-\infty, T], \quad j = 0, 1, 2, 3, \tag{23}$$

where α is the constant defined in (12).

[2]No smallness condition on h is needed for this when $\tilde{V}$ is defined on the whole of the real axis: the function $\mathcal{F} = \mu f + \varepsilon\tilde{V}'$ is then defined on $\mathbb{R}$ and the definition of $\tilde{\xi}^u$ can be propagated from $(-\infty, -T^*)$ to $(-\infty, -T^* + h)$ and then to any interval $(-\infty, -T^* + nh)$, $n \geq 1$, by rewriting equation (17) as $\tilde{\xi}^u(t) = \mathcal{F}\big(\tilde{\xi}^u(t - h)\big) - \tilde{\xi}^u(t - 2h)$. In the general case, the smallness of h ensures that $\tilde{\xi}^u(t)$ remains in the domain of definition of $\tilde{V}'$ for $t \in (-\infty, T]$ when using the same argument.

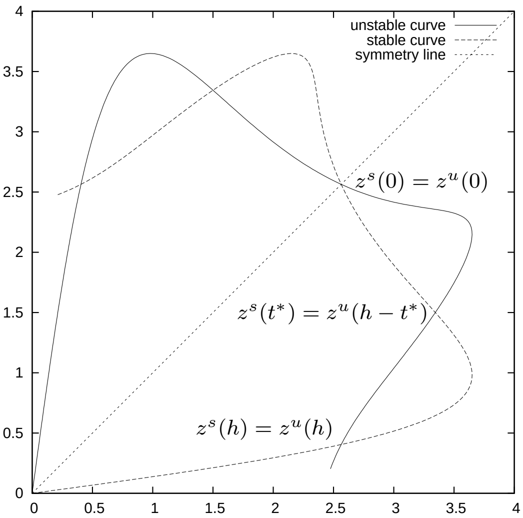

FIGURE 2. Invariant curves for $\tilde{V}'(y) = y^3$, $h = 2$ and $\varepsilon = 0.025$. Thanks to A. Delshams and R. Ramírez-Ros.

Consider the function

$$D(t) = \xi^u(t) - \xi^u(-t). \tag{24}$$

There exist real analytic functions c_1, c_2, ν_1, ν_2 defined in $[-T, T]$ such that

$$D = \alpha^{-1}(c_1\nu_1 + c_2\nu_2) \ \text{ on } [-T, T] \tag{25}$$

and

- *c_1 and c_2 are h-periodic and*

$$\sup_{t\in\mathbb{R}} \left| \frac{d^j}{dt^j} c_1(t) \right| \le \frac{C_0|\varepsilon|}{h^{4+j}} e^{-\frac{\pi^2}{h}}, \qquad j = 0, 1, 2, \tag{26}$$

$$\sup_{t\in\mathbb{R}} \left| \frac{d^j}{dt^j} \Big(c_2(t) - c_2^N(t) \Big) \right| \le C_N |\varepsilon| e^{-\frac{\pi^2}{h}} h^{2N+1-j}, \qquad j = 0, 1, 2, \tag{27}$$

where

$$c_2^N(t) = -\frac{2\varepsilon}{h} e^{-\pi^2/h} \left(\sum_{k=0}^{N} h^{2k} B_k^+(\varepsilon) \right) \sin \frac{2\pi t}{h}, \tag{28}$$

with real-analytic functions B_k^+, holomorphic for $|\varepsilon| < \varepsilon_0$, satisfying (10),

• *ν_1 and ν_2 satisfy*

(29) $$\begin{vmatrix} \nu_1(t) & \nu_2(t) \\ \nu_1(t+h) & \nu_2(t+h) \end{vmatrix} = 1, \qquad t \in [-T, T-h],$$

(30) $$\sup_{t\in[-T,T]} \left| \frac{d^j}{dt^j} \Big(\nu_1 - \frac{d}{dt} \xi^{N,\mathrm{out}} \Big)(t) \right| \le C_N |\varepsilon| h^{2N+3}, \qquad j = 0,1,2,\ N \in \mathbb{N},$$

(31) $$\sup_{t\in[-T,T]} \left| \frac{d^j}{dt^j} \nu_2(t) \right| \le \frac{C_0}{h^2}, \qquad j = 0,1,2.$$

The proof of Theorem 1.5 will start in Section 2. Observe that, in view of (23), the defect of evenness measured by $D(t)$ has to be $O(h^n)$ for any n; in fact it is exponentially small, as shown by the exact formula (25) and the information on c_1, c_2, ν_1, ν_2 provided in (26)–(31), and $\alpha^{-1} c_2^N(t)$ will account for the dominant part of the splitting phenomenon.

The functions ν_1 and ν_2 will be obtained as particular solutions of a certain *linear* second-order difference equation. In the theory of linear difference equations, the determinant

(32) $$W_h(\phi_1, \phi_2)(t) := \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1(t+h) & \phi_2(t+h) \end{vmatrix}$$

is called discrete Wronskian (or Casoratian), and it is constant for a pair of solutions of the kind of equations we are interested in—see Section 4.

1.5. **Deduction of Theorem 1.1 from Theorem 1.5.** Let $h \in (0, h_0)$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $\xi^u(t)$ be as in Theorem 1.5 and set $\xi^s(t) = \xi^u(-t)$. We denote by $z^u = (x^u, y^u)$ and $z^s = (x^s, y^s)$ the natural parametrizations of the unstable and stable invariant manifolds defined by

(33) $$x^{u,s}(t) = \xi^{u,s}(t - h/2), \quad y^{u,s}(t) = \xi^{u,s}(t + h/2).$$

In view of (7) the Lazutkin invariant at the homoclinic point $z^u(0) = z^s(0)$ can be written

$$\omega = \frac{d}{dt} \det(z^s - z^u, \dot{z}^u)_{|t=0}.$$

On the other hand, since $\xi^u - \xi^s = D$, (33) yields

(34) $$\det(z^s - z^u, \dot{z}^u)(t) = W_h(\dot{\xi}^u, D)(t - h/2),$$

whence

(35) $$\omega = \frac{d}{dt} W_h(\dot{\xi}^u, D)(t)_{|t=-h/2}.$$

Lemma 1.6. *For any $N \in \mathbb{N}$,*

(36) $$W_h(\dot{\xi}^u, D) = \alpha^{-2} c_2^N + E_N$$

with a function E_N (depending on N, h, ε) such that

$$E_N = O\big(\varepsilon h^{2N+1} e^{-\pi^2/h}\big), \quad \dot{E}_N = O\big(\varepsilon h^{2N} e^{-\pi^2/h}\big) \quad \text{on } [-T,T], \tag{37}$$

where the notation $g = O(f)$ means that there exists a constant $C_N > 0$, that may depend on N but it is independent of h and ε, such that $|g| \le C_N |f|$ on the considered interval.

Proof. For any $N \in \mathbb{N}$, writing the estimates (23) and (30) at $N+1$, we have

$$\frac{d^j}{dt^j}(\dot{\xi}^u - \alpha^{-1}\nu_1) = O(\varepsilon h^{2N+5}) \quad \text{on } [-T,T], \qquad j = 0,1,2. \tag{38}$$

We thus define

$$E = W_h(\dot{\xi}^u - \alpha^{-1}\nu_1, D)$$

and, using (25), (29) and the h-periodicity of c_1 and c_2, we get

$$W_h(\dot{\xi}^u, D)(t) = \alpha^{-2} c_2(t) + E(t).$$

Let $N \in \mathbb{N}$. Formula (36) holds with $E_N = E + \alpha^{-2}(c_2 - c_2^N)$ and, since (27) yields
$c_2 - c_2^N = O(\varepsilon h^{2N+1} e^{-\pi^2/h})$ and $\dot{c}_2 - \dot{c}_2^N = O(\varepsilon h^{2N} e^{-\pi^2/h})$, it is sufficient to control E and $\dot{E}$.

We have

$$\frac{d^j D}{dt^j} = O\big(\varepsilon h^{-3-j} e^{-\pi^2/h}\big) \quad \text{on } [-T,T], \qquad j = 0,1,2 \tag{39}$$

as a consequence of the bounds $\frac{d^j c_1}{dt^j} = O\big(h^{-4-j} e^{-\pi^2/h}\big)$ (as stated in (26)), $\frac{d^j \nu_1}{dt^j} = O(\varepsilon h)$ (which follows from (1) and (30) with $N = 0$), $\frac{d^j c_2}{dt^j} = O\big(\varepsilon h^{-1-j} e^{-\pi^2/h}\big)$ (which follows from (27) and (28) with $N = 0$) and $\frac{d^j \nu_2}{dt^j} = O(h^{-2})$ (as stated in (31)). Together with (38), this implies

$$\frac{d^j E}{dt^j} = O\big(\varepsilon^2 h^{2N+2-j} e^{-\pi^2/h}\big) \quad \text{on } [-T,T], \qquad j = 0,1,2, \tag{40}$$

and the conclusion follows. □

The asymptotic expansion (11) for ω follows from (35) and Lemma 1.6, since we can write $\omega = \alpha^{-2}\dot{c}_2^N(-h/2) + \dot{E}_N(-h/2)$ and (28) shows that $\dot{c}_2^N(-h/2) = 4\pi\varepsilon h^{-2} e^{-\pi^2/h} \sum_{k=0}^N h^{2k} B_k^+(\varepsilon)$.

We now assume that there exists N_0 such that (15) holds, with a constant c_{N_0} that we shall specify later, and we proceed to show that there is only one primary homoclinic orbit other than the orbit of $z^u(0) = z^s(0)$ and compute the lobe area. To this end, we shall use a linear change of variables, so as to make the manifolds appear as graphs over the first coordinate, and a reparametrization of $\mathcal{W}^s_{h,\varepsilon}$.

Figure 1 suggests the linear symplectic change of variables

$$\tilde{x} = \frac{\sqrt{2}}{2}(x-y),\qquad \tilde{y} = \frac{\sqrt{2}}{2}(x+y). \tag{41}$$

We define $\tilde{z}^u = (\tilde{x}^u, \tilde{y}^u)$ and $\tilde{z}^s = (\tilde{x}^s, \tilde{y}^s)$ by means of the above relations. By (33) and (1), at first order in ε one finds

(42)

$$\tilde{x}^s_{|\varepsilon=0}(t) = \tilde{x}^u_{|\varepsilon=0}(t) = \frac{\sqrt{2}}{2}(\xi^0(t-h/2) - \xi^0(t+h/2)) = \gamma\sqrt{2}\frac{\sinh\frac{h}{2}\sinh t}{\cosh^2(\frac{h}{2}) + \sinh^2(t)}.$$

Since $\dot{\tilde{x}}^u - \alpha^{-1}\dot{\tilde{x}}^u{}_{|\varepsilon=0}$ and $\dot{\tilde{x}}^s - \alpha^{-1}\dot{\tilde{x}}^s{}_{|\varepsilon=0}$ are $O(\varepsilon h^3)$ (because of (23) with $N=0$), we can find $K>1$ and $t_0>0$ independent of h and ε such that, for $t \in [-t_0, t_0]$,

$$K^{-1}h^2 < \dot{\tilde{x}}^u(t) < Kh^2, \quad K^{-1}h^2 < \dot{\tilde{x}}^s(t) < Kh^2. \tag{43}$$

In particular, $\tilde{x}^u$ and $\tilde{x}^s$ are invertible in $[-t_0, t_0]$ and the manifolds $\tilde{z}^u$, $\tilde{z}^s$ are graphs over the $\tilde{x}$ variable. Moreover, setting $t_1 = t_0/K^2$,

$$\tilde{x}^u\big((-t_1,t_1)\big) \subset (-Kh^2t_1, Kh^2t_1) = (-K^{-1}h^2t_0, K^{-1}h^2t_0) \subset \tilde{x}^s\big((-t_0,t_0)\big),$$

consequently, the function

$$\phi = (\tilde{x}^s)^{-1} \circ \tilde{x}^u \tag{44}$$

is well defined in $(-t_1, t_1)$ and a piece of $\mathcal{W}^s_{h,\varepsilon}$ can be reparametrized as

$$\tilde{z}^s\big(\phi(t)\big) = \Big(\tilde{x}^u(t), \tilde{y}^s\big(\phi(t)\big)\Big).$$

Observe that $\tilde{x}^u(0) = \tilde{x}^s(0) = 0$, thus $\phi(0) = 0$ (and more generally $\phi(kh) = kh$ for $k \in \mathbb{Z}$, $|kh| < t_1$, since $\tilde{x}^u$ and $\tilde{x}^s$ coincide on $h\mathbb{Z}$).

Homoclinic points correspond to solutions of the equation

$$\tilde{y}^u(t) - \tilde{y}^s(\phi(t)) = 0. \tag{45}$$

We know that any $t \in h\mathbb{Z} \cap (-t_1, t_1)$ is solution of this equation, and we need to prove that (45) admits only one solution in the interval $(0,h)$. If this is the case and if we denote by t^* the unique solution of (45) in $(0,h)$, then there will be exactly two primary homoclinic orbits, the orbits of $z^u(0) = z^s(0)$ and $z^u(t^*) = z^s\big(\phi(t^*)\big)$, and according to the definition of Section 1.3 the lobe area will be given by

$$A = \int_0^{t^*} \Delta(t)dt, \qquad \Delta(t) = \Big(\tilde{y}^u(t) - \tilde{y}^s\big(\phi(t)\big)\Big)\dot{\tilde{x}}^u(t) \tag{46}$$

(because the change of variables (41) preserves algebraic area; notice that we'll have $\phi(t^*) = h - t^*$ as a consequence of the computation of Section 1.3).

Let us study equation (45) or, equivalently, the equation $\Delta(t) = 0$.

Lemma 1.7. *For any $N \in \mathbb{N}$, there exist a positive constant C_N (independent of h and ε) and a function F_N (depending on N, h, ε) such that*

$$\Delta(t) = \alpha^{-2} c_2^N (t - h/2) + F_N(t) \tag{47}$$

with

$$|F_N| \le C_N |\varepsilon| h^{2N+1} e^{-\pi^2/h}, \quad |\dot{F}_N| \le C_N |\varepsilon| h^{2N} e^{-\pi^2/h} \quad \text{on } [-t_1, t_1]. \tag{48}$$

Proof. We first compute $\psi = \phi - \mathrm{Id}$ in terms of the functions

$$f = \big(\tilde{x}^s\big)^{-1} \quad \text{and} \quad \tilde{D}(t) = \tilde{x}^u(t) - \tilde{x}^s(t) = \frac{\sqrt{2}}{2}\big(D(t - h/2) - D(t + h/2)\big) \tag{49}$$

(the latter function is exponentially small, according to (39)). By Taylor's formula, since $f' \circ \tilde{x}^s = \frac{1}{\dot{\tilde{x}}^s} = \frac{1}{\dot{\tilde{x}}^u} + \frac{\dot{\tilde{D}}^2}{\dot{\tilde{x}}^s \dot{\tilde{x}}^u}$, we have

$$\psi = f \circ (\tilde{x}^s + \tilde{D}) - \mathrm{Id} = \frac{\tilde{D}}{\dot{\tilde{x}}^u} + \chi \tilde{D}^2,$$

$$\chi = \frac{1}{\dot{\tilde{x}}^s \dot{\tilde{x}}^u} + \int_0^1 (1 - \theta)\, f'' \circ (\tilde{x}^s + \theta \tilde{D})\, d\theta.$$

Thus, again by Taylor's formula,

$$\Delta = \big(\tilde{y}^u - \tilde{y}^s \circ (\mathrm{Id} + \psi)\big)\dot{\tilde{x}}^u = (\tilde{y}^u - \tilde{y}^s)\dot{\tilde{x}}^u - \dot{\tilde{y}}^s \tilde{D} - G,$$

$$G = \dot{\tilde{x}}^u \dot{\tilde{y}}^s \chi \tilde{D}^2 + \dot{\tilde{x}}^u \psi^2 \int_0^1 (1 - \theta)\, \ddot{\tilde{y}}^s \circ (\mathrm{Id} + \theta\psi)\, d\theta.$$

Now, $(\tilde{y}^u - \tilde{y}^s)\dot{\tilde{x}}^u - \dot{\tilde{y}}^s \tilde{D} = \det(\dot{\tilde{z}}^u, \tilde{z}^u - \tilde{z}^s) = \det(\dot{z}^u, z^u - z^s)$ because (41) is symplectic and, by (34) and Lemma 1.6, for any $N \in \mathbb{N}$ the value at a point t of this determinant coincides with the value of $\alpha^{-2} c_2^N + E_N$ at $t - h/2$. We thus get (47) with

$$F_N(t) = E_N(t - h/2) - G(t).$$

The term $E_N(t - h/2)$ and its derivative are controlled by (37). We are thus left with the question of estimating G and its derivative; the result will follow from

$$G = O\big(\varepsilon^2 h^{-6} e^{-2\pi^2/h}\big), \quad \dot{G} = O\big(\varepsilon^2 h^{-7} e^{-2\pi^2/h}\big) \quad \text{on } [-t_1, t_1]. \tag{50}$$

To derive (50), we first bound ψ, χ and their derivatives. By (39) and (49), we have

$$\tilde{D} = O\big(\varepsilon h^{-3} e^{-\pi^2/h}\big), \quad \dot{\tilde{D}} = O\big(\varepsilon h^{-4} e^{-\pi^2/h}\big).$$

Inequalities (23) with $N = 0$ entail $\frac{d^j}{dt^j}\big(\tilde{x}^{s,u} - \alpha^{-1} \tilde{x}^{s,u}_{|\varepsilon=0}\big) = O(\varepsilon h^3)$ for $j = 0, 1, 2, 3$, together with (42) this yields

$$\frac{d^j \tilde{x}^{s,u}}{dt^j} = O(h^2), \qquad j = 0, 1, 2, 3.$$

Then, because of (43), $f'' = -\frac{\ddot{x}^s}{(\dot{x}^s)^3} \circ f = O(h^{-4})$ and $f''' = O(h^{-6})$. This yields

$$\chi, \dot{\chi} = O(h^{-4}), \quad \psi = O\big(\varepsilon h^{-5} e^{-\pi^2/h}\big), \quad \dot{\psi} = O\big(\varepsilon h^{-6} e^{-\pi^2/h}\big).$$

Thus, (50) is a consequence of the definition of G and of the bound $\frac{d^j \tilde{y}^s}{dt^j} = O(h^2)$ ($j = 0, 1, 2, 3$) for $\tilde{y}^s(t) = \frac{\sqrt{2}}{2}\big(\xi^s(t - h/2) + \xi^s(t + h/2)\big)$. □

In view of (28), Lemma 1.7 yields, for any $N \in \mathbb{N}$,

$$\Delta(t) = b_N \sin\frac{2\pi t}{h} + F_N(t), \qquad b_N = 2\varepsilon\alpha^{-2}h^{-1}e^{-\pi^2/h}\big(B_0^+(\varepsilon) + \cdots + h^{2N}B_N^+(\varepsilon)\big). \tag{51}$$

Moreover, $\Delta(0) = \Delta(h) = 0$ (in fact, Δ vanishes on all integer multiples of h). By choosing appropriately N, we shall be in a position to apply

Lemma 1.8. *Suppose $\Delta(t) = b\sin\frac{2\pi t}{h} + F(t)$ for $t \in [0, h]$ with a C^1 function F such that $F(0) = F(h) = 0$ and*

$$|F| < b/2 \quad \textit{and} \quad [\dot{F}| < \pi b/h \quad \textit{on } [0, h].$$

Then Δ has a unique zero in $(0, h)$; this zero t^* satisfies $|t^* - \frac{h}{2}| < \frac{h}{8}$.

Proof. On the intervals $[0, \frac{h}{8}]$ and $[\frac{7h}{8}, h]$, we have $\cos\frac{2\pi t}{h} \geq \sqrt{2}/2$, hence $\Delta'(t) \geq \frac{b\pi}{h}(\sqrt{2} - 1) > 0$ and the function Δ cannot have other zeroes than 0 and h since it is increasing.

On $[\frac{h}{8}, \frac{3h}{8}]$, we have $\Delta(t) \geq \frac{b}{2}(\sqrt{2}-1) > 0$, while on $[\frac{5h}{8}, \frac{7h}{8}]$, $\Delta(t) \leq -\frac{b}{2}(\sqrt{2}-1) < 0$, therefore there is no zero in these intervals and there must be at least one in $(\frac{3h}{8}, \frac{5h}{8})$. But in this interval the zero must be unique because $\Delta'(t) \leq -\frac{b\pi}{h}(\sqrt{2}-1) < 0$. □

Assuming that condition (15) holds for a certain integer N_0 with the constant c_{N_0} defined as

$$c_{N_0} := \frac{1}{2C_{N_0}},$$

we get $|b_{N_0}| > 2C_{N_0}|\varepsilon|e^{-\pi^2/h}h^{2N_0+1}$ (because $\alpha^2 < 2$) and we can apply Lemma 1.8 to (51) with $N = N_0$: inequalities (48) guarantee the existence and uniqueness of a zero of Δ in $(0, h)$.

Now, for any $N \in \mathbb{N}$, the zero $t^* \in (\frac{3h}{8}, \frac{5h}{8})$ of Δ, which depends on h and ε but not on N, satisfies

$$\Big|\Big(t^* - \frac{h}{2}\Big)b_N\Big| < C_N|\varepsilon|e^{-\pi^2/h}h^{2N+2}, \tag{52}$$

since $\frac{2\sqrt{2}}{\pi}\left|\frac{2\pi}{h}\left(t^*-\frac{h}{2}\right)\right| \le \left|\sin\frac{2\pi}{h}\left(t^*-\frac{h}{2}\right)\right|$ and $b_N \sin\frac{2\pi}{h}\left(t^*-\frac{h}{2}\right) = F_N(t^*)$. The lobe area is thus given by

$$A = \int_0^{h/2} b_N \sin\frac{2\pi t}{h} dt + \int_{h/2}^{t^*} b_N \sin\frac{2\pi t}{h} dt + \int_0^{t^*} F_N(t) dt$$

and (14) follows, since the value of the first integral is precisely $\frac{1}{\pi} b_N h$, the second integral has absolute value $< |b_N(t^*-\frac{h}{2})| = O\big(\varepsilon h^{2N+2} e^{-\pi^2/h}\big)$ and the third integral is $O\big(\varepsilon h^{2N+2} e^{-\pi^2/h}\big)$.

1.6. **Description of the Proof of the Analytic Theorem 1.5.** The rest of the paper is devoted to the proof of the Analytic Theorem 1.5. Here we give an informal description of the proof, pointing out the main steps.

The lengthiest and most cumbersome part consists in proving the existence of a suitable solution of the invariance equation (17) satisfying boundary conditions (19) and (21), ξ^u, and obtaining a meaningful asymptotic formula for the difference between $\xi^u(t)$ and $\xi^s(t) = \xi^u(-t)$. This is accomplished in several steps, which are listed in the form of theorems, in Section 2. The proof of those theorems, for the sake of clarity, is postponed to subsequent sections and [MSS08].

The scheme of this first part of the proof is the following.

First of all, in Proposition 2.1 we perform a scaling which allows to assume that the perturbation $\widetilde{V}'$ is of order 5 instead of 3. This amounts for the constant α in the formulas of Theorems 1.1 and 1.5.

Next, in Section 2.2, we introduce the different domains where we shall work. It is clear that, in order to measure the area between the unstable and stable manifolds, the domains where their natural parametrizations are defined need to have a large enough intersection. On the other hand, the arguments to obtain an exponentially small term in the asymptotic formula rely on finding these natural parametrizations in the widest possible complex strip in t in which these parametrizations are holomorphic. The width of this strip is limited by the functions that appear in the approximations we use. Since the first term in these approximations will be ξ^0, the function that gives the separatrix in the integrable case, and its singularities closest to the real line are located at $\pm i\pi/2$, the largest strip we shall be able to deal with is $\{|\mathrm{Im}\, t| < \pi/2\}$.

We will divide the domain in which we need to find ξ^u in two parts, the *outer domain* and the *inner domain* (see Sections 2.2 and 2.8). The outer domain comprises points up to a distance δ of $i\pi/2$, where δ is some value larger than h. The inner domain contains the points at a distance between δ and h of $i\pi/2$. (It will be sufficient to choose $\delta = \sqrt{h}$ at the end.)

The final objective of this first part consists in finding good enough approximations of ξ^u, ξ^s and their difference in the upper part of the domain, that is, at points whose imaginary part is $\pi/2 - h$. To achieve such approximations in the inner domain, we need to start with good approximations in the outer domain.

The good approximations in the outer domain will be given by the asymptotic expansion of ξ^u in powers of h. We find it indirectly by first expanding in an auxiliary parameter in Section 2.4 (see Proposition 2.2) and expanding in h each coefficient of this auxiliary series, in Section 2.5, by means of the Euler-MacLaurin formula.

It turns out that the asymptotic series in h for ξ^u is the same as the one for ξ^s, which implies that the difference between the invariant manifolds is smaller than any power of h (see Corollary 2.9). However, these approximations are not longer accurate at points close to $i\pi/2$. To study the behavior of ξ^u there, we need to use different approximations.

The formal approach, in Section 2.6, consists in introducing a new variable

$$t = i\pi/2 + hz,$$

and expand again in h and z, reordering the series obtained in the outer part. This procedure yields a new formal series

$$\xi^u(i\pi/2 + hz) \sim \sum_{j\geq 0} h^{2j}\tilde{\phi}_j(z),$$

where $\tilde{\phi}_j(z)$ are well defined formal power series in z.

The tool we use to give rigor to these formal expansions is the so-called *resurgence theory*. After the introduction of the new variable z, suggested by the above expansions, we expand the invariance equation (17) in powers of h to obtain a family of *inner equations*. In Section 2.7 we will claim the existence of two families of solutions of the full hierarchy of equations, with prescribed expansions in z, $\tilde{\phi}_j$, one corresponding to ξ^u and the other to ξ^s, and an asymptotic formula for their difference (see Theorem 2.17). This study relies on very different techniques than those used here, and the proofs of the results we quote here are given in [MSS08].

Once we have the solutions of the inner equations, in Section 2.9 we will find the continuation of the function ξ^u up to points with $\operatorname{Im} t = \pi/2 - h$ by matching the outer and inner series (see Theorem 2.18).

At this point, we shall have obtained two different approximations of ξ^u and ξ^s. The outer one will be good enough in the outer region, but without enough precision in the inner region to capture the exponentially small phenomena we want to study. The inner one will be more accurate; moreover, in the inner region, we shall have refined information on the difference between ξ^u and ξ^s at our disposal.

In parallel to this work, we will claim in Theorems 2.4 and 2.20 the existence and list some properties of an appropriate set of solutions of equation (71), which is the linearization of the invariance equation (17) around ξ^u. This information is not used till the next step, but since the techniques used to prove these theorems are the same as those used to prove Theorems 2.3 and 2.18, this is why we have chosen to group them together.

In Section 3 we prove Theorem 1.5. We use the results of Section 2 to obtain the asymptotic formula for $D = \xi^u - \xi^s$ on the real line. Instead of introducing flow box coordinates as in [DRR98, Gel99], in Section 3.1 we take advantage of the fact that D satisfies a linear homogeneous second order difference equation,

$$D(t+h) + D(t-h) = m(t)D(t),$$

and we find a suitable set of fundamental solutions of this equation, $\{\nu_1, \nu_2\}$, using the fact that it is close to equation (71).

To estimate $D(t)$ we use that any solution of such an equation must be of the form $c_1(t)\nu_1(t) + c_2(t)\nu_2(t)$, where c_i, $i = 1, 2$, are h-periodic functions and we will use the already known asymptotic formula for D to obtain an asymptotic expression of the functions c_1 and c_2. Finally, since c_i will be analytic and periodic, we will bound their Fourier coefficients to obtain the desired formula.

We have placed after Section 3 the actual proofs of most of the results. They are rather technical and may be omitted at first reading.

2. Approximation of the manifolds

In this section we find a particular solution ξ^u of equation (17) satisfying boundary conditions (19) and (21), as well as different approximations of this function. More concretely, we will provide three different approximations of ξ^u. The first two are related to the asymptotic expansion of ξ^u in powers of h, and will give arbitrarily good approximations of ξ^u at points far from $i\pi/2$, the first singularity of ξ^0 in the upper half plane, but they will fail whenever t is $O(h)$ close to $i\pi/2$. The third approximation, which formally appears from a suitable reordering of the asymptotic expansion in h of ξ^u — which is divergent —, will provide the necessary approximation at points t close to $i\pi/2$.

2.1. **Rescaling.** We are interested in finding the solution of the equation (17) with boundary conditions (19) and (21). We first perform a scaling in order to make the perturbative terms in ε of order five in ξ instead of order three.[3]

Proposition 2.1. *Let* $\varepsilon_0 < 1/|2V_2|$. *Define* α *as in* (12) *and*

$$V'(y,h,\varepsilon) = \frac{1}{\varepsilon}\big(\mu\alpha f(y/\alpha) - \mu f(y)\big) + \alpha\tilde{V}'(y/\alpha). \tag{53}$$

Then there exist $h_0, y_0, C > 0$ *such that* V' *extends holomorphically to*

$$B = \{(y,h,\varepsilon) \in \mathbb{C}^3 \mid |y| < y_0, |\varepsilon| < \varepsilon_0, |h| < h_0\},$$

[3] This also makes that the limit flow defined by (17) coincides with the limit flow of the integrable equation, that is, the equation obtained when $\varepsilon = 0$.

the function V' is odd with respect to y and even with respect to h,

$$|V'(y,h,\varepsilon)| \leq C|y|^5 \tag{54}$$

$$|V''(y,h,\varepsilon)| \leq C|y|^4 \tag{55}$$

for all $(y,h,\varepsilon) \in B$, and the change $\tilde{\xi} = \alpha\xi$ transforms equation (17) into

$$\tilde{\xi}(t+h) + \tilde{\xi}(t-h) = \mu f(\tilde{\xi}(t)) + \varepsilon V'(\tilde{\xi}(t), h, \varepsilon). \tag{56}$$

The proof is a straightforward computation. What we denote by V'' is the function $\frac{\partial}{\partial y} V'$.

Hereafter, we shall write again ξ instead of $\tilde{\xi}$.

2.2. **Outer domain.** Let $T > 0$. We are interested in finding a solution $\xi^u(t)$ of the equation (56) with boundary conditions (19) and (21) in some large complex domain, D^u. Since $F_{h,\varepsilon}$ is reversible, this will imply the existence of the solution $\xi^s(t) = \xi^u(-t)$ of the equation (56) with boundary conditions (22) and (21) in some domain $D^s = -D^u$. The domains will be chosen in such a way that $D^u \cap D^s$ is nonempty and large enough to contain the given real interval $[-T, T]$.

The domain D^u where ξ^u will be defined splits in two domains, $D^{u,\text{out}}_\delta$ and $D^{u,\text{in}}_h$, where ξ^u will have different approximations:

$$D^u = D^{u,\text{out}}_\delta \cup D^{u,\text{in}}_h.$$

The *outer domain* $D^{u,\text{out}}_\delta$ depends on a parameter $\delta \in (0, \pi/2)$ and is depicted on Figure 3. It is defined as follows:

$$\begin{aligned} D^{u,\text{out}}_\delta =& \{t \in \mathbb{C} \mid \operatorname{Re} t \leq -1\} \\ & \cup \{t \in \mathbb{C} \mid -1 \leq \operatorname{Re} t \leq 0,\ -\frac{\pi}{2} \leq \operatorname{Im} t \leq \frac{\pi}{2},\ \left|t - \frac{\pi}{2} i\right| \geq \delta,\ \left|t + \frac{\pi}{2} i\right| \geq \delta\} \\ & \cup \{t \in \mathbb{C} \mid 0 \leq \operatorname{Re} t \leq T+1,\ 0 \leq \operatorname{Im} t \leq \frac{\pi}{2} - \delta,\ \arg\left(t - \frac{\pi}{2} i\right) < -\beta\} \\ & \cup \{t \in \mathbb{C} \mid 0 \leq \operatorname{Re} t \leq T+1,\ -\frac{\pi}{2} + \delta \leq \operatorname{Im} t \leq 0,\ \arg\left(t + \frac{\pi}{2} i\right) > \beta\} \end{aligned}$$

where $\beta = \arctan \frac{\pi}{2(T+1)}$ (so that $(-\infty, T]$ is well inside $D^{u,\text{out}}_\delta$).

The *inner domain* $D^{u,\text{in}}_h = D^{u,\text{in}}_h(R)$, which corresponds to a region of the complex plane closer to the singularities $\pm i\pi/2$ of ξ^0, will be rigorously defined in Section 2.8 for any $R > 0$ such that $Rh < \delta$ (this domain will also depend on δ).

In the end, δ will be chosen appropriately as a function of h. In fact, $\delta = \sqrt{h}$ will be enough for our purposes. The choice of the constant R will be dictated by Theorem 2.14.

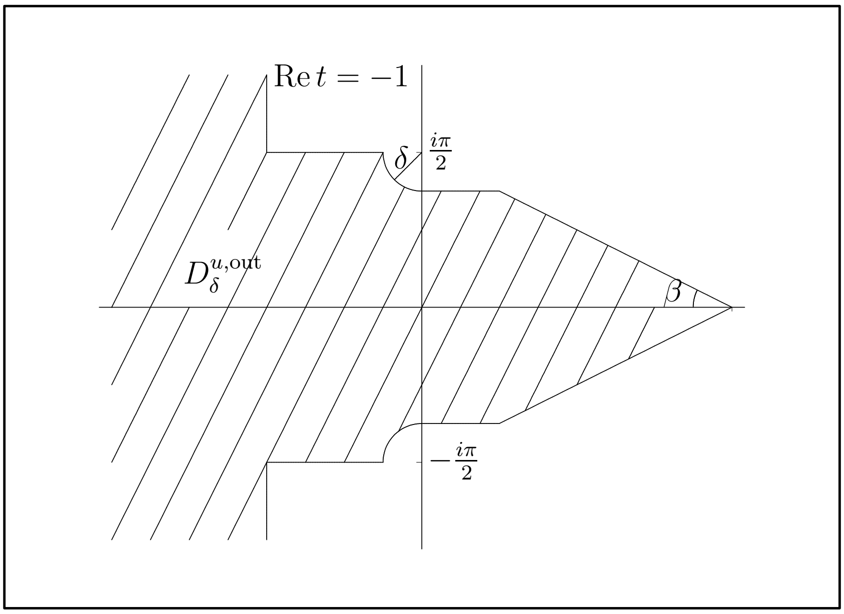

FIGURE 3. The outer domain $D_\delta^{u,\mathrm{out}}$.

2.3. **Unperturbed linearized invariance equation.** One of the important points in the arguments used in this work will be to control the solutions of the linearization of the invariance equation (17) around the unstable solution, ξ^u. See Section 4 for a general exposition of the basic techniques in solving linear second order difference equations.

For $\varepsilon = 0$, the linearization of (18) around ξ^0 is

$$\eta(t+h) + \eta(t-h) = f'(\xi^0(t))\eta(t). \tag{57}$$

A fundamental set of solutions of this equation is $\{\eta_1, \eta_2^c\}$ for any $c \in \mathbb{C}$, with

$$\eta_1(t) = \frac{d}{dt}\xi^0(t) = -\gamma\frac{\sinh t}{\cosh^2 t}, \tag{58}$$

$$\eta_2^c(t) = \frac{A_1 + A_2\sinh^2 t + A_3(t-c)\tanh t}{\gamma^2\cosh t}, \tag{59}$$

where

$$A_1 = \mu^2, \quad A_2 = -\frac{1}{2}, \quad A_3 = -\frac{3\gamma\mu}{2h}$$

(see [DRR98, p. 335]). We remark that $\eta_2^c = \eta_2^0 + c\frac{A_3}{\gamma^3}\eta_1$ and that $W_h(\eta_1, \eta_2^c) = 1$ for all t, independently of c, where this Wronskian is defined according to (32).

We will be particularly interested in η_2^0, since it is real analytic, but also in

$$\eta_2^{i\pi/2} = \eta_2^0 + A\eta_1, \qquad A = -\frac{3i\pi\mu}{4h\gamma^2}, \tag{60}$$

because $\eta_2^{i\pi/2}$ has better bounds around $i\pi/2$. We will list the properties we will need about this set of functions in Lemma 5.1.

2.4. **Outer approximation.** Here we deal with the approximation of ξ^u in $D_\delta^{u,out}$, i.e. far from the singularities of ξ^0.

Equation (56) can be written $\xi(t+h)+\xi(t-h)=\mathcal{F}\big(\xi(t),h,\varepsilon\big)$ with

$$\mathcal{F}(y,h,\varepsilon)=\mu f(y)+\varepsilon V'(y,h,\varepsilon). \tag{61}$$

We shall determine a solution through a sequence of approximating functions, to be obtained by expanding the equation in powers of some auxiliary parameter. Hence, we introduce a new parameter $\underline{\varepsilon}$ and replace $\mathcal{F}$ by

$$\mathcal{F}(y,h,\varepsilon,\underline{\varepsilon})=\mu f(y)+\underline{\varepsilon}V'(y,h,\varepsilon) \tag{62}$$

(i.e. we *freeze* the dependence on ε inside V'); we shall find a solution of the new equation

$$\xi(t+h)+\xi(t-h)=\mathcal{F}\big(\xi(t),h,\varepsilon,\underline{\varepsilon}\big) \tag{63}$$

and restore the relation $\underline{\varepsilon}=\varepsilon$ at the end. From now on, we will not write explicitly the dependence on h, ε and $\underline{\varepsilon}$.

We look for a solution of (63) of the form $\xi=\sum_{k\geq 0}\underline{\varepsilon}^k\xi_k$. Substituting into (56) and collecting the terms of each order, we get equation (18) for the first term and an inductive system of equations for the coefficients ξ_k, $k\geq 1$, namely

$$\xi_k(t+h)+\xi_k(t-h)-\mu f'\big(\xi_0(t)\big)\xi_k(t)=f_k(t) \tag{64}$$

with f_k depending only on $\xi_0,\dots,\xi_{k-1}$:

$$f_1=V'\circ\xi_0 \tag{65}$$

$$f_k=\mu\sum_{r=2}^{k}\frac{1}{r!}f^{(r)}\circ\xi_0\sum_{\substack{j_1+\cdots+j_r=k\\1\leq j_1,\dots,j_r\leq k}}\xi_{j_1}\cdots\xi_{j_r} \tag{66}$$
$$+\sum_{r=1}^{k-1}\frac{1}{r!}V^{(r+1)}\circ\xi_0\sum_{\substack{j_1+\cdots+j_r=k-1\\1\leq j_1,\dots,j_r\leq k-1}}\xi_{j_1}\cdots\xi_{j_r},\qquad k\geq 2.$$

Let us use the notation $\mathbb{D}(0,\rho)=\{z\in\mathbb{C}\mid |z|<\rho\}$.

Proposition 2.2. *Consider the sequence of equations given by* (18) *and* (64)*, for* $k\geq 1$*. Let* $\varepsilon_0<1/|2V_2|$*. There exists* $h_0>0$*, such that for any* $h\in(0,h_0)$ *and* $\delta\in(h,\pi/2)$*, there exists a unique sequence of real analytic functions* $(\xi_k^u)_{k\geq 0}$ *defined for* $(t,\varepsilon)\in D_\delta^{u,\text{out}}\times\mathbb{D}(0,\varepsilon_0)$ *and* $i\pi$*-antiperiodic in* t *such that*

(i) ξ_0^u *is a solution of* (18) *satisfying* $\lim_{t\to-\infty}\xi_0^u(t)=0$*,* $\xi_0^u(t)>0$ *for* $-t$ *large enough and* $\xi_0^u(-h/2)=\xi_0^u(h/2)$*,*

(ii) *for* $k\geq 1$*,* ξ_k^u *is a solution of* (64) *satisfying* $\lim_{t\to-\infty}\xi_k^u(t)=0$ *and* $\xi_k^u(-h/2)=\xi_k^u(h/2)$*.*

In fact, $\xi_0^u = \xi^0$, given in (1), *and, for each $k \geq 0$, there exists $C_k > 0$, independent of h and δ, such that, for any $(t, \varepsilon) \in D_\delta^{u,\mathrm{out}} \times \mathbb{D}(0, \varepsilon_0)$,*

$$|\xi_k^u(t, h, \varepsilon)| \leq \begin{cases} C_k h^{2k+1} e^{\mathrm{Re}\, t} & \text{for } \mathrm{Re}\, t \leq -1, \\ C_k \dfrac{h^{2k+1}}{|\cosh t|^{2k+1}} & \text{for } \mathrm{Re}\, t \geq -1. \end{cases} \tag{67}$$

The proof of this proposition can be found in Section 5.4 (see also the preliminary results in Section 5.3).

Now, we put $\underline{\varepsilon} = \varepsilon$ and define the *first outer approximation of order N* as

$$\xi^{u,N} = \sum_{k=0}^{N} \varepsilon^k \xi_k^u(t, h, \varepsilon), \qquad N \in \mathbb{N}. \tag{68}$$

Theorem 2.3. *Let $\varepsilon_0 < 1/|2V_2|$. There exist $h_0, \rho_0, C_0 > 0$ such that, for any $h \in (0, h_0)$ and $\delta \in (\rho_0 h, \pi/2)$, there exists a unique real analytic function ξ^u, holomorphic for $(t, \varepsilon) \in D_\delta^{u,\mathrm{out}} \times \mathbb{D}(0, \varepsilon_0)$, πi-antiperiodic in t, solution of* (56), *verifying boundary conditions* (19), (21) *and*

$$|\xi^u(t, h, \varepsilon) - \xi^0(t, h)| \leq C_0 |\varepsilon| h^3 e^{\mathrm{Re}\, t} \quad \text{for } \mathrm{Re}\, t < -1.$$

Moreover, for any $N \geq 1$, there exist $h_N, \rho_N, C_N > 0$ such that, for $h \in (0, h_N)$, $\delta \in (\rho_N h, \pi/2)$ and $(t, \varepsilon) \in D_\delta^{u,\mathrm{out}} \times \mathbb{D}(0, \varepsilon_0)$,

$$|\xi^u(t, h, \varepsilon) - \xi^{u,N}(t, h, \varepsilon)| \leq \begin{cases} C_N |\varepsilon|^{N+1} h^{2N+3} e^{\mathrm{Re}\, t} & \text{for } \mathrm{Re}\, t \leq -1, \\ C_N |\varepsilon|^{N+1} \dfrac{h^{2N+3}}{|\cosh^{2N+3} t|} & \text{for } \mathrm{Re}\, t \geq -1. \end{cases} \tag{69}$$

The proof of this theorem is placed in Sections 5.5–5.7.

At this point, we can define

$$\xi^s(t, h, \varepsilon) = \xi^u(-t, h, \varepsilon), \tag{70}$$

which will provide a parametrization of the invariant stable manifold, being solution of the invariance equation (56) and satisfying the boundary conditions (22) and (21).

The linearization of the invariance equation (56) around ξ^u is the equation

$$\eta(t + h) + \eta(t - h) = \Big(\mu f'\big(\xi^u(t, h, \varepsilon)\big) + \varepsilon V''\big(\xi^u(t, h, \varepsilon), h, \varepsilon\big)\Big)\eta(t). \tag{71}$$

In the forthcoming arguments, we will need the two systems of fundamental solutions of this equation provided by the following

Theorem 2.4. *Let $\varepsilon_0, h_0, \rho_0$ as in Theorem 2.3. For $h \in (0, h_0)$ and $\delta \in (\rho_0 h, \pi/2)$, there exist functions $\eta_1^u, \eta_2^{u,0}, \eta_2^{u,i\pi/2}$ holomorphic for $(t, \varepsilon) \in D_\delta^{u,\mathrm{out}} \times \mathbb{D}(0, \varepsilon_0)$, solutions of* (71), *satisfying the following properties:*

- $\eta_1^u = \frac{d}{dt}\xi^u$, $\eta_2^{u,i\pi/2} = \eta_2^{u,0} + A\eta_1^u$ *with A as in* (60).

- $W_h(\eta_1^u, \eta_2^{u,0}) = W_h(\eta_1^u, \eta_2^{u,i\pi/2}) = 1$.
- *For any* $N \geq 0$*, consider*

$$\eta_1^{u,N} = \frac{d}{dt}\xi^{u,N}, \tag{72}$$

where $\xi^{u,N}$ *is given in* (68)*, and the functions* η_2^0*,* $\eta_2^{i\pi/2}$ *defined by* (58) *and* (59)*. Then, there exist* $h_N, C, C_N > 0$*, independent of* h *and* δ*, such that, for* $h \in (0, h_N)$*,* $\delta \in (\rho_N h, \pi/2)$ *(where* ρ_N *is given in Theorem 2.3) and* $(t, \varepsilon) \in D_\delta^{u,\mathrm{out}} \times \mathbb{D}(0, \varepsilon_0)$ *with* $\operatorname{Re} t \geq -1$*,*

$$|\eta_1^u(t, h, \varepsilon) - \eta_1^{u,N}(t, h, \varepsilon)| \leq C_N \frac{|\varepsilon| h^{2N+3}}{|\cosh^{2N+4} t|}, \tag{73}$$

$$|\eta_2^{u,0}(t, h, \varepsilon) - \eta_2^0(t, h)| \leq C \frac{|\varepsilon|}{|\cosh^4 t|}. \tag{74}$$

If moreover $\operatorname{Im} t \geq 0$*, then*

$$|\eta_2^{u,i\pi/2}(t, h, \varepsilon) - \eta_2^{i\pi/2}(t, h)| \leq C \frac{|\varepsilon|}{|\cosh^2 t|}. \tag{75}$$

The proof of this theorem is placed in Sections 5.6–5.7.

Remark 2.5. All the results in this section, in particular the existence of ξ^u, η_1^u, η_2^u and inequalities (67), (69), (73), (74) and (75) will be established in a sectorial domain $U(\beta_1, \beta_2, r_1, r_2, \delta)$ which is larger than $D_\delta^{u,\mathrm{out}}$—see Figure 6 and Section 5.1 for its precise definition.

2.5. **Outer expansion of ξ^u and ξ^s.** The purpose of this section and the next one is to compute the asymptotic expansion in h of the function $\xi^{u,N}(t, h, \varepsilon)$ of (68). In view of (69) this provides an asymptotic expansion for $\xi^u(t, h, \varepsilon)$ up to order $2N + 1$. It turns out that the coefficients of this asymptotic expansion are even functions of t, thus the approximation properties are equally valid for the stable solution $\xi^s(t, h, \varepsilon)$.

More precisely, we will construct a finite sequence of functions $(\xi_k^N)_{k=0,\ldots,N}$ holomorphic in

$$\mathcal{U}^{out}_{\varepsilon_0,h_0} = \{(t, h, \varepsilon) \in \mathbb{C}^3 \mid h \in \mathbb{D}(0, h_0),\ \operatorname{dist}(t, \tfrac{i\pi}{2} + i\pi\mathbb{Z}) > |h|,\ \varepsilon \in \mathbb{D}(0, \varepsilon_0)\}, \tag{76}$$

which will contain the asymptotic expansion of the functions $\xi_k^u(t, h, \varepsilon)$ of Proposition 2.2 up to order h^{2N+1}. Even though $\xi_k^N(t, h, \varepsilon)$ will have an infinite expansion in powers of h which depends on N, the terms of degree $\leq 2N+1$ will not depend on N (provided $0 \leq k \leq N$).

Proposition 2.6. *Let* ε_0, h_0 *as in Theorem 2.3. For any* $N \geq 0$*, there exist a constant* $C_N > 0$ *and a sequence of real analytic functions* $\big(\xi_k^N(t, h, \varepsilon)\big)_{k=0,\ldots,N}$ *such that*

- $\xi_0^N(t,h,\varepsilon) = \xi^0(t,h)$,
- *each* ξ_k^N *is holomorphic in* $\mathcal{U}^{out}_{\varepsilon_0,h_0}$, *even and* $i\pi$*-antiperiodic with respect to* t, *odd with respect to* h *and satisfies*

$$|\xi_k^N(t,h,\varepsilon)| \le \begin{cases} C_N |h|^{2k+1} e^{\mathrm{Re}\, t}, & |\mathrm{Re}\, t| \ge 1, \\ C_N \dfrac{|h|^{2k+1}}{|\cosh t|^{2k+1}}, & |\mathrm{Re}\, t| \le 1, \end{cases} \tag{77}$$

- *if* h *is real, with* $0 < h < h_N$, *and* $\delta \in (\rho_N h, \pi/2)$ *(with* ρ_N *and* h_N *as in Theorem 2.3), then, for* $(t,\varepsilon) \in D_\delta^{u,\mathrm{out}} \times \mathbb{D}(0,\varepsilon_0)$,

$$|\xi_k^u(t,h,\varepsilon) - \xi_k^N(t,h,\varepsilon)| \le \begin{cases} C_k h^{2N+2k+1} e^{\mathrm{Re}\, t}, & \mathrm{Re}\, t \le -1, \\ C_k \dfrac{h^{2N+2k+1}}{|\cosh^{2N+2k+1} t|}, & \mathrm{Re}\, t \ge -1. \end{cases} \tag{78}$$

The proof of this proposition is placed in Section 6, where explicit expressions are given for the functions ξ_k^N (see formulas (196) and (197)). They are obtained by solving approximately the sequence of equations (18) and (64).

Corollary 2.7. *Defining the* outer expansion *as*

$$\xi^{N,\mathrm{out}} = \sum_{k=0}^{N} \varepsilon^k \xi_k^N(t,h,\varepsilon), \qquad (t,h,\varepsilon) \in \mathcal{U}^{out}_{\varepsilon_0,h_0}, \tag{79}$$

we have that

$$|\xi^{u,N}(t,h,\varepsilon) - \xi^{N,\mathrm{out}}(t,h,\varepsilon)| \le \begin{cases} C_N |\varepsilon| h^{2N+3} e^{\mathrm{Re}\, t}, & \mathrm{Re}\, t \le -1 \\ C_N |\varepsilon| \dfrac{h^{2N+3}}{|\cosh^{2N+3} t|}, & \mathrm{Re}\, t \ge -1 \end{cases} \tag{80}$$

for $h \in (0, h_N)$ *and* $(t,\varepsilon) \in D_\delta^{u,\mathrm{out}} \times \mathbb{D}(0,\varepsilon_0)$, *with* $\delta \in (\rho_N h, \pi/2)$. *Furthermore, the function*

$$\eta_1^N = \frac{d}{dt} \xi^{N,\mathrm{out}} \tag{81}$$

satisfies

$$\left| \frac{d^j}{dt^j} \left(\eta_1^u(t,h,\varepsilon) - \eta_1^N(t,h,\varepsilon) \right) \right| \le j! C_N |\varepsilon| h^{2N+3}, \qquad j \in \mathbb{N}, \tag{82}$$

for real $t \le T$, $0 < h < h_N$ *and for* $\varepsilon \in \mathbb{D}(0,\varepsilon_0)$.

Proof. The first part is obtained by plugging inequalities (78) into $\xi^{u,N} - \xi^{N,\mathrm{out}} = \sum_{k=1}^{N} \varepsilon^k (\xi_k^u - \xi_k^N)$, using the condition $|t \pm i\pi/2| \ge \delta > \rho_N h$ to control the negative powers of $|\cosh t|$. The second part is an immediate consequence of inequalities (73), (80) and Cauchy estimates. □

Remark 2.8. At this stage, the first statements of Theorem 1.5 about ξ^u are proved, as a consequence of Theorem 2.3 and Corollary 2.7, taking into account the scaling by α performed in Section 2.1. In particular, inequality (23) follows from (69) and (80) (the passage from $0 < h < h_N$ to $0 < h < h_0$ is innocuous, since the ratio of the left-hand side of (23) with $|\varepsilon| h^{2N+3}$ is bounded for $h \in [h_N, h_0]$ and $\varepsilon \in \mathbb{D}(0, \varepsilon_0)$).

We remark that in the first statement of Proposition 2.6 the parameter h is complex, while inequality (80) only makes sense if h is real, since the functions ξ_k^u which are involved in $\xi^{u,N}$ are only defined for real and positive h.

In fact, the function $\xi^{N,\mathrm{out}}(t, h, \varepsilon)$ in (79) collects all the terms up to order $2N + 1$ of the asymptotic expansion in h of the first outer approximation $\xi^{u,N}$. Moreover, by Theorem 2.3, $\xi^{N,\mathrm{out}}(t, h, \varepsilon)$ also contains all the terms up to order $2N+1$ of the asymptotic expansion in h of the function ξ^u. Since $\xi^{N,\mathrm{out}}$ is even in t and ξ^s is defined through $\xi^s(t, h, \varepsilon) = \xi^u(-t, h, \varepsilon)$, we have that the asymptotic expansions of ξ^u and ξ^s coincide up to order $2N + 1$. As N can be any natural number, ξ^u and ξ^s have the same asymptotic expansion in powers of h. We can summarize these facts in the following corollary.

Corollary 2.9. *Let $\varepsilon_0 < 1/|2V_2|$ as in Theorem 2.3. For any $N \geq 0$ there exist $h_N > 0$, $\rho_N \geq 0$ and $C_N \geq 0$ such that, for any $\varepsilon \in \mathbb{D}(0, \varepsilon_0)$, $0 < h < h_N$, if $\rho_N h < \delta < \pi/2$, the difference between ξ^u and ξ^s can be bounded as*

$$|\xi^u(t, h, \varepsilon) - \xi^s(t, h, \varepsilon)| \leq C_N |\varepsilon| \frac{h^{2N+3}}{|\cosh^{N+3} t|}, \qquad t \in D_\delta^{u,\mathrm{out}} \cap (-D_\delta^{u,\mathrm{out}}). \tag{83}$$

Proof. This is an immediate consequence of inequalities (69), (80), the fact that $\xi^{N,\mathrm{out}}$ is even with respect to t and that $\xi^s(t, h, \varepsilon) = \xi^u(-t, h, \varepsilon)$. □

In the analytic context, this contact beyond all orders is related to exponentially small phenomena. In order to compute an asymptotic formula of the difference between ξ^u and ξ^s we will need to have good approximations of the two functions up to distance $O(h \ln(1/h))$ of $\pm i\pi/2$.

2.6. **Asymptotic expansions and inner equations.** The functions $\xi_k^N(t, h, \varepsilon)$ are holomorphic with respect to their three arguments in the domain $\mathcal{U}^{out}_{\varepsilon_0, h_0}$ and, with respect to t, even and $i\pi$-antiperiodic. Being holomorphic in $\mathcal{U}^{out}_{\varepsilon_0, h_0}$, they can be expanded in Taylor series with respect to h around 0, and then in Laurent series with respect to t around $\frac{i\pi}{2}$. We now state a result about the structure of these expansions (which entails in particular that each coefficient of the h-expansion is meromorphic in t):

Proposition 2.10. *Let $N \geq 0$. Then the functions $(\xi_k^N)_{k=0\dots N}$ given in Proposition 2.6 verify:*

$$\xi_k^N(t, h, \varepsilon) = \sum_{m \geq 0} h^{2m+2k+1} \Xi_{k,m}^N(t, \varepsilon), \qquad (t, h, \varepsilon) \in \mathcal{U}^{out}_{\varepsilon_0, h_0},$$

with real analytic functions $\Xi^N_{k,m}(t,\varepsilon)$*, even,* $i\pi$*-antiperiodic and meromorphic in* $t \in \mathbb{C}$*, with poles located in* $\frac{i\pi}{2} + i\pi\mathbb{Z}$*. Moreover:*

$$\Xi^N_{k,m}(t,\varepsilon) = \sum_{\ell \ge -m-k-1} a^N_{k,m,\ell}(\varepsilon)(t - \tfrac{i\pi}{2})^{2\ell+1},$$

the coefficients $a^N_{k,m,\ell}$ *being holomorphic in* $\varepsilon \in \mathbb{D}(0,\varepsilon_0)$ *and purely imaginary whenever* ε *is real. Equivalently,*

(84)

$$\xi^N_k(\tfrac{i\pi}{2} + hz, h, \varepsilon) = \sum_{n\ge 0} h^{2n}\phi^N_{k,n}(z,\varepsilon) \quad \text{for } h \in \mathbb{D}(0,h_0),\ 1 < |z| < \tfrac{\pi}{|h|} - 1,\ \varepsilon \in \mathbb{D}(0,\varepsilon_0),$$

with

$$\phi^N_{k,n}(z,\varepsilon) = \sum_{m\ge 0} a^N_{k,m,n-k-m-1}(\varepsilon) z^{-2(m+k-n)-1}$$

holomorphic in $\{|z| > 1,\ |\varepsilon| < \varepsilon_0\}$*.*

Furthermore, for $0 \le k \le N$*,*

$$m < N < N' \ \Rightarrow\ \Xi^N_{k,m} = \Xi^{N'}_{k,m}, \tag{85}$$

$$N < N' \ \Rightarrow\ \phi^{N'}_{k,n}(z,\varepsilon) - \phi^N_{k,n}(z,\varepsilon) = O(z^{-2(N+k-n)-1}) \ \text{ for all } n. \tag{86}$$

The proof of Proposition 2.10 is given in Section 7.1.

For example, since $\xi^N_0 = \xi^0 = \frac{\sinh h}{\cosh t}$, we get $\Xi^N_{0,m}(t,\varepsilon) = \frac{1}{(2m+1)!\cosh t}$ for every m, hence $\phi^N_{0,0}(z) = -iz^{-1}$ in (84).

The property (85) allows us to define, for each $k, m \ge 0$, the meromorphic function

$$\Xi_{k,m} = \Xi^N_{k,m} \quad \text{for any } N \ge \max\{k, m+1\}, \tag{87}$$

which, in view of (78), turns out to be a coefficient of the *asymptotic expansion* of $\xi^u_k(t,h,\varepsilon)$ with respect to h:

Corollary 2.11. *For each* $k \ge 0$ *and* $\delta \in (0, \frac{\pi}{2})$*, the function* ξ^u_k *of Proposition 2.2 admits the asymptotic expansion*

$$\xi^u_k(t,\varepsilon,h) \sim \sum_{m\ge 0} h^{2m+2k+1}\Xi_{k,m}(t,\varepsilon), \qquad h \to 0$$

where the coefficients $\Xi_{k,m}$ *are defined by* (87) *and the asymptotic property is uniform with respect to* $(t,\varepsilon) \in D^{u,\text{out}}_\delta \times \mathbb{D}(0,\varepsilon_0)$*.*

Proof. Let $N \ge k$, so that $\Xi_{k,m} = \Xi^N_{k,m}$ for $0 \le m \le N-1$. On the one hand, Proposition 2.6 yields $\xi^u_k(t,h,\varepsilon) - \xi^N_k(t,h,\varepsilon) = O(h^{2N+2k+1})$ uniformly in t and ε (using the fact that $e^{\operatorname{Re} t}$ and $1/\cosh t$ are bounded in $D^{u,\text{out}}_\delta$). On the other hand, $\xi^N_k(t,h,\varepsilon) - \sum_{m=0}^{N-1} h^{2m+2k+1}\Xi_{k,m}(t,\varepsilon) = O(h^{2N+2k+1})$ in view of the Taylor h-expansion of ξ^N_k in Proposition 2.10. □

As for the property (86), it shows that, for each $k, n \geq 0$, the sequence of Laurent series $(\operatorname{Laur}\phi^N_{k,n})_{N\geq 0}$ is "formally convergent". What we mean is the following: denoting by $\operatorname{Laur}\phi^N_{k,n}(z,\varepsilon) \in z^{2(n-k)-1}\mathbb{C}[[z^{-1}]]$ the Laurent expansion around ∞ of the meromorphic function $\phi^N_{k,n}$, we observe that for each $p \in \mathbb{Z}$ the coefficient of z^{-p} in this formal series does not depend on N provided N is large enough; in fact, only odd powers are needed, and if $p = 2(m+k-n)+1$ we get a well-defined coefficient

$$A_{k,n,m}(\varepsilon) = a^N_{k,m,n-k-m-1}(\varepsilon) \tag{88}$$

as soon as $N > m$; the formal limit[4] can thus be defined as

$$\tilde{\phi}_{k,n}(z,\varepsilon) = \sum_{m\geq 0} A_{k,n,m}(\varepsilon) z^{-2(m+k-n)-1} \in z^{2(n-k)-1}\mathbb{C}[[z^{-1}]],$$

and it is characterized by the fact that

$$\tilde{\phi}_{k,n}(z,\varepsilon) - \operatorname{Laur}\phi^N_{k,n}(z,\varepsilon) \in z^{-2(N+k-n)-1}\mathbb{C}[[z^{-1}]], \qquad N \geq 0.$$

Of course these formal series $\tilde{\phi}_{k,n}(z,\varepsilon)$ need not be convergent for any value of z; on the contrary, the analysis of their divergence through resurgence theory will be at the heart of our method, as indicated in next section.

We can also set

$$\tilde{\phi}_n(z,\varepsilon) = \sum_{k\geq 0} \varepsilon^k \tilde{\phi}_{k,n}(z,\varepsilon) \in z^{2n-1}\mathbb{C}[[z^{-1}]]. \tag{89}$$

Indeed, this series of formal series makes sense, since the coefficient of each power z^{-p} is made up of finitely many terms only (because the valuation of $\tilde{\phi}_{k,n}(z)$ increases with k). More concretely, from (89),

$$\tilde{\phi}_n(z,\varepsilon) = \sum_{\ell\geq 0} \frac{B_{\ell,n}(\varepsilon)}{z^{2(\ell-n)+1}} \tag{90}$$

where

$$B_{\ell,n}(\varepsilon) = \sum_{k=0}^{\ell} \varepsilon^k A_{k,n,\ell-k}(\varepsilon), \tag{91}$$

which thus depend holomorphically on ε in $\mathbb{D}(0,\varepsilon_0)$. For instance,

$$\tilde{\phi}_0(z,\varepsilon) = -iz^{-1} + (A_{0,0,1}(\varepsilon) + \varepsilon A_{1,0,0}(\varepsilon))z^{-3} + \cdots \tag{92}$$

with coefficients which are purely imaginary when $\varepsilon \in \mathbb{R}$. The variable (or indeterminate) z is called "inner variable" and all the previous formal series, $\tilde{\phi}_{k,n}(z,\varepsilon)$ or $\tilde{\phi}_n(z,\varepsilon)$, are called "inner expansions". We now introduce the "inner equations" inherited from the invariance equation (56), which they satisfy.

[4] This is simply convergence in the sense of the *Krull topology* of $z^{2(n-k)-1}\mathbb{C}[[z^{-1}]]$ (the topology induced by a metric which can be defined from the valuations of the formal series).

Recall that (56) was rewritten

$$\xi(t+h)+\xi(t-h)=\mathcal{F}(\xi(t),h,\varepsilon),$$

with $\mathcal{F}$ defined in (61). Introducing the new unknown $\phi(z)=\xi(i\frac{\pi}{2}+hz)$, we get

$$\phi(z+1)+\phi(z-1)=\mathcal{F}(\phi(z),h,\varepsilon). \tag{93}$$

We have

$$\mathcal{F}(y,h,\varepsilon)=\sum_{n\geq 0}h^{2n}\mathcal{F}_n(y,\varepsilon) \tag{94}$$

(expanding $\mu=\cosh h$ and $V'(y,h)$ in powers of h). Expanding the unknown in powers of h, i.e. setting

$$\phi(z)=\sum_{n\geq 0}h^{2n}\phi_n(z), \tag{95}$$

and inserting this expansion into (93), we obtain a sequence of equations; the first one is non-linear:

$$\phi_0(z+1)+\phi_0(z-1)=\mathcal{F}(\phi_0(z),0,\varepsilon)=\frac{2\phi_0(z)}{1+\phi_0(z)^2}+\varepsilon V'(\phi_0(z),0,\varepsilon) \tag{96}$$

and is called the *first inner equation*, while the subsequent ones read:

$$\phi_n(z+1)+\phi_n(z-1)-\partial_y\mathcal{F}(\phi_0(z),0,\varepsilon)\phi_n(z)=f_n[\phi_0,\dots,\phi_{n-1},\varepsilon](z),\qquad n\geq 1, \tag{97}$$

where the right-hand sides are determined inductively:

$$f_n[\phi_0,\dots,\phi_{n-1},\varepsilon]=\mathcal{F}_n(\phi_0,\varepsilon)+\sum\frac{1}{r!}\mathcal{F}_{n_0}^{(r)}(\phi_0,\varepsilon)\phi_{n_1}\dots\phi_{n_r}. \tag{98}$$

where the sum in (98) is taken over all $n_0\geq 0$, $r\geq 1$, $n_0+r\geq 2$, $1\leq n_1,\dots,n_r\leq n-1$, $n_0+n_1+\cdots+n_r=n$.

In fact, f_n is the coefficient of h^{2n} in $\mathcal{F}(\phi_0+h^2\phi_1+\cdots+h^{2(n-1)}\phi_{n-1},h,\varepsilon)$. Thus, the nth of these *secondary inner equations* (97) is linear non-homogeneous in the nth unknown ϕ_n, with a right-hand side determined by $\phi_0,\dots,\phi_{n-1}$.

The first inner equation (96) makes sense in the differential ring $z^{-1}\mathbb{C}[[z^{-1}]]$ (i.e. both sides of the equation are defined for an unknown $\phi_0(z)\in z^{-1}\mathbb{C}[[z^{-1}]]$), and if such a solution ϕ_0 is given, the secondary inner equations (97) make sense in the field of fractions of this ring, $\mathbb{C}[[z^{-1}]][z]$. Indeed, the only operations involved in the equations are multiplication, substitution of ϕ_0 in the $\mathcal{F}_n'$s (or rather in their Taylor expansions in y) and their derivatives, and the shift operator

$$\phi(z)\ \mapsto\ \phi(z+1):=\sum_{r\geq 0}\tfrac{1}{r!}\phi^{(r)}(z),$$

which is well-defined in $\mathbb{C}[[z^{-1}]][z]$ because the above series of formal series is formally convergent (the valuations increase).

Proposition 2.12. *The formal series* $\tilde\phi_0(z), \tilde\phi_1(z), \dots$ *defined by* (89) *are odd formal solutions of the system of inner equations* (96)–(97). *Their coefficients are pure imaginary whenever* ε *is real.*

The proof of this proposition is given in Section 7.2.

Remark 2.13. These formal series are not the only odd formal solutions: it turns out (see [MSS08]) that $\tilde\phi_0$ and $-\tilde\phi_0$ are the only odd formal solutions of the first inner equation, and that, for each of these choices and for any sequence of complex numbers $(b_n)_{n\ge1}$, there is a sequence of odd formal solutions $(\phi_n)_{n\ge1}$ such that b_n is the coefficient of z^3 in $\phi_n(z)$ (observe that according to (89) the coefficient of z^3 in $\tilde\phi_1(z)$ is zero).

2.7. **Solutions of the inner equations.** The present section is devoted to statements about the inner equations, the proofs of which rely on Écalle's resurgence theory and are given in the article [MSS2]. Roughly speaking, the approach of [MSS2] consists in checking the Borel summability of the formal series $\tilde\phi_n(z)$ and, more than this, studying the domain of holomorphy of their Borel transforms $\widehat\phi_n(\zeta)$ and then analyzing the singularities of these holomorphic functions by means of the so-called alien calculus. The presence of singularities in the ζ-plane implies that the $\tilde\phi_n(z)$'s do not converge for any value of z, because of a "factorial" divergence (the modulus of the coefficient of z^{-p} is larger than $M^{p+1}p!$ for some $M > 0$). In particular, the $\tilde\phi_n(z)$'s are Gevrey-1 series and the asymptotic expansion properties satisfied by their Borel sums $\phi_n^u(z)$ are of Gevrey-1 type. But in the present section, we content ourselves with extracting from [MSS2] the minimum information which is needed for going on with the proof of Theorem 1.5; for instance the statements we give for the $\phi_n^u(z)$'s are only formulated in the framework of the usual (Poincaré) asymptotic expansion theory.

Recall that the angle $\beta \in \left(0, \frac{\pi}{2}\right)$ was fixed in the definition of the outer domain $D_\delta^{u,\text{out}}$ in Section 2.2.

Theorem 2.14. *Consider the sequence of odd formal solutions* $(\tilde\phi_n(z,\varepsilon))_{n\ge0}$ *of the inner equations defined by* (89). *Then there exist an increasing sequence of numbers* $R_n \ge 1$ *and a unique sequence of functions* $\phi_n^u(z,\varepsilon)$ *which are holomorphic in* $\mathcal{D}_{in}^u(R_n) \times \mathbb{D}(0,\varepsilon_0)$*, where*

$$\mathcal{D}_{in}^u(R_n) = \{\, z \in \mathbb{C} \mid |z| \ge R_n,\ \beta/2 < arg(z) < 2\pi - \beta/2 \,\}, \tag{99}$$

which satisfy the system of inner equations (96)–(97) *for* $n \ge 0$ *and for any* $(z,\varepsilon) \in \mathcal{D}_{in}^u(R_n) \times \mathbb{D}(0,\varepsilon_0)$ *such that* $z-1, z+1 \in \mathcal{D}_{in}^u(R_n)$*, and which satisfy*

$$\phi_n^u(z,\varepsilon) \sim \tilde\phi_n(z,\varepsilon), \qquad |z| \to \infty,\ \ z \in \mathcal{D}_{in}^u(R_n), \quad \textit{uniformly in } \varepsilon \in \mathbb{D}(0,\varepsilon_0)$$

for $n \ge 0$.

Remark 2.15. It is shown in [MSS08] how to obtain the function ϕ_n^u from $\tilde{\phi}_n$ by Borel Laplace summation around the direction of $\mathbb{R}^-$: by (90), we can write

$$\tilde{\phi}_n(z,\varepsilon) = \sum_{1\le m\le n} B_{n-m,n}(\varepsilon) z^{2m-1} + \sum_{p\ge 0} B_{n+p,n}(\varepsilon) z^{-2p-1}$$

and it turns out that the Borel transform $\widehat{\phi}_n(\zeta,\varepsilon) = \sum_{p\ge 0} B_{n+p,n}(\varepsilon)\zeta^{2p}/(2p)!$ defines a holomorphic function for ζ near 0 which extends analytically to the half-planes $\{\operatorname{Re}\zeta < 0\}$ and $\{\operatorname{Re}\zeta > 0\}$, and that one can define

$$\phi_n^u(z,\varepsilon) = \sum_{1\le m\le n} B_{n-m,n}(\varepsilon) z^{2m-1} + \int_0^{e^{i\theta}\infty} \widehat{\phi}_n(\zeta,\varepsilon)\, e^{-z\zeta} d\zeta$$

with $\theta \in (\pi/2+\beta, 3\pi/2-\beta)$ chosen according to $\arg z$. Since, by Proposition 2.12, $\overline{B_{\ell,n}(\bar{\varepsilon})} = -B_{\ell,n}(\varepsilon)$ for any ℓ, n, this entails

$$\overline{\phi_n^u(\bar{z},\bar{\varepsilon})} = -\phi_n^u(z,\varepsilon), \qquad z \in \mathcal{D}_{in}^u(R_n). \tag{100}$$

Later, in Section 2.8, we shall introduce the inner domain $D_h^{u,\text{in}}(R_n)$ in such a way that $t \in D_h^{u,\text{in}}(R_n) \Rightarrow \frac{t-i\pi/2}{h} \in \mathcal{D}_{in}^u(R_n)$ and the function $\sum_{n=0}^N h^{2n}\phi_n^u(\frac{t-i\pi/2}{h},\varepsilon)$ will be the relevant approximation of $\xi^u(t,h,\varepsilon)$ for $t \in D_h^{u,\text{in}}(R_N)$. We set

$$\mathcal{D}_{in}^s(R_n) = -\mathcal{D}_{in}^u(R_n) = \{\, z \in \mathbb{C} \mid |z| \ge R_n,\ -\pi+\beta/2 < arg(z) < \pi-\beta/2 \,\}$$

(see Figure 4). The symmetries of the problem imply that the formulas

$$\phi_n^s(z,\varepsilon) = -\phi_n^u(-z,\varepsilon), \qquad 0 \le n \le N \tag{101}$$

define a sequence of solutions which are holomorphic in $\mathcal{D}_{in}^s(R_n) \times \mathbb{D}(0,\varepsilon_0)$. We also define, for any $N \neq 0$, the functions

$$\phi^{u,N}(z,h,\varepsilon) = \sum_{n=0}^N h^{2n}\phi_n^u(z,\varepsilon), \qquad \phi^{s,N}(z,h,\varepsilon) = -\phi^{u,N}(-z,h,\varepsilon), \tag{102}$$

which are holomorphic in the domains $\mathcal{D}_{in}^{u,s}(R_N) \times \mathbb{C} \times \mathbb{D}(0,\varepsilon_0)$, with $\overline{\phi^{u,N}(\bar{z},\bar{\varepsilon})} = -\phi^{u,N}(z,\varepsilon)$ and $\overline{\phi^{s,N}(\bar{z},\bar{\varepsilon})} = -\phi^{s,N}(z,\varepsilon)$ for real h.

Since the formal solutions we started with are odd in z, the functions $\phi_n^s(z,\varepsilon)$ will have the same asymptotic expansions $\tilde{\phi}_n(z,\varepsilon)$, but one should not believe that they coincide with the functions $\phi_n^u(z,\varepsilon)$ (i.e. that the ϕ_n^u's are odd in z). On the contrary, there is a discrepancy, exponentially small with respect to z and thus invisible from the viewpoint of the usual asymptotic expansion theory, which resurgence theory will allow us to analyze.

The difference $\phi^{s,N} - \phi^{u,N}$ is defined in the intersection $\mathcal{D}_{in}^s(R_N) \cap \mathcal{D}_{in}^u(R_N)$, which has two connected components; we shall study it in the lower one, which we denote $\mathcal{D}_{in}(R_N)$ (see Figure 4). In order to state our main result about this difference, we need to introduce solutions of the linearization of the first inner equation (96).

Proposition 2.16. *The linear difference equation*

$$\psi(z+1)+\psi(z-1)=\partial_y\mathcal{F}(\tilde{\phi}_0(z,\varepsilon),0,\varepsilon)\psi(z) \tag{103}$$

admits formal solutions in $\mathbb{C}[[z^{-1}]][z]$ *(with coefficients depending holomorphically on* $\varepsilon\in\mathbb{D}(0,\varepsilon_0)$*) of the form*

$$\tilde{\psi}_1(z,\varepsilon)=\tilde{\phi}_0'(z,\varepsilon)=iz^{-2}+O(z^{-4}),\quad \tilde{\psi}_2(z,\varepsilon)=-\tfrac{i}{5}z^3+O(z), \tag{104}$$

such that $\tilde{\psi}_1(z,\varepsilon)$ *is even,* $\tilde{\psi}_2(z,\varepsilon)$ *is odd and* $W_1(\tilde{\psi}_1,\tilde{\psi}_2)=1$.

Moreover, the linear difference equation

$$\psi(z+1)+\psi(z-1)=\partial_y\mathcal{F}(\phi_0^u(z,\varepsilon),0,\varepsilon)\psi(z),$$

where $\phi_0^u(z,\varepsilon)$ *is the Borel sum of* $\tilde{\phi}_0(z,\varepsilon)$ *described in Theorem 2.14, admits solutions* $\psi_1^u(z,\varepsilon)$ *and* $\psi_2^u(z,\varepsilon)$ *which are holomorphic in* $\mathcal{D}^u_{in}(R_0)\times\mathbb{D}(0,\varepsilon_0)$ *and satisfy* $W_1(\psi_1^u,\psi_2^u)=1$ *and*

$$\psi_j^u(z,\varepsilon)\sim\tilde{\psi}_j(z,\varepsilon),\qquad |z|\to\infty,\ z\in\mathcal{D}^u_{in}(R_0),\quad \textit{uniformly in } \varepsilon\in\mathbb{D}(0,\varepsilon_0)$$

for $j=1,2$.

Since $\tilde{\phi}_0(z,\varepsilon)$ satisfies (96), it is obvious that its derivative $\tilde{\psi}_1(z,\varepsilon)$ is a solution of the linearized equation (103). As in Section 2.3, the reader is referred to Section 4 for the theory of linear second-order difference equations, in particular for the construction of an independent solution $\tilde{\psi}_2(z,\varepsilon)$ and for the properties of the Wronskian

$$W_1(\psi_1,\psi_2)=\begin{vmatrix}\psi_1(z) & \psi_2(z)\\ \psi_1(z+1) & \psi_2(z+1)\end{vmatrix}.$$

The functions $\psi_1^u=\frac{d}{dz}\phi_0^u$ and ψ_2^u are Borel sums of $\tilde{\psi}_1$ and $\tilde{\psi}_2$.

Theorem 2.17. 1) *There exist functions* $\psi^u_{1,n}(z,\varepsilon)$, $\psi^u_{2,n}(z,\varepsilon)$, $n\geq 0$, *generated by the secondary inner equations* (97), *holomorphic in* $\mathcal{D}^u_{in}(R_n)\times\mathbb{D}(0,\varepsilon_0)$ *such that*

a) $\psi^u_{i,0}=\psi^u_i$, $i=1,2$, *are the functions given by Proposition 2.16,*
b) $\psi^u_{1,n}=\frac{d}{dz}\phi^u_n$, $n\geq 0$, *where the functions* ϕ^u_n *are given by Theorem 2.14,*
c) *for* $z\in\mathcal{D}^u_{in}(R_n)$ *and for* $n\geq 0$,

$$|\psi^u_{1,n}(z)|=O(z^{2n-2}) \tag{105}$$

$$|\psi^u_{2,n}(z)|=O(z^{2n+3}), \tag{106}$$

d) $W_1(\psi^u_{1,0},\psi^u_{2,0})=1$, $\sum_{k=0}^{n}W_1(\psi^u_{1,n-k},\psi^u_{2,k})=0$ *if* $n\geq 1$.

2) *There exist complex numbers* A_n^+ *and* B_n^+, $n\geq 0$, *which depend holomorphically on* $\varepsilon\in\mathbb{D}(0,\varepsilon_0)$, *such that*

$$B_0^+=4\pi^2\widehat{V}(2\pi)+O(\varepsilon) \tag{107}$$

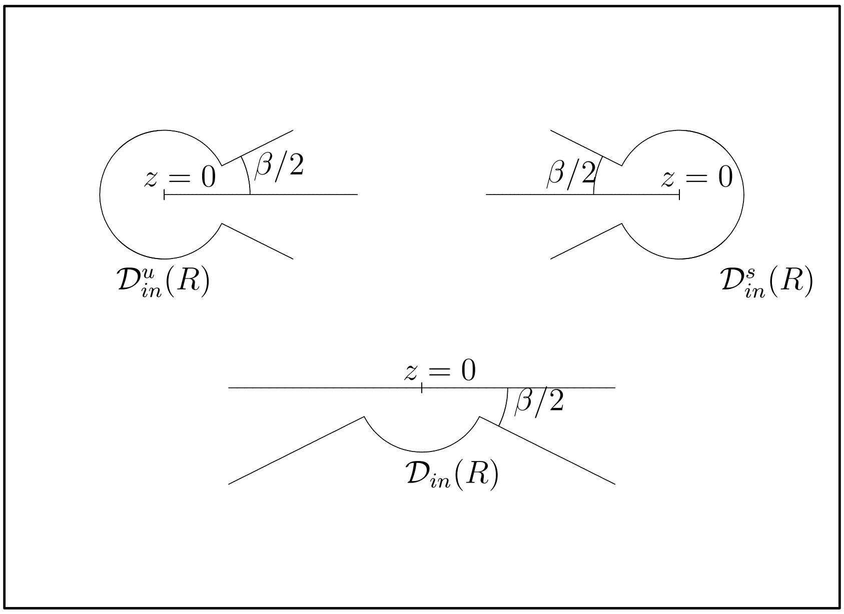

FIGURE 4. The domains $\mathcal{D}^u_{in}(R)$, $\mathcal{D}^s_{in}(R)$ and $\mathcal{D}_{in}(R)$.

(with the entire function $\widehat{V}(\zeta)$ introduced in (9)*) and, for any $N \geq 0$, the function*

$$D^{N,inn}(z,\varepsilon) = -\varepsilon\, e^{-2\pi i z} \sum_{n=0}^{N} h^{2n} \sum_{n_1+n_2=n} \left(A^+_{n_1} \psi^u_{1,n_2}(z,\varepsilon) + i B^+_{n_1} \psi^u_{2,n_2}(z,\varepsilon) \right) \tag{108}$$

satisfies the following: if one denotes by $\mathcal{D}_{in}(R_N)$ the lower connected component of $\mathcal{D}^s_{in}(R_N) \cap \mathcal{D}^u_{in}(R_N)$, then, for any $\theta \in (0,1)$, the difference of the functions $\phi^{u,N}$ and $\phi^{s,N}$ defined by (102) *can be written*

$$\phi^{u,N}(z,\varepsilon,h) - \phi^{s,N}(z,\varepsilon,h) = D^{N,inn}(z,\varepsilon) + O(\varepsilon\, e^{-2\pi(1+\theta)|\operatorname{Im} z|}), \tag{109}$$

uniformly in $z \in \mathcal{D}_{in}(R_N)$, $h \in \mathbb{D}(0,1)$, $\varepsilon \in \mathbb{D}(0,\varepsilon_0)$.

As already mentioned, *the proofs of Theorems 2.14 and 2.17 and of Proposition 2.16 are in [MSS2].*

The constants B^+_n are the ones appearing in Theorems 1.1 and 1.5. The constants A^+_n and B^+_n have a resurgent origin: the Borel transforms of the formal series $\tilde{\phi}_n(z,\varepsilon)$ give rise to holomorphic functions in the ζ-plane which extend analytically to the universal cover of $\mathbb{C} \setminus 2\pi i \mathbb{Z}$, with singularities at $2\pi i$ which account for the main part of the asymptotic expansion of $\phi^s_n(z,\varepsilon) - \phi^u_n(z,\varepsilon)$ when z lies in $\mathcal{D}_{in}(R_n)$, while the singularities at $-2\pi i$ are related with the asymptotic expansion in the upper part of $\mathcal{D}^s_{in}(R_n) \cap \mathcal{D}^u_{in}(R_n)$ (the singularities at $2\pi i m$ with $|m| \geq 2$ correspond to exponentially small corrections of higher orders).

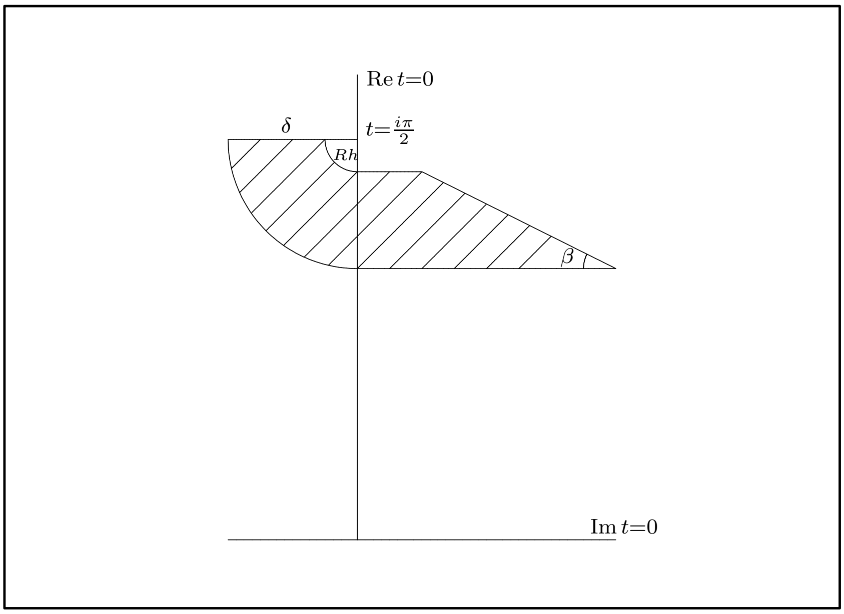

FIGURE 5. The upper half of the inner domain $D_h^{u,\mathrm{in}}(R)$.

2.8. **Inner domain.** We now define the inner domain, a region in the complex plane closer to the singularities $\pm i\pi/2$ of $\xi^0(t,h)$ than the outer domain $D_\delta^{u,\mathrm{out}}$, where the functions $\xi^u(t,h,\varepsilon)$ and $\xi^s(t,h,\varepsilon)$ will be well approximated by making $z=(t-i\pi/2)/h$ in the functions $\phi^{u,N}(z,h,\varepsilon)$ and $\phi^{s,N}(z,h,\varepsilon)$ defined in (102).

Given $R>0$, for any $\delta \in (Rh, \frac{\pi}{2})$, we set

(110)
$$\begin{aligned}
D_h^{u,\mathrm{in}}(R) = &\{t\in\mathbb{C} \mid \operatorname{Re} t \le 0,\ \operatorname{Im} t \le \frac{\pi}{2},\ Rh \le |t-\frac{\pi}{2}i| \le \delta\} \\
&\cup \{t\in\mathbb{C} \mid \operatorname{Re} t \ge 0,\ \frac{\pi}{2}-\delta \le \operatorname{Im} t \le \frac{\pi}{2}-Rh,\ \arg(t-\frac{\pi}{2}i) < -\beta\} \\
&\cup \{t\in\mathbb{C} \mid \operatorname{Re} t \le 0,\ \operatorname{Im} t \ge -\frac{\pi}{2},\ Rh \le |t+\frac{\pi}{2}i| \le \delta\} \\
&\cup \{t\in\mathbb{C} \mid \operatorname{Re} t \ge 0,\ -\frac{\pi}{2}+Rh \le \operatorname{Im} t \le -\frac{\pi}{2}+\delta,\ \arg(t+\frac{\pi}{2}i) > \beta\}.
\end{aligned}$$

Observe that this domain is symmetric with respect to the real axis and that, if $R \ge R_N$ and $t \in D_h^{u,\mathrm{in}}(R)$, then $\frac{t-i\pi/2}{h} \in \mathcal{D}_{in}(R_N)$.

2.9. **Matching of the outer and inner approximations.** In this section we use the information obtained from the study of the first inner equation (96) and the full hierarchy of equations (97) to improve our knowledge of the functions ξ^u and ξ^s given by Theorem 2.3 and formula (70). This will be achieved by matching the approximation of these functions found in the outer domain $D_\delta^{u,\mathrm{out}}$ with the appropriate approximations in the inner domain $D_h^{u,\mathrm{in}}(R)$.

Theorem 2.18. *Let $\varepsilon_0 < 1/|2V_2|$ and consider the function ξ^u of Theorem 2.3, solution of (56) verifying boundary conditions (19) and (21). For any $N \geq 0$, besides the constants ρ_N and R_N introduced in Theorems 2.3 and 2.14, there exist constants $h_N, \kappa_N, C_N > 0$ such that, if $|\varepsilon| < \varepsilon_0$, $0 < h < h_N$, $\max\{\rho_N, 2R_N\}h < \delta < \pi/2$ and*

$$A_\delta^N = \left(\frac{h}{\delta}\right)^{2N+1} + \frac{\delta^{2N+3}}{h} < \kappa_N, \tag{111}$$

then ξ^u admits an analytic continuation to

$$D^u(2R_N) = D_\delta^{u,\mathrm{out}} \cup D_h^{u,\mathrm{in}}(2R_N) \tag{112}$$

and, for $t \in D_h^{u,\mathrm{in}}(2R_N)$, $\mathrm{Im}\, t > 0$,

$$\left|\xi^u(t) - \phi^{u,N}\left(\tfrac{t-i\pi/2}{h}\right)\right| \leq C_N|\varepsilon|\frac{A_\delta^N}{\left|\frac{t-i\pi/2}{h}\right|^2}, \tag{113}$$

where $\phi^{u,N}$ is the function introduced in (102).

The proof of this Theorem is placed in Section 8. Observe that if we choose $\delta = h^{1/2}$ (which is licit, if h is small enough), then the constant A_δ^N defined in (111) becomes

$$A_{h^{1/2}}^N = 2h^{N+1/2}. \tag{114}$$

Corollary 2.19. *With the same hypotheses as in Theorem 2.18, the function ξ^s defined in (70) satisfies*

$$|\xi^s(t) - \phi^{s,N}\left(\tfrac{t-i\pi/2}{h}\right)| \leq C_N|\varepsilon|\frac{A_\delta^N}{\left|\frac{t-i\pi/2}{h}\right|^2}, \tag{115}$$

for $t \in D_h^{s,\mathrm{in}}(2R_N) = -D_h^{u,\mathrm{in}}(2R_N)$, $\mathrm{Im}\, t \geq 0$.

Proof. Take h real and $t \in D_h^{s,\mathrm{in}}(2R_N)$. By (102), $\xi^s(t,h,\varepsilon) - \phi^{s,N}\left(\frac{t-i\pi/2}{h}, h, \varepsilon\right) = \xi^u(-t,h,\varepsilon) + \phi^{u,N}\left(\frac{-t+i\pi/2}{h}, h, \varepsilon\right)$, and this quantity is the complex conjugate of $\xi^u(-\bar{t},h,\bar{\varepsilon}) - \phi^{u,N}\left(\frac{-\bar{t}-i\pi/2}{h}, h, \bar{\varepsilon}\right)$. We have $-\bar{t} \in D_h^{u,\mathrm{in}}(2R_N)$; if $\mathrm{Im}\, t \geq 0$, then $\mathrm{Im}\,(-\bar{t}) \geq 0$ and, by Theorem 2.18, the modulus is bounded by $C_N A_\delta^N \left|\frac{-\bar{t}-i\pi/2}{h}\right|^{-2} = C_N A_\delta^N \left|\frac{t-i\pi/2}{h}\right|^{-2}$. □

In the forthcoming arguments, we will need the extension to the inner domain of the functions η_1^u, $\eta_2^{u,0}$, and $\eta_2^{u,i\pi/2}$, solutions of the linearized equation (71), given in Theorem 2.4.

Theorem 2.20. *Let $\varepsilon_0 < 1/|2V_2|$ and consider the functions η_1^u, $\eta_2^{u,0}$ and $\eta_2^{u,i\pi/2}$ of Theorem 2.4. For any $N \geq 0$, under the same conditions as in Theorem 2.18 for ε, h, δ and A_δ^N, these functions admit an analytic continuation to $D_\delta^{u,\mathrm{out}} \cup$*

$D_h^{u,\mathrm{in}}(2R_N)$ and there exist $C>0$ and $C_N>0$ such that, for $t\in D_h^{u,\mathrm{in}}(2R_N)$, $\operatorname{Im} t>0$,

$$|\eta_1^u(t,h,\varepsilon)-\frac{1}{h}\psi_1^{u,N}\big(\tfrac{t-i\pi/2}{h},h,\varepsilon\big)|\le C_N\frac{A_\delta^N h^2}{|\cosh t|^3}, \tag{116}$$

$$|\eta_2^{u,i\pi/2}(t,h,\varepsilon)-h\psi_2^u\big(\tfrac{t-i\pi/2}{h},\varepsilon\big)|\le C\frac{1}{|\cosh t|^2}, \tag{117}$$

where

$$\psi_1^{u,N}=\sum_{n=0}^{N}h^{2n}\psi_{1,n}^u \quad \text{and} \quad \psi_2^u=\psi_{2,0}^u \tag{118}$$

(with the notation of Theorem 2.17).

The proof of Theorem 2.20 is given in Section 8.

Corollary 2.21. *There exists $C>0$ such that, for $t\in D_\delta^{u,\mathrm{out}}\cup D_h^{u,\mathrm{in}}(R_0)$,*

$$|\eta_1^u(t)|\le C\frac{h}{|\cosh t|^2}, \tag{119}$$

$$|\eta_2^{u,0}(t)|\le \frac{C}{h^2|\cosh t|^2}. \tag{120}$$

Proof. In $D_\delta^{u,\mathrm{out}}$, we can use Theorem 2.4: inequality (73) with $N=0$ yields $\eta_1^u=\eta_1+O(\varepsilon h^3/\cosh^4 t)$ while $\eta_2^{u,0}=\eta_2^0+O(\varepsilon/\cosh^4 t)$ by (74), and η_1 and η_2^0 are easy to bound in view of (58)–(59); in $D_h^{u,\mathrm{in}}(R_0)$ we use Theorem 2.20. In both cases, the condition $|t\pm i\pi/2|\ge R_0 h$ allows one to control the negative powers of $\cosh t$. □

3. Proof of the Analytic Theorem 1.5

In this section, we use the results presented in the previous section to prove the Analytic Theorem 1.5.

We introduce

$$D(t)=\xi^u(t)-\xi^s(t), \tag{121}$$

which is the function defined in (24), scaled by α (see Proposition 2.1). By Theorem 2.3, it is real analytic, holomorphic in

$$\mathcal{R}=D^u(R_0)\cap D^s(R_0), \tag{122}$$

where $D^s(R)=-D^u(R)$ and $D^u(R)$ was introduced in (112). Observe that

$$\mathcal{R}=\{-\pi+\beta<\arg\big(t-i\tfrac{\pi}{2}\big)<-\beta \text{ and } \beta<\arg\big(t+i\tfrac{\pi}{2}\big)<\pi-\beta\}\cap\{|\operatorname{Im} t|<\tfrac{\pi}{2}-R_0h\}$$

is the intersection of a lozenge (with vertices at $\pm i\pi/2$ and $\pm(T+1)$) and a horizontal strip.

Since ξ^u and ξ^s satisfy the invariance equation (56), the function $D(t)$ satisfies the linear second order difference equation

$$D(t+h) + D(t-h) = m(\xi^u, \xi^s)(t) D(t), \tag{123}$$

where

$$\begin{aligned} m(\chi_1, \chi_2)(t) \\ = \int_0^1 \big(\mu f'(s\chi_1(t) + (1-s)\chi_2(t)) + \varepsilon V''(s\chi_1(t) + (1-s)\chi_2(t), h, \varepsilon)\big) ds. \end{aligned} \tag{124}$$

The importance of D being a solution of this equation is the following. If ν_1 and ν_2 are two solutions of (123) such that $W_h(\nu_1, \nu_2) = 1$, then

$$D(t) = c_1(t)\nu_1(t) + c_2(t)\nu_2(t), \tag{125}$$

with $c_1 = W_h(D, \nu_2)$ and $c_2 = W_h(\nu_1, D)$ h-periodic. Hence if we find such solutions ν_1 and ν_2 real analytic and satisfying certain bounds in $\mathcal{R}$, then, using that $D(t)$ is bounded in $\mathcal{R}$ and that the coefficients $c_i(t)$ are real analytic h-periodic functions, we will be able to deduce exponentially small bounds for $c_i(t)$ and then for $D(t)$ for real t by means of Lemma 3.3 below.

3.1. **Solutions of the linear equation** (123)**.**

The definition (124) implies that

$$m(\xi^u, \xi^u)(t) = \mu f'(\xi^u(t)) + \varepsilon V''(\xi^u(t), h, \varepsilon), \tag{126}$$

thus equation (123) is close to equation (71), for which we already have a fundamental system of real analytic solutions, $\{\eta_1^u, {\eta_2^u}^{,0}\}$, with precise estimates, by Theorem 2.4 and Theorem 2.20. It is thus natural to look for the solutions ν_1, ν_2 of equation (123) as small perturbations of η_1^u and ${\eta_2^u}^{,0}$.

However, we will not be able to find them in the whole domain $\mathcal{R}$, but in a slightly smaller one. More concretely, we define

$$\mathcal{R}_\sigma = \mathcal{R} \cap \{|\operatorname{Im} t| \le \tfrac{\pi}{2} - \tfrac{\sigma}{2\pi} h |\ln h|\} \tag{127}$$

for $\sigma > 0$. Notice that $\mathcal{R}_\sigma \cap \mathbb{R}$ does not depend on σ:

$$\mathcal{R}_\sigma \cap \mathbb{R} = \mathcal{R} \cap \mathbb{R} = (-T-1, T+1),$$

nor does $\mathcal{R}_\sigma \cap \{|\operatorname{Im} t| \le 1\} = \mathcal{R} \cap \{|\operatorname{Im} t| \le 1\}$.

Theorem 3.1. *Let $\varepsilon_0 < 1/|2V_2|$. For any $\sigma > 13$ and $N \ge 0$, under the same conditions as in Theorem 2.18 for ε, h, δ and A_δ^N, there exist $\tilde\rho_N, C_N > 0$ such that, if $A_\delta^N < \tilde\rho_N h^{13}$, then there exist $\nu_1, \nu_2 \colon \mathcal{R}_\sigma \to \mathbb{C}$, real analytic solutions of equation* (123) *satisfying $W_h(\nu_1, \nu_2) = 1$.*

Moreover, for $t \in \mathcal{R}$ such that $|\operatorname{Im} t| \le 1$,

$$|\nu_1(t) - \eta_1^u(t)| \le C_N |\varepsilon| \Big(\frac{A_\delta^N}{h^3 |\ln h|^3} + h^{\sigma-3} |\ln h|^2 \Big), \tag{128}$$

$$|\nu_2(t) - {\eta_2^u}^{,0}(t)| \le C_N |\varepsilon| \Big(\frac{A_\delta^N}{h^6 |\ln h|^3} + h^{\sigma-6} |\ln h|^2 \Big), \tag{129}$$

with η_1^u *and* $\eta_2^{u,0}$ *as in Theorem 2.4, while for* $t \in \mathcal{R}_\sigma$ *with* $\operatorname{Im} t = \pi/2 - \frac{\sigma}{2\pi} h|\ln h|$,

$$|\nu_1(t) - \frac{1}{h}\psi_1^{u,N}\big(\tfrac{t-i\pi/2}{h}\big)| \leq C_N\Big(\frac{A_\delta^N}{h^{12}|\ln h|^{12}} + \frac{h^{\sigma-12}}{|\ln h|^7}\Big) \tag{130}$$

$$|\nu_2(t) - h\psi_2^u\big(\tfrac{t-i\pi/2}{h}\big) + \frac{A}{h}\psi_1^{u,N}\big(\tfrac{t-i\pi/2}{h}\big)| \leq \frac{C_N}{h^2|\ln h|^2}, \tag{131}$$

where $\psi_1^{u,N}$ *and* ψ_2^u *are defined in* (118) *and* A *is the constant introduced in* (60).

Theorem 3.1 is proven in Section 9.

In view of Remark 2.8 and Theorem 3.1, only inequalities (26)–(27) and (30)–(31) remain to be proved to complete the proof of Theorem 1.5 (the passage from $0 < h < h_N$ to $0 < h < h_0$ can be dealt with as in Remark 2.8 and will not be commented further).

3.2. **Proof of inequalities** (26)–(27) **of Theorem 1.5.** The proof will consist in two steps. First we will bound the function $D(t)$ in (121) and, consequently, the coefficients $c_i(t)$, when $\operatorname{Im} t = \pi/2 - \frac{\sigma}{2\pi} h|\ln h|$. Later, using that these coefficients are h-periodic real analytic functions, we will deduce exponentially small bounds on the real line. Moreover, as we want to obtain not only exponentially small bounds for $D(t)$ but asymptotic expressions, we will write explicitly the dominant terms.

For the sake of simplicity, from now on we fix $\theta = 1/2$ in Theorem 2.17 and $\delta = h^{1/2}$, hence $A_\delta^N = 2h^{N+1/2}$ as in (114). In particular, for any $N \geq 13$, there exists $h_N > 0$ such that all the hypotheses of Theorems 2.3–3.1 are satisfied for any $0 < h < h_N$.

Lemma 3.2. *Let* $\varepsilon_0 < 1/|2V_2|$. *For any* $\sigma > 13$ *and* $N \geq 13$, *there exist* $h_N, C_N > 0$ *such that, if* $0 < h < h_N$, $|\varepsilon| < \varepsilon_0$ *and* $t \in \mathcal{R}_\sigma$ *with* $\operatorname{Im} t = \pi/2 - \frac{\sigma}{2\pi} h|\ln h|$, *then*

$$\big|D(t) - D^{N,inn}\big(\tfrac{t-i\pi/2}{h}\big)\big| \leq C_N|\varepsilon|\Big(\frac{h^{N+1/2}}{|\ln h|^2} + h^{3\sigma/2}\Big), \tag{132}$$

where the function $D^{N,inn}$ *was defined in Theorem 2.17.*

Proof. For such a value of t, Theorem 2.18 and Corollary 2.19 imply

$$\big|D(t) - (\phi^{u,N} - \phi^{s,N})\big(\tfrac{t-i\pi/2}{h}\big)\big| \leq K_N|\varepsilon|\frac{h^{N+1/2}}{|\ln h|^2}$$

with a suitable $K_N > 0$ (taking into account that $A_\delta^N = 2h^{N+1/2}$). On the other hand, since $\operatorname{Im}\big(\frac{t-i\pi/2}{h}\big) = -\frac{\sigma}{2\pi}|\ln h|$ and $\theta = 1/2$, formula (109) of Theorem 2.17 yields

$$\big|(\phi^{u,N} - \phi^{s,N})\big(\tfrac{t-i\pi/2}{h}\big) - D^{N,inn}\big(\tfrac{t-i\pi/2}{h}\big)\big| \leq \tilde{K}_N|\varepsilon|h^{3\sigma/2}$$

with a suitable $\tilde{K}_N > 0$ and the conclusion follows. □

The rest of the proof of Theorem 1.5 only requires the following

Lemma 3.3. *Let $0 < h \le r$ and $M > 0$, and suppose that a real analytic h-periodic function f extends holomorphically to a complex strip $\{t \in \mathbb{C} \mid -r < \operatorname{Im} t < r\}$ and continuously to the closure of this strip, with $|f(t)| \le M$ on the line $\{\operatorname{Im} t = r\}$. Then, for any $t_0, t \in \mathbb{R}$,*

$$|f(t) - f(t_0)| \le 5M \exp\Big(-\frac{2\pi r}{h}\Big),$$
$$\Big|\frac{d^j}{dt^j} f(t)\Big| \le 50M \frac{2^j j!}{h^j} \exp\Big(-\frac{2\pi r}{h}\Big), \qquad j \in \mathbb{N}^*.$$

Proof. We expand f in Fourier series,

$$f(t) = \sum_{k\in\mathbb{Z}} f_k e^{2k\pi i t/h}.$$

Since f is holomorphic in a strip, we can compute the coefficients of the Fourier series along different horizontal lines $\{\operatorname{Im} t = \rho\}$: for any $\rho \in [-r, r]$,

$$f_k = \frac{1}{h}\int_{i\rho}^{i\rho+h} f(\tau) e^{-2\pi i k\tau/h} d\tau = \frac{e^{2\pi k\rho/h}}{h}\int_0^h f(t+i\rho) e^{-2\pi i k t/h} dt. \tag{133}$$

By choosing $\rho = r$ for $k \le -1$, we get

$$|f_k| \le M e^{-2\pi|k|r/h}, \tag{134}$$

and this inequality holds as well for $k \ge 1$, since $f_k = \overline{f_{-k}}$. Thus, for any $t \in \mathbb{R}$,

$$|f(t) - f_0| = \Big|\sum_{k\in\mathbb{Z}^*} f_k e^{2k\pi i t/h}\Big| \le 2M \sum_{k\ge1} e^{-2k\pi r/h} \le 2M e^{-2\pi r/h} \sum_{k\ge0} e^{-2k\pi}$$

and

$$\frac{1}{j!}\Big|\frac{d^j}{dt^j} f(t)\Big| \le \frac{2M}{h^j}\sum_{k\ge1}\frac{(2\pi k)^j}{j!} e^{-2k\pi r/h}$$
$$\le \frac{2^{j+1}M}{h^j}\sum_{k\ge1} e^{\pi k - 2\pi k r/h} \le \frac{2^{j+1}M}{h^j} e^{\pi - 2\pi r/h}\sum_{k\ge0} e^{-k\pi},$$

whence the conclusion follows. □

Now, we only need to use that $D(t) = c_1(t)\nu_1(t) + c_2(t)\nu_2(t)$, where ν_i are given in Theorem 3.1 and $c_1 = W_h(D, \nu_2)$, $c_2 = W_h(\nu_1, D)$, and we shall get inequalities which are sufficient to deduce (26)–(27).

Lemma 3.4. *Let $\varepsilon_0 < 1/|2V_2|$. For each $N_0 \in \mathbb{N}$ there exist $h_{N_0}, C_{N_0} > 0$ such that, for any $j \in \mathbb{N}$,*

- *if $0 < h < h_0$, $-\varepsilon_0 < \varepsilon < \varepsilon_0$ and $t \in \mathbb{R}$, then*

$$\Big|\frac{d^j}{dt^j}\Big(c_1(t) + \frac{3\pi\varepsilon B_0^+\mu}{2(h\gamma)^2} e^{-\frac{\pi^2}{h}}\big(\cos\tfrac{2\pi t}{h} + 1\big)\Big)\Big| \le 2^j j! C_0 |\varepsilon| e^{-\frac{\pi^2}{h}} h^{-2-j} |\ln h|^3, \tag{135}$$

- *if* $N_0 \in \mathbb{N}$, $0 < h < h_{N_0}$, $-\varepsilon_0 < \varepsilon < \varepsilon_0$ *and* $t \in \mathbb{R}$, *then*

$$\Big|\frac{d^j}{dt^j}\Big(c_2(t) + \frac{2\varepsilon}{h}e^{-\frac{\pi^2}{h}}\big(\sin\tfrac{2\pi t}{h}\big)\sum_{k=0}^{N_0} h^{2k}B_k^+\Big)\Big| \le 2^j j! C_{N_0}|\varepsilon| e^{-\frac{\pi^2}{h}} h^{2N_0+1-j} \tag{136}$$

where the B_k^+*'s are the coefficients given by Theorem 2.17.*

Proof. First we note that, since both ξ^u and ξ^s satisfy the boundary condition (21), we have $D(-h/2) = D(h/2) = 0$, hence $c_1(h/2) = c_2(h/2) = 0$.

We begin with choosing any $N \ge 15$ and $\sigma = N - 3/2$. Inequality (135) will follow from the fact that, on the segment $\big\{0 \le \operatorname{Re} t \le h$ and $\operatorname{Im} t = \pi/2 - \frac{\sigma}{2\pi}h|\ln h|\big\}$,

$$\big|c_1(t) + \frac{i\varepsilon B_0^+ A}{h}\exp\big(-2\pi i(\tfrac{t-i\pi/2}{h})\big)\big| \le K_N|\varepsilon| h^{\sigma-2}|\ln h|^3, \tag{137}$$

for a suitable $K_N > 0$, where A is the constant introduced in (60); indeed, since $\frac{i\varepsilon B_0^+ A}{h}\exp\big(-2\pi i(\frac{t-i\pi/2}{h})\big) = \frac{3\pi\varepsilon B_0^+\mu}{4(h\gamma)^2}e^{-\frac{\pi^2}{h}}e^{-2\pi i t/h}$ and $\operatorname{Im} t = \pi/2 - \frac{\sigma}{2\pi}h|\ln h| \Rightarrow |e^{2\pi i t/h}| = h^\sigma e^{-\frac{\pi^2}{h}}$ (exponentially smaller than $e^{-2\pi i t/h}$), inequality (137) will entail

$$|c_1(t) + \frac{3\pi\varepsilon B_0^+\mu}{2(h\gamma)^2}e^{-\frac{\pi^2}{h}}\cos\tfrac{2\pi t}{h}| \le K_N|\varepsilon|h^{\sigma-2}|\ln h|^3$$

on the same segment, and it will then be sufficient to apply Lemma 3.3 with $t_0 = h/2$ and $r = \pi/2 - \frac{\sigma}{2\pi}h|\ln h|$ (so that $e^{-2\pi r/h} = h^{-\sigma}e^{-\pi^2/h}$).

To prove (137), since $c_1 = W_h(D, \nu_2)$, we use the estimates given in Lemma 3.2 for $D(t)$ and in Theorem 3.1 (with $\delta = h^{1/2}$) for $\nu_2(t)$, with t such that

$$-h \le \operatorname{Re} t \le h, \quad \operatorname{Im} t = \pi/2 - \frac{\sigma}{2\pi}h|\ln h|. \tag{138}$$

We get

$$D(t) = D^{N,inn}(z) + O(\varepsilon h^{N+1/2}|\ln h|^{-2} + h^{3\sigma/2}), \tag{139}$$

$$\nu_2(t) = -\frac{A}{h}\psi_1^{u,N}(z) + h\psi_2^{u,N}(z) + O(h^{-2}|\ln h|^{-2}), \tag{140}$$

with $z = \frac{t-i\pi/2}{h}$; our assumption on t implies $|\operatorname{Re} z| \le 1$ and $\operatorname{Im} z = -\frac{\sigma}{2\pi}|\ln h|$, thus the estimates (105) and (106) yield

$$\psi_{1,n}^u(z) = O(|\ln h|^{2n-2}), \quad \psi_{2,n}^u(z) = O(|\ln h|^{2n+3}), \tag{141}$$

which implies, together with $|e^{-2\pi i z}| = h^\sigma$ and $A = O(h^{-3})$,

$$D(t) = -\varepsilon e^{-2\pi i z}(A_0^+\psi_{1,0}^u + iB_0^+\psi_{2,0}^u) + |\varepsilon|O(h^{\sigma+2}|\ln h|^5 + h^{N+1/2}|\ln h|^{-2} + h^{3\sigma/2}),$$

$$\nu_2(t) = -\frac{A}{h}\psi_{1,0}^u(z) + O(h^{-2}).$$

Inequality (137) follows from $\sigma + 2 = N + 1/2 < 3\sigma/2$, $e^{-2\pi i z}\big(A_0^+\psi_{1,0}^u(z) + iB_0^+\psi_{2,0}^u(z)\big) = O(h^\sigma|\ln h|^3)$, $\frac{A}{h}\psi_{1,0}^u(z) = O(h^{-4}|\ln h|^{-2})$ and $W_1(\psi_{1,0}^u, \psi_{2,0}^u) = 1$.

Let us now prove inequality (136). Given $N_0 \in \mathbb{N}$, we choose $N = 6N_0 + 21$ and $\sigma = 4N_0 + 19$. Inequality (136) will follow from the fact that, on the segment $\{0 \le \operatorname{Re} t \le h \text{ and } \operatorname{Im} t = \pi/2 - \frac{\sigma}{2\pi} h|\ln h|\}$,

$$|c_2(t) + i\frac{\varepsilon}{h} e^{-2\pi i z} \sum_{k=0}^{N_0} h^{2k} B_k^+| \le |\varepsilon| K_N h^{N-1}, \tag{142}$$

for a suitable $K_N > 0$, with $z = \frac{t - i\pi/2}{h}$; indeed, using that $e^{2\pi i t/h}$ is exponentially smaller than $e^{-2\pi i t/h}$ on this segment, one can replace $ie^{-2\pi i z} = ie^{-\pi^2/h} e^{-2\pi i t/h}$ with $2e^{-\pi^2/h} \sin \frac{2\pi t}{h}$ in (142) and then apply Lemma 3.3 with $t_0 = h/2$ and $r = \pi/2 - \frac{\sigma}{2\pi} h|\ln h|$ as previously: the right-hand side of (142) gets multiplied by $e^{-2\pi r/h} = h^{-\sigma} e^{-\pi^2/h}$ and $N - 1 - \sigma$ coincides with $2N_0 + 1$.

To prove (142), since $c_2 = W_h(\nu_1, D)$, we use (139) and the estimate for ν_1 given by (130) in Theorem 3.1, for any t in the range (138). We get

$$\nu_1(t) = h^{-1} \psi_1^{u,N}(z) + O(h^{\sigma-12} |\ln h|^{-7}),$$
$$D(t) = D^{N,inn}(z) + O(\varepsilon h^{N+1/2} |\ln h|^{-2}),$$

with $z = \frac{t - i\pi/2}{h}$ for which the estimates (141) still hold (we used $\sigma < N + 1/2 < 3\sigma/2$ to simplify the error terms). Since $h^{-1}\psi_1^{u,N}(z) = O(h^{-1}|\ln h|^{-2})$ and $D^{N,inn}(z) = O(\varepsilon h^{\sigma} |\ln h|^3)$, we obtain

$$c_2(t) = W_1(h^{-1}\psi_1^{u,N}, D^{N,inn})(z) + O(\varepsilon h^{N-1/2} |\ln h|^{-4}).$$

We can write $D^{N,inn}(z) = -\varepsilon e^{-2\pi i z} \chi^N(z)$ with

$$\chi^N = \sum_{n=0}^{N} h^{2n} \chi_n, \qquad \chi_n = \sum_{n_1+n_2=n} \left(A_{n_1}^+ \psi_{1,n_2}^u + i B_{n_1}^+ \psi_{2,n_2}^u \right),$$

while $\psi_1^{u,N} = \sum_{n=0}^{N} h^{2n} \psi_{1,n}^u$ and, by (1.d) in Theorem 2.17,

$$0 \le n \le N \ \Rightarrow \sum_{\substack{0 \le n', n'' \le n \\ n'+n''=n}} W_1(\psi_{1,n'}^u, \chi_{n''}) = iB_n^+,$$

hence

$$W_1(\psi_1^{u,N}, \chi^N) = \sum_{n=0}^{N} i h^{2n} B_n^+ + \sum_{n=N+1}^{2N} h^{2n} \sum_{\substack{0 \le n', n'' \le N \\ n'+n''=n}} W_1(\psi_{1,n'}^u, \chi_{n''}).$$

Since $\chi_n(z) = O(|\ln h|^{2n+3})$, we have $W_1(\psi_{1,n'}^u, \chi_{n''}) = O(|\ln h|^{2n+1})$ in the above sum, therefore

$$W_1(h^{-1}\psi_1^{u,N}, D^{N,inn})(z) = -i\frac{\varepsilon}{h} e^{-2\pi i z} \sum_{n=0}^{N} h^{2n} B_n^+ + O(\varepsilon h^{2N+1+\sigma} |\ln h|^{2N+3}),$$

which is sufficient to conclude. □

3.3. **Proof of inequalities** (30)–(31) **of Theorem 1.5.**

Lemma 3.5. *Let $\varepsilon_0 < 1/|2V_2|$. For each $N_0 \in \mathbb{N}$ there exist $h_{N_0}, C_{N_0} > 0$ such that, if $N_0 \in \mathbb{N}$, $0 < h < h_{N_0}$, $-\varepsilon_0 < \varepsilon < \varepsilon_0$ and $t \in [-T, T]$, then*

$$\left|\frac{d^j}{dt^j}\big(\nu_1 - \eta_1^{N_0}\big)(t)\right| \le C_{N_0} j! |\varepsilon| h^{2N_0+3}, \qquad j \in \mathbb{N},$$

where $\eta_1^{N_0} = \frac{d}{dt}\xi^{N_0,\mathrm{out}}$ as in (81)*.*

Proof. Let $N_0 \in \mathbb{N}$. We choose $N = 2N_0 + 6$ and $\sigma = 2N_0 + 13$. By virtue of Cauchy inequalities and inequality (128) in Theorem 3.1, we have

$$\frac{d^j}{dt^j}\nu_1(t) = \frac{d^j}{dt^j}\eta_1^u(t) + |\varepsilon| j! O\big(h^{N-5/2}|\ln h|^{-1} + h^{\sigma-3}|\ln h|\big),$$

while inequality (82) in Corollary 2.7 yields

$$\frac{d^j}{dt^j}\eta_1^u(t) = \frac{d^j}{dt^j}\eta_1^{N_0}(t) + j! O(\varepsilon h^{2N_0+3}).$$

The conclusion follows since both $N - 5/2$ and $\sigma - 3$ are larger than $2N_0 + 3$. □

This gives inequality (30). Finally, inequality (31) follows from Cauchy inequalities, (129) and (120).

4. Some notes on linear second order difference equations

Here we review some standard definitions and results about the theory of linear second order difference equations.

Definition 4.1. *Given $h > 0$, we define the first order difference operator Δ_h by the formula*

$$\Delta_h f(t) = f(t+h) - f(t),$$

for a function $f\colon U \subset \mathbb{C} \to \mathbb{C}$.

According to (32), the Wronskian of two functions $f, g\colon U \subset \mathbb{C} \to \mathbb{C}$ can thus be written

$$W_h(f,g) = \begin{vmatrix} f & g \\ \Delta_h f & \Delta_h g \end{vmatrix}.$$

Notice that $\Delta_h f(t)$ and $W_h(f,g)(t)$ are defined only for those $t \in U$ such that $t + h \in U$.

In what follows, we shall consider only two types of domains:

(I) Given a function $r_+\colon (a,b) \subset \mathbb{R} \to \mathbb{R}$, we define

$$U^u_{r_+} = \{t \in \mathbb{C} \mid a < \operatorname{Im} t < b,\ \operatorname{Re} t < r_+(\operatorname{Im} t)\},$$

$$U^s_{r_+} = \{t \in \mathbb{C} \mid a < \operatorname{Im} t < b,\ \operatorname{Re} t > r_+(\operatorname{Im} t)\}$$

Observe that, for all $\lambda > 0$, if $t \in U^u_{r_+}$, then $t - \lambda \in U^u_{r_+}$, while if $t \in U^s_{r_+}$, then $t + \lambda \in U^s_{r_+}$.

(II) Given two functions $r_+, r_- \colon (a,b) \subset \mathbb{R} \to \mathbb{R}$ such that $r_- < r_+$, we define

$$U_{r_-,r_+} = \{t \in \mathbb{C} \mid a < \operatorname{Im} t < b,\ r_-(\operatorname{Im} t) \le \operatorname{Re} t < r_+(\operatorname{Im} t)\}.$$

Observe that the closure of the domain $D_\delta^{u,\mathrm{out}}$ defined in Section 2.2 is the closure of a domain of type (I), while the closure of the domain $D_h^{u,\mathrm{in}}$ of Section 2.8 is the disjoint union of the closures of two domains of type (II).

Given some complex function g, we will need to solve the equation

$$\Delta_h f = g. \tag{143}$$

The method to solve this equation will depend, essentially, on the domain of definition of g. If g is defined on a domain $U_{r_+}^u$, then the formula

$$\Delta_{h,u}^{-1} g(t) = \sum_{k=1}^{\infty} g(t-kh) \tag{144}$$

defines a solution $\Delta_{h,u}^{-1} g$ of equation (143) which tends to 0 as $\operatorname{Re} t \to -\infty$ provided that this series is normally convergent, in which case the general solution is obtained by adding any h-periodic function to the particular solution $\Delta_{h,u}^{-1} g$. We can consider $\Delta_{h,u}^{-1}$ as a right inverse of operator Δ_h when both operators are defined in suitable spaces.

On the other hand, if $g \colon U_{r_+}^s \to \mathbb{C}$, a right inverse of Δ_h is given by

$$\Delta_{h,s}^{-1} g(t) = \sum_{k=0}^{\infty} g(t+kh), \tag{145}$$

provided this series is normally convergent.

The next three lemmas summarize some elementary results about second order linear difference equations which we will use and whose proofs we omit (see however Section 2.1 and Appendix A.2 of [MSS08]).

Lemma 4.2. *Given a domain of type (I), $U = U_{r_+}^u$, resp. $U = U_{r_+}^s$, on which a function G is defined, consider the linear second order difference equation*

$$u(t+h) + u(t-h) - G(t)u(t) = 0, \qquad t \in U, \tag{146}$$

where the unknown u is required to be defined on $U_{r_+ + h}^u$, resp. $U_{r_+ - h}^s$. Then:

1. *For any two solutions u_1 and u_2, the function $W_h(u_1,u_2)$ is h-periodic.*
2. *If u_1 is a solution which does not vanish, then*

$$u_2 = cu_1 \text{ solution such that } W_h(u_1,u_2) \equiv 1 \iff \Delta_h c(t) = \tfrac{1}{u_1(t+h)u_1(t)}.$$

3. *For any two solutions u_1 and u_2 such that $W_h(u_1,u_2)$ does not vanish, the set of solutions of (146) is*

$$\{u = c_1 u_1 + c_2 u_2,\ c_1 \text{ and } c_2\ h\text{-periodic functions}\}.$$

Once the solutions of a homogeneous difference equation are found, it is possible to obtain the solutions of the non-homogeneous one. In the case of an unbounded domain extending to the left, we have the following

Lemma 4.3. *Let* $r_+\colon (a,b) \to \mathbb{R}$ *be a function and consider the corresponding unbounded domain* $U^u_{r_+}$ *of type (I), on which two functions* G *and* H *are supposed to be defined. Assume that* $u_1, u_2\colon U^u_{r_++h} \to \mathbb{C}$ *are two solutions of* (146) *such that* $W_h(u_1,u_2) \equiv 1$*. Then a solution of the equation*

$$u(t+h) + u(t-h) - G(t)u(t) = H(t), \qquad t \in U^u_{r_+}, \tag{147}$$

is given by

$$t \in U^u_{r_++h} \;\mapsto\; \sum_{k=1}^{\infty} \big(u_1(t-kh)u_2(t) - u_1(t)u_2(t-kh)\big)H(t-kh) \tag{148}$$

if this series is absolutely convergent.

In Section 8, we shall have to deal with bounded domains and to find solutions that satisfy some given initial conditions:

Lemma 4.4. *Let* $r_-, r_+\colon (a,b) \to \mathbb{R}$ *be functions and consider the corresponding domain* U_{r_-,r_+} *of type (II), on which two functions* G *and* H *are supposed to be defined. Assume that* $u_1, u_2\colon U_{r_--h,r_++h} \to \mathbb{C}$ *are two solutions of* (146) *(for* $t \in U_{r_-,r_+}$*) such that* $W_h(u_1,u_2) \equiv 1$ *(on* U_{r_--h,r_+}*). Then, for any function* $u^*\colon U_{r_--h,r_-+h} \to \mathbb{C}$*, the equation*

$$u(t+h) + u(t-h) - G(t)u(t) = H(t), \qquad t \in U_{r_-,r_+} \tag{149}$$

admits a unique solution which is defined on U_{r_--h,r_++h} *and satisfies*

$$u(t) = u^*(t), \qquad t \in U_{r_--h,r_-+h}. \tag{150}$$

This solution is $u = u_p + u_h$*, where*

$$u_h(t) = c_1(t)u_1(t) + c_2(t)u_2(t), \qquad t \in U_{r_--h,r_++h},$$

c_1*,* c_2 *are the* h*-periodic functions uniquely determined by*

$$\left.\begin{array}{rcl} c_1(t) & = & W_h(u^*,u_2)(t) \end{array}\right\}, \qquad t \in U_{r_--h,r_-},$$

and

$$u_p(t) = \begin{cases} 0 & \text{for } t \in U_{r_--h,r_-+h}, \\ \displaystyle\sum_{k=1}^{k^*(t)} \big(u_1(t-kh)u_2(t) - u_1(t)u_2(t-kh)\big)H(t-kh) & \text{for } t \in U_{r_-+h,r_++h}, \end{cases}$$

with $k^*(t) = \left\lfloor \frac{\operatorname{Re} t - r_-(\operatorname{Im} t)}{h} \right\rfloor$ *(so that* $t - k^*(t)h \in U_{r_-,r_-+h}$ *in the last case).*

Observe that, in this lemma, the existence and uniqueness of the solution u is obvious, since equation (149) can be written

$$u(t) = -u(t-2h) + G(t-h)u(t-h) + H(t-h), \qquad t \in U_{r_-+h,r_++h},$$

so that the values of u^* on U_{r_--h,r_-+h} uniquely determine the values of u on U_{r_-+h,r_-+2h}, and then on U_{r_-+2h,r_-+3h}, and so on until the domain $U_{r_-+h,r_++h} \cap \bigcup_{k\geq 1} U_{r_-+kh,r_-+(k+1)h}$ is covered. We call the domain U_{r_--h,r_-+h} a "boundary layer". In fact, the function u_h is the unique solution of the homogeneous equation (146) whose restriction to the boundary layer is u^*, while u_p is the unique solution of the non-homogeneous equation (149) whose restriction to the boundary layer vanishes identically.

Remark 4.5. If G, H and u^* are analytic, this does not imply that the solution u is itself analytic: there are possible failures of analyticity (or even discontinuities) on the curves $\{\operatorname{Re} t = r_-(\operatorname{Im} t) + kh\}$, $k \geq 1$. However the above chain of reasoning shows that

> *if G and H admit a continuation which is holomorphic in a neighborhood of U_{r_-,r_+} and if u^* admits a continuation which is holomorphic in a neighborhood of the closure of U_{r_--h,r_++h} and which satisfies equation* (149) *in a neighborhood of the curve $\{\operatorname{Re} t = r_-(\operatorname{Im} t)\}$, then the solution u admits a continuation which is holomorphic in a neighborhood of U_{r_--h,r_++h}.*

We shall give more details when using a non-linear variant of this in Section 8.5.

5. Outer approximations. Proof of Proposition 2.2 and Theorems 2.3 and 2.4

5.1. **Extended domains.** Our intention is to find analytic functions defined in the domain $D_\delta^{u,\mathrm{out}}$ introduced in Section 2.2. In fact, as announced in Remark 2.5, we shall find these functions in larger domains, defined as follows. Given $0 < \beta_1 \leq \beta$, $0 \leq \beta_2 < \pi/2$, $0 \leq r_1 < 1/2$, $1/2 < r_2 \leq 1$ and $\delta \geq 0$, we set

(151)
$$\begin{aligned}
&U(\beta_1,\beta_2,r_1,r_2,\delta)\\
&= \{t \in \mathbb{C} \mid \operatorname{Re} t \leq -1 + r_1\}\\
&\cup \{t \in \mathbb{C} \mid -1 + r_1 \leq \operatorname{Re} t \leq 0,\ |\operatorname{Im} t| \leq \frac{\pi}{2},\ \left|t - \frac{\pi}{2} i\right| \geq r_2\delta,\ \left|t + \frac{\pi}{2} i\right| \geq r_2\delta\}\\
&\cup \{t \in \mathbb{C} \mid -1 + r_1 \leq \operatorname{Re} t \leq -r_2\delta,\ \operatorname{Im} t \geq \frac{\pi}{2},\ -\pi - \beta_2 \leq \arg\left(t - \frac{\pi}{2} i\right) \leq -\pi\}\\
&\cup \{t \in \mathbb{C} \mid 0 \leq \operatorname{Re} t \leq T_1 + 1,\ 0 \leq \operatorname{Im} t \leq \frac{\pi}{2} - r_2\delta,\ \arg\left(t - \frac{\pi}{2} i\right) \leq -\beta_1\}\\
&\cup \{t \in \mathbb{C} \mid -1 + r_1 \leq \operatorname{Re} t \leq -r_2\delta,\ \operatorname{Im} t \leq -\frac{\pi}{2},\ \pi \leq \arg\left(t + \frac{\pi}{2} i\right) \leq \pi + \beta_2\}\\
&\cup \{t \in \mathbb{C} \mid 0 \leq \operatorname{Re} t \leq T_1 + 1,\ -\frac{\pi}{2} + r_2\delta \leq \operatorname{Im} t \leq 0,\ \arg\left(t + \frac{\pi}{2} i\right) \leq \beta_1\},
\end{aligned}$$

where $T_1 = \frac{\pi}{2}\cot(\beta_1) - 1$.

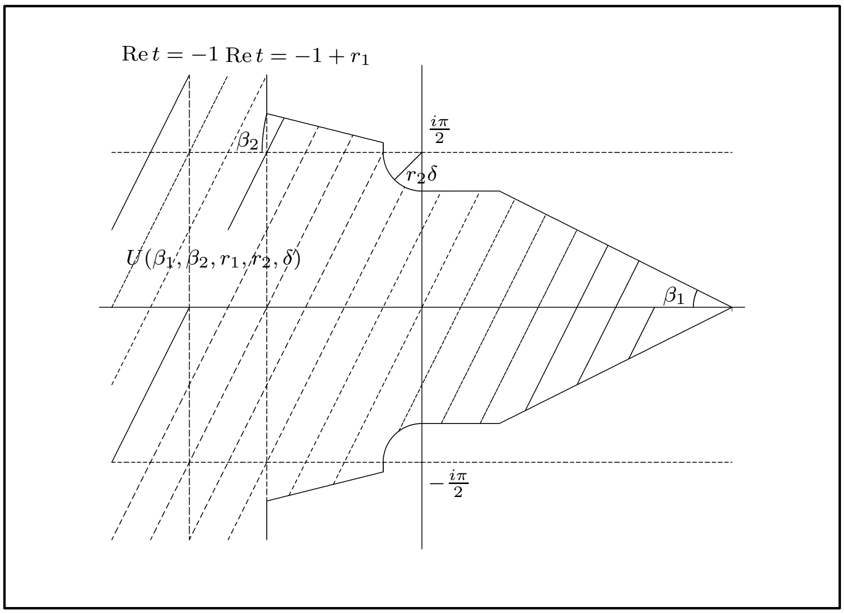

FIGURE 6. The extended outer domain $U(\beta_1, \beta_2, r_1, r_2, \delta)$. It is symmetrical with respect the real axis.

Observe that $D_\delta^{u,\mathrm{out}} = U(\beta, 0, 0, 1, \delta) \subsetneq U(\tilde\beta_1, \tilde\beta_2, \tilde r_1, \tilde r_2, \delta) \subsetneq U(\beta_1, \beta_2, r_1, r_2, \delta)$ for $\beta_1 < \tilde\beta_1 < \beta$, $0 < \tilde\beta_2 < \beta_2$, $0 < \tilde r_1 < r_1$ and $r_2 < \tilde r_2 < 1$; see Figure 6. Moreover, for any $t \in U(\tilde\beta_1, \tilde\beta_2, \tilde r_1, \tilde r_2, \delta)$, if we use the notation $\tau = \min\{|t - i\pi/2|, |t + i\pi/2|\}$ and $D(t, \rho)$ for the disc of radius ρ centered at t, we have

$$\begin{aligned} D(t, \kappa\tau) &\subset U(\beta_1, \beta_2, r_1, r_2, \delta) && \text{if } \mathrm{Re}\, t > -1 + \tilde r_1, \\ D(t, \kappa) &\subset U(\beta_1, \beta_2, r_1, r_2, \delta) && \text{if not,} \end{aligned}$$

with a certain $\kappa > 0$ depending on $\beta_1, \tilde\beta_1, \beta_2, \tilde\beta_2, r_2, \tilde r_2$. This will allow us to use Cauchy inequalities to estimate the derivatives of the functions we want to describe at the price of passing from one of these domains to a smaller one (such a reduction of domain will be performed N times, where N is fixed but arbitrary).

5.2. **The linearized equation.** In this section we will prove the existence of a solution ξ^u of the invariance equation (56), satisfying boundary conditions (19) and (21), as well as the properties of the sequence of approximating functions given by Proposition 2.2. Furthermore, we will find suitable solutions of the linearization of invariance equation (56) around ξ^u that will be needed in the proof of the Analytic Theorem 1.5.

To prove all these results, we will need to solve equations of the form

$$\mathcal{L}(\eta) = g,$$

(for instance, see (64)) where the linear map $\mathcal{L}$ is defined by

$$\mathcal{L}(\eta)(t) = \eta(t+h) + \eta(t-h) - \mu f'(\xi^0(t))\eta(t). \tag{152}$$

This linear map appears when one linearizes the unperturbed invariance equation (18) around ξ^0.

By Lemmas 4.3 and 4.4, solutions of $\mathcal{L}(\eta) = g$ can be obtained if a fundamental set of solutions of the homogeneous equation $\mathcal{L}(\eta) = 0$ is known. This is our case, due to the integrability of the unperturbed invariance equation. In next lemma we list some properties of a family of fundamental solutions of $\mathcal{L}(\eta) = 0$.

Lemma 5.1. *The functions η_1 and η_2^c, defined in* (58) *and* (59) *verify that*

$$W(\eta_1, \eta_2^c)(t) = 1,$$

and $\mathcal{L}(\eta) = 0$, where $\mathcal{L}$ is the linear operator defined in (152)*. Moreover, for any $\beta_1 \le \beta$, $0 \le \beta_2 < \pi/2$, $0 \le r_1 < 1/2$, $1/2 < r_2 \le 1$, there exists $C > 0$ such that η_1 can be bounded as follows in the domain $U(\beta_1, \beta_2, r_1, r_2, 0)$, defined in* (151)*,*

$$|\eta_1(t)| \le Che^{\operatorname{Re} t}, \qquad t \in U(\beta_1, \beta_2, r_1, r_2, 0),\ \operatorname{Re} t \le -1, \tag{153}$$

$$|\eta_1(t)| \le C\frac{h}{|\cosh t|^2}, \qquad t \in U(\beta_1, \beta_2, r_1, r_2, 0),\ -1 \le \operatorname{Re} t. \tag{154}$$

If $c = 0$, then η_2^0 is real analytic and satisfies

$$|\eta_2^0(t)| < C\frac{e^{-\operatorname{Re} t}}{h^2}, \qquad t \in U(\beta_1, \beta_2, r_1, r_2, 0),\ \operatorname{Re} t \le -1. \tag{155}$$

If $\alpha = i\pi/2$, then $\eta_2^{i\pi/2} = \eta_2^0 + A\eta_1$, being A the constant introduced in (60)*, and satisfies*

$$|\eta_2^{\pm i\pi/2}(t)| < C\frac{|\cosh t|^3}{h^2}, \qquad t \in U(\beta_1, \beta_2, r_1, r_2, 0),\ -1 \le \operatorname{Re} t. \tag{156}$$

Furthermore, η_1 is $i\pi$-antiperiodic and odd, while η_2^0 is even. Both are meromorphic, with singularities at $i(\pi/2 + k\pi)$, $k \in \mathbb{Z}$.

The proof of this lemma is a detailed study of the functions η_1 and η_2^c, given by (58) and (59), and it is performed in [DRR98].

5.3. **Banach spaces and technical lemmas.** Here we place the definition of the spaces of functions we will use along the proofs of Proposition 2.2 and Theorems 2.3 and 2.4, as well as some technical lemmas concerning these spaces and operators between them.

In order to make the proofs more readable, we will use the following convention: we will say that $g_1 = O(g_2)$ in some domain U if there exists some positive constant C, that may depend on β, N and other constants, but does not depend on δ nor h, such that $|g_1| \le C|g_2|$ in U.

For $l, m \in \mathbb{R}$, we define the spaces

(157) $$\mathcal{X}_{l,m}(\beta_1,\beta_2,r_1,r_2,\delta) = \{\xi\colon U(\beta_1,\beta_2,r_1,r_2,\delta) \to \mathbb{C} \text{ real analytic, such that } \|\xi\|_{l,m} < \infty\},$$

where

(158) $$\|\xi\|_{l,m} = \max\{\sup_{\mathrm{Re}\, t \le -1} e^{-l\,\mathrm{Re}\, t}\,|\xi(t)|, \sup_{-1\le \mathrm{Re}\, t} |\cosh t|^m\,|\xi(t)|\}.$$

With this norm, they are Banach spaces. In what follows, we shall not write explicitly the dependence of $\mathcal{X}_{l,m}$ on β_1, β_2, r_1, r_2 and δ, unless it is essential for the statements.

Some properties of these spaces, that we will use hereafter, are the following.

Lemma 5.2. *Let $R > 0$. For any $\beta_1 < \beta$, $0 < \beta_2 < \pi/2$, $0 < r_1 < 1/2$, $1/2 < r_2 < 1$ and $\delta > Rh$,*

(a) *If $\zeta_1 \in \mathcal{X}_{l_1,m_1}(\beta_1,\beta_2,r_1,r_2,\delta)$ and $\zeta_2 \in \mathcal{X}_{l_2,m_2}(\beta_1,\beta_2,r_1,r_2,\delta)$, then $\zeta_1\zeta_2 \in \mathcal{X}_{l_1+l_2,m_1+m_2}(\beta_1,\beta_2,r_1,r_2,\delta)$ and*

$$\|\zeta_1\zeta_2\|_{l_1+l_2,m_1+m_2} \le \|\zeta_1\|_{l_1,m_1}\|\zeta_2\|_{l_2,m_2}.$$

(b) *If $\zeta \in \mathcal{X}_{l_1,m_1}(\beta_1,\beta_2,r_1,r_2,\delta)$, then, for any $l_2 \le l_1$ and $m_2 \le m_1$, $\zeta \in \mathcal{X}_{l_2,m_2}(\beta_1,\beta_2,r_1,r_2,\delta)$ and*

$$\|\zeta\|_{l_2,m_2} \le O(\delta^{m_2-m_1})\|\zeta\|_{l_1,m_1}.$$

Moreover, for any $m_1 \le m_3$, $\zeta \in \mathcal{X}_{l_1,m_3}(\beta_1,\beta_2,r_1,r_2,\delta)$ and

$$\|\zeta\|_{l_1,m_3} \le O(1)\|\zeta\|_{l_1,m_1}.$$

(c) *If $\zeta \in \mathcal{X}_{l,m}(\beta_1,\beta_2,r_1,r_2,\delta)$, with $\|\zeta\|_{l,m} = O(h^m)$, and $g(y) = O(y^k)$ is an analytic function around the origin, then $g\circ\zeta \in \mathcal{X}_{kl,km}(\beta_1,\beta_2,r_1,r_2,\delta)$ and*

$$\|g\circ\zeta\|_{kl,km} = O(h^{km}).$$

(d) *If $\zeta \in \mathcal{X}_{l,m}(\beta_1,\beta_2,r_1,r_2,\delta)$, then, for any $\beta_1 < \tilde\beta_1 < \beta$, $0 < \tilde\beta_2 < \beta_2$, $0 < \tilde r_1 < r_1$ and $1 > \tilde r_2 > r_2$, $\zeta^{(j)} \in \mathcal{X}_{l,m+j}(\tilde\beta_1,\tilde\beta_2,\tilde r_1,\tilde r_2,\delta)$ and $\|\zeta^{(j)}\|_{l,m+j} \le O(1)\|\zeta\|_{l,m}$ for all j.*

Proof. Part (a), (b) and (c) are straightforward. Part (d) is obtained by means of Cauchy inequalities in sectorial domains. The constant involved depends neither on j nor on δ. □

Remark 5.3. In part (d) of the preceding lemma we have not explicitly written the constant $O(1)$ involved because we will use it at most N times, with N arbitrary but fixed. Hence, the resulting derivative will still be defined in $D^{u,\mathrm{out}}_\delta$. If we were interested in using part (d) in an iterative process, which is not the case, it would be necessary a more careful control of the value of this constant.

As we have already noted, along this section we will have to solve equations of the form $\mathcal{L}(\eta) = g$. It will be accomplished by using a right inverse of the operator $\mathcal{L}$. Notice that several right inverses are possible. Following (148) in Lemmas 4.3, two of them are defined as follows. We introduce the linear operators $\tilde{\mathcal{G}}$ and $\mathcal{G}$ given formally by

$$\tilde{\mathcal{G}}(g)(t) = \sum_{k=1}^{\infty} V_k(t) g(t - kh), \tag{159}$$

where

$$V_k(t) = \eta_2^0(t)\eta_1(t - kh) - \eta_2^0(t - kh)\eta_1(t), \tag{160}$$

and

$$\mathcal{G}(g) = \tilde{\mathcal{G}}(g) + \Delta_{h,u}^{-1}(\eta_2^0 g)(h/2)\eta_1. \tag{161}$$

We recall that η_1 and η_2^0 are defined in (58) and (59), and are a fundamental set of solutions of $\mathcal{L}(\eta) = 0$ and the operator $\Delta_{h,u}^{-1}$ was defined in (144).

We remark that V_k is real analytic and πi-periodic. Furthermore, since $\eta_2^{i\pi/2} = \eta_2^0 + A\eta_1$, one also has

$$V_k(t) = \eta_2^{i\pi/2}(t)\eta_1(t - kh) - \eta_2^{i\pi/2}(t - kh)\eta_1(t). \tag{162}$$

We will use this fact to have better bounds of V_k at points close to $i\pi/2$.

Lemma 5.4. *Let $R > 0$, $l > 1$ and $m > 4$. For any $\beta_1 \leq \beta$, $0 \leq \beta_2 < \pi/2$, $0 \leq r_1 < 1/2$, $1/2 < r_2 \leq 1$ and $\delta > Rh$,*

(a) *$\tilde{\mathcal{G}}\colon \mathcal{X}_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta) \to \mathcal{X}_{l,m-2}(\beta_1, \beta_2, r_1, r_2, \delta)$ is a right inverse of $\mathcal{L}$, that is, satisfies $\mathcal{L} \circ \tilde{\mathcal{G}} = \mathrm{Id}$, with $\|\tilde{\mathcal{G}}\| = O(1/h^2)$. If $g \in \mathcal{X}_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta)$ is $i\pi$-antiperiodic, then so is $\tilde{\mathcal{G}}(g)$.*

(b) *Let $g \in \mathcal{X}_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta)$. Then*

$$\Delta_h \tilde{G}(g)(-h/2) = -\Delta_h \eta_1(-h/2) \Delta_{h,u}^{-1}(\eta_2^0 g)(h/2),$$

and $|\Delta_{h,u}^{-1}(\eta_2^0 g)(h/2)| = O(h^{-3})\|g\|_{l,m}$.

(c) *$\mathcal{G}\colon \mathcal{X}_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta) \to \mathcal{X}_{1,m-2}(\beta_1, \beta_2, r_1, r_2, \delta)$ is a right inverse of $\mathcal{L}$ and $\|\mathcal{G}\| = O(1/h^2)$. Moreover, for any $i\pi$-antiperiodic $g \in \mathcal{X}_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta)$, $\mathcal{G}(g)$ is the only $i\pi$-antiperiodic solution of the equation $\mathcal{L}(\eta) = g$ which is analytic in $U(\beta_1, \beta_2, r_1, r_2, \delta)$ and satisfies the boundary conditions $\lim_{t \to -\infty} \eta(t) = 0$ and $\eta(-h/2) = \eta(h/2)$.*

Proof. We start by proving (a). First of all notice that, since V_k is $i\pi$-periodic, if $g \in \mathcal{X}_{l,m}$ is $i\pi$-antiperiodic and the sum is uniformly convergent, $\tilde{\mathcal{G}}(g)$ is $i\pi$-antiperiodic and analytic.

Next, from (153), (154), (155) and (156) we have that there exists $C > 0$ such that for any $t, t-kh \in U(\beta_1, \beta_2, r_1, r_2, \delta)$

$$|V_k(t)| \leq \begin{cases} Ce^{kh}/h, & \text{if } \operatorname{Re} t \leq -1, \\ Ce^{kh}/(h\tau^2), & \text{if } \operatorname{Re} t - kh \leq -1 \text{ and } -1 \leq \operatorname{Re} t, \\ C\dfrac{1}{h}\Big(\dfrac{\tau^3}{\tau_k^2} + \dfrac{\tau_k^3}{\tau^2}\Big), & \text{if } -1 \leq \operatorname{Re} t, \operatorname{Re} t - kh, \end{cases} \tag{163}$$

where $\tau = |\cosh t|$ and $\tau_k = |\cosh(t-kh)|$.

Let $g \in \mathcal{X}_{l,m}$. It satisfies

$$|g(t)| \leq \begin{cases} e^{l \operatorname{Re} t}\|g\|_{l,m}, & \text{if } \operatorname{Re} t \leq -1, \\ |\cosh t|^{-m}\|g\|_{l,m}, & \text{if } -1 \leq \operatorname{Re} t. \end{cases} \tag{164}$$

Then, if $\operatorname{Re} t \leq -1$, we have that

$$|e^{-l\operatorname{Re} t}\tilde{\mathcal{G}}(g)(t)| \leq C \sum_{k\geq 1} e^{-l\operatorname{Re} t}\frac{e^{kh}}{h} e^{l\operatorname{Re} t} e^{-lkh}\|g\|_{l,m} \leq \frac{C}{h^2}\|g\|_{l,m},$$

and, if $-1 \leq \operatorname{Re} t$,

$$\begin{aligned} |\tau^{m-2}\tilde{\mathcal{G}}(g)(t)| &\leq \tau^{m-2}\sum_{k\geq 1}|V_k(t)||g(t-kh)| \\ &\leq \tau^{m-2}\Big(\sum_{-1\leq \operatorname{Re} t - kh}|V_k(t)||g(t-kh)| + \sum_{\operatorname{Re} t - kh \leq -1}|V_k(t)||g(t-kh)|\Big) \\ &\leq C\sum_{k\geq 1}\frac{1}{h}\Big(\frac{\tau^{m+1}}{\tau_k^{m+2}} + \frac{\tau^{m-4}}{\tau_k^{m-3}}\Big)\|g\|_{l,m} + \tau^{m-4} C \sum_{k\geq 1}\frac{e^{-(l-1)kh}}{h}\|g\|_{l,m} \\ &\leq \frac{C}{h^2}\|g\|_{l,m}, \end{aligned}$$

which proves that $\tilde{\mathcal{G}}\colon \mathcal{X}_{l,m} \to \mathcal{X}_{l,m-2}$ with $\|\tilde{\mathcal{G}}\| = O(h^{-2})$. Since the series defining $\tilde{G}$ is uniformly convergent for any $g \in \mathcal{X}_{l,m}$, by Lemma 4.3, $\tilde{\mathcal{G}}$ is a right inverse of $\mathcal{L}$ on $\mathcal{X}_{l,m}$. This proves (a).

Now we prove (b). For $g \in \mathcal{X}_{l,m}$, using inequalities (155) and (164), we have that

$$|\Delta_{h,u}^{-1}(\eta_2^0 g)(h/2)| \leq \sum_{k\geq 1}|\eta_2^0(h/2-kh)g(h/2-kh)| \leq O(h^{-3})\|g\|_{l,m}.$$

Moreover, using that η_1 is odd and η_2^0 is even, we have that

$$\begin{aligned}
\Delta_h\tilde{G}(g)(-h/2)& \\
&= \tilde{G}(g)(h/2) - \tilde{G}(g)(-h/2) \\
&= \big(\eta_2^0(h/2)\eta_1(-h/2) - \eta_2^0(-h/2)\eta_1(h/2)\big)g(-h/2) \\
&\quad + \sum_{k=2}^{\infty}\big(\eta_2^0(h/2)\eta_1(h/2-kh) - \eta_2^0(h/2-kh)\eta_1(h/2)\big)g(h/2-kh) \\
&\quad - \sum_{k=2}^{\infty}\big(\eta_2^0(-h/2)\eta_1(h/2-kh) - \eta_2^0(h/2-kh)\eta_1(-h/2)\big)g(h/2-kh) \\
&= -2\eta_1(h/2)\eta_2^0(-h/2)g(-h/2) - 2\eta_1(h/2)\sum_{k=2}^{\infty}\eta_2^0(h/2-kh)g(h/2-kh) \\
&= -2\eta_1(h/2)\sum_{k=1}^{\infty}\eta_2^0(h/2-kh)g(h/2-kh) \\
&= -\Delta_h\eta_1(-h/2)\Delta_{h,u}^{-1}(\eta_2^0 g)(h/2),
\end{aligned}$$

which proves (b).

To prove (c), we start by observing that since $\eta_1 \in \mathcal{X}_{1,2}$, with $\|\eta_1\|_{1,2} = O(h)$, $m > 4$ and Lemma 5.2,

$$\|\Delta_{h,u}^{-1}(\eta_2^0 g)(h/2)\eta_1\|_{1,m-2} \leq \frac{C}{h^3}\|g\|_{l,m}\|\eta_1\|_{1,m-2} \leq O(h^{-2})\|g\|_{l,m}.$$

Using this inequality and (a), it follows that $\mathcal{G}\colon \mathcal{X}_{l,m} \to \mathcal{X}_{1,m}$ is a right inverse of $\mathcal{L}$ with $\|\mathcal{G}\| \leq O(1/h^2)$. If $g \in \mathcal{X}_{l,m}$ is $i\pi$-antiperiodic, since so is η_1, $\mathcal{G}(g)$ is $i\pi$-antiperiodic.

Moreover, for any $g \in \mathcal{X}_{l,m}$, by (b),

$$\mathcal{G}(g)(h/2) - \mathcal{G}(g)(-h/2) = \Delta_h\tilde{\mathcal{G}}(g)(-h/2) + \Delta_{h,u}^{-1}(\eta_2^0 g)(h/2)\Delta_h\eta_1(-h/2) = 0.$$

Finally we prove the uniqueness statement. Let $g \in \mathcal{X}_{l,m}$ be $i\pi$-antiperiodic. By Lemma 4.2, the set of solutions of $\mathcal{L}(\eta) = g$ is given by

$$\{\mathcal{G}(g) + c_1\eta_1 + c_2\eta_2^0 \mid c_1, c_2 \ h\text{-periodic}\}.$$

Let $\eta = \mathcal{G}(g) + \tilde{c}_1\eta_1 + \tilde{c}_2\eta_2^0$ be an analytic $i\pi$-antiperiodic solution of $\mathcal{L}(\eta) = g$, satisfying the boundary conditions $\lim_{t\to-\infty}\eta(t) = 0$ and $\eta(-h/2) = \eta(h/2)$. The analyticity of η, $\mathcal{G}(g)$, η_1 and η_2^0 implies that $\tilde{c}_1$ and $\tilde{c}_2$ are analytic, since $\tilde{c}_1 = W_h(\eta - \mathcal{G}(g), \eta_2^0)$ and $\tilde{c}_2 = -W_h(\eta - \mathcal{G}(g), \eta_1)$. Because of the growth of $|\eta_2^0|$ as $\operatorname{Re} t \to -\infty$, the first boundary condition implies $\tilde{c}_2 \equiv 0$. Since $\mathcal{G}(g)$ and η_1 are $i\pi$-antiperiodic, $\tilde{c}_1$ is $i\pi$-periodic; having already a real period, $\tilde{c}_1$ must be constant. Since η_1 is odd, the second boundary condition implies $\tilde{c}_1 \equiv 0$. □

5.4. **Proof of Proposition 2.2.** Using the notation introduced in the previous section, we can rewrite Proposition 2.2 as

Proposition 5.5. *Consider the sequence of equations given by* (18)*, for* $k = 0$*, and* (64)*, for* $k \geq 1$*. There exist* $\rho_0 > 0$ *and a unique sequence of functions* $(\xi_k^u)_{k\geq 1}$ *such that for any* $\delta > \rho_0 h$

(i) ξ_k^u *is a solution of* (18)*, for* $k = 0$*, and of* (64)*, for* $k \geq 1$*,*
(ii) $\lim_{t\to-\infty} \xi_k^u(t) = 0$ *and* $\xi_k^u(-h/2) = \xi_k^u(h/2)$*; moreover, for* $k = 0$*,* $\xi_0^u(t) > 0$ *for* $-t$ *large enough.*
(iii) *for any* $\beta_1 \leq \beta$*,* $0 \leq \beta_2 < \pi/2$*,* $0 \leq r_1 < 1/2$*,* $1/2 < r_2 \leq 1$*,* $\xi_k^u \in \mathcal{X}_{1,2k+1}(\beta_1, \beta_2, r_1, r_2, \delta)$ *and is* $i\pi$*-antiperiodic.*

Moreover, for any $N \geq 0$*, for* $0 \leq k \leq N$*,*

$$\|\xi_k^u\|_{1,2k+1} \leq O(h^{2k+1}). \tag{165}$$

Proof. We start by considering equation (18). It is, in fact, the invariance equation for the integrable McMillan map. In [DRR98], it is proven that $\xi_0^u = \xi^0$ is its only solution satisfying (i), (ii) and (iii). It clearly verifies inequality (165) for $k = 0$.

To prove the claim for $k \geq 1$, we proceed by induction.

Using the notation of the preceding section, we rewrite equations (64) as

$$\mathcal{L}(\xi_k) = f_k, \tag{166}$$

where $\mathcal{L}$ is defined in (152) and f_k are given by (65), for $k = 1$, and by (66), for $k \geq 2$.

We start by checking the case $k = 1$.

Let $\rho_0 > B^{-1}$, where B is the radius of convergence of the function V defined in (53).

First of all, we claim that $f_1 = V' \circ \xi^0 \in \mathcal{X}_{5,5}$ and that $\|f_1\|_{5,5} = O(h^5)$. Indeed, this follows from the fact that $\xi^0 \in \mathcal{X}_{1,1}$, with $\|\xi^0\|_{1,1} \leq O(h)$, the composition is well defined for $t \in D_\delta^{u,\mathrm{out}}$ if $\delta > \rho_0 h$, $V'(y) = O(y^5)$ and (a) in Lemma 5.2. Moreover, f_1 is $i\pi$-antiperiodic.

Then, by (c) in Lemma 5.4, $\xi_1^u = \mathcal{G}(f_1)$ is the only solution of $\mathcal{L}(\xi) = f_1$ satisfying (i), (ii), (iii) and inequality (165).

Now we prove the step k of the induction process.

We assume, by induction, that there exists a unique sequence of functions, $(\xi_j^u)_{j=0,\dots,k-1}$, $\xi_j^u \in \mathcal{X}_{1,2j+1}$, verifying (i), (ii), (iii), and inequality (165).

Then, we claim that $f_k^u = f_k(\xi_0, \xi_1^u, \dots, \xi_{k-1}^u) \in \mathcal{X}_{3,2k+3}$ and $\|f_k^u\|_{3,2k+3} = O(h^{2k+3})$. Indeed, we recall that

$$f_k^u = \sum_{n=2}^{k} \frac{1}{n!} f^{(n)} \circ \xi_0 \sum_{\substack{j_1+\cdots+j_n=k\\ 1\leq j_1,\dots,j_n\leq k}} \xi_{j_1}^u \cdots \xi_{j_n}^u + \sum_{n=1}^{k-1} \frac{1}{n!} V^{(n+1)} \circ \xi_0 \sum_{\substack{j_1+\cdots+j_n=k-1\\ 1\leq j_1,\dots,j_n\leq k-1}} \xi_{j_1}^u \cdots \xi_{j_n}^u.$$

By the induction hypothesis, f_k^u is $i\pi$-antiperiodic. Now, using (a) in Lemma 5.2, we have that in the first sum in n above, for $i_1+\cdots+i_j=k$, the product $\xi_{j_1}^u\cdots\xi_{j_n}^u$ belongs to $\mathcal{X}_{n,2k+n}$ and $\|\xi_{j_1}^u\cdots\xi_{j_n}^u\|_{n,2k+n}=O(h^{2k+n})$.

On the other hand, if $n=2$, since $f''(y)=O(y)$, by (c) in Lemma 5.2 we have that $f''\circ\xi_0\in\mathcal{X}_{1,1}$, with $\|f''\circ\xi_0\|_{1,1}=O(h)$, and if $n\geq 3$, $f^{(n)}(y)=O(1)$.

Hence, for $n=2$ and $j_1+j_2=k$, $(f''\circ\xi_0)\xi_{j_1}^u\xi_{j_2}^u$ belongs to $\mathcal{X}_{3,2k+3}$ and

$$\|(f''\circ\xi_0)\xi_{j_1}^u\xi_{j_2}^u\|_{3,2k+3}=O(h^{2k+3}).$$

For $n\geq 3$, $(f^{(n)}\circ\xi_0)\xi_{j_1}^u\cdots\xi_{j_n}^u$ belongs to $\mathcal{X}_{n,2k+n}$, with norm

$$\|(f^{(n)}\circ\xi_0)\xi_{j_1}^u\cdots\xi_{j_n}^u\|_{n,2k+n}=O(h^{2k+n}).$$

By (b) in Lemma 5.2, we have that $\|(f^{(n)}\circ\xi_0)\xi_{j_1}^u\cdots\xi_{j_n}^u\|_{3,2k+3}=O(h^{2k+3})$.

The terms in the second sum in n can be treated analogously. In this case, $V^{(n+1)}(y)=O(y^{5-n})$, for $0\leq n\leq 5$, and, hence $V^{(n+1)}\circ\xi_0$ belongs to $\mathcal{X}_{5-n,5-n}$, with norm $\|V^{(n+1)}\circ\xi_0\|_{5-n,5-n}=O(h^{5-n})$. When $n\geq 5$, $V^{(n+1)}(y)=O(1)$.

Then, since $j_1+\cdots j_n=k-1$, we have that $\|\xi_{j_1}^u\cdots\xi_{j_n}^u\|_{n,2k+n-2}=O(h^{2k+n-2})$.

Finally, since $n\geq 1$, by (a) and (b) of Lemma 5.2, $\|(V^{(n+1)}\circ\xi_0)\xi_{j_1}^u\cdots\xi_{j_n}^u\|_{3,2k+3}=O(h^{2k+3})$, which proves the claim.

Hence, by (c) in Lemma 5.4, $\xi_k^u=\mathcal{G}(f_k^u)$ is the unique function we are looking for. □

5.5. **Fixed point equation for ξ^u.** For any $N\geq 0$ and provided that $\delta>\rho_0 h$, from the functions ξ_k^u, $k=0,\dots,N$, given by Proposition 5.5, we define the first outer approximation, $\xi^{u,N}=\sum_{k=0}^N\underline{\varepsilon}^k\xi_k^u$ (see also (68)). Now we claim

Proposition 5.6. *For any $N\geq 0$, $\beta_1<\beta$, $0<\beta_2<\pi/2$, $0<r_1<1/2$, $1/2<r_2<1$, there exist $h_N>0$ and $\rho_N>0$ such that, if $\delta>\rho_N h$ and $0<h<h_N$, the equation* (56) *has a unique solution $\tilde{\xi}^u\in\mathcal{X}_{1,1}(\beta_1,\beta_2,r_1,r_2,\delta)$, $i\pi$-antiperiodic, verifying*

$$\|\tilde{\xi}^u-\xi^{u,N}\|_{3,2N+3}\leq O(\varepsilon^{N+1}h^{2N+3}).$$

The proof of this proposition will follow from the following three technical lemmas, 5.8, 5.9 and 5.10, and is placed after them.

Remark 5.7. Notice that, although the function $\tilde{\xi}^u$ given by Proposition 5.6 satisfies inequality (69) and the boundary condition (19), it is not the one claimed in Theorem 2.3 because it does not necessarily satisfy the boundary condition (21).

We introduce the new unknown η defined by $\xi=\xi^{u,N}+\eta$. The invariance equation (56) now reads

$$\eta(t+h)+\eta(t-h)=\mathcal{F}\circ(\xi^{u,N}+\eta)(t)-\big(\xi^{u,N}(t+h)+\xi^{u,N}(t-h)\big),\tag{167}$$

where $\mathcal{F}(y,h,\varepsilon,\underline{\varepsilon},)$ was defined in (62). We rewrite this equation as

$$\mathcal{L}(\eta)=\mathcal{H}(\eta),\tag{168}$$

where

$$\mathcal{H}(\eta) = G_N + \ell(\eta) + \mathcal{N}(\eta), \tag{169}$$

with

$$G_N(t) = \mathcal{F} \circ \xi^{u,N}(t) - \xi^{u,N}(t+h) - \xi^{u,N}(t-h), \tag{170}$$

and

$$\ell(\eta) = (D\mathcal{F} \circ \xi^{u,N} - \mu Df \circ \xi_0)\eta, \tag{171}$$

$$\mathcal{N}(\eta) = \mathcal{F} \circ (\xi^{u,N} + \eta) - \mathcal{F} \circ \xi^{u,N} - D\mathcal{F} \circ \xi^{u,N}\eta. \tag{172}$$

Now, the proof requires three auxiliary lemmas summarizing some properties of the function G_N and the maps ℓ and $\mathcal{N}$.

Lemma 5.8. *For any $N \geq 0$, $\beta_1 < \beta$, $0 < \beta_2 < \pi/2$, $0 < r_1 < 1/2$, $1/2 < r_2 < 1$, there exists $\tilde{\rho}_N > 0$ such that if $\delta > \tilde{\rho}_N h$, then $G_N \in \mathcal{X}_{3,2N+5}(\beta_1, \beta_2, r_1, r_2, \delta)$, is $i\pi$-antiperiodic and $\|G_N\|_{3,2N+5} = O(\varepsilon^{N+1}h^{2N+5})$.*

Proof. By Proposition 5.5 and (b) in Proposition 5.2, $\|\xi^{u,N}\|_{1,1} \leq O(h)$. Hence, there exists $\tilde{\rho}_N > 0$ such that if $\delta > \tilde{\rho}_N h$, $\mathcal{F} \circ \xi^{u,N}$ is well defined.

By definition of $\xi^{u,N}$, we have that

$$\frac{\partial^k G_N}{\partial \underline{\varepsilon}^k}(t)_{|\underline{\varepsilon}=0} = 0, \qquad k = 0, \ldots, N.$$

Hence, we can bound $G_N(t)$ by $\varepsilon^{N+1} \sup_{|\underline{\varepsilon}|<\varepsilon} |\frac{\partial^{N+1} G_N}{\partial \underline{\varepsilon}^{N+1}}(t)|$.

We remark that $\xi^{u,N}$ is a polynomial of degree N in $\underline{\varepsilon}$. Hence,

$$\frac{\partial^{N+1} G_N}{\partial \underline{\varepsilon}^{N+1}} = \mu \frac{\partial^{N+1}}{\partial \underline{\varepsilon}^{N+1}}(f \circ \xi^{u,N}) + \underline{\varepsilon} \frac{\partial^{N+1}}{\partial \underline{\varepsilon}^{N+1}}(V' \circ \xi^{u,N}) + \frac{\partial^N}{\partial \underline{\varepsilon}^N}(V' \circ \xi^{u,N}).$$

For the first term in the right hand side above, since $\partial^{N+1}\xi^{u,N}/\partial\underline{\varepsilon}^{N+1} = 0$, we have

$$\frac{\partial^{N+1}}{\partial \underline{\varepsilon}^{N+1}}(f \circ \xi^{u,N}) = \sum_{j=2}^{N+1} f^{(j)} \circ \xi^{u,N} \sum_{\substack{i_1+\cdots+i_j=N+1 \\ 1\leq i_1,\ldots,i_j \leq N}} \sigma^N_{i_1,\ldots,i_j} \frac{\partial^{i_1}\xi^{u,N}}{\partial \underline{\varepsilon}^{i_1}} \cdots \frac{\partial^{i_j}\xi^{u,N}}{\partial \underline{\varepsilon}^{i_j}} \tag{173}$$

where $\sigma^N_{i_1,\ldots,i_j}$ are combinatorial coefficients. Notice that, for $0 \leq k \leq N$,

$$\frac{\partial^k \xi^{u,N}}{\partial \underline{\varepsilon}^k} = \sum_{l=k}^{N} \frac{l!}{(l-k)!} \underline{\varepsilon}^{l-k} \xi^u_l.$$

Hence, by Proposition 5.5 and (a) and (b) in Lemma 5.2, we have that $\partial^k\xi^{u,N}/\partial\underline{\varepsilon}^k$ belongs to $\mathcal{X}_{1,2k+1}$, if $0 \leq k \leq N$, with norm bounded by $O(h^{2k+1})$, which implies that, in (173), for $2 \leq j \leq N+1$,

$$\left\| \frac{\partial^{i_1}\xi^{u,N}}{\partial \underline{\varepsilon}^{i_1}} \cdots \frac{\partial^{i_j}\xi^{u,N}}{\partial \underline{\varepsilon}^{i_j}} \right\|_{j,2N+2+j} = O(h^{2N+2+j}). \tag{174}$$

Since $f(y)=y-y^3+O(y^5)$, we have that by (c) in Lemma 5.2, $\|f''\circ\xi^{u,N}\|_{1,1}=O(h)$. Then, if we set $j=2$ in the sum in (173) and in (174), by (b) in Lemma 5.2, we have that, for $i_1+i_2=N+1$,

$$\Big\|f''\circ\xi^{u,N}\frac{\partial^{i_1}\xi^{u,N}}{\partial\underline{\varepsilon}^{i_1}}\frac{\partial^{i_2}\xi^{u,N}}{\partial\underline{\varepsilon}^{i_2}}\Big\|_{3,2N+5}=O(h^{2N+5}).$$

Moreover, for $j>2$ in (173), since $f^{(j)}\circ\xi^{u,N}=O(1)$, we also have that, by (b) in Lemma 5.2,

$$\Big\|f^{(j)}\circ\xi^{u,N}\frac{\partial^{i_1}\xi^{u,N}}{\partial\underline{\varepsilon}^{i_1}}\cdots\frac{\partial^{i_j}\xi^{u,N}}{\partial\underline{\varepsilon}^{i_j}}\Big\|_{3,2N+5}=O(h^{2N+5}).$$

which proves the claim for (173).

The terms

$$\underline{\varepsilon}\frac{\partial^{N+1}}{\partial\underline{\varepsilon}^{N+1}}(V'\circ\xi^{u,N})\qquad\text{and}\qquad\frac{\partial^{N}}{\partial\underline{\varepsilon}^{N}}(V'\circ\xi^{u,N}),$$

using that $V'(y)=O(y^5)$, are bounded analogously in $\mathcal{X}_{3,2N+5}$ by $O(h^{2N+5})$. □

Lemma 5.9. *Under the hypotheses of Lemma 5.8, the operator ℓ in* (171) *is a bounded linear map from $\mathcal{X}_{3,2N+3}(\beta_1,\beta_2,r_1,r_2,\delta)$ to $\mathcal{X}_{3,2N+5}(\beta_1,\beta_2,r_1,r_2,\delta)$, and $\|\ell\|=O(h^4/\delta^2)$. Moreover, if $\eta\in\mathcal{X}_{3,2N+3}(\beta_1,\beta_2,r_1,r_2,\delta)$ is $i\pi$-antiperiodic, so is $\ell(\eta)$.*

Proof. The fact that ℓ preserves $i\pi$-antiperiodicity follows immediately from its definition.

By (62), we have that

$$D\mathcal{F}\circ\xi^{u,N}-\mu Df\circ\xi_0=\mu(Df\circ\xi^{u,N}-Df\circ\xi_0)+\underline{\varepsilon}V''\circ\xi^{u,N}.$$

Also, $D^2f(y)=O(y)$ and $V''(y)=O(y^4)$. Moreover, $\|\xi^{u,N}\|_{1,1}\le O(h)$ and $\|\xi^{u,N}-\xi_0\|_{1,3}\le O(h^3)$. Combining these facts with Lemma 5.2, we have that $D\mathcal{F}\circ\xi^{u,N}-\mu Df\circ\xi_0\in\mathcal{X}_{4,4}$, and that $\|D\mathcal{F}\circ\xi^{u,N}-\mu Df\circ\xi_0\|_{4,4}\le O(h^4)$. Hence, by (b) in Lemma 5.2, if $\eta\in\mathcal{X}_{3,2N+3}$,

$$\begin{aligned}\|\ell(\eta)\|_{3,2N+5}&\le\|D\mathcal{F}\circ\xi^{u,N}-\mu Df\circ\xi_0\|_{0,2}\|\eta\|_{3,2N+3}\\&\le O(\delta^{-2})\|D\mathcal{F}\circ\xi^{u,N}-\mu Df\circ\xi_0\|_{4,4}\|\eta\|_{3,2N+3}\\&\le O(h^4\delta^{-2})\|\eta\|_{3,2N+3}.\end{aligned}$$

□

Lemma 5.10. *Let $B_\kappa\subset\mathcal{X}_{3,2N+3}(\beta_1,\beta_2,r_1,r_2,\delta)$ denote the ball of radius κ. Under the hypotheses of Lemma 5.8, there exists $0<\kappa_N<h$ such that for any $0<\kappa<\kappa_N$, the map $\mathcal{N}$ in* (172) *is well defined and Lipschitz from B_κ to $\mathcal{X}_{3,2N+5}(\beta_1,\beta_2,r_1,r_2,\delta)$. Its Lipschitz constant is bounded by $O(\kappa h/\delta^{2N+2})$. Moreover, $\mathcal{N}$ preserves $i\pi$-antiperiodicity.*

Proof. From the definition of $\mathcal{N}$, it is easily verified that $i\pi$-antiperiodicity is preserved.

We write $\mathcal{N}$ as

$$\mathcal{N}(\eta) = \int_0^1 \big(D\mathcal{F}(\xi^{u,N} + t\eta) - D\mathcal{F}(\xi^{u,N})\big)dt\eta.$$

Notice that $D\mathcal{F}(y+z) - D\mathcal{F}(y) \le O(y)O(z)$. Since $\|\xi^{u,N}\|_{1,1} \le O(h)$, there exists $\kappa_N < h$ such that for all $0 < \kappa < \kappa_N$, if $\|\eta\|_{3,2N+3} \le \kappa$ and $t \in [0,1]$, $D\mathcal{F}(\xi^{u,N} + t\eta)$ is well defined and $\|D\mathcal{F}(\xi^{u,N} + t\eta) - D\mathcal{F}(\xi^{u,N})\|_{4,2N+4} \le O(h\kappa)$. Hence, if $\eta \in B_\kappa$,

$$\begin{aligned}\|\mathcal{N}(\eta)\|_{3,2N+5} &\le \Big\| \int_0^1 \big(D\mathcal{F}(\xi^{u,N} + t\eta) - D\mathcal{F}(\xi^{u,N})\big)dt\Big\|_{0,2} \|\eta\|_{3,2N+3} \\ &\le O(h\kappa/\delta^{2N+2})\|\eta\|_{3,2N+3} \\ &\le O(h\kappa^2/\delta^{2N+2}).\end{aligned}$$

We finally compute the Lipschitz constant of the map $\mathcal{N}$. For $\eta, \tilde\eta \in B_\kappa$,

$$\begin{aligned}\mathcal{N}(\eta) - \mathcal{N}(\tilde\eta) &= \int_0^1 \big(D\mathcal{F}(\xi^{u,N} + t\eta) - D\mathcal{F}(\xi^{u,N} + t\tilde\eta)\big)\eta dt \\ &\quad + \int_0^1 \big(D\mathcal{F}(\xi^{u,N} + t\tilde\eta) - D\mathcal{F}(\xi^{u,N})\big)(\eta - \tilde\eta)dt.\end{aligned} \tag{175}$$

Since $D^2\mathcal{F}(y) = O(y)$ and $\|\xi^{u,N}\|_{1,1} \le O(h)$, by Lemma 5.2, we have that for η and $\tilde\eta$ in B_κ,

$$\|D\mathcal{F}(\xi^{u,N} + t\eta) - D\mathcal{F}(\xi^{u,N} + t\tilde\eta)\|_{4,2N+4} \le O(h)\|\eta - \tilde\eta\|_{3,2N+3}$$

Then we can bound the first integral in (175) by

$$\begin{aligned}\Big\| \int_0^1 \big(D\mathcal{F}(\xi^{u,N} + t\eta) - D\mathcal{F}(\xi^{u,N} + t\tilde\eta)\big)\eta dt\Big\|_{3,2N+5} &\le O(h)\|\eta\|_{0,1}\|\eta - \tilde\eta\|_{3,2N+3} \\ &\le O\Big(\frac{h\kappa}{\delta^{2N+2}}\Big)\|\eta - \tilde\eta\|_{3,2N+3}.\end{aligned}$$

With the same argument, we obtain the same bound for the second integral in (175). □

Proof of Proposition 5.6. With the introduction of the unknown η defined by $\tilde\xi = \xi^{u,N} + \eta$ we have transformed invariance equation (56) into equation (168). Since $\tilde{\mathcal{G}}$, defined in (159), by Lemma 5.4, is a right inverse of $\mathcal{L}$, we can rewrite equation (168) as a fixed point equation as

$$\eta = \tilde{\mathcal{G}} \circ \mathcal{H}(\eta).$$

We claim that the above equation has a unique $i\pi$-antiperiodic fixed point in the ball of radius $O(\varepsilon^{N+1}h^{2N+3})$ in $\mathcal{X}_{3,2N+3}$, which implies the proposition. We prove the claim by checking that $\tilde{\mathcal{G}} \circ \mathcal{H}$ is a contraction in the in the ball of radius $O(\varepsilon^{N+1}h^{2N+3})$ in $\mathcal{X}_{3,2N+3}$. Since, by Lemmas 5.8, 5.9 and 5.10 and (a) in Lemma 5.4, $\tilde{\mathcal{G}} \circ \mathcal{H}$ sends the subspace of $i\pi$-antiperiodic functions in $\mathcal{X}_{3,2N+3}$ to itself, the fixed point of the contraction will be $i\pi$-antiperiodic.

First of all, by Lemmas 5.4 and 5.8, we have that $\tilde{\mathcal{G}} \circ \mathcal{H}(0) = \tilde{\mathcal{G}}(G_N) \in \mathcal{X}_{3,2N+3}$, and

$$\|\tilde{\mathcal{G}} \circ \mathcal{H}(0)\|_{3,2N+3} \leq O(\varepsilon^{N+1}h^{2N+3}). \tag{176}$$

Also, by the Lemmas 5.9, 5.10 and 5.4, if $0 < \kappa < h$, we have that $\tilde{\mathcal{G}} \circ \mathcal{H}$: $B_\kappa \subset \mathcal{X}_{3,2N+3} \to \mathcal{X}_{3,2N+3}$ is a well defined Lipschitz map with

$$\operatorname{lip} \tilde{\mathcal{G}} \circ \mathcal{H} \leq \max\{O(h^2/\delta^2), O(\kappa/(h\delta^{2N+2}))\},$$

where B_κ denotes the ball of radius κ. We take $\kappa = 2\|\tilde{\mathcal{G}} \circ \mathcal{H}(0)\|_{3,2N+3}$. Then, there exists $\rho_N \geq \tilde{\rho}_N$ such that if $\delta > \rho_N h$, we have that

$$\operatorname{lip} \tilde{\mathcal{G}} \circ \mathcal{H}_{|B_\kappa} \leq \max\{O(h^2/\delta^2), O(h^{2N+2}/\delta^{2N+2})\} < 1.$$

Moreover, by (176), there exists $h_N > 0$ such that, if $0 < h < h_N$, we have that $\tilde{\mathcal{G}} \circ \mathcal{H}(B_\kappa) \subset B_\kappa$, which proves the claim. □

5.6. **Linearized invariance equation around $\tilde{\xi}^u$.** This section is a preliminary step in the proof of Theorem 2.4. Here we look for solutions of the invariance equation (56) linearized around $\tilde{\xi}^u$, the solution of the invariance equation (56) given by Proposition 5.6,

$$\eta(t+h) + \eta(t-h) = \big(\mu f'(\tilde{\xi}^u(t)) + \varepsilon V''(\tilde{\xi}^u(t), h, \varepsilon)\big)\eta(t). \tag{177}$$

Proposition 5.11. *Let h_0 and ρ_0 be the constants given by Proposition 5.6. Then, for any $0 < h < h_0$ and $\rho_0 h < \delta < \pi/2$, equation (177) has three solutions, $\tilde{\eta}_1^u$, $\tilde{\eta}_2^{u,0}$, and $\tilde{\eta}_2^{u,i\pi/2}$ such that, for any $\beta_1 < \beta$, $0 < \beta_2 < \pi/2$, $0 < r_1 < 1/2$, $1/2 < r_2 < 1$,*
a) *$\tilde{\eta}_1^u, \tilde{\eta}_2^{u,0}$: $U(\beta_1, \beta_2, r_1, r_2, \delta) \to \mathbb{C}$ are real analytic, $\tilde{\eta}_1^u = (\tilde{\xi}^u)'$, $\eta_1 - \tilde{\eta}_1^u \in \mathcal{X}_{3,4}(\beta_1, \beta_2, r_1, r_2, \delta)$ with*

$$\|\eta_1 - \tilde{\eta}_1^u\|_{3,4} \leq O(\varepsilon h^3), \tag{178}$$

and $\eta_2^0 - \tilde{\eta}_2^{u,0} \in \mathcal{X}_{3,4}(\beta_1, \beta_2, r_1, r_2, \delta)$ with

$$\|\eta_2^0 - \tilde{\eta}_2^{u,0}\|_{3,4} \leq O(\varepsilon), \tag{179}$$

where η_1 and η_2^0 are defined in (58) and (59).
b) *$\tilde{\eta}_2^{u,i\pi/2}$: $U(\beta_1, \beta_2, r_1, r_2, \delta) \to \mathbb{C}$ is an analytic function and, for $t \in U(\beta_1, \beta_2, r_1, r_2, \delta)$ with $-1 \leq \operatorname{Re} t$ and $0 \leq \operatorname{Im} t$,*

$$|\eta_2^{i\pi/2}(t) - \tilde{\eta}_2^{u,i\pi/2}(t)| \leq \frac{O(\varepsilon)}{|\cosh t|^2}, \tag{180}$$

with $\eta_2^{i\pi/2}$ defined in (59).

Moreover, the following relations hold:
c) $\tilde\eta_2^{u,i\pi/2} = \tilde\eta_2^{u,0} + A\tilde\eta_1^u$, *where A is the constant introduced in* (60),
d) $W_h(\tilde\eta_1^u, \tilde\eta_2^{u,0})(t) = W_h(\tilde\eta_1^u, \tilde\eta_2^{u,i\pi/2})(t) = 1$, *for $t \in U(\beta_1, \beta_2, r_1, r_2, \delta)$.*

Some statements of this proposition will be proven following the lines of the proof of Proposition 5.6. More concretely, to prove the existence of a real analytic solution of (177) close to η_2^0, we introduce the new unknown u by setting $\eta = \eta_2^0 + u$. Then, equation (177) reads

$$\mathcal{L}(u) = \mathcal{H}_1(u), \tag{181}$$

where

$$\mathcal{H}_1(u) = (m_\varepsilon(\tilde\xi^u) - m_0(\xi^0))(\eta_2^0 + u) \tag{182}$$

with

$$m_\varepsilon(\xi)(t) = \mu f'(\xi(t)) + \varepsilon V''(\xi(t), h, \varepsilon). \tag{183}$$

The following auxiliary lemmas summarize the properties we will need of m_ε and $\mathcal{H}_1$.

Lemma 5.12. *For any $\beta_1 < \beta$, $0 < \beta_2 < \pi/2$, $0 < r_1 < 1/2$, $1/2 < r_2 < 1$ the function $m_\varepsilon(\tilde\xi^u) - m_0(\xi^0) \in \mathcal{X}_{4,4}(\beta_1, \beta_2, r_1, r_2, \delta)$ and*

$$\|m_\varepsilon(\tilde\xi^u) - m_0(\xi^0)\|_{4,4} \le O(\varepsilon h^4). \tag{184}$$

Proof. Indeed, we remark that

$$m_\varepsilon(\tilde\xi^u)(t) - m_0(\xi^0)(t) = \int_0^1 \mu f''(\xi_0(t) + s(\tilde\xi^u(t) - \xi_0(t))\, ds\, (\tilde\xi^u(t) - \xi_0(t)) \\ + \varepsilon V''(\tilde\xi^u(t), h, \varepsilon).$$

Then, by Proposition 5.6, we have that $\xi_0, \tilde\xi^u \in \mathcal{X}_{1,1}$ with $\|\xi_0\|_{1,1}, \|\tilde\xi^u\|_{1,1} \le O(h)$, $\tilde\xi^u - \xi_0 \in \mathcal{X}_{3,3}$ and $\|\tilde\xi^u - \xi_0\|_{3,3} \le O(\varepsilon h^3)$. Hence, since $f''(y) = O(y)$, by (c) in Lemma 5.2 we have that $f''(\xi_0(t) + s(\tilde\xi^u(t) - \xi_0(t))) \in \mathcal{X}_{1,1}$ with norm bounded by $O(h)$ and, hence,

$$\Big\| \int_0^1 \mu f''(\xi_0(t) + s(\tilde\xi^u(t) - \xi_0(t))\, ds\, (\tilde\xi^u(t) - \xi_0(t)) \Big\|_{4,4} = O(\varepsilon h^4).$$

Finally, since $V''(y, h, \varepsilon) = O(y^4)$, the claim follows. □

Lemma 5.13. *For any $\beta_1 < \beta$, $0 < \beta_2 < \pi/2$, $0 < r_1 < 1/2$, $1/2 < r_2 < 1$, the map $\mathcal{H}_1$ defined in* (182) *is affine from $\mathcal{X}_{3,4}(\beta_1, \beta_2, r_1, r_2, \delta)$ to $\mathcal{X}_{3,6}(\beta_1, \beta_2, r_1, r_2, \delta)$, $\|\mathcal{H}_1(0)\|_{3,6} \le O(\varepsilon h^2)$ and $\operatorname{lip} \mathcal{H}_1 \le O(\varepsilon h^4 \delta^{-2})$.*

Proof. First notice that, as a consequence of Lemma 5.12, since $\eta_2^0 \in \mathcal{X}_{-1,2}$ with $\|\eta_2^0\|_{-1,2} \le O(h^{-2})$, we have that $\mathcal{H}_1(0) = (m(\tilde{\xi}^u) - m_0(\xi^0))\eta_2^0 \in \mathcal{X}_{3,6}$ and $\|\mathcal{H}_1(0)\|_{3,6} \le O(\varepsilon h^2)$.

On the other hand, if $u_1, u_2 \in \mathcal{X}_{3,4}$, again by Lemmas 5.12 and 5.2, we have that $\mathcal{H}_1(u_1), \mathcal{H}_1(u_2) \in \mathcal{X}_{7,8}$ and

$$\begin{aligned}\|\mathcal{H}_1(u_1) - \mathcal{H}_1(u_2)\|_{3,6} &\le O(\delta^{-2})\|\mathcal{H}_1(u_1) - \mathcal{H}_1(u_2)\|_{3,8}\\ &\le O(\delta^{-2})\|m_\varepsilon(\tilde{\xi}^u) - m_0(\xi^0)\|_{4,4}\|u_1 - u_2\|_{3,4}\\ &\le O(\varepsilon h^4\delta^{-2})\|u_1 - u_2\|_{3,4}.\end{aligned}$$ □

On the other hand, to prove the existence of a solution of equation (177) close to $\eta_2^{i\pi/2}$, we redefine u by $\eta = \eta_2^{i\pi/2} + u$. Then, η is a solution of equation (177) if and only if u satisfies

$$\mathcal{L}(u) = \mathcal{H}_2(u), \tag{185}$$

where

$$\mathcal{H}_2(u) = (m_\varepsilon(\tilde{\xi}^u) - m_0(\xi^0))(\eta_2^{i\pi/2} + u) \tag{186}$$

where $m_\varepsilon(\xi)$ was defined in (183).

To deal with equation (185), for any $\beta_1 < \beta$, $0 < \beta_2 < \pi/2$, $0 < r_1 < 1/2$, $1/2 < r_2 < 1$, $\delta > 0$, we introduce the spaces

$$\begin{aligned}\tilde{\mathcal{X}}_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta) =&\\ &\{u\colon U(\beta_1, \beta_2, r_1, r_2, \delta) \cap \{\operatorname{Im} t > 0\} \to \mathbb{C} \mid \text{ analytic, } \|u\|_{l,m} < \infty\},\end{aligned} \tag{187}$$

where the norm $\|\cdot\|_{l,m}$ was defined in (158) and the domain U was introduced in (151). They are Banach spaces. It is clear that if $u \in \mathcal{X}_{l,m}$, its restriction to $\{\operatorname{Im} t > 0\}$ belongs to $\tilde{\mathcal{X}}_{l,m}$, with smaller or equal norm. On the other hand, Lemma 5.2 also holds for $\tilde{\mathcal{X}}_{l,m}$, and we will use it without further notice. Moreover, it is not difficult to see that Lemma 5.4 holds in $\tilde{\mathcal{X}}_{l,m}$, that is, the operator $\tilde{\mathcal{G}}$ defined in (159) satisfies $\tilde{\mathcal{G}}\colon \tilde{\mathcal{X}}_{l,m} \to \tilde{\mathcal{X}}_{l,2}$, with norm bounded by $O(h^{-2})$.

Lemma 5.14. *For any $\beta_1 < \beta$, $0 < \beta_2 < \pi/2$, $0 < r_1 < 1/2$, $1/2 < r_2 < 1$, the map $\mathcal{H}_2$ in (186) is affine from $\tilde{\mathcal{X}}_{3,2}(\beta_1, \beta_2, r_1, r_2, \delta)$ to $\tilde{\mathcal{X}}_{3,6}(\beta_1, \beta_2, r_1, r_2, \delta)$, $\|\mathcal{H}_2(0)\|_{3,2} = O(1)$, $\|\mathcal{H}_2(0)\|_{3,6} \le O(\varepsilon h^2)$ and $\operatorname{lip}\mathcal{H}_2 = O(\varepsilon h^4)$.*

Proof. By inequalities (155) and (156) we have that $\eta_2^{i\pi/2} \in \tilde{\mathcal{X}}_{-1,-3}$, and

$$\|\eta_2^{i\pi/2}\|_{-1,-3} \le O(h^{-2}).$$

Hence, by Lemma 5.12, $\mathcal{H}_2(0) = (m_\varepsilon(\tilde{\xi}^u) - m_0(\xi^0))\eta_2^0 \in \tilde{\mathcal{X}}_{3,1}$, with $\|\mathcal{H}_2(0)\|_{3,1} \le O(\varepsilon h^2)$. Therefore, $\mathcal{H}_2(0)$ belongs to $\tilde{\mathcal{X}}_{3,2}$ and $\tilde{\mathcal{X}}_{3,6}$ with the same norm.

On the other hand, if u_1 and u_2 belong to $\tilde{\mathcal{X}}_{3,2}$, by Lemma (5.12), we have that

$$\|\mathcal{H}_2(u_1) - \mathcal{H}_2(u_2)\|_{3,6} \le O(\varepsilon h^4)\|u_1 - u_2\|_{3,2}.$$ □

Proof of Proposition 5.11. Clearly, a solution of equation (177) is simply $\tilde{\eta}_1^u = (\tilde{\xi}^u)'$. This function, by Propositions 5.5 and 5.6, satisfies inequality (178). Indeed, by Proposition 5.6 and, by (d) in Lemma 5.2 we have that, for any smaller $0 < \beta_2$, $0 < r_1$ and larger $\beta_1 < \beta$, $r_2 < 1$,

$$\|\eta_1 - \tilde{\eta}_1^u\|_{3,4} = \|\xi_0' - (\tilde{\xi}^u)'\|_{3,4} = O(\varepsilon h^3).$$

Now we prove the existence of a solution $\tilde{\eta}_2^{u,0}$ of equation (177) close to η_2^0. By the introduction of u by $\eta = \eta_2^0 + u$, it is equivalent to find a solution of equation (181). By Lemmas 5.4 and 5.13, $\tilde{\mathcal{G}} \circ \mathcal{H}_1$ is a well defined map from $\mathcal{X}_{3,4}$ to itself with $\operatorname{lip} \tilde{\mathcal{G}} \circ \mathcal{H}_1 \leq O(\varepsilon h^2 \delta^{-2}) < 1$. Hence, it has a unique fixed point, $\tilde{u}_2^0$. Moreover, since $\|\tilde{\mathcal{G}} \circ \mathcal{H}_1(0)\|_{3,4} \leq O(\varepsilon)$, the fixed point also satisfies $\|\tilde{u}_2^0\|_{3,4} \leq O(\varepsilon)$. Hence, $\tilde{\eta}_2^{u,0} = \eta_2^0 + \tilde{u}_2^0$ is a solution of equation (177) satisfying inequality (179).

Now we proceed to prove the existence of a solution of equation (177) close to $\eta_2^{i\pi/2}$. We start by considering the new unknown u defined by $\eta = \eta_2^{i\pi/2} + u$ and finding a solution of equation (185) in $\tilde{\mathcal{X}}_{3,2}$. Afterwards, we will extend the solution thus obtained to $U(\beta_1, \beta_2, r_1, r_2, \delta)$.

To obtain a solution of equation (185), notice first that $\tilde{\mathcal{G}} \circ \mathcal{H}_2$ is a well defined map from $\tilde{\mathcal{X}}_{3,2}$ to $\tilde{\mathcal{X}}_{3,2}$. Indeed, if $u \in \tilde{\mathcal{X}}_{3,2}$, then $\mathcal{H}_2(u)$ belongs to $\tilde{\mathcal{X}}_{3,6}$ and, hence, by Lemma 5.4, $\tilde{\mathcal{G}} \circ \mathcal{H}_2(u)$ belongs to $\tilde{\mathcal{X}}_{3,4}$, which, by Lemma 5.2, is continuously injected into $\tilde{\mathcal{X}}_{3,2}$. Moreover, when we consider $\tilde{\mathcal{G}} \circ \mathcal{H}_2$ as a map from $\tilde{\mathcal{X}}_{3,2}$ to $\tilde{\mathcal{X}}_{3,2}$, it satisfies that $\operatorname{lip} \tilde{\mathcal{G}} \circ \mathcal{H}_2 \leq O(\varepsilon h^2 \delta^{-2})$. Indeed, if $u_1, u_2 \in \tilde{\mathcal{X}}_{3,2}$, using Lemmas 5.2 and 5.4,

$$\begin{aligned}
\|\tilde{\mathcal{G}} \circ \mathcal{H}_2(u_1) - \tilde{\mathcal{G}} \circ \mathcal{H}_2(u_2)\|_{3,2} &\leq O(\delta^{-2}) \|\tilde{\mathcal{G}} \circ \mathcal{H}_2(u_1) - \tilde{\mathcal{G}} \circ \mathcal{H}_2(u_2)\|_{3,4} \\
&\leq O(h^{-2}\delta^{-2}) \|(m(\tilde{\xi}^u) - m_0(\xi^0))(u_1 - u_2)\|_{3,6} \\
&\leq O(\varepsilon h^2 \delta^{-2}) \|u_1 - u_2\|_{3,2}.
\end{aligned}$$

In particular, $\tilde{\mathcal{G}} \circ \mathcal{H}_2$ is a contraction in $\tilde{\mathcal{X}}_{3,2}$. Let $\tilde{u}_2^{i\pi/2}$ be its unique fixed point. Since $\|\tilde{\mathcal{G}} \circ \mathcal{H}_2(0)\|_{3,2} = O(\varepsilon)$, we have that $\|\tilde{u}_2^{i\pi/2}\|_{3,2} = O(\varepsilon)$. Hence $\tilde{\eta}_2^{u,i\pi/2} = \eta_2^{i\pi/2} + \tilde{u}_2^{i\pi/2}$ is a solution of equation (177), satisfying inequality (180). However, we remark that, up to this point, $\tilde{\eta}_2^{u,i\pi/2}$ is defined only in $U(\beta_1, \beta_2, r_1, r_2, \delta) \cap \{\operatorname{Re} t \geq 0\}$.

Now we prove that $W_h(\tilde{\eta}_1^u, \tilde{\eta}_2^{u,0}) = 1$. Indeed, since $W_h(\eta_1, \eta_2^0) \equiv 1$, we have that

$$\begin{aligned}
|W_h(\tilde{\eta}_1^u, \tilde{\eta}_2^{u,0})(t) - 1| &\leq |W_h(\tilde{\eta}_1^u, \tilde{u}_2^0)(t)| + |W_h(\xi_0 - (\tilde{\xi}^u)', \eta_2^0)(t)| \\
&\leq O(\varepsilon h^2 e^{4 \operatorname{Re} t}) + O(\varepsilon h^2 e^{2 \operatorname{Re} t}),
\end{aligned}$$

which implies that $\lim_{\operatorname{Re} t \to -\infty} W_h(\tilde{\eta}_1^u, \tilde{\eta}_2^{u,0})(t) = 1$. But, by Lemma 4.2, $W_h(\tilde{\eta}_1^u, \tilde{\eta}_2^{u,0})$ is a h-periodic function. Hence, it is constant 1.

With the same argument one checks that $W_h(\tilde{\eta}_1^u, \tilde{\eta}_2^{u,i\pi/2})(t) = 1$, for $t \in U(\beta_1, \beta_2, r_1, r_2, \delta) \cap \{\operatorname{Im} t > 0\}$.

To prove that $\tilde{\eta}_2^{u,i\pi/2} = \tilde{\eta}_2^{u,0} + A\tilde{\eta}_1^u$, in $U(\beta_1, \beta_2, r_1, r_2, \delta) \cap \{\operatorname{Im} t \geq 0\}$, where A was the constant introduced in (60), notice that, by Lemma 4.2, we can write $\tilde{\eta}_2^{u,i\pi/2} = c_1\tilde{\eta}_1^u + c_2\tilde{\eta}_2^{u,0}$, with c_1 and c_2 are h-periodic functions. Since $W_h(\tilde{\eta}_1, \tilde{\eta}_2^{u,0}) = 1$, we have that $c_2 = W_h(\tilde{\eta}_1, \tilde{\eta}_2^{u,i\pi/2}) = 1$, and, since $W_h(\eta_2^{i\pi/2}, \eta_2^0) = A$, we have that, also for $t \in U(\beta_1, \beta_2, r_1, r_2, \delta)$, $\operatorname{Im} t \geq 0$,

$$\begin{aligned}|c_2(t) - A| &\leq |W_h(\eta_2^0, \tilde{u}_2^{i\pi/2})(t)| + |W_h(\tilde{u}_2^0, \eta_2^{i\pi/2})(t)| + |W_h(\tilde{u}_2^0, \tilde{u}_2^{i\pi/2})(t)| \\ &\leq O(\frac{\varepsilon}{h} e^{2\operatorname{Re} t}) + O(\varepsilon^2 h e^{6\operatorname{Re} t}),\end{aligned}$$

which implies that $c_2(t) = A$. Hence, $\tilde{\eta}_2^{u,i\pi/2} = \tilde{\eta}_2^{u,0} + A\tilde{\eta}_1^u$ in $U(\beta_1, \beta_2, r_1, r_2, \delta) \cap \{\operatorname{Im} t \geq 0\}$, and the same formula provides an analytic extension of $\tilde{\eta}_2^{u,i\pi/2}$ to $U(\beta_1, \beta_2, r_1, r_2, \delta)$. □

5.7. **Proof of Theorems 2.3 and 2.4.** Up to this point, in Propositions 5.6 and 5.11, we have established the existence of a solution of the invariance equation (56), $\tilde{\xi}^u$, and, related to this solution, three solutions of equation (177), the linearized invariance equation around $\tilde{\xi}^u$, which we called $\tilde{\eta}_1^u$, $\tilde{\eta}_2^{u,0}$ and $\tilde{\eta}_1^{u,i\pi/2}$. Furthermore, this solution $\tilde{\xi}^u$ is real analytic, πi-antiperiodic, and satisfies the boundary condition (19) but not necessarily the boundary condition (21).

In order to find the solution that also satisfies the boundary condition (21), notice that, for any real T, the function $\tilde{\xi}^u(t-T)$ is also a πi-antiperiodic real analytic solution of the invariance equation that satisfies the boundary condition (19). Hence, we look for T such that

$$\tilde{\xi}^u(h/2 - T) = \tilde{\xi}^u(-h/2 - T). \tag{188}$$

Lemma 5.15. *There exists a real number, $T(h,\varepsilon) = O(\varepsilon^{N+1}h^{2N+2})$, such that equation* (188) *is satisfied. Moreover, $T(h,\varepsilon)$ is the unique solution of equation* (188) *in the ball of radius* $\min\{O(1), O(\varepsilon^{N+1}h^{-2N-1})\}$.

Proof. Equation (188) is equivalent to

$$\xi^{u,N}(-h/2 - T) - \xi^{u,N}(h/2 - T) = v^u(h/2 - T) - v^u(-h/2 - T), \tag{189}$$

where $v^u = \tilde{\xi}^u - \xi^{u,N}$. In order to solve equation (189), we consider some fixed small neighborhood of the origin, B, the ball of radius 1/2, for instance. Let $p(T)$ denote the left hand side of (189), and $q(T)$ the right hand side. In this way, equation (189) can be written as $p(T) = q(T)$, which we will treat as a fixed point equation after inverting p.

We recall that

$$p(T) = \xi_0(-h/2 - T) - \xi_0(h/2 - T) + \tilde{p}(T),$$

where $\xi_0(t) = \gamma \sec t$ was introduced in (1).

By Proposition 5.6, we have that $\sup_{t\in B}|\tilde{p}'(t)| = O(\varepsilon h^3)$. Hence, $|p'(t)|$ is bounded by below by $O(h^2)$. Besides, since $p(0) = 0$, we have that p^{-1} is defined in the ball of radius $O(h^2)$, and $(p^{-1})'$ is bounded by $O(h^{-2})$.

On the other hand, $|q'(t)|$ is bounded from above in B by $O(\varepsilon^{N+1}h^{2N+3})$ and $q(0) = O(\varepsilon^{N+1}h^{2N+4})$. Therefore, the composition $p^{-1}\circ q$ is well defined in the ball of radius $\min\{O(1), O(\varepsilon^{N+1}h^{-2N-1})\}$ to itself and its derivative $(p^{-1}\circ q)'$ is bounded by $O(\varepsilon^{N+1}h^{2N+1})$. Since it is a contraction, it has a unique fixed point, $T(h)$, which is the solution of equation (188).

Moreover, and $p^{-1}\circ q$ sends the ball of radius $O(\varepsilon^{N+1}h^{2N+2})$ to itself, which implies that the fixed point satisfies $T(h,\varepsilon) = O(\varepsilon^{N+1}h^{2N+2})$. □

We finally define $\xi^u(t) = \tilde{\xi}^u(t - T(h,\varepsilon))$. By Proposition 8.10 and Lemma 5.15, it satisfies boundary conditions (19) and (21). To finish the proof of Theorem 2.3, we only need to check that ξ^u satisfies inequality (69). Indeed, by Proposition 8.10, we have that

$$\begin{aligned}\|\xi^u - \xi^{u,N}\|_{1,2N+3} &= \|\xi^{u,N}_T - \xi^{u,N}\|_{1,2N+3} + \|\tilde{\xi}^u - \xi^{u,N}\|_{1,2N+3}\\ &\le \|\xi^{u,N}_T - \xi^{u,N}\|_{1,2N+3} + O(\varepsilon^{N+1}h^{2N+3}),\end{aligned}$$

where $\xi^{u,N}_T(t) = \xi^{u,N}(t-T)$. Finally, since $\xi^{u,N}\in\mathcal{X}_{1,1}$, with $\|\xi^{u,N}\|_{1,1}\le O(h)$, by (d) in Lemma 5.2, we have that

$$\|\xi^{u,N}_T - \xi^{u,N}\|_{1,2N+3} \le \|\xi^{u,N\prime}\|_{1,2}O(\varepsilon^{N+1}h^{2N+2}) \le O(\varepsilon^{N+1}h^{2N+3}).$$

It is analogously checked that the functions η_1^u, $\eta_2^{u,0}$ and $\eta_2^{u,i\pi/2}$ defined by $\eta_1^u(t) = \tilde{\eta}_1^u(t-T)$, etc., satisfy bounds (73), (74) and (75), respectively. □

6. Outer asymptotic expansion. Proof of Proposition 2.6

6.1. **Euler-MacLaurin Formula and first order difference operators.** The Euler-MacLaurin summation formula states that, given a C^∞ function $g\colon [0,\infty)\to\mathbb{R}$, with $g^{(k)}\in L^1(\mathbb{R})$ for all $k\ge 0$, for any $N\ge 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} g(n) = \int_0^\infty g(x)dx + \frac{1}{2}g(0) - \sum_{j=1}^{N}\frac{B_{2j}}{(2j)!}g^{(2j-1)}(0)\\ - \int_0^\infty \frac{\tilde{B}_{2N}(x)}{(2N)!}g^{(2N)}(x)dx,\end{aligned}\tag{190}$$

where B_{2j} are the Bernoulli numbers and $\tilde{B}_{2N}(x)$ are periodic functions related to the Bernoulli polynomials (see, for instance, [Olv74]). It provides a way to approximate sums by integrals, which will be very convenient in our problem.

Given ϕ a function, two solutions of the equation $\Delta_h\psi = \phi$ are given by $\Delta_{h,u}^{-1}\phi$ and $\Delta_{h,s}^{-1}\phi$, defined in (144) and (145), respectively, provided that the sums in the operators are absolutely convergent.

We apply the Euler-MacLaurin formula to obtain integral expressions of the operators $\Delta_{h,u}^{-1}$ and $\Delta_{h,s}^{-1}$.

Lemma 6.1. *If ϕ is a C^∞ function with exponential decay at $-\infty$, then, for any $N \geq 1$,*

$$\Delta_{h,u}^{-1}\phi(t) = -\frac{1}{2}\phi(t) + \frac{1}{h}\int_{-\infty}^{t} \phi(x)dx + \sum_{j=1}^{N} \frac{B_{2j}}{(2j)!}h^{2j-1}\phi^{(2j-1)}(t) - h^{2N}\int_0^\infty \frac{\tilde{B}_{2N}(x)}{(2N)!}\phi^{(2N)}(t-xh)dx. \tag{191}$$

If ϕ is a C^∞ function with exponential decay at ∞, then, for any $N \geq 1$,

$$\Delta_{h,s}^{-1}\phi(t) = -\frac{1}{2}\phi(t) - \frac{1}{h}\int_{t}^{\infty} \phi(x)dx + \sum_{j=1}^{N} \frac{B_{2j}}{(2j)!}h^{2j-1}\phi^{(2j-1)}(t) + h^{2N}\int_0^\infty \frac{\tilde{B}_{2N}(x)}{(2N)!}\phi^{(2N)}(t+xh)dx. \tag{192}$$

If ϕ is an even C^∞ function with exponential decay at $-\infty$, then, for any $N \geq 1$,

$$\Delta_{h,u}^{-1}\phi(h/2) = \frac{1}{h}\int_{-\infty}^{0} \phi(x)dx - \frac{h^{2N}}{2(2N)!}\int_0^\infty \tilde{B}_{2N}(x)\big(\phi^{(2N)}(-\frac{h}{2} - xh) + \phi^{(2N)}(-\frac{h}{2} + xh)\big)dx. \tag{193}$$

Proof. Relations (191) and (192) follow from using Euler-MacLaurin formula (190) in (144) and (145). To prove formula (193), note that, if both operators are defined,

$$\big(\Delta_{h,u}^{-1}\phi - \Delta_{h,s}^{-1}\phi\big)(t) = \sum_{k\in\mathbb{Z}} \phi(t-kh)$$

and hence, if ϕ is even,

$$\Delta_{h,u}^{-1}\phi(h/2) = \frac{1}{2}\big(\Delta_{h,u}^{-1}\phi - \Delta_{h,s}^{-1}\phi\big)(-h/2).$$

Applying formulas (191) and (192) to the above relation we obtain (193). □

6.2. **Introducing the asymptotic expansion.** We will obtain the asymptotic expansion of the outer approximation of order N, $\xi^{u,N} = \sum_{k=0}^{N}\varepsilon^k\xi_k^u$, computing the asymptotic expansion of the functions ξ_k^u. We recall that these functions were constructed in the following way:

(1) $\xi_0^u = \xi_0$ is the function defined in (1),

(2) for $k \geq 1$,

$$\xi_k^u = \mathcal{G}(f_k^u), \tag{194}$$

where f_k^u is obtained by substituting recursively $\xi_0^u, \ldots, \xi_{k-1}^u$ into (65), for $k = 1$, and into (66), for $k \geq 2$, and $\mathcal{G}$, defined in (161), is a right inverse of the operator $\mathcal{L}$ given by (152). We remark that we can write operator $\mathcal{G}$ in formula (194), using the operator $\Delta_{h,u}^{-1}$ defined in (144), as

$$\mathcal{G}(g)(t) = \eta_2^0 \Delta_{h,u}^{-1}(\eta_1 g)(t) - \eta_1 \Delta_{h,u}^{-1}(\eta_2^0 g)(t) + \eta_1 \Delta_{h,u}^{-1}(\eta_2^0 g)(h/2). \tag{195}$$

In order to obtain the asymptotic expansion in h of ξ_k^u, we modify the above scheme by substituting $\mathcal{G}$ by its asymptotic expansion in h up to order h^{2N}, provided by the Euler-MacLaurin formula.

More concretely, for a fixed $N \geq 0$, we define the sequence of functions ξ_0^N, $\ldots$, ξ_N^N as follows:

(1) $\xi_0^N = \xi_0$ is the function defined in (1),
(2) for $k \geq 1$,

$$\xi_k^N = \mathcal{T}_N(f_k^N), \tag{196}$$

where f_k^N is obtained by substituting recursively $\xi_0^N, \ldots, \xi_{k-1}^N$ into (65), for $k = 1$, and into (66), for $k \geq 2$, and $\mathcal{T}_N$ is defined by

$$\begin{aligned} \mathcal{T}_N(g)(t) &= \frac{1}{h}\eta_2^0(t)\int_{-\infty}^t \eta_1(x)g(x)dx - \frac{1}{h}\eta_1(t)\int_0^t \eta_2^0(x)g(x)dx \\ &\quad + \sum_{j=1}^N \frac{B_{2j}}{(2j)!}h^{2j-1}\big(\eta_2^0(t)(\eta_1 g)^{(2j-1)}(t) - \eta_1(t)(\eta_2^0 g)^{(2j-1)}(t)\big). \end{aligned} \tag{197}$$

Notice that $\mathcal{T}_N$ is obtained formally replacing in (195) the operator $\Delta_{h,u}^{-1}$ by formula (191), computing $\Delta_{h,u}^{-1}(\eta_2^0 g)(h/2)$ with formula (193) and dropping the error terms.

We remark also that the operator $\mathcal{T}_N$, unlike the operator $\mathcal{G}$ in (195), is defined for complex h. In Lemma 6.3 we will give a precise description of the dependence of $\mathcal{T}_N$ with respect to h.

6.3. **The operator $\mathcal{T}_N$ and its approximating properties.** In order to prove that the sequence $(\xi_k^N)_{k=0,\ldots,N}$ defined by the recurrence (196) is indeed well defined and that each function ξ_k^N contains all the terms up to degree $2N+1$ of the asymptotic expansion of ξ_k^u in powers of h, we will prove that the operator $\mathcal{T}_N$ is well defined between suitable Banach spaces and that it is indeed a good approximation of the operator $\mathcal{G}$ used in the recurrence (194) that defines the functions ξ_k^u.

We introduce, for $l, m \in \mathbb{R}$, the spaces

$$\mathcal{X}_{l,m}^e(\beta_1, \beta_2, r_1, r_2, \delta) = \{\xi \in \mathcal{X}_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta) \mid \xi \text{ is even}\}. \tag{198}$$

They are Banach spaces with the norm defined in $\mathcal{X}_{l,m}$ (see (157) and (158) for the definition of $\mathcal{X}_{l,m}$ and its norm). It is clear that $\mathcal{X}^e_{l,m} \subset \mathcal{X}_{l,m}$, with the same norm.

Analogously to the function $V_k(t)$ introduced in (160), we define

$$V(t,s) = \eta_2^0(t)\eta_1(t+s) - \eta_1(t)\eta_2^0(t+s). \tag{199}$$

We remark that, like $V_k(t)$, the map $t \to V(t,s)$ is $i\pi$-periodic.

Using this function V and performing the change of variables $x = t+s$ in (197), the operator $\mathcal{T}_N$ can be written as

$$\begin{aligned}\mathcal{T}_N(g)(t) = \frac{1}{h}\int_{-\infty}^{0} V(t,s)g(t+s)ds + \frac{1}{h}\eta_1(t)\int_{-\infty}^{0}\eta_2^0(s)g(s)ds \\ + \sum_{j=1}^{N}\frac{B_{2j}}{(2j)!}h^{2j-1}\frac{\partial^{2j-1}}{\partial s^{2j-1}}\big(V(t,s)g(t+s)\big)_{|s=0},\end{aligned} \tag{200}$$

which is the equivalent expression to $\mathcal{G}$ in (161) with integrals instead of sums and the correcting term given by the Euler-MacLaurin formula (see also (159) for the definition of $\tilde{\mathcal{G}}$).

Lemma 6.2. *We denote $\tau = |t - i\pi/2|$, $\tau_s = |t - s - i\pi/2|$. Given $\beta_1 < \beta$, $0 < \beta_2 < \pi/2$, $0 < r_1 < 1/2$, $1/2 < r_2 < 1$, $\delta > 0$, for any $N \geq 0$, $\beta_1 < \tilde\beta_1 < \beta$, $0 < \tilde\beta_2 < \beta_2$, $0 < \tilde r_1 < r_1$, $r_2 < \tilde r_2 < 1$, there exists $C_N > 0$ such that for any $g \in \mathcal{X}^e_{l,m}(\beta_1,\beta_2,r_1,r_2,\delta)$, any $0 \leq j \leq 2N$, $|h| < h_0$, any $t \in U(\tilde\beta_1,\tilde\beta_2,\tilde r_1,\tilde r_2,\delta)$, $\operatorname{Im} t \geq 0$, and any $s \in \mathbb{R}^+$, we have that*

$$\left|\frac{\partial^j}{\partial s^j}V(t,s)g(t+s)\right| \leq \begin{cases} C_N e^{l\operatorname{Re} t}e^{(l-1)\operatorname{Re} s}\|g\|_{l,m}/|h|, & \text{if } \operatorname{Re} t, \operatorname{Re} t+s \leq -1,\\ C_N e^{l\operatorname{Re} t}e^{(l-1)\operatorname{Re} s}\|g\|_{l,m}/(|h|\tau^2), & \text{if } \operatorname{Re} t+s \leq -1 \leq \operatorname{Re} t,\\ \dfrac{C_N}{|h|}\Big(\dfrac{\tau^3}{\tau_s^{m+j+2}} + \dfrac{1}{\tau^2\tau_s^{m+j-3}}\Big)\|g\|_{l,m}, & \text{if } -1 \leq \operatorname{Re} t, \operatorname{Re} t+s,\end{cases}$$

Proof. We remark that $V(t,s) = \eta_2^{i\pi/2}(t)\eta_1(t+s) - \eta_1(t)\eta_2^{i\pi/2}(t+s)$. Then, from (153), (154), (155) and (156) we have that, for $t \in U(\beta_1,\beta_2,r_1,r_2,\delta)$, $\operatorname{Im} t \geq 0$, $s \in \mathbb{R}^+$

$$|V(t,s)| \leq \begin{cases} K_N e^{-\operatorname{Re} s}/|h|, & \text{if } \operatorname{Re} t, \operatorname{Re} t+s \leq -1,\\ K_N e^{-\operatorname{Re} s}/(|h|\tau^2), & \text{if } \operatorname{Re} t+s \leq -1 \leq \operatorname{Re} t,\\ \dfrac{K_N}{|h|}\Big(\dfrac{\tau^3}{\tau_s^2} + \dfrac{\tau_s^3}{\tau^2}\Big), & \text{if } -1 \leq \operatorname{Re} t, \operatorname{Re} t+s,\end{cases} \tag{201}$$

for some suitable constant $K_N > 0$. Hence, since if $g \in \mathcal{X}^e_{l,m}(\beta_1,\beta_2,r_1,r_2,\delta)$, it satisfies

$$|g(t)| \leq \begin{cases} e^{l\operatorname{Re} t}\|g\|_{l,m}, & \text{if } \operatorname{Re} t \leq -1,\\ \tau^{-m}\|g\|_{l,m}, & \text{if } -1 \leq \operatorname{Re} t,\end{cases}$$

we have that

$$|V(t,s)g(t+s)| \leq \begin{cases} C_N e^{l\mathrm{Re}\,t} e^{(l-1)\mathrm{Re}\,s} \|g\|_{l,m}/|h|, & \text{if } \mathrm{Re}\,t, \mathrm{Re}\,t+s \leq -1, \\ C_N e^{l\mathrm{Re}\,t} e^{(l-1)\mathrm{Re}\,s} \|g\|_{l,m}/(|h|\tau^2), & \text{if } \mathrm{Re}\,t+s \leq -1 \leq \mathrm{Re}\,t, \\ \dfrac{C_N}{|h|}\Big(\dfrac{\tau^3}{\tau_s^{m+2}} + \dfrac{1}{\tau^2\tau_s^{m-3}}\Big)\|g\|_{l,m}, & \text{if } -1 \leq \mathrm{Re}\,t, \mathrm{Re}\,t+s, \end{cases}$$

for some suitable constant $C_N > 0$. We can take derivatives above as in (d) in Lemma 5.2 to obtain the claimed bound in the restricted domain $U(\tilde\beta_1, \tilde\beta_2, \tilde r_1, \tilde r_2, \delta)$. □

Lemma 6.3. *Assume that $l > 1$ and $m > 4$. For any $N \geq 0$, $\beta_1 < \tilde\beta_1 < \beta$, $0 < \tilde\beta_2 < \beta_2$, $0 < \tilde r_1 < r_1$, $r_2 < \tilde r_2 < 1$, $\mathcal{T}_N$ is a bounded linear map from $\mathcal{X}^e_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta)$ to $\mathcal{X}^e_{1,m-2}(\tilde\beta_1, \tilde\beta_2, \tilde r_1, \tilde r_2, \delta)$, and $\|\mathcal{T}_N\| = O(h^{-2})$. Moreover, if $g \in \mathcal{X}^e_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta)$, depends also on (ε, h) and is analytic in the domain $\mathcal{U}^{out}_{\varepsilon_0,h_0}$ defined in (76), then $h^2\mathcal{T}_N(g)$ is analytic in $\mathcal{U}^{out}_{\varepsilon_0,h_0}$. Furthermore, $\mathcal{T}_N$ preserves oddness with respect to h and $i\pi$-antiperiodicity with respect to t.*

Proof. It is clear that if g is real with respect to t, so is $\mathcal{T}_N(g)$. On the other hand, since η_1 is odd and η_2^0 is even, we deduce from (197) that $\mathcal{T}_N$ preserves evenness with respect to t.

Since the function $t \to V(t,s)$ is πi-periodic and η_1 is πi-antiperiodic, from (200) we have that $\mathcal{T}_N$ preserves πi-antiperiodicity.

Now we check that $\mathcal{T}_N$ is a well defined operator from $\mathcal{X}^e_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta)$ to $\mathcal{X}^e_{1,m-2}(\tilde\beta_1, \tilde\beta_2, \tilde r_1, \tilde r_2, \delta)$, with norm bounded by $O(h^{-2})$. We introduce $\mathcal{I}_N$ and $\mathcal{J}_N$, with $\mathcal{T}_N = \mathcal{I}_N + \mathcal{J}_N$, where $\mathcal{I}_N$ is the first line of the right hand side in (200), that is, is the integral part of $\mathcal{T}_N$, while $\mathcal{J}_N$ is the second line of the right hand side of (200).

We claim that both operators, $\mathcal{I}_N$ and $\mathcal{J}_N$ are bounded linear maps from the space $\mathcal{X}^e_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta)$ to $\mathcal{X}^e_{1,m-2}(\tilde\beta_1, \tilde\beta_2, \tilde r_1, \tilde r_2, \delta)$ with norm bounded by $O(h^{-2})$. We prove this claim for each operator separately.

We deal with first with $\mathcal{I}_N$. We follow the arguments used in the proof of Lemma 5.4. Furthermore, we remark that, since real analyticity is preserved, we only need to consider t with $\mathrm{Im}\,t \geq 0$.

Let $g \in \mathcal{X}^e_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta)$. Since $l > 1$ and $\eta_2^0 \in \mathcal{X}_{-1,2}(\beta_1, \beta_2, r_1, r_2, \delta)$, we have that

$$\Big|\int_{-\infty}^0 \eta_2^0(x) g(x) dx\Big| \leq O(1)\|g\|_{l,m}.$$

Then, by Lemma 6.2, and using that $\eta_1 \in \mathcal{X}_{1,2}(\beta_1, \beta_2, r_1, r_2, \delta)$, with $\|\eta_1\|_{1,2} \leq O(h)$, if $\mathrm{Re}\, t \leq -1$, we have that

$$\begin{aligned}|e^{-\mathrm{Re}\, t}\mathcal{I}_N(g)(t)| &\leq \frac{1}{|h|}\int_{-\infty}^{0} e^{-\mathrm{Re}\, t}\frac{e^{-\mathrm{Re}\, s}}{h}e^{l\mathrm{Re}\, t}e^{l\mathrm{Re}\, s}\|g\|_{l,m}ds + \|g\|_{l,m}\\ &\leq O(h^{-2})\|g\|_{l,m},\end{aligned}$$

and, if $-1 \leq \mathrm{Re}\, t$,

$$\begin{aligned}|\tau^{m-2}\mathcal{I}_N(g)(t)| &\leq \frac{\tau^{m-2}}{|h|}\int_{-\infty}^{0}|V(t,s)g(t+s)|ds + \tau^{m-4}\|g\|_{l,m}\\ &\leq \frac{\tau^{m-2}}{|h|}\Big(\int_{-1\leq \mathrm{Re}\, t+s}|V(t,s)g(t+s)|ds\\ &\quad + \int_{\mathrm{Re}\, t+s\leq -1}|V(t,s)g(t+s)|ds\Big)\\ &\leq \int_{-\infty}^{0}\frac{1}{h^2}\big(\frac{\tau^{m+1}}{\tau_s^{m+2}} + \frac{\tau^{m-4}}{\tau_s^{m-3}}\big)\|g\|_{l,m}ds\\ &\quad + \tau^{m-4}\int_{-\infty}^{0}\frac{e^{(l-1)\mathrm{Re}\, s}}{h^2}\|g\|_{l,m} + \tau^{m-4}\|g\|_{l,m}\\ &\leq O(h^{-2})\|g\|_{l,m},\end{aligned}$$

which proves the claim for $\mathcal{I}_N$.

Now we check that $\mathcal{J}_N$ also satisfies the claim.

Using (201), if $g \in \mathcal{X}^e_{l,m}(\beta_1, \beta_2, r_1, r_2, \delta)$, we have that

$$\frac{\partial^{2j-1}}{\partial s^{2j-1}}\big(V(t,s)g(t+s)\big)_{|s=0} \in \mathcal{X}^e_{l,2j+m-2}(\tilde\beta_1, \tilde\beta_2, \tilde r_1, \tilde r_2, \delta),$$

and,

$$\Big\|\frac{\partial^{2j-1}}{\partial s^{2j-1}}\big(V(t,s)g(t+s)\big)_{|s=0}\Big\|_{l,2j+m-2} \leq O(h^{-1})\|g\|_{l,m}.$$

Hence, by Lemma 5.2,

$$\frac{\partial^{2j-1}}{\partial s^{2j-1}}\big(V(t,s)g(t+s)\big)_{|s=0} \in \mathcal{X}^e_{1,m-2}(\tilde\beta_1, \tilde\beta_2, \tilde r_1, \tilde r_2, \delta)$$

and

$$\Big\|\frac{\partial^{2j-1}}{\partial s^{2j-1}}\big(V(t,s)g(t+s)\big)_{|s=0}\Big\|_{l,m-2} \leq O(h^{-2})\|g\|_{l,m},$$

which proves the claim for $\mathcal{J}_N$.

Finally, if g is analytic in $\mathcal{U}^{out}_{\varepsilon_0,h_0}$, since $h^2\eta_1(t)\eta_2^0(t)$ is meromorphic with respect to t and entire with respect to h, $\mathcal{T}_N(g)$ is analytic in $\mathcal{U}^{out}_{\varepsilon_0,h_0}$. Moreover, since $\eta_1(t)\eta_2^0(t)/h$ is even with respect to h, $\mathcal{T}_N$ preserves parity with respect to h. □

Lemma 6.4. *Let $l>1$, $m>4$. For any $N\geq 0$, $\beta_1<\beta$, $0<\beta_2<\pi/2$, $0<r_1<1/2$, $1/2<r_2<1$, $0\leq k\leq N$, $0<h<h_0$ and for any $g\in\mathcal{X}^e_{l,m}(\beta_1,\beta_2,r_1,r_2,\delta)$,*

$$\|\mathcal{G}(g)-\mathcal{T}_N(g)\|_{1,m+2N-2}=O(h^{2N-2})\|g\|_{l,m}.$$

Proof. For $g\in\mathcal{X}^e_{l,m}$, using (191), (193), and the function $V(t,s)$ introduced in (199), we can write the difference of the operator as

$$\mathcal{G}(g)-\mathcal{T}_N(g)=E_1(g)+E_2^1(g)+E_2^2(g), \tag{202}$$

where

$$E_1(g)(t)=-h^{2N-1}\int_{-\infty}^0\frac{\tilde{B}_{2N}((t-s)/h)}{(2N)!}\frac{\partial^{2N}}{\partial s^{2N}}\big(V(t,s)g(t+s)\big)ds, \tag{203}$$

$$E_2^1(g)(t)=-\frac{h^{2N}}{2}\eta_1(t)\int_0^\infty\frac{\tilde{B}_{2N}(x)}{(2N)!}(\eta_2 g)^{(2N)}(-h/2-xh)dx, \tag{204}$$

$$E_2^2(g)(t)=\frac{h^{2N}}{2}\eta_1(t)\int_0^\infty\frac{\tilde{B}_{2N}(x)}{(2N)!}(\eta_2^0 g)^{(2N)}(-h/2+xh)dx, \tag{205}$$

We consider each term of the right hand of (202) separately. Moreover, since both $\mathcal{G}$ and $\mathcal{T}_N$ preserve real analyticity, it is enough to compute their bounds for $\operatorname{Im}t\geq 0$.

Now we bound $\|E_1(g)\|_{1,m+2N-2}$. We recall that $\tilde{B}_{2N}$ is a bounded periodic function. Hence, we can assume that $|\tilde{B}_{2N}(x)/h)/(2N)!|$ is bounded by some constant C. Using Lemma 6.2, we have that, for $\operatorname{Re}t\leq -1$,

$$|e^{-\operatorname{Re}t}E_1(g)(t)|\leq Ce^{-\operatorname{Re}t}h^{2N-1}\int_{-\infty}^0\frac{e^{l\operatorname{Re}t}e^{(l-1)\operatorname{Re}s}\|g\|_{l,m}}{h}ds\leq Ch^{2N-2}\|g\|_{l,m},$$

and, for $-1\leq\operatorname{Re}t$,

$$\begin{aligned}|\frac{E_1(g)(t)}{\tau^{2-m-2N}}|&\leq C\frac{h^{2N-1}}{\tau^{2-m-2N}}\int_{\operatorname{Re}t+s\leq -1}|\frac{\partial^{2N}}{\partial s^{2N}}\big(V(t,s)g(t+s)\big)|ds\\&\quad+C\frac{h^{2N-1}}{\tau^{2-m-2N}}\int_{-1\leq\operatorname{Re}t+s}|\frac{\partial^{2N}}{\partial s^{2N}}\big(V(t,s)g(t+s)\big)|ds\\&\leq C\frac{h^{2N-1}}{\tau^{2-m-2N}}\int_{-\infty}^0 e^{l\operatorname{Re}t}\frac{e^{(l-1)\operatorname{Re}s}}{h\tau^2}\|g\|_{l,m}ds\\&\quad+C\frac{h^{2N-2}}{\tau^{2-m-2N}}\int_{-\infty}^0\big(\frac{\tau^3}{\tau_s^{m+2N+2}}+\frac{1}{\tau^2\tau_s^{m+2N-3}}\big)\|g\|_{l,m}ds\\&\leq 2Ch^{2N-2}\|g\|_{l,m}.\end{aligned}$$

Consequently,

$$\|E_1(g)\|_{1,m+2N-2}\leq O(h^{2N-2})\|g\|_{l,m}.$$

Now we obtain a suitable bound for $E_2^1(g)$. From (155), since $g \in \mathcal{X}_{l,m}$, we have that, for $\mathrm{Re}\, t \leq -1$,

$$|\eta_2^0(t)g(t)| \leq C\frac{e^{2\mathrm{Re}\, t}}{h^2}\|g\|_{l,m},$$

and, consequently, for $-1 \leq \mathrm{Re}\, t$,

$$|(\eta_2^0(t)g(t))^{(2N)}| \leq C\frac{e^{2\mathrm{Re}\, t}}{h^2}\|g\|_{l,m}.$$

Hence, since the integral in $E_2^1(g)$ is computed along the real line, using (153) and (154), we have that,

$$\|E_2^1(g)\|_{1,m+2N-2} \leq O(h^{2N-2})\|g\|_{l,m}.$$

The same bounds apply to $E_2^2(g)$. □

6.4. **The outer asymptotic expansion.** Applying the results of the previous section, here we prove the existence of the outer asymptotic expansion and its approximating properties.

First we claim existence.

Lemma 6.5. *For any $N \geq 0$, $\beta_1 < \beta$, $0 < \beta_2 < \pi/2$, $0 < r_1 < 1/2$, $1/2 < r_2 < 1$, the sequence $(\xi_k^N)_{k=0,\dots,N}$, obtained by the recurrence* (196)*, is indeed well defined and ξ_k^N belongs to $\mathcal{X}^e_{1,2k+1}(\beta_1,\beta_2,r_1,r_2,\delta)$ with $\|\xi_k^N\|_{1,2k+1} = O(h^{2k+1})$. Moreover, they are analytic in $\mathcal{U}^{out}_{\varepsilon_0,h_0}$ (see 76), odd with respect to h and $i\pi$-antiperiodic with respect to t.*

Proof. Since $\xi_0^N = \xi^0$ is the function defined in (1), the claim is trivial for $k = 0$. We prove the claim for $k \geq 1$ by induction.

First we consider the case $k = 1$. Let $\beta_1^* < \beta_1$, $\beta_2 < \beta_2^* < \pi/2$, $r_1 < r_1^* < 1/2$, $0 < r_2^* < r_2$. By (66), we have that $f_1^N = V' \circ \xi_0^N$. Since $V'(y) = O(y^5)$, by (c) in Lemma 5.2, we have that $f_1^N \in \mathcal{X}^e_{5,5}(\beta_1^*,\beta_2^*,r_1^*,r_2^*,\delta)$, f_1^N is odd with respect to h and that

$$\|f_1^N\|_{5,5} \leq O(h^5).$$

Furthermore, f_1^N is analytic in $\mathcal{U}^{out}_{\varepsilon_0,h_0}$. Hence, by Lemma 6.3, $\xi_1^N = \mathcal{T}_N(f_1^N)$ belongs to $\mathcal{X}^e_{1,3}(\tilde\beta_1,\tilde\beta_2,\tilde r_1,\tilde r_2,\delta)$, for any $\beta_1^* < \tilde\beta_1 < \beta_1$, $\beta_2 < \tilde\beta_2 < \beta_2^* < \pi/2$, $r_1 < \tilde r_1 < r_1^* < 1/2$, $0 < r_2^* < \tilde r_2 < r_2$ and $\|\xi_1^N\|_{1,3} = O(h^3)$, and is analytic in $\mathcal{U}^{out}_{\varepsilon_0,h_0}$ and odd with respect to h.

Now we assume, by induction, that the functions ξ_k^N, defined by the above recurrence, exist and verify the claimed properties for $k = 1,\dots,j-1$. We recall that

$$f_j^N = \mu \sum_{n=2}^{j} \frac{1}{n!} f^{(n)} \circ \xi_0 \sum_{\substack{j_1+\cdots+j_n=j \\ 1\leq j_1,\ldots,j_n\leq j}} \xi_{j_1}^N \cdots \xi_{j_n}^N +$$

$$+ \sum_{n=1}^{j-1} \frac{1}{n!} V^{(n+1)} \circ \xi_0 \sum_{\substack{j_1+\cdots+j_n=j-1 \\ 1\leq j_1,\ldots,j_n\leq j-1}} \xi_{j_1}^N \cdots \xi_{j_n}^N.$$

Note that, since, by the induction hypothesis, ξ_k^N, $k = 0, \ldots, j-1$, are even with respect to t, so is f_j^N. Besides, since by the induction hypotheses, ξ_k^N, $k = 0, \ldots, j-1$, are odd with respect to h and $f(y)$ and $V'(y)$ are odd with respect to y, f_j^N is odd with respect to h. Moreover, as was already pointed out in Section 5.3, in the proof of Proposition 5.5, when we proved analogous properties of the sequence of functions f_k^u, we have that $f_j^N \in \mathcal{X}^e_{3,2j+3}(\tilde\beta_1^j, \tilde\beta_2^j, \tilde r_1^j, \tilde r_2^j, \delta)$, for $\beta_1^* < \tilde\beta_1^j < \beta_1$, $\beta_2 < \tilde\beta_2^j < \beta_2^*$, $r_1 < \tilde r_1^j < r_1^*$, $r_2^* < \tilde r_2^j < r_2$ and $\|f_j^N\|_{3,2j+3} = O(h^{2j+3})$. It is clearly analytic in $\mathcal{U}^{out}_{\varepsilon_0,h_0}$. Then, by Lemma 6.3, $\xi_j^N = \mathcal{T}_N(f_j^N)$ belongs to $\mathcal{X}^e_{1,2j+1}(\tilde\beta_1^{j+1}, \tilde\beta_2^{j+1}, \tilde r_1^{j+1}, \tilde r_2^{j+1}, \delta)$, with $\tilde\beta_1^j < \tilde\beta_1^{j+1} < \beta_1$, $\beta_2 < \tilde\beta_2^{j+1} < \tilde\beta_2^j$, $r_1 < \tilde r_1^{j+1} < \tilde r_1^j$, $\tilde r_2^j < \tilde r_2^{j+1} < r_2$ and $\|\xi_j^N\|_{1,2j+1} = O(h^{2j+1})$, and analytic in $\mathcal{U}^{out}_{\varepsilon_0,h_0}$ and odd with respect to h. □

Finally, we claim the approximating properties.

Lemma 6.6. *Under the hypothesis of Proposition 2.6, the functions* $(\xi_k^u)_{0\leq k\leq N}$ *and* $(\xi_k^N)_{0\leq k\leq N}$ *verify*

(206) $$\|\xi_k^u - \xi_k^N\|_{1,2k+2N+1} = O(h^{2N+2k+1}), \qquad k = 0, \ldots, N.$$

Proof. The claim is proved by induction.

Note that the case $k = 0$ is trivial, since $\xi_0^u = \xi_0^N = \xi_0$.

Now we consider the case $k = 1$. By recurrences (194) and (196), we have that

$$\xi_1^u = \mathcal{G}(f_1^u), \qquad \xi_1^N = \mathcal{T}_N(f_1^N),$$

where, in this case $f_1^u = f_1^N = f_1(\xi^0) = V' \circ \xi^0$. Since $V'(y) = O(y^5)$, $f_1(\xi^0) \in \mathcal{X}_{5,5}$ and $\|f_1(\xi^0)\|_{5,5} \leq O(h^5)$. Hence, by Lemma 6.4 with $l = m = 5$,

$$\begin{aligned} \|\xi_1^u - \xi_1^N\|_{1,2N+3} &\leq \|\mathcal{G}(f_1) - \mathcal{T}_N(f_1)\|_{1,2N+3} \\ &\leq O(h^{2N-2})\|f_1^N\|_{5,5} \\ &\leq O(h^{2N+3}), \end{aligned}$$

which proves the claim for k=1.

Now we proceed by induction. We assume that, for $0 \leq j \leq k-1$, the functions ξ_j^u and ξ_j^N satisfy (206). We recall that $\xi_j^u \in \mathcal{X}_{1,2j+1}$, $\xi_j^N \in \mathcal{X}^e_{1,2j+1}$ and $\|\xi_j^u\|_{1,2j+1} = \|\xi_j^N\|_{1,2j+1} = O(h^{2j+1})$.

First we claim that, under the induction hypothesis, the functions f_k^u and f_k^N obtained by substituting $(\xi_j^u)_{0\le j\le k-1}$ and $(\xi_j^N)_{0\le j\le k-1}$, respectively, into (66) satisfy

$$\|f_k^u - f_k^N\|_{3,2N+2j+3} \le O(h^{2n+2j+3}). \tag{207}$$

Indeed, we have that

$$f_k^u - f_k^N = \mu \sum_{n=2}^{k} \frac{1}{n!} f^{(n)} \circ \xi_0 \sum_{\substack{j_1+\cdots+j_n=k\\ 1\le j_1,\ldots,j_n\le k}} (\xi_{j_1}^u \cdots \xi_{j_n}^u - \xi_{j_1}^N \cdots \xi_{j_n}^N) +$$

$$+ \sum_{n=1}^{k-1} \frac{1}{n!} V^{(n+1)} \circ \xi_0 \sum_{\substack{j_1+\cdots+j_n=k-1\\ 1\le j_1,\ldots,j_n\le k-1}} (\xi_{j_1}^u \cdots \xi_{j_n}^u - \xi_{j_1}^N \cdots \xi_{j_n}^N).$$

By the induction hypothesis, in the first sum, since $j_1 + \cdots + j_n = k$,

$$\|\xi_{j_1}^u \cdots \xi_{j_n}^u - \xi_{j_1}^N \cdots \xi_{j_n}^N\|_{n,2k+2N+n} = O(h^{2k+2N+n}).$$

On the other hand, for $n = 2$, $\|f'' \circ \xi_0\|_{1,1} = O(h)$, which implies that, in the first sum,

$$\|f'' \circ \xi_0(\xi_{j_1}^u \cdots \xi_{j_n}^u - \xi_{j_1}^N \cdots \xi_{j_n}^N)\|_{3,2k+2N+3} = O(h^{2N+2k+3}).$$

For $n \ge 2$, using Lemma 5.2, we have that

$$\|f^{(n)} \circ \xi_0(\xi_{j_1}^u \cdots \xi_{j_n}^u - \xi_{j_1}^N \cdots \xi_{j_n}^N)\|_{3,2k+3} = O(h^{2k+2N+3}).$$

The terms in the second sum can be bounded analogously taking into account that $\|V^{(n+1)} \circ \xi_0\|_{5-n,5-n} = O(h^{5-n})$, for $n = 1, \ldots, 5$, and $V^{(n+1)} \circ \xi_0$ is bounded for $n \ge 5$. Hence inequality (207) is proven.

Finally, using Lemma 5.4, inequality (207), Lemma 6.4 and the fact that $f_k^N \in \mathcal{X}^e_{3,2k+3}$, with $\|f_k^N\|_{3,2k+3} = O(h^{2k+3})$, we have that

$$\begin{aligned}
\|\xi_k^u - \xi_k^N\|_{1,2k+2N+1} &= \|\mathcal{G}(f_k^u) - \mathcal{T}_N(f_k^N)\|_{1,2k+2N+1} \\
&\le \|\mathcal{G}(f_k^u) - \mathcal{G}(f_k^N)\|_{1,2k+2N+1} + \|\mathcal{G}(f_k^N) - \mathcal{T}_N(f_k^N)\|_{1,2k+2N+1} \\
&\le O(h^{-2})\|f_k^u - f_k^N\|_{1,2k+2N+3} + O(h^{2N-2})\|f_k^N\|_{3,2k+3} \\
&\le O(h^{2k+2N+1})
\end{aligned}$$

□

Proposition 2.6 follows immediately from Lemmas 6.5 and 6.6.

7. Asymptotic expansion in the inner variable

Here we give the proofs of Propositions 2.10 and 2.12, which are closely related.

7.1. **Proof of Proposition 2.10.** We recall that, by Proposition 2.6, the functions $(\xi_k^N)_{k=0,\dots,N}$ are analytic in $\mathcal{U}^{out}_{\varepsilon_0,h_0}$ with respect to (t,h,ε), even and πi-antiperiodic with respect to t and odd with respect to h.

We expand ξ_k^N in powers of h. Since it is odd with respect to h, we have that

$$\xi_k^N(t,h,\varepsilon) = \sum_{j\geq 0} h^{2j+1}\chi_{k,j}^N(t,\varepsilon), \qquad \chi_{k,j}^N(t,\varepsilon) = \frac{1}{2\pi i}\int_{\gamma_H} \frac{\xi_k^N(t,h,\varepsilon)}{h^{2j+2}}dh, \tag{208}$$

where γ_H is the positively oriented circumference of radius H around $h=0$, with $0<H<h_0$.

We remark that, by the definition of $\mathcal{U}^{out}_{\varepsilon_0,h_0}$ in (76), the coefficients $\chi_{k,j}^N$ can be computed by formula (208) for any t such that $H<|t-i\pi/2|<\pi$. It is clear that they are $i\pi$-antiperiodic and even with respect to t. Moreover, since they do not depend on h, their only singularity in $\{t \mid \; |t-i\pi/2|<\pi\}$ is $i\pi/2$.

Moreover, by Lemma 6.5, for any t with $H<|t-i\pi/2|<\pi$, $0<H<h_0$,

$$|\chi_{k,j}^N(t,\varepsilon)\cosh^{2k+1} t| \leq \frac{1}{2\pi}\Big|\int_{\gamma_H}\frac{\xi_k^N(t,h,\varepsilon)\cosh^{2k+1} t}{h^{2j+2}}dh\Big| \leq O(H^{2(k-j)}).$$

Hence, if $0\leq j<k$, $\chi_{k,j}^N\equiv 0$.

With the same argument, if $k\leq j$, for $0<H<|t-i\pi/2|<2H$, we have that

$$|\chi_{k,j}^N(t,\varepsilon)\cosh^{2j+1} t| \leq O(H^{2(k-j)})|\cosh^{2(j-k)} t| \leq O(1),$$

which implies that $\chi_{k,j}^N$ has a pole of order at most $2j+1$.

Defining

$$\Xi_{k,m}^N = \chi_{k,k+m}^N,$$

we have proven the first part of Proposition 2.10.

Formula (84) follows from the Laurent expansion of $\Xi_{k,m}^N$,

$$\Xi_{k,m}^N(t,\varepsilon) = \sum_{l\geq -m-k-1} a_{k,m,l}^N(\varepsilon)(t-\frac{i\pi}{2})^{2l+1},$$

making the change $t=i\pi/2+hz$, and reordering the absolutely convergent series

$$\begin{aligned}\xi_k^N(\frac{i\pi}{2}+hz,\varepsilon,h) &= \sum_{m\geq 0} h^{2m+2k+1}\Xi_{k,m}^N(\frac{i\pi}{2}+hz,\varepsilon)\\ &= \sum_{m\geq 0} h^{2m+2k+1}\sum_{l\geq -m-k-1} a_{k,m,l}^N(\varepsilon)(hz)^{2l+1}\\ &= \sum_{n\geq 0} h^{2n}\sum_{m\geq 0} a_{k,m,n-k-m-1}^N(\varepsilon)z^{-2(m+k-n)-1}.\end{aligned}$$

We finally check claim (85). We fix $N<N'$. Assume that there exists m such that $m\leq N-1$ and $\Xi_{k,m}^N-\Xi_{k,m}^{N'}\neq 0$ and take m minimal. In particular, there

exist $t, C \in \mathbb{R}$ such that, for $0 < h < h_0$,

$$|\xi_k^N(t) - \xi_k^{N'}(t)| \geq Ch^{2m+2k+1}.$$

But, by Lemma 6.6, we have that

$$|\xi_k^N(t) - \xi_k^{N'}(t)| \leq |\xi_k^N(t) - \xi_k^u(t)| + |\xi_k^u(t) - \xi_k^{N'}(t)| \leq O(h^{2N+2k+1}).$$

Hence, $\Xi_{k,m}^N = \Xi_{k,m}^{N'}$, for all $0 \leq m \leq N-1$.

Finally, after substituting t by $i\pi/2 + hz$ and reordering the series, we obtain claim (86).

The fact that the coefficients $a_{k,m,l}^N$ are purely imaginary whenever ε is real is an immediate consequence of $i\pi$-antiperiodicity and real analyticity. □

7.2. **Proof of Proposition 2.12.** By Proposition 2.6, the outer asymptotic expansion $\xi^{N,\text{out}} = \sum_{k=0}^N \varepsilon^k \xi_k^N$ introduced in (79) is analytic in $\mathcal{U}_{\varepsilon_0,h_0}^{out}$, even and πi-antiperiodic with respect to t and odd with respect to h. Moreover, $\|\xi^{N,\text{out}}\|_{1,1} \leq O(h)$.

We define by E^N the error function obtained by substituting $\xi^{N,\text{out}}$ into the invariance equation (56), that is,

$$E^N(t,\varepsilon,h) = \xi^{N,\text{out}}(t+h/2,h,\varepsilon) + \xi^{N,\text{out}}(t-h/2,h,\varepsilon) - \mathcal{F}(\xi^{N,\text{out}}(t),h,\varepsilon), \tag{209}$$

where $\mathcal{F}$ was introduced in (61).

We first remark that, since $\xi^{N,\text{out}}$ is analytic in $\mathcal{U}_{\varepsilon_0,h_0}^{out}$, E^N is analytic in

$$\tilde{\mathcal{U}}^{out} = \{(t,h,\varepsilon) \in \mathbb{C}^3 \mid |\varepsilon| < \varepsilon_0, \ \text{dist}(t, ik\pi/2) > \tfrac{3}{2}|h|, \ k \in \mathbb{Z}, \ |h| < h_0\}. \tag{210}$$

Moreover, since $\xi^{N,\text{out}}$ is even and $i\pi$-antiperiodic with respect to t and $\mathcal{F}(y,h,\varepsilon)$ is odd with respect to y and even with respect to h, E^N is even and $i\pi$-antiperiodic with respect to t, and odd with respect to h. Hence

$$E^N(t,\varepsilon,h) = \sum_{k\geq 0} h^{2k+1} E_k^N(t,\varepsilon), \qquad E_k^N(t,\varepsilon) = \frac{1}{2\pi i}\int_{\gamma_H} \frac{E^N(t,h,\varepsilon)}{h^{2k+2}}dh, \tag{211}$$

where γ_H is the positively oriented circumference of radius H around $h=0$ and $0 < H < h_0$.

Using that the function ξ^u, given by Theorem 2.3, is a solution of the invariance equation (56) and inequalities (69) and (80), we have that, for $0 < h < h_0$ and $t \in D_\delta^{u,\text{out}}$ (skipping the dependence on ε),

$$\begin{aligned}|E^N(t,h)| &\leq |\xi^{N,\text{out}}(t+h/2,h) - \xi^u(t+h/2)| + |\xi^{N,\text{out}}(t-h/2,h) - \xi^u(t-h/2)| \\ &\quad + |\mathcal{F}(\xi^{N,\text{out}}(t,h),h) - \mathcal{F}(\xi^u(t),h),h)| \\ &\leq C(t)h^{2N+3},\end{aligned}$$

which implies that $E_k^N \equiv 0$ for $k = 1,\ldots,N$.

On the other hand, since $\|\xi^{N,\text{out}}\|_{1,1} \le O(h)$, we have that $\|E^N\|_{1,1} \le O(h)$. Hence, we have that, for $0 < H < |t - i\pi/2| < 2H$,

$$|E_k^N(t,\varepsilon)\cosh^{2k+1} t| \le O(H^{-2k})|\cosh^{2k} t| \le O(1),$$

from which we can deduce that E_k^N has a pole of order at most $2k+1$ at $i\pi/2$. Therefore, we can write E^N in the form

$$E^N(t,\varepsilon,h) = \sum_{k\ge N+1} h^{2k+1} \sum_{\ell\ge -k} E_{k,\ell}^N(\varepsilon)(t - i\pi/2)^{2\ell-1}. \tag{212}$$

Hence, by substituting $t = i\pi/2 + hz$ into (212), we have that

$$E^N(i\frac{\pi}{2} + hz, \varepsilon, h) = \sum_{n\ge 0} h^{2n}\mathcal{E}_n^N(z,\varepsilon)$$

where

$$\mathcal{E}_n^N(z,\varepsilon) = \sum_{k\ge N+1} E_{k,n-k}^N(\varepsilon)\frac{1}{z^{2k-2n+1}}. \tag{213}$$

Finally, we remark that the formal series $\tilde{\phi}_n$ introduced in (89) are the formal limit of

$$\phi_n^N(z,\varepsilon) = \sum_{k=0}^{N} \varepsilon^k \phi_{k,n}^N(z,\varepsilon) \tag{214}$$

where the functions $\phi_{k,n}^N$ are given by Proposition 2.10, and "formal limit" means each coefficient of the z-expansion of ϕ_n^N is a finite sum of holomorphic functions in ε.

Notice that, in fact, $\mathcal{E}_n^N$ is the error function when substituting $\phi_0^N, \dots, \phi_n^N$ into equations (96), for $n = 0$, and (97), for $n \ge 1$. Hence, by equation (213), we have that

$$\mathcal{E}_n^N(z,\varepsilon) = O\big(\frac{1}{z^{2(N-n)+3}}\big),$$

which implies that $\mathcal{E}_n^N(z,\varepsilon)$ tends formally in the above sense to 0, when $N \to +\infty$, and, consequently, that $\tilde{\phi}_n$ are formal solutions of equations (96), for $n = 0$, and (97), for $n \ge 1$. □

8. Matching inner and outer approximations. Proofs of Theorems 2.18 and 2.20

8.1. **Starting point and domains in the inner variable.** Let $N \in \mathbb{N}^*$, $h \in (0, h_N)$ and $\delta \in (\max\{\rho_N, R_N\}h, \pi/2)$, so that we can apply Theorem 2.3: the solution ξ^u of equation (56) is known to be holomorphic for $t \in D_\delta^{u,\text{out}}$ and $\varepsilon \in \mathbb{D}(0,\varepsilon_0)$, for which values it is well approximated by $\xi^{u,N}$ according to inequalities (69), while $\xi^{u,N}$ is itself well approximated by $\xi^{N,\text{out}} = \sum_{k=0}^{N} \varepsilon^k \xi_k^N(t,h,\varepsilon)$ according to inequalities (80).

Moreover, according to Remark 2.5, all this is valid for t in a domain larger than $D_\delta^{u,\mathrm{out}}$; the domain $U(\beta/2,\beta/2,0,\cos(\beta/2),\delta)$ will be sufficient for our purpose.

Let us use the inner variable and set

$$\phi^u(z) = \xi^u(i\pi/2 + hz). \tag{215}$$

The function ϕ^u satisfies equation (93) and is holomorphic at least in the domain

$$\begin{aligned} \tilde{\mathcal{D}}_{\mathrm{out}} =& \Big\{ z\in\mathbb{C} \mid \operatorname{Re} z \le 0,\ \frac{\delta}{h} \le |z| \le 3\frac{\delta}{h},\ -\pi-\frac{\beta}{2} \le \arg(z) \le -\frac{\pi}{2} \Big\} \\ & \cup \Big\{ z\in\mathbb{C} \mid \operatorname{Re} z \ge 0,\ -3\frac{\delta}{h} \le \operatorname{Im} z \le -\frac{\delta}{h},\ -\frac{\pi}{2} \le \arg(z) \le -\frac{\beta}{2} \Big\}, \end{aligned} \tag{216}$$

which is only a part of the domain which corresponds to $U(\beta/2,\beta/2,0,\cos(\beta/2),\delta)$ by the change of variable $t \mapsto z = \frac{t-i\pi/2}{h}$.

Our aim is to follow the analytic continuation of ξ^u in $D_h^{u,\mathrm{in}}(R_N)$. Since ξ^u is real analytic, we can restrict ourselves to the upper half-plane $\{\operatorname{Im} t > 0\}$; we thus need to show that ϕ^u admits an analytic continuation in

$$\begin{aligned} \tilde{\mathcal{D}}^*_{\mathrm{in}} =& \Big\{ z\in\mathbb{C} \mid \operatorname{Re} z \le 0,\ \operatorname{Im} z \le 0,\ 2R_N < |z| \ \le \frac{\delta}{h} \Big\} \\ & \cup \Big\{ z\in\mathbb{C} \mid \operatorname{Re} z \ge 0,\ -\frac{\delta}{h} \le \operatorname{Im} z \le -2R_N,\ -\frac{\pi}{2} \le \arg(z) < -\beta \Big\} \end{aligned}$$

(observe that $\tilde{\mathcal{D}}_{\mathrm{out}} \cup \tilde{\mathcal{D}}^*_{\mathrm{in}}$ is connected—see Figure 7). Moreover, we want to estimate $\phi^u - \phi^{u,N}$ in $\tilde{\mathcal{D}}^*_{\mathrm{in}}$, with $\phi^{u,N} = \sum_{n=0}^N h^{2n}\phi_n^u(z,\varepsilon)$ holomorphic in $\mathcal{D}^u_{in}(R_N)$ (according to (102) and Theorem 2.14—observe that both $\tilde{\mathcal{D}}_{\mathrm{out}}$ and $\tilde{\mathcal{D}}^*_{\mathrm{in}}$ are contained in $\mathcal{D}^u_{in}(R_N)$). Up to the factor $|\varepsilon|$ in the right-hand side of inequality (113), Theorem 2.18 is a consequence of

Proposition 8.1. *The function*

$$\Psi = \phi^u - \phi^{u,N},$$

which is known to be holomorphic in $\tilde{\mathcal{D}}_{\mathrm{out}}$*, admits an analytic continuation in* $\tilde{\mathcal{D}}_{\mathrm{out}} \cup \tilde{\mathcal{D}}^*_{\mathrm{in}}$ *which satisfies*

$$|\Psi(z)| = O(A_\delta^N |z|^{-2}), \qquad |\frac{d}{dz}\Psi(z)| = O(A_\delta^N |z|^{-3}), \qquad z\in\tilde{\mathcal{D}}^*_{\mathrm{in}}, \tag{217}$$

provided that h *and* A_δ^N *are small enough.*

The following pages until Section 8.5 are devoted to proving this proposition and, finally, incorporating the missing factor $|\varepsilon|$ in the right-hand side of (217), so as to complete the proof of Theorem 2.18. The proof of Theorem 2.20 will be addressed using similar tools in Sections 8.6–8.7.

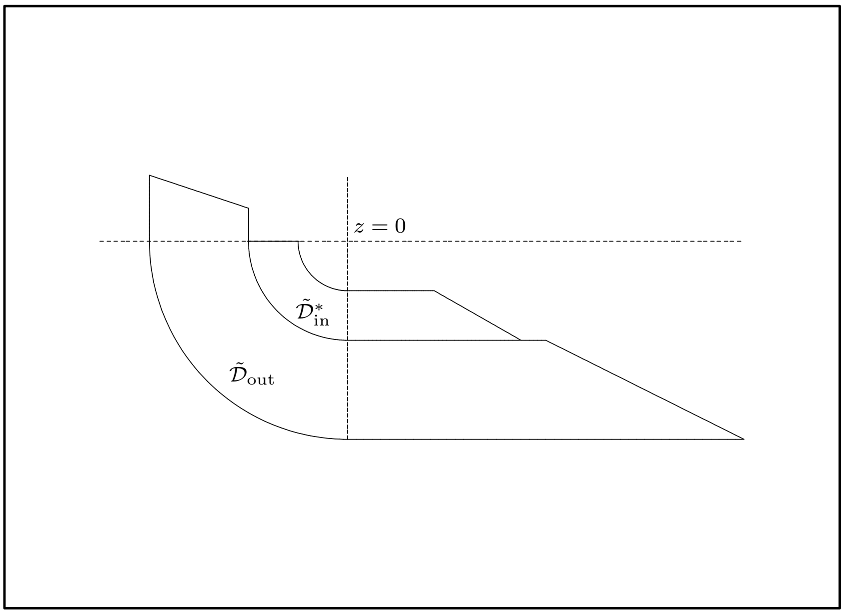

FIGURE 7. The domains $\tilde{\mathcal{D}}_{\text{out}}$ and $\tilde{\mathcal{D}}_{\text{in}}^*$.

With a view to applying the techniques of Section 4 for difference equations in bounded domains, we introduce a domain which is larger than $\tilde{\mathcal{D}}_{\text{in}}^*$:

(218)

$$\tilde{\mathcal{D}}_{\text{in}}^{**} = \Big\{ z \in \mathbb{C} \mid \operatorname{Re} z \le 0,\ R_N < |z| \ \le 2\frac{\delta}{h},\ -\pi - \frac{\beta}{4} < \arg(z) \le -\frac{\pi}{2} \Big\} \cup \Big\{ z \in \mathbb{C} \mid \operatorname{Re} z \ge 0,\ -2\frac{\delta}{h} < \operatorname{Im} z \le -R_N,\ -\frac{\pi}{2} \le \arg(z) < -\frac{3}{4}\beta \Big\}.$$

This is a domain of type (II), which we can write as

$$\tilde{\mathcal{D}}_{\text{in}}^{**} = U_{r_-,r_+}, \tag{219}$$

with certain piecewise analytic functions $r_\pm$, the precise form of which is of no interest here. Using these functions $r_\pm$, we also introduce

$$\tilde{\mathcal{D}}_{\text{in}} = U_{r_- - 2, r_+}. \tag{220}$$

The domain

$$U = \tilde{\mathcal{D}}_{\text{in}} \setminus \tilde{\mathcal{D}}_{\text{in}}^{**} = U_{r_- - 2, r_-} \tag{221}$$

will play the role of a boundary layer: since $U \subset \tilde{\mathcal{D}}_{\text{out}} \cap \tilde{\mathcal{D}}_{\text{in}} \subset \mathcal{D}_{in}^u(R_N)$, the function ϕ^u is holomorphic in a neighborhood of the closure of $U = U_{r_- - 2, r_-}$, while $\phi^{u,N}$ is holomorphic in a neighborhood of $\tilde{\mathcal{D}}_{\text{in}} = U_{r_- - 2, r_+}$, and the difference equation (93) will allow us to follow the continuation of ϕ^u in $\tilde{\mathcal{D}}_{\text{in}}$ (see Figures 7 and 8).

The starting point for the matching method is

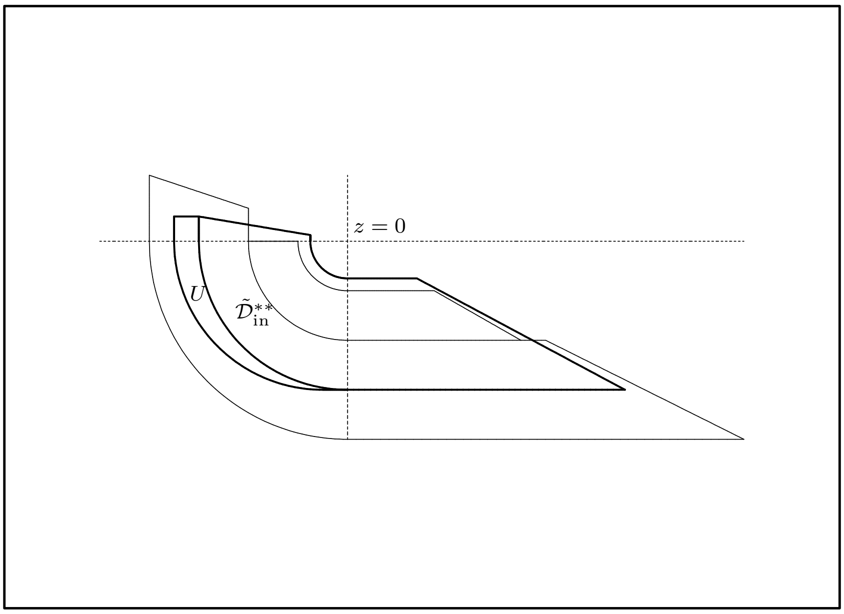

FIGURE 8. The domain $\tilde{\mathcal{D}}_{\text{in}} = \tilde{\mathcal{D}}^{**}_{\text{in}} \cup U$ superimposed over $\tilde{\mathcal{D}}_{\text{out}} \cup \tilde{\mathcal{D}}^{*}_{\text{in}}$.

Lemma 8.2. *Let ψ^* denote the restriction of the function $\phi^u - \phi^{u,N}$ to the boundary layer U. Then*

$$|\psi^*(z)| = O\big(\big(\tfrac{h}{\delta}\big)^2 A_\delta^N\big), \qquad |\frac{d}{dz}\psi^*(z)| = O\big(\big(\tfrac{h}{\delta}\big)^3 A_\delta^N\big), \qquad z \in U.$$

Moreover, ψ^* admits a continuation which is holomorphic in a neighborhood of the closure of U and which satisfies the nonlinear difference equation

$$\psi(z+1) + \psi(z-1) = \mathcal{F} \circ (\phi^{u,N} + \psi)(z) - \big(\phi^{u,N}(z+1) + \phi^{u,N}(z-1)\big) \tag{222}$$

in a neighborhood of the curve $\big\{z \in \mathbb{C} \mid \operatorname{Re} z = r_-(\operatorname{Im} z) - 1\big\}$.

Notice that equation (222) makes sense for $z \in U_{r_- -1, r_+ -1}$ provided that the unknown function ψ has a sufficiently small modulus in this domain (so that $\mathcal{F} \circ (\phi^{u,N} + \psi)$ be defined) and is defined in $\tilde{\mathcal{D}}_{\text{in}} = U_{r_- -2, r_+}$. The matching method will consist in proving that there is a unique such solution whose restriction to U is ψ^*; this function will necessarily be analytic and it will provide the desired continuation of Ψ.

Proof of Lemma 8.2. Equation (222) is just a rephrasing with the unknown $\psi = \phi - \phi^{u,N}$ of the invariance equation (93), which is indeed satisfied by ϕ^u.

By (69) and (80), for $z \in \tilde{\mathcal{D}}_{\text{out}}$,

$$|\phi^u(z) - \xi^{N,\text{out}}(i\pi/2 + hz)| \le O\big(\varepsilon\big(\tfrac{h}{\delta}\big)^{2N+3}\big) \tag{223}$$

and $\big(\frac{h}{\delta}\big)^2 A_\delta^N = \big(\frac{h}{\delta}\big)^{2N+3} + h\delta^{2N+1}$. Hence, to prove the claim we only need to bound $|\xi^{N,\text{out}}(i\pi/2 + hz) - \phi^{u,N}(z)|$.

On the one hand, by Theorem 2.14, $\phi_n^u \sim \tilde{\phi}_n$, where $\tilde{\phi}_n$ is given in (90). Hence

$$\Big|\phi_n^u(z) - \sum_{l=0}^{N} \frac{B_{\ell,n}}{z^{2(\ell-n)+1}}\Big| \leq O\Big(\frac{1}{z^{2(N-n)+3}}\Big),$$

where $B_{\ell,n}$, defined in (91), are the coefficients of the asymptotic expansion of ϕ_n^u. Consequently,

$$\Big|\phi^{u,N}(z) - \sum_{n=0}^{N} h^{2N} \sum_{l=0}^{N} \frac{B_{\ell,n}}{z^{2(\ell-n)+1}}\Big| \leq O\Big(\frac{1}{z^{2N+3}}\Big). \tag{224}$$

On the other hand, we recall that, by Proposition 2.10 and formula (88), we have that

$$\Big|\phi_{k,n}^N(z) - \sum_{m=0}^{N} \frac{A_{k,n,m}}{z^{2(m+k-n)+1}}\Big| \leq O\Big(\frac{1}{z^{2(N+k-n)+3}}\Big). \tag{225}$$

In particular,

$$|\phi_{k,n}^N(z)| \leq O\Big(\frac{1}{z^{2(k-n)+1}}\Big). \tag{226}$$

Hence, by (79) and (84), and using (226) with $n = N+1$, we have that

$$\Big|\xi^{N,\mathrm{out}}(i\pi/2 + hz) - \sum_{n=0}^{N} h^{2n} \sum_{k=0}^{N} \varepsilon^k \phi_{k,n}^N(z)\Big| \leq O(h^{2N+2} z^{2N+1}).$$

Now, using this last inequality and (225), we get

$$\Big|\xi^{N,\mathrm{out}}(i\pi/2+hz) - \sum_{n=0}^{N} h^{2n} \sum_{k=0}^{N} \varepsilon^k \sum_{m=0}^{N} \frac{A_{k,n,m}}{z^{2(m+k-n)+1}}\Big| \leq O(h^{2N+2} z^{2N+1}) + O\Big(\frac{1}{z^{2N+3}}\Big). \tag{227}$$

Finally, by (91), we have

$$\Big|\sum_{n=0}^{N} h^{2n} \sum_{k=0}^{N} \varepsilon^k \sum_{m=0}^{N} \frac{A_{k,n,m}}{z^{2(m+k-n)+1}} - \sum_{n=0}^{N} h^{2n} \sum_{\ell=0}^{N} \frac{B_{\ell,n}}{z^{2(\ell-n)+1}}\Big| \leq O\Big(\frac{1}{z^{2N+3}}\Big). \tag{228}$$

Hence, by (224), (227) and (228), we obtain

$$|\xi^{N,\mathrm{out}}(i\pi/2 + hz) - \phi^{u,N}(z)| \leq O(h^{2N+2} z^{2N+1}) + O\Big(\frac{1}{z^{2N+3}}\Big).$$

The bound for $|\psi^*|$ follows since $z \in \tilde{\mathcal{D}}_{\mathrm{out}} \Rightarrow \delta/h \leq |z| \leq 3\sec(\beta/2)\,\delta/h$. Moreover, $z \in U \Rightarrow D\big(z, \sin(\beta/4)|z|\big) \subset \tilde{\mathcal{D}}_{\mathrm{out}}$, hence the Cauchy inequalities yield the claimed bounds for $|\frac{d}{dz}\psi^*|$. □

8.2. **Definition of the operators involved in the matching method.** Let us introduce the intermediary set

$$\tilde{\mathcal{D}}'_{\text{in}} = U_{r_- -1, r_+ -1}. \tag{229}$$

As already mentioned, the matching method will rely on considering equation (222), to be satisfied in $\tilde{\mathcal{D}}'_{\text{in}}$ by an unknown ψ defined in $\tilde{\mathcal{D}}_{\text{in}}$, with boundary condition

$$\psi_{|U} = \psi^* \tag{230}$$

(forgetting for a while that the function ψ^* already has a continuation to a neighborhood of U).

Equation (222) can be written

$$\mathcal{L}_{\text{in}}(\psi) = \mathcal{H}_{\text{in}}(\psi), \tag{231}$$

with a linear map

$$\mathcal{L}_{\text{in}}(\psi)(z) = \psi(z+1) + \psi(z-1) - \partial_y \mathcal{F}(\phi_0^u, 0, \varepsilon)\psi(z) \tag{232}$$

(where ϕ_0^u is the solution of the first inner equation (96) given by Theorem 2.14) and a functional

$$\mathcal{H}_{\text{in}}(\psi) = G_{\text{in},N} + \ell_{\text{in}}(\psi) + \mathcal{N}_{\text{in}}(\psi), \tag{233}$$

where

$$G_{\text{in},N}(z) = \mathcal{F}(\phi^{u,N}(z), h, \varepsilon) - \phi^{u,N}(z+1) - \phi^{u,N}(z-1), \tag{234}$$

$$\ell_{\text{in}}(\psi) = (\partial_y \mathcal{F}(\phi^{u,N}, h, \varepsilon) - \partial_y \mathcal{F}(\phi_0^u, 0, \varepsilon))\psi, \tag{235}$$

$$\mathcal{N}_{\text{in}}(\psi) = \mathcal{F}(\phi^{u,N} + \psi, h, \varepsilon) - \mathcal{F}(\phi^{u,N}, h, \varepsilon) - \partial_y \mathcal{F}(\phi^{u,N}, h, \varepsilon)\psi. \tag{236}$$

We shall have to solve equations of the form $\mathcal{L}_{\text{in}}(\psi) = \Phi$, with a given function Φ defined in $\tilde{\mathcal{D}}'_{\text{in}}$. As for the homogeneous equation $\mathcal{L}_{\text{in}}(\psi) = 0$ in $\tilde{\mathcal{D}}'_{\text{in}}$, we already know, by Proposition 2.16, a fundamental set of solutions $\{\psi_1^u, \psi_2^u\}$ defined in $\tilde{\mathcal{D}}_{\text{in}}$ with Wronskian 1. We can thus apply Lemma 4.4 and get

Lemma 8.3. *Let*

$$\psi_h = c_1 \psi_1^u + c_2 \psi_2^u, \tag{237}$$

where c_1 and c_2 are the unique 1-periodic functions whose restrictions to $U' = U_{r_- -2, r_- -1}$ are given by

$$c_{1|U'} = W_1(\psi^*, \psi_2^u)_{|U'}, \quad c_{2|U'} = W_1(\psi_1^u, \psi^*)_{|U'}.$$

Then, for any function Φ defined in $\tilde{\mathcal{D}}'_{\text{in}}$, there is a unique function ψ defined in $\tilde{\mathcal{D}}_{\text{in}}$ which satisfies the linear difference equation $\mathcal{L}_{\text{in}}(\psi) = \Phi$ in $\tilde{\mathcal{D}}'_{\text{in}}$ and the boundary condition (230)*. This solution is*

$$\psi = \psi_h + \mathcal{G}_{\text{in}}(\Phi),$$

where

$$\mathcal{G}_{\rm in}(\Phi)(z) = \sum_{k=1}^{k^*(z)} \big(\psi_1^u(z-k)\psi_2^u(z) - \psi_1^u(z)\psi_2^u(z-k)\big)\Phi(z-k), \qquad z \in \tilde{\mathcal{D}}_{\rm in},$$

with $k^*(z) = \lfloor \operatorname{Re} z - r_-(\operatorname{Im} z) + 1 \rfloor \geq -1$ *and with the convention that* $\mathcal{G}_{\rm in}(\Phi)(z) = 0$ *when* $k^*(z) = -1$ *or* 0 *(i.e. when* $z \in U$*).*

Observe that the function ψ_h does not depend on Φ: it is the unique solution of the homogeneous equation which satisfies the boundary condition (230).

As a result, the problem of finding ψ defined in $\tilde{\mathcal{D}}_{\rm in}$ satisfying equation (222) in $\tilde{\mathcal{D}}'_{\rm in}$ and the boundary condition (230) is equivalent to the fixed point problem

$$\psi = \psi_h + \mathcal{G}_{\rm in} \circ \mathcal{H}_{\rm in}(\psi). \tag{238}$$

We shall prove that the right hand side of this equation defines a contraction of a certain Banach space $\mathcal{Y}_2$.

8.3. **Banach spaces and technical lemmas.** For $m \in \mathbb{Z}$, we define the spaces

$$\begin{aligned} \mathcal{Y}_m &= \{\phi\colon \tilde{\mathcal{D}}_{\rm in} \to \mathbb{C} \text{ such that } \|\phi\|_m < \infty\}, \\ \mathcal{Y}'_m &= \{\phi\colon \tilde{\mathcal{D}}'_{\rm in} \to \mathbb{C} \text{ such that } \|\phi\|_m < \infty\}, \end{aligned} \tag{239}$$

where

$$\|\phi\|_m = \sup |z^m \phi(z)|. \tag{240}$$

With this norm, they are Banach spaces. Since $\tilde{\mathcal{D}}'_{\rm in} \subset \tilde{\mathcal{D}}_{\rm in}$ and

$$z \in \tilde{\mathcal{D}}_{\rm in} \;\Rightarrow\; 1 < |z| \leq \max\{2 + 2\delta/h, 2\sec(3\beta/4)\,\delta/h\} \leq c\delta/h$$

with $c = \max\{4, 2\sec(3\beta/4)\}$, we have the following lemma (analogous to Lemma 5.2), whose proof is straightforward.

Lemma 8.4. *Let* ζ, ζ_1, ζ_2 *be functions defined on* $\tilde{\mathcal{D}}_{\rm in}$*, resp.* $\tilde{\mathcal{D}}'_{\rm in}$*, and* $m, m_1, m_2 \in \mathbb{Z}$*. Then*

$$\|\zeta_1\zeta_2\|_{m_1+m_2} \leq \|\zeta_1\|_{m_1}\|\zeta_2\|_{m_2},$$

and

$$m_1 \leq m_2 \;\Rightarrow\; \|\zeta\|_{m_1} \leq \|\zeta\|_{m_2} \leq (c\delta/h)^{m_2-m_1}\|\zeta\|_{m_1}.$$

If moreover $\|\zeta\|_m = O(h^m)$ *and* $g(y) \in y^k\mathbb{C}\{y\}$*, then for* h *small enough the functions* $g \circ \zeta$ *is well defined on* $\tilde{\mathcal{D}}_{\rm in}$*, resp.* $\tilde{\mathcal{D}}'_{\rm in}$*, with*

$$\|g \circ \zeta\|_{km} = O(h^{km}).$$

The linear map $\mathcal{G}_{\rm in}$ and the function ψ_h defined in Lemma 8.3 satisfy the following

Lemma 8.5. *For* $m \in \mathbb{Z}$*,* $\mathcal{G}_{\rm in}$ *induces a bounded linear map* $\mathcal{G}_{\rm in}\colon \mathcal{Y}'_m \to \mathcal{Y}_2$*, with*

$$m \leq 3 \;\Rightarrow\; \|\mathcal{G}_{\rm in}\| = O\big((\delta/h)^{4-m}\big), \qquad m \geq 5 \;\Rightarrow\; \|\mathcal{G}_{\rm in}\| = O(1).$$

On the other hand, $\psi_h \in \mathcal{Y}_2$ *and* $\|\psi_h\|_2 = O(A_\delta^N)$*.*

(One could deal with the case $m = 4$ as well: an adaptation of the proof shows that $\|\mathcal{G}_{\rm in}\| = O(\ln(\delta/h))$ in this case.)

Proof. By Proposition 2.16, $\psi_1^u \in \mathcal{Y}_2$ and $\psi_2^u \in \mathcal{Y}_{-3}$ with constants $C_1, C_2 > 0$ such that $\|\psi_1\|_2 \le C_1$ and $\|\psi_2\|_{-3} \le C_2$. Thus, for $\Phi \in \mathcal{Y}'_m$ and $z \in \tilde{\mathcal{D}}_{\rm in}$,

$$|z^2 \mathcal{G}_{\rm in}(\Phi)(z)| \le C_1 C_2 \|\Phi\|_m \sum_{k=1}^{k^*(z)} a_k(z), \qquad a_k(z) = |z-k|^{-m-2}|z|^5 + |z-k|^{3-m}.$$

We take into account that, for $1 \le k \le k^*(z)$,

$$|z-k| \ge |z|\sin(3\beta/4), \quad 1 \le |z|, |z-k| \le c\delta/h, \quad k^*(z) \le c'\delta/h$$

with $c' = 2 + 2\cot(3\beta/4)$ (the first inequality is obtained by distinguishing the case $\operatorname{Re} z \le 0$, for which it is obvious, from the case $\operatorname{Re} z > 0$, for which $|z-k| \ge |\operatorname{Im} z| \ge |z|\sin(3\beta/4)$) and we distinguish three cases:

(1) If $m \le -3$, then $-m-2 > 0$ and $3-m \ge 0$, hence $a_k(z) \le 2(c\delta/h)^{3-m}$,
$$\|\mathcal{G}_{\rm in}(\Phi)\|_2 \le 2c^{3-m}c' C_1 C_2 (\delta/h)^{4-m}\|\Phi\|_m.$$

(2) If $-2 \le m \le 3$, then $-m-2 \le 0$ and $3-m \ge 0$ imply
$$a_k(z) \le |z|^{3-m}(\sin(3\beta/4))^{-m-2} + |z-k|^{3-m} \le c_m(\delta/h)^{3-m}$$
with $c_m = c^{3-m}\big(1 + (\sin(3\beta/4))^{-m-2}\big)$, whence
$$\|\mathcal{G}_{\rm in}(\Phi)\|_2 \le c_m c' C_1 C_2 (\delta/h)^{4-m}\|\Phi\|_m.$$

(3) If $m \ge 5$, then $-m-2 < 0$ and $3-m \le -2$ imply $a_k(z) \le |z-k|^{3-m}\big(1 + (\sec(3\beta/4))^5\big)$, whence
$$\|\mathcal{G}_{\rm in}(\Phi)\|_2 \le \big(1 + (\sec(3\beta/4))^5\big) C_1 C_2 \|\Phi\|_m \sum_{k=1}^{k^*(z)} |z-k|^{3-m}$$
and we can write $\operatorname{Re} z = \rho + K$ with $-\frac{1}{2} < \rho \le \frac{1}{2}$ and $K \in \mathbb{Z}$, $|z-k| \ge |\rho + K - k| \ge |K-k| - \frac{1}{2}$ and
$$\sum_{k=1}^{k^*(z)} |z-k|^{3-m} \le 1 + \sum_{k\in\mathbb{N}^*,\ k\ne K} \big(|K-k| - \tfrac{1}{2}\big)^{3-m} < \infty.$$

As for ψ_h, we use the formulas (237) and the bounds on ψ^* and $\frac{d}{dz}\psi^*$ in the boundary layer U given in Lemma 8.2: for $z \in U'$,

$$|\psi^*(z)| \le C^* A_\delta^N (\delta/h)^{-2}, \quad |\Delta_1 \psi^*(z)| \le C^* A_\delta^N (\delta/h)^{-3}$$

and (using Proposition 2.16, Cauchy inequalities and the fact that $|z|, |z+1| \in [\delta/h, 3\delta/h]$)

$$\begin{aligned} |\psi_1^u(z)| &\le C_1(\delta/h)^{-2}, \qquad |\Delta_1\psi_1^u(z)| \le C_1(\delta/h)^{-3} \\ |\psi_2^u(z)| &\le C_2(\delta/h)^{3}, \qquad |\Delta_1\psi_2^u(z)| \le C_2(\delta/h)^{2}, \end{aligned} \tag{241}$$

with suitable constants C_1, C_2, whence

$$\|c_1\|_0 \le C^* C_2 A_\delta^N, \quad \|c_2\|_0 \le C^* C_1 A_\delta^N (\delta/h)^{-5},$$

by periodicity of c_1 and c_2. Now, $\|\psi_1^u\|_2 \le C_1$ and, in view of Lemma 8.4, $\|\psi_2^u\|_2 \le c^5 C_2 (\delta/h)^5$ thus $\|\psi_h\|_2 \le (1+c^5) C^* C_1 C_2 A_\delta^N$. □

8.4. **Solution of the fixed point equation** (238) **in** $\tilde{\mathcal{D}}_{\text{in}}$**.** We shall look for a solution of (238) with small enough norm in $\mathcal{Y}_2$. For this, we shall study separately the three terms which form $\mathcal{G}_{\text{in}} \circ \mathcal{H}_{\text{in}}(\psi) = \mathcal{G}_{\text{in}}(G_{\text{in},N}) + \mathcal{G}_{\text{in}} \circ \ell_{\text{in}}(\psi) + \mathcal{G}_{\text{in}} \circ \mathcal{N}_{\text{in}}(\psi)$.

Lemma 8.6. *The function* $G_{\text{in},N}$ *defined in* (234) *belongs to* $\mathcal{Y}'_{-2N+1}$ *and satisfies* $\|G_{\text{in},N}\|_{-2N+1} = O(h^{2N+2})$. *As a consequence*

$$\mathcal{G}_{\text{in}}(G_{\text{in},N}) \in \mathcal{Y}_2, \quad \|\mathcal{G}_{\text{in}}(G_{\text{in},N})\|_2 = O(A_\delta^N). \tag{242}$$

Proof. Since $\phi^{u,N} = \sum_{n=0}^N h^{2n}\phi_n^u$ and the functions ϕ_n^u are solutions of the first and secondary inner equations up to order N, we have that

$$\frac{\partial^k G_{\text{in},N}}{\partial (h^2)^k}(z)_{h=0} = 0, \qquad k = 0, \dots, N.$$

Hence, we can bound $G_{\text{in},N}(z)$ by $\frac{h^{2N+2}}{(N+1)!}\sup_{\bar{h}\in[0,h]} |\frac{\partial^{N+1} G_{\text{in},N}}{\partial (h^2)^{N+1}}(z)|$. We remark that $\phi^{u,N}$ is a polynomial of degree $\le N$ in h^2. Hence,

$$\frac{\partial^{N+1} G_{\text{in},N}}{\partial (h^2)^{N+1}} = \frac{\partial^{N+1}}{\partial (h^2)^{N+1}} \mathcal{F}(\phi^{u,N}, h, \varepsilon) = \frac{\partial^{N+1}}{\partial (h^2)^{N+1}} \sum_{\ell \ge 0} h^{2\ell} \mathcal{F}_\ell(\phi^{u,N}, \varepsilon),$$

where the functions $\mathcal{F}_\ell$ where introduced in (94).

In order to bound this derivative, first we observe that, by Theorem 2.14 and by formula (90), $\frac{\partial^i}{\partial (h^2)^i}\left(\phi^{u,N}\right) = O(\phi_i^u) \in \mathcal{Y}_{-2i+1}$, and that $\|\frac{\partial^i}{\partial (h^2)^i}\left(\phi^{u,N}\right)\|_{-2i+1}$ is bounded independently of h.

We remark that the functions $\mathcal{F}_\ell$ verify:

$$\frac{\partial^j}{\partial z^j}\mathcal{F}_\ell(z,\varepsilon) = O(1), \quad \text{if } j \text{ is odd}, \tag{243}$$

$$\frac{\partial^j}{\partial z^j}\mathcal{F}_\ell(z,\varepsilon) = O(z), \quad \text{if } j \text{ is even}. \tag{244}$$

With all these bounds, using the Faa-di-Bruno formula,

$$\begin{aligned} |\frac{\partial^k}{\partial (h^2)^k}(\mathcal{F}_\ell \circ \phi^{u,N})| &= |\sum_{j=1}^k \frac{\partial^k \mathcal{F}_\ell}{\partial z^k} \circ \phi^{u,N} \sum_{\substack{i_1+\dots+i_j=k \\ 1\le i_1,\dots,i_j \le k}} \sigma^k_{i_1,\dots,i_j} \frac{\partial^{i_1}\phi^{u,N}}{\partial (h^2)^{i_1}} \cdots \frac{\partial^{i_j}\phi^{u,N}}{\partial (h^2)^{i_j}}| \\ &\le \sum_{j=1}^k |\frac{\partial^k \mathcal{F}_\ell}{\partial z^k} \circ \phi^{u,N}||z|^{2k-j} \end{aligned}$$

taking into account that, since $\phi^{u,N} = O(z^{-1})$, the largest term corresponds to $j = 1$, we have, if $k \leq N$

$$|\frac{\partial^k}{\partial(h^2)^k}(\mathcal{F}_\ell \circ \phi^{u,N}) \leq |z|^{2k-1}.$$

In the case $k = N+1$, one needs to take into account that $\partial^{N+1}\phi^{u,N}/\partial(h^2)^{N+1} = 0$, so, the term corresponding to $j = 1$ is zero and the largest term corresponds to $j = 2$, obtaining

$$|\frac{\partial^k}{\partial(h^2)^k}(\mathcal{F}_\ell \circ \phi^{u,N}) \leq |z|^{2N-1}.$$

Once we have bound the derivatives of the functions $\mathcal{F}_\ell \circ \phi^{u,N}$, one can apply the Leibnitz rule obtaining

$$\begin{aligned}
&\Big|\frac{\partial^{N+1}}{\partial(h^2)^{N+1}}\left(h^{2\ell}\mathcal{F}_\ell \circ \phi^{u,N}\right)\Big| \\
&\qquad = \Big|\sum_{k=0}^{N+1}\binom{N+1}{k}\frac{\partial^k}{\partial(h^2)^k}\left(\mathcal{F}_\ell \circ \phi^{u,N}\right)\frac{\partial^{N+1-k}}{\partial(h^2)^{N+1-k}}\left(h^{2\ell}\right)\Big| \\
&\qquad \leq \begin{cases} |z|^{2N-1} & \text{if } \ell \leq N+1 \\ |z|^{2N-1}h^{2(\ell-N-1)} & \text{if } \ell > N+1 \end{cases}
\end{aligned}$$

which gives the final bound on $G_{\mathrm{in},N}$.

Then, by Lemma 8.5 with $m = -2N+1 < 3$, we get $\mathcal{G}_{\mathrm{in}}(G_{\mathrm{in},N}) \in \mathcal{Y}_2$ and $\|\mathcal{G}_{\mathrm{in}}(G_{\mathrm{in},N})\|_2 \leq O(\delta^{2N+3}/h) \leq O(A_\delta^N)$. □

Lemma 8.7. *The operator ℓ_{in} defined in* (235) *induces a bounded linear map from $\mathcal{Y}_2$ to $\mathcal{Y}_2$ with $\|\ell_{\mathrm{in}}\| = O(h^2)$. As a consequence, $\mathcal{G}_{\mathrm{in}} \circ \ell_{\mathrm{in}}$ induces a bounded linear operator of $\mathcal{Y}_2$ with $\|\mathcal{G}_{\mathrm{in}} \circ \ell_{\mathrm{in}}\| = O(\delta^2)$.*

Proof. We write ℓ_{in} as

$$\ell_{\mathrm{in}}(\psi) = (\partial_y\mathcal{F}(\phi^{u,N}, h, \varepsilon) - \partial_y\mathcal{F}(\phi^{u,N}, 0, \varepsilon) - \partial_y\mathcal{F}(\phi^{u,N}, 0, \varepsilon) - \partial_y\mathcal{F}(\phi_0^u, 0, \varepsilon))\psi.$$

Taking into account that, by Theorem 2.14, $|\phi_0^u(z)| \leq \frac{C}{|z|}$, and $|\phi^{u,N}(z) - \phi_0^u(z)| \leq O(h^2|z|)$, and using again formula (94) and inequalities (243) and (244), we have that

$$|\ell_{\mathrm{in}}(\psi)(z)| \leq C_N h^2|\psi(z)|, \qquad z \in \tilde{\mathcal{D}}_{\mathrm{in}}$$

with a suitable constant $C_N > 0$.

Then, Lemma 8.5 with $m = 2$ shows that the operator $\mathcal{G}_{\mathrm{in}} \circ \ell_{\mathrm{in}} \colon \mathcal{Y}_2 \to \mathcal{Y}_2$ has norm $O(\delta^2)$. □

Lemma 8.8. *There exists $\lambda_N > 0$ such that, for any $0 < \lambda < \lambda_N$, the functional $\mathcal{N}_{\rm in}$ of* (236) *induces a Lipschitz map from the closed ball $B_2(\lambda) = \{\psi \in \mathcal{Y}_2 \mid \|\psi\|_2 \le \lambda\}$ to $\mathcal{Y}_5$, with Lipschitz constant $O(\lambda)$. As a consequence, $\mathcal{G}_{\rm in} \circ \mathcal{N}_{\rm in}$ induces a Lipschitz map from $B_2(\lambda)$ to $\mathcal{Y}_2$ with Lipschitz constant $O(\lambda)$.*

Proof. To bound the non-linear operator $\mathcal{N}_{\rm in}$, let be $\psi \in B_2(\lambda) \subset \mathcal{Y}_2$. We write $\mathcal{N}_{\rm in}$ as

$$\mathcal{N}_{\rm in}(\psi) = \int_0^1 \big(\partial_y \mathcal{F}(\phi^{u,N} + t\psi, h, \varepsilon) - \partial_y \mathcal{F}(\phi^{u,N}, h, \varepsilon)\big) dt\, \psi.$$

Since $\|\phi^{u,N}\|_1 \le O(1)$ and $\|\psi\|_2 \le \lambda$, using formula (94) and inequalities (243) and (244), we have that $\|\partial_y \mathcal{F}(\phi^{u,N} + t\psi) - \partial_y \mathcal{F}(\phi^{u,N})\|_3 = O(\lambda)$. Hence, if $\psi \in B_2(\lambda)$,

$$\begin{aligned}\|\mathcal{N}_{\rm in}(\psi)\|_5 &\le \Big\| \int_0^1 \big(\partial_y \mathcal{F}(\phi^{u,N} + t\psi, h, \varepsilon) - \partial_y \mathcal{F}(\phi^{u,N}, h, \varepsilon)\big) dt \Big\|_3 \|\psi\|_2 \\ &\le O(\lambda)\|\psi\|_2 \le O(\lambda^2).\end{aligned}$$

We finally compute the Lipschitz constant of the map $\mathcal{N}_{\rm in}$. For $\psi, \tilde{\psi} \in B_2(\lambda)$,

$$\begin{aligned}\mathcal{N}_{\rm in}(\psi) - \mathcal{N}_{\rm in}(\tilde{\psi}) &= \int_0^1 \big(\partial_y \mathcal{F}(\phi^{u,N} + t\psi, h, \varepsilon) - \partial_y \mathcal{F}(\phi^{u,N} + t\tilde{\psi}, h, \varepsilon)\big)\psi dt \\ &\quad + \int_0^1 \big(\partial_y \mathcal{F}(\phi^{u,N} + t\tilde{\psi}, h, \varepsilon) - \partial_y \mathcal{F}(\phi^{u,N}, h, \varepsilon)\big)(\psi - \tilde{\psi}) dt\end{aligned}$$

Since $D^2\mathcal{F}(y) = O(y)$ and $\|\phi^{u,N}\|_1 = O(1)$, by Lemma 5.2, we have that for ψ and $\tilde{\psi}$ in $B_2(\lambda)$,

$$\|D\mathcal{F}(\phi^{u,N} + t\psi) - D\mathcal{F}(\phi^{u,N} + t\tilde{\psi})\|_3 = O(1)\|\psi - \tilde{\psi}\|_2$$

Then

$$\begin{aligned}&\Big\| \int_0^1 \big(D\mathcal{F}(\phi^{u,N} + t\psi) - D\mathcal{F}(\phi^{u,N} + t\tilde{\psi})\big)\psi dt \Big\|_5 \\ &\qquad\qquad = O(1)\|\psi\|_2\|\psi - \tilde{\psi}\|_2 = O(\lambda)\|\psi - \tilde{\psi}\|_2.\end{aligned}$$

With the same argument, we obtain the same bound for the last integral.

Then Lemma 8.5 with $m = 5$ shows that the map $\mathcal{G}_{\rm in} \circ \mathcal{N}_{\rm in} \colon B_2(\lambda) \to \mathcal{Y}_2$ has Lipschitz constant $O(\lambda)$. □

Collecting the results of Lemmas 8.5–8.8, we arrive at the solution of our fixed point problem:

Lemma 8.9. *If h and A_δ^N are small enough, then equation* (238) *admits a solution ψ in $\tilde{\mathcal{D}}_{\rm in}$, with*

$$|\psi(z)| = O(A_\delta^N |z|^{-2}), \qquad z \in \tilde{\mathcal{D}}_{\rm in}.$$

Proof. We first observe that $h/\delta < A^{\frac{1}{2N+1}}$ by definition of A_δ^N, hence $\delta < (A_\delta^N)^{\alpha_N}$ with $\alpha_N = \frac{1}{2N+2}(1+\frac{1}{2N+1})$ (because $A_\delta^N > \delta^{2N+2}\cdot\frac{\delta}{h}$) and δ can be made arbitrarily small by requiring A_δ^N to be small enough.

The map $\psi \mapsto \psi_h + \mathcal{G}_{\rm in} \circ \mathcal{H}_{\rm in}(\psi)$ which appears in the right-hand side of (238) can be written $\mathcal{E}(\psi)$, with

$$\mathcal{E}(0) = \psi_h + \mathcal{G}_{\rm in}(G_{{\rm in},N}) \in \mathcal{Y}_2, \quad \|\mathcal{E}(0)\|_2 \le C_N A_\delta^N$$

(using Lemmas 8.5 and 8.6) and $\mathcal{E}$ induces a Lipschitz map from $B_2(\lambda)$ to $\mathcal{Y}_2$ with Lipschitz constant $\le C'_N(\delta^2+\lambda)$ for any positive $\lambda < \lambda_N$ (using Lemmas 8.7 and 8.8). We thus impose

$$C_N C'_N A_\delta^N \le 1/8, \quad C'_N\delta^2 \le 1/4$$

and choose $\lambda = 2C_N A_\delta^N$, so that the Lipschitz constant of $\mathcal{E}$ on $B_2(\lambda)$ is $\le 1/2$,

$$\|\psi\|_2 \le \lambda \;\Rightarrow\; \|\mathcal{E}(\psi)\|_2 \le \tfrac{1}{2}\|\psi\|_2 + \|\mathcal{E}(0)\|_2 \le \lambda$$

and we get a unique fixed point $\psi = \sum_{k\in\mathbb{N}}\big(\mathcal{E}^{k+1}(0) - \mathcal{E}^k(0)\big)$, which satisfies $\|\psi\|_2 \le 2\|\mathcal{E}(0)\|_2$ (because $\|\mathcal{E}^{k+1}(0) - \mathcal{E}^k(0)\|_2 \le 2^{-k}\|\mathcal{E}(0)\|_2$ for each k). □

8.5. **Proof of Proposition 8.1 and end of the proof of Theorem 2.18.** The function ψ that we just found in Lemma 8.9 is defined in $\tilde{\mathcal{D}}_{\rm in} = U_{r_--2,r_+}$ and, by definition of $\psi_h, \mathcal{G}_{\rm in}, \mathcal{H}_{\rm in}$, it satisfies the boundary condition $\psi_{|U} = \psi^*$ with $U = U_{r_--2,r_-}$ and equation (222) in $\tilde{\mathcal{D}}'_{\rm in}$, which can be rewritten

(245)

$$\psi(z) = -\psi(z-2) + \mathcal{F}\circ(\phi^{u,N}+\psi)(z-1) - \big(\phi^{u,N}(z) + \phi^{u,N}(z-2)\big), \qquad z \in U_{r_-,r_+}$$

(in particular, it follows from our computations that, for all $z \in U_{r_-,r_+}$, $|\psi(z-1)|$ is so small that $|(\phi^{u,N}+\psi)(z-1)| < y_0$).

Knowing that

- the function $\phi^{u,N}$ is analytic in U_{r_--2,r_+},
- the function $\mathcal{F}(y)$ is analytic for $|y| < y_0$,
- the function ψ^* admits a continuation which is analytic in a neighborhood of the closure of U_{r_--2,r_-} and satisfies equation (245) for z in a neighborhood of the curve $\{\operatorname{Re} z = r_-(\operatorname{Im} z)\}$ (as claimed in Lemma 8.2),

we deduce that ψ admits a continuation which is analytic in a neighborhood of U_{r_--2,r_+}.

Indeed, denoting by $S(\psi)$ the right-hand side of (245), we can argue by induction and suppose that, for a $k \in \mathbb{N}$, the restriction $\psi_{|U_{r_--2,r_-+k}\cap U_{r_--2,r_+}}$ admits a continuation ψ_k which is analytic in an open set

$$W_k \supset \big\{r_-(\operatorname{Im} z) - 2 \le \operatorname{Re} z \le \min\{r_-(\operatorname{Im} z) + k, r_+(\operatorname{Im} z)\}\big\}$$

and satisfies $\psi_k = S(\psi_k)$ in an open set

$$V_k \supset \big\{r_-(\operatorname{Im} z) \le \operatorname{Re} z \le \min\{r_-(\operatorname{Im} z) + k, r_+(\operatorname{Im} z)\}\big\}.$$

The function $\tilde{\psi}_k = S(\psi_k)$ is then analytic in an open set

$$\tilde{W}_k \supset \big\{r_-(\operatorname{Im} z) \le \operatorname{Re} z \le \min\{r_-(\operatorname{Im} z) + k + 1, r_+(\operatorname{Im} z)\}\big\}$$

(choosing $\tilde{W}_k$ small enough so that $|(\phi^{u,N} + \psi_k)(z-1)| < y_0$ there) and coincides with ψ_k in V_k; by gluing ψ_k and $\tilde{\psi}_k$ we obtain a function ψ_{k+1} analytic in the open set

$$W_k \cup \tilde{W}_k \supset \big\{r_-(\operatorname{Im} z) - 2 \le \operatorname{Re} z \le \min\{r_-(\operatorname{Im} z) + k + 1, r_+(\operatorname{Im} z)\}\big\}.$$

Moreover, the restrictions of ψ_{k+1} and ψ to $U_{r_-,r_-+k+1} \cap U_{r_-,r_+}$ coincide (because $\psi_{k+1} = S(\psi_k) = S(\psi) = \psi$ in this domain) and

$$S(\psi_{k+1})_{|\tilde{W}_k} = S(\psi_{k+1|W_k}) = S(\psi_k) = \psi_{k+1|\tilde{W}_k},$$

i.e. $S(\psi_{k+1}) = \psi_{k+1}$ on $V_{k+1} := \tilde{W}_k$, which yields the next step of the induction.

The claim on Ψ of Proposition 8.1 follows, since the functions ψ and $\Psi = \phi^u - \phi^{u,N}$ both coincide with ψ^* in the boundary layer $U = U_{r_--2,r_-}$ and $\tilde{\mathcal{D}}^*_{\rm in} \subset \tilde{\mathcal{D}}_{\rm in} = U_{r_--2,r_+}$.

The claim on $\frac{d}{dz}\Psi$ follows from applying Cauchy estimates to Ψ for $z \in \tilde{\mathcal{D}}^*_{\rm in}$ and the bounds of Ψ on $\tilde{\mathcal{D}}_{\rm in}$.

Moreover, the above arguments show that Ψ thus extended is also holomorphic in ε: in fact, $\Psi(z,\varepsilon)$ is holomorphic for $(z,\varepsilon) \in (\tilde{\mathcal{D}}_{\rm out} \cup \tilde{\mathcal{D}}^*_{\rm in}) \times \mathbb{D}(0,\varepsilon_0)$, with $|\Psi(z,\varepsilon)| \le C_N A_\delta^N |z|^{-2}$ in this domain. But $\Psi(z,0) = 0$, hence the Schwarz lemma yields

$$|\Psi(z,\varepsilon)| \le \Big(\max_{\tilde{\varepsilon}\in\mathbb{D}(0,\varepsilon_0)} |\Psi(z,\tilde{\varepsilon})|\Big)\frac{|\varepsilon|}{\varepsilon_0} \le \frac{C_N A_\delta^N |\varepsilon|}{\varepsilon_0 |z|^2},$$

which is equivalent to the desired inequality (113).

8.6. **Rephrasing of Theorem 2.20 in the inner variable. Proof in the case of η_1^u.** We keep using the same notations as in Sections 8.1–8.3 and define the functions

$$\Psi_1^u(z) = \eta_1^u(i\pi/2 + hz), \qquad \Psi_2^{u,i\pi/2}(z) = \eta_2^{u,i\pi/2}(i\pi/2 + hz), \tag{246}$$

which are holomorphic at least for $z \in \tilde{\mathcal{D}}_{\rm out}$. According to the first statement in Theorem 2.4, $\eta_1^u = \frac{d}{dt}\xi^u$, hence

$$\Psi_1^u = h^{-1}\frac{d\phi^u}{dz}, \tag{247}$$

while the function $\eta_2^{u,0}$ differs from $\eta_2^{u,i\pi/2}$ by $A\eta_1^u$. Proving Theorem 2.20 is thus equivalent to following the analytic continuation of Ψ_1^u and $\Psi_2^{u,i\pi/2}$ in $\tilde{\mathcal{D}}^*_{\rm in}$ (this will provide the analytic continuation for η_1^u and $\eta_2^{u,i\pi/2}$ in $D_h^{u,{\rm in}}(R_N)$, and thus for $\eta_2^{u,0}$ as well) and verifying that in this domain

$$|\Psi_1^u(z) - h^{-1}\psi_1^{u,N}(z)| \le \text{const } h^{-1}A_\delta^N|z|^{-3}, \quad |\Psi_2^{u,i\pi/2}(z) - h\psi_2^u(z)| \le \text{const } h^{-2}|z|^{-2}$$

(taking into account that $|\cosh(i\pi/2 + hz)| \ge \text{const } h|z|$ for $z \in \tilde{\mathcal{D}}^*_{\rm in}$).

The existence of the analytic continuation of Ψ_1^u in $\tilde{\mathcal{D}}_{\rm in}$ follows from formula (247) and Proposition 8.1. Moreover, since $\psi_1^{u,N} = \frac{d}{dz}\phi^{u,N}$,

$$|\Psi_1^u(z) - h^{-1}\psi_1^{u,N}(z)| = h^{-1}\left|\tfrac{d}{dz}(\phi^u - \phi^{u,N})(z)\right| \le \text{const}\,\frac{A_\delta}{h|z|^3}, \qquad z \in \tilde{\mathcal{D}}_{\rm in}^*,$$

still by Proposition 8.1. The case of Ψ_1^u is thus settled.

In the case of $\Psi_2^{u,i\pi/2}$, we can use a linear difference equation to find the analytic continuation: $\Psi_2^{u,i\pi/2}$ is holomorphic at least in a neighborhood of the closure of the boundary layer $U = U_{r_- -2, r_-}$ and, since $\eta_2^{u,i\pi/2}$ is solution of the linearized equation (71), it satisfies

$$\Psi(z+1) + \Psi(z-1) = G(z)\Psi(z), \qquad G(z) = \partial_y \mathcal{F}\big(\phi^u(z), h, \varepsilon\big), \tag{248}$$

for z in a neighborhood of the curve $\{\operatorname{Re} z = r_-(\operatorname{Im} z) - 1\}$. Since G is holomorphic in $\tilde{\mathcal{D}}_{\rm in} = U_{r_- -2, r_+}$ (by virtue of Proposition 8.1), we can define a continuation of $\Psi_2^{u,i\pi/2}$ which is holomorphic in $\tilde{\mathcal{D}}_{\rm in}$ by reasoning as in Remark 4.5 or Section 8.5.

Therefore, also $\Psi_2^{u,i\pi/2}$ is holomorphic in $\tilde{\mathcal{D}}_{\rm out} \cup \tilde{\mathcal{D}}_{\rm in}^*$ and what remains to be proved in Theorem 2.20 can be rewritten in the variable z as

Proposition 8.10. *The restriction of $\Psi_2^{u,i\pi/2}$ to $\tilde{\mathcal{D}}_{\rm in}^*$ satisfies $\Psi_2^u - h\psi_2^u \in \mathcal{Y}_2$ and*

$$\|\Psi_2^{u,i\pi/2} - h\psi_2^{u,i\pi/2}\|_2 = O(1/h^2)$$

provided that h and A_δ^N are small enough.

This will be proved in Section 8.7 along the lines of the proof of Proposition 8.1.

8.7. **End of the proof of Theorem 2.20: the case of $\eta_2^{u,i\pi/2}$.** All we need to do is to prove Proposition 8.10. Let us consider

$$\varphi = \Psi_2^{u,i\pi/2} - h\psi_2^u$$

as new unknown, so that equation (248) becomes

$$\varphi(z+1) + \varphi(z-1) = \partial_y\mathcal{F}(\phi^u(z), h, \varepsilon)(h\psi_2^u(z) + \varphi(z)) - \partial_y\mathcal{F}(\phi_0^u(z), 0, \varepsilon)h\psi_2^u(z)$$

which we write as

$$\mathcal{L}_{\rm in}(\varphi) = \mathcal{H}_{\rm in,1}(\varphi), \tag{249}$$

where

$$\mathcal{H}_{\rm in,1}(\varphi) = (\mathcal{M}_h(\phi^u) - \mathcal{M}_0(\phi_0^u))(\varphi + h\psi_2^u), \tag{250}$$

with the notation

$$\mathcal{M}_h(\phi)(z) = \partial_y\mathcal{F}(\phi, h, \varepsilon). \tag{251}$$

Denoting by φ^* the restriction of $\Psi_2^{u,i\pi/2} - h\psi_2^u$ to the boundary layer U, we can view the analytic continuation of $\Psi_2^{u,i\pi/2} - h\psi_2^u$ to $\tilde{\mathcal{D}}_{\mathrm{in}}$ as the unique solution of equation (249) satisfying the boundary condition

$$\varphi_{|U} = \varphi^*.$$

Equivalently, it can be obtained as the unique solution of the fixed point problem

$$\varphi = \mathcal{G}_{\mathrm{in}} \circ \mathcal{H}_{\mathrm{in},1}(\varphi) + \varphi_h,$$

where $\varphi_h = c_1\psi_1^u + c_2\psi_2^u$ in $\tilde{\mathcal{D}}_{\mathrm{in}}$ and the 1-periodic functions c_1 and c_2 are uniquely determined by their restriction to $U' = U_{r_- -2, r_- -1}$,

$$c_{1|U'} = W_1(\varphi^*, \psi_2^u)_{|U'}, \quad c_{2|U'} = W_1(\psi_1^u, \varphi^*)_{|U'}.$$

We shall see $\varphi \mapsto \mathcal{G}_{\mathrm{in}} \circ \mathcal{H}_{\mathrm{in},1}(\varphi) + \varphi_h$ as a map from $\mathcal{Y}_2$ to itself. To estimate $\|\varphi_h\|_2$, we begin by controlling $|\varphi^*|$.

Lemma 8.11. *The function φ^* satisfies*

$$|\varphi^*(z)| \le O(\frac{1}{\delta^2}), \qquad z \in U.$$

Proof. If $z \in U$, then $\delta/h \le |z| \le 3\delta/h$ and, by Theorem 2.4,

$$|\Psi_2^{u,i\pi/2}(z) - \eta_2^{i\pi/2}(i\pi/2 + hz)| \le C\frac{\varepsilon}{(h|z|)^2} \le C\frac{\varepsilon}{\delta^2}. \tag{252}$$

From the exact formula for $\eta_2^{i\pi/2}$ given in (59), we can compute the asymptotic behavior of $\eta_2^{i\pi/2}(i\pi/2 + hz)$ for $z \in U$: since $|hz| << 1$ in this domain, we get

$$\eta_2^{i\pi/2}(i\pi/2 + hz) = -\frac{i}{5}hz^3 + O\Big(zh, \frac{h^3}{z}, h^3z^5\Big),$$

hence

$$|\eta_2^{i\pi/2}(i\pi/2 + hz) + \frac{i}{5}hz^3| \le O\Big(\delta + \frac{h^4}{\delta} + \frac{\delta^5}{h^2}\Big), \qquad z \in U. \tag{253}$$

On the other hand, by Proposition 2.16,

$$|\psi_2^u(z) + \frac{i}{5}z^3| \le O(z) \le O\Big(\frac{\delta}{h}\Big), \qquad z \in U. \tag{254}$$

Putting together (252), (254) and (253), we get

$$|\Psi_2^{u,i\pi/2}(z) - h\psi_2(z)| \le O\Big(\frac{\varepsilon}{\delta^2} + \delta + \frac{h^4}{\delta} + \frac{\delta^5}{h^2}\Big) \le O(\frac{1}{\delta^2}), \qquad z \in U. \qquad \square$$

Lemma 8.12. *The function φ_h (which is the unique solution of the homogeneous equation $\mathcal{L}_{\mathrm{in}}(\varphi) = 0$ such that $\varphi_{|U} = \varphi^*$) verifies $\varphi_h(z) \in \mathcal{Y}_2$ and $\|\varphi_h\|_2 \le O(h^{-2})$.*

Proof. Using Lemma 8.11 and Cauchy estimates in U, we have $|\varphi^*(z+1) - \varphi^*(z)| = O(h/\delta^3)$ for $z \in U'$. Then, by (241),

$$|c_1(z)| = |W_1(\varphi^*, \psi_2^u)(z)| = O\Big(\frac{1}{\delta^2}\Big(\frac{\delta}{h}\Big)^2\Big) = O\Big(\frac{1}{h^2}\Big)$$

$$|c_2(z)| = |W_1(\psi_1^u, \varphi^*)(z)| = O\Big(\frac{1}{\delta^2}\Big(\frac{h}{\delta}\Big)^3\Big) = O\Big(\frac{h^3}{\delta^5}\Big)$$

Now, for $z \in \tilde{\mathcal{D}}_{\mathrm{in}}$, we use $R_N \leq |z| \leq \delta/h$ and get

$$|\varphi^*(z)z^2| \leq O\Big(\frac{1}{h^2}\Big) + |z|^5 O\Big(\frac{h^3}{\delta^5}\Big) \leq O\Big(\frac{1}{h^2}\Big). \qquad \square$$

We now need to study the operator $\mathcal{H}_{\mathrm{in},1}$, which is defined with the help of $\mathcal{M}_h$.

Lemma 8.13. *$\mathcal{M}_h(\phi^u) - \mathcal{M}_0(\phi_0^u) \in \mathcal{Y}_3$ and $\|\mathcal{M}_h(\phi^u) - \mathcal{M}_0(\phi_0^u)\|_3 \leq O(A_\delta^0)$.*

Proof. We recall that $\mathcal{F}(y, h, \varepsilon)$ is even with respect to h. Hence,

$$\begin{aligned} &\mathcal{M}_h(\phi^u)(z) - \mathcal{M}_0(\phi_0^u)(z) \\ &\quad = \mathcal{M}_h(\phi^u)(z) - \mathcal{M}_0(\phi^u) + \mathcal{M}_0(\phi_0^u) - \mathcal{M}_0(\phi_0^u) \\ &\quad = h^2 \int_0^1 D^2_{(h^2)y}\mathcal{F}(\phi^u, hs, \varepsilon)ds + (\phi^u - \phi_0^u)\int_0^1 \partial^2_{yy}\mathcal{F}(\phi^u + s\phi_0^u, 0, \varepsilon)ds. \end{aligned}$$

Using formula (94), inequalities (243) and (244) and Theorem 2.18 for $N = 0$, we obtain

$$|\mathcal{M}_h(\phi^u)(z) - \mathcal{M}_0(\phi_0^u)(z)| \leq O(h^2 + \frac{A_\delta^0}{|z|^3}).$$

Using that $|z| = O(\delta/h)$ for $z \in \tilde{\mathcal{D}}_{\mathrm{in}}$, we obtain the claimed result. $\square$

Lemma 8.14. *The map $\mathcal{H}_{\mathrm{in},1}$, introduced in* (250) *verifies:*

(1) *$\mathcal{H}_{\mathrm{in},1} : \mathcal{Y}_2 \mapsto \mathcal{Y}_5$ is an affine map and* $\operatorname{lip}\mathcal{H}_{\mathrm{in},1} = O(A_\delta^0)$.
(2) *$\mathcal{H}_{\mathrm{in},1}(0) \in \mathcal{Y}_0$ and $\|\mathcal{H}_{\mathrm{in},1}(0)\|_0 \leq O(hA_\delta^0)$*

Proof. By Lemma 8.13, $\|\mathcal{M}_h(\phi^u) - \mathcal{M}_0(\phi_0^u)\|_3 \leq O(A_\delta^0)$. On the other hand, the function $h\psi_2^u \in \mathcal{Y}_{-3}$ with $\|h\psi_2^u\|_{-3} \leq O(h)$. Hence, by Lemma 8.4, $\|h\psi_2^u\|_2 \leq O(h(\delta/h)^5)$. Then, if $\varphi \in \mathcal{Y}_2$, we have that $h\psi_2 + \varphi \in \mathcal{Y}_2$ and, consequently, $\mathcal{H}_{\mathrm{in},1}(\varphi) = (\mathcal{M}(\phi^u)(z) - \mathcal{M}_0(z))\,(h\psi_2 + \varphi) \in \mathcal{Y}_5$. Then, if we take $\varphi_i \in \mathcal{Y}_2$, $i = 1, 2$, we have

$$\|\mathcal{H}_{\mathrm{in},1}(\varphi_1) - \mathcal{H}_{\mathrm{in},1}(\varphi_2)\|_5 \leq \|\mathcal{M}_h(\phi^u) - \mathcal{M}_0(\phi_0^u)\|_3\|\varphi_1 - \varphi_2)\|_2 \leq O(A_\delta^0)\|\varphi_1 - \varphi_2)\|_2.$$

On the other hand,in a natural way $\mathcal{H}_{\mathrm{in},1}(0) = (\mathcal{M}_h(\phi^u) - \mathcal{M}_0(\phi_0^u))\,h\psi_2^u \in \mathcal{Y}_0$, and

$$\|\mathcal{H}_{\mathrm{in},1}(0)\|_0 = \|\mathcal{M}(\phi^u) - \mathcal{M}_0(\phi_0^u)\|_3\|h\psi_2\|_{-3} \leq O(A_\delta^0 h) \qquad \square$$

End of the proof of Proposition 8.10. The map $\mathcal{G}_{\rm in} \circ \mathcal{H}_{\rm in,1}$ is well defined from $\mathcal{Y}_2$ to itself. Indeed, when $\varphi \in \mathcal{Y}_2$, by Lemma 8.14, $\mathcal{H}_{\rm in,1}(\varphi) \in \mathcal{Y}_5$ and then, by Lemma 8.5 with $m = 5$, $\mathcal{G}_{\rm in} \circ \mathcal{H}_{\rm in,1}(\varphi) \in \mathcal{Y}_2$. Moreover, if $\varphi_1, \varphi_2 \in \mathcal{Y}_2$,

$$\|\mathcal{G}_{\rm in} \circ \mathcal{H}_{\rm in,1}(\varphi_1) - \mathcal{G}_{\rm in} \circ \mathcal{H}_{\rm in,1}(\varphi_2)\|_2 \le \|\mathcal{H}_{\rm in,1}(\varphi_1) - \mathcal{H}_{\rm in,1}(\varphi_2)\|_5 \le O(A_\delta^0)\|\varphi_1 - \varphi_2\|_2.$$

Therefore, when considered as a map from $\mathcal{Y}_2$ to itself, the map $\mathcal{G}_{\rm in} \circ \mathcal{H}_{\rm in,1} + \varphi_h$ has Lipschitz constant $O(A_\delta^0)$, which is smaller than 1 with the standing hypotheses.

The analytic continuation φ of $\Psi_2^{u,i\pi/2} - h\psi_2^u$ is thus obtained by iterating this map and the first approximation is $\mathcal{G}_{\rm in} \circ \mathcal{H}_{\rm in,1}(0) + \varphi_h$. Applying Lemma 8.5 with $m = 0$ and Lemma 8.14, we get

$$\|\mathcal{G}_{\rm in} \circ \mathcal{H}_{\rm in,1}(0)\|_2 \le C(\delta/h)^4 \|\mathcal{H}_{\rm in,1}(h\psi_2)\|_0 \le O((\delta^4/h^3)A_\delta^0) \le O(1/h^2).$$

Then, using Lemma 8.12 for φ_h and the above inequality, we obtain $\|\mathcal{G}_{\rm in} \circ \mathcal{H}_{\rm in,1}(0) + \varphi_h\|_2 \le O(1/h^2)$. This is sufficient to conclude that $\|\varphi\|_2 \le O(1/h^2)$. □

9. Proof of Theorem 3.1

9.1. **Obtaining independent solutions.** We want to obtain two independent real analytic solutions ν_1 and ν_2 of equation (123) with Wronskian 1. We will proceed in two steps. In a first one, we will find two solutions of (123) close to the two real analytic solutions of equation (71), η_1^u, $\eta_2^u = \eta_2^{u,0}$, obtained in Theorem 2.20. The Wronskian of these solutions will not necessarily be the constant function 1. In a second step, we will modify these solutions in order that their Wronskian be 1.

We start by introducing the new unknowns u_i, $i = 1, 2$, defined by $\nu_i = \eta_i^u + u_i$, $i = 1, 2$. With these new unknowns, equation (123) reads, for $i = 1, 2$,

$$u_i(t + h) + u_i(t - h) = m(\xi^u, \xi^u)(t)u_i(t) + \tilde{m}(t)(\eta_i^u(t) + u_i(t)), \tag{255}$$

where m was introduced in (124), and

$$\tilde{m} = m(\xi^u, \xi^s) - m(\xi^u, \xi^u) \tag{256}$$

is defined in the domain $\mathcal{R}$ (see (122)).

Lemma 9.1. *Let $\tau_\pm = \tau_\pm(t) = t \mp i\pi/2$. For any $\theta \in (0, 1)$, there exists a sequence of positive constants $(C_N)_{N\in\mathbb{N}}$ such that, for any $N \ge 0$ and $t \in \mathcal{R}$,*

$$|\tilde{m}(t)| \le \begin{cases} C_N \frac{|\varepsilon| h^{2N+4}}{|\cosh t|^{2N+4}}, & |\tau_\pm(t)| > \delta, \\ C_N|\varepsilon|\Big(\frac{A_\delta^N h^3}{|\tau_\pm|^3} + \frac{|\tau_\pm|^2}{h^2} e^{\frac{2\pi}{h}\operatorname{Im}\tau_\pm}\big(1 + \frac{h^3}{|\tau_\pm|^3} e^{\frac{2\pi}{h}\theta \operatorname{Im}\tau_\pm}\big)\Big), & |\tau_\pm(t)| \le \delta. \end{cases} \tag{257}$$

Proof. By the definition of m in (124), since $f(y) = y - y^3 + O(y^5)$ and $V'(y) = O(y^5)$, and provided that ξ^u and ξ^s are small enough, we have that there exists some positive constant C such that

$$|\tilde{m}(t)| \leq C(|\xi^u(t)| + |\xi^s(t)|)|\xi^u(t) - \xi^s(t)|. \tag{258}$$

The bound for $\tau_\pm > \delta$ follows from inequality (83) for $\xi^u - \xi^s$ and inequalities (67) and (69), with $N = 0$, for $\xi^u + \xi^s$.

For $\tau_\pm \leq \delta$, we write

$$\begin{aligned} |\xi^u(t) - \xi^s(t)| \leq{}& |\xi^u(t) - \phi^{u,N}((t - i\pi/2)/h)| \\ &+ |\phi^{u,N}((t - i\pi/2)/h) - \Phi^{s,N}((t - i\pi/2)/h)| \\ &+ |\Phi^{s,N}((t - i\pi/2)/h) - \xi^s(t)|, \end{aligned}$$

where $\phi^{u,N}$ and $\phi^{s,N}$ were introduced in (102). Then, the first and third differences above can be bounded using inequality (113). The second one is bounded using inequality (109), recalling that $z = (t - i\pi/2)/h$ and inequality (104). □

We remark that, up to now, the difference between ξ^u and ξ^s is not small at a distance $O(h)$ of $\pm i\pi/2$ and, hence, $\tilde{m}$ is not small in $\mathcal{R}$. In fact, from formula (257), we have that $|\tilde{m}(t)| = O(1)$ at a distance $O(h)$ of $\pm i\pi/2$. By this reason, we will find solutions of equation (123) only defined in the smaller domain $\mathcal{R}_\sigma = \{t \in \mathcal{R} \mid |\mathrm{Im}\,(t \mp i\pi/2)| \geq \frac{\sigma}{2\pi} h|\ln h|\}$, $\sigma > 1$, introduced in (127). This restriction, together with inequality (257), implies that in the upper side of the domain, that is, if $\mathrm{Im}\,(t - i\pi/2) = -\frac{\sigma}{2\pi} h|\ln h|$, for $\sigma > 1$, we have that

$$|\tilde{m}(t)| \leq O\big(\varepsilon\sigma^2 h^\sigma |\ln h|^2\big) + O\big(\varepsilon \frac{A_\delta^N}{|\ln h|^3}\big). \tag{259}$$

In order to solve equation (255), we introduce the linear operator

$$\mathcal{L}^u(u)(t) = u(t + h) + u(t - h) - m(\xi^u, \xi^u)(t)u(t). \tag{260}$$

Then, equation (255) can be written as

$$\mathcal{L}^u(u_i) = \tilde{m}(\eta_i^u + u_i) \tag{261}$$

Our purpose is to solve equation (261) as a fixed point equation. In order to do so, we need to define a right inverse of $\mathcal{L}^u$ in some suitable spaces. Hence, we introduce the spaces

$$\mathcal{Z}_\mu = \{u\colon \mathcal{R}_\sigma \to \mathbb{C} \text{ real analytic, such that } \|u\|_\mu < \infty\}, \tag{262}$$

where

$$\|u\|_\mu = \sup_{t \in \mathcal{R}_\sigma} |u(t) \cosh^\mu t|. \tag{263}$$

They are Banach spaces. Moreover,

Lemma 9.2. *Let $\mu > -2$. Then the operator $\mathcal{L}^u$ admits a bounded right inverse $\mathcal{G}^u \colon \mathcal{Z}_\mu \to \mathcal{Z}_{\mu+4}$ such that $\|\mathcal{G}^u\| \leq \frac{1}{h^2}|\ln h|$.*

We postpone the proof of this lemma to the end of this section.

Lemma 9.3. *Under the hypothesis of Theorem 2.20, if $\sigma > 4$ then for any $N \geq 0$ there exists a constant $\rho_N^1 > 0$ such that, if $A_\delta^N < \rho_N^1 h^4 |\ln h|^7$, then equations* (261)*, $i = 1, 2$, have two real analytic solutions, $u_i \in \mathcal{Z}_9$, $i = 1, 2$ satisfying*

$$\|u_1\|_9 \leq O(\varepsilon h^2 A_\delta^N |\ln h|) + O(\varepsilon h^{2+\sigma} |\ln h|^6), \tag{264}$$

$$\|u_2\|_9 \leq O(\varepsilon \frac{1}{h} A_\delta^N |\ln h| + O(\varepsilon h^{\sigma-1} |\ln h|^6). \tag{265}$$

Proof. We start by solving equation (261) for $i = 1$. We rewrite it by using the operator $\mathcal{G}^u$ as

$$u = \mathcal{G}^u(\tilde{m}(\eta_1^u + u)). \tag{266}$$

By inequalities (119) and (257) we have that for $t \in \mathcal{R}_\sigma$

$$|\tilde{m}(t)\eta_1^u(t)\cosh^5 t| \leq \begin{cases} O(\varepsilon \frac{h^{2N+5}}{\delta^{2N+1}}), & \tau_\pm > \delta, \\ O(\varepsilon A_\delta^N h^4) + O(\varepsilon h^{4+\sigma} |\ln h|^5), & \tau_\pm \leq \delta, \end{cases}$$

which, since $(h/\delta)^{2N+1} \leq A_\delta^N$, can be summarized as

$$\|\tilde{m}\eta_1^u\|_5 \leq O(\varepsilon h^4 A_\delta^N) + O(\varepsilon h^{4+\sigma} |\ln h|^5). \tag{267}$$

Hence, by Lemma 9.2,

$$\|\mathcal{G}^u(\tilde{m}\eta_1^u)\|_9 \leq O(\varepsilon h^2 A_\delta^N |\ln h|) + O(\varepsilon h^{2+\sigma} |\ln h|^6). \tag{268}$$

Moreover the map $u \to \mathcal{G}^u(\tilde{m}(\eta_1^u + u))$, considered as a map from $\mathcal{Z}_9$ to itself, is Lipschitz with a constant less than $O(\varepsilon h^{-4} |\ln h|^{-7} (A_\delta^N + h^\sigma))$, since, for $u \in \mathcal{Z}_9$,

$$\begin{aligned} \|\mathcal{G}^u(\tilde{m}u)\|_9 &\leq O(\varepsilon h^{-2} |\ln h|) \|\tilde{m}u\|_5 \\ &\leq O(\varepsilon h^{-2} |\ln h|) \|\tilde{m}\|_{-4} \|u\|_9 \\ &\leq O(\varepsilon h^{-4} |\ln h|^{-7} (A_\delta^N + h^\sigma)) \|u\|_9 \end{aligned}$$

(we have used Lemma 9.1 to bound $\|\tilde{m}\|_{-4}$). Hence, since $\sigma > 4$, by the standing hypotheses on A_δ^n, equation (266) has a unique solution in $\mathcal{Z}_9$. Furthermore, by inequality (268) this solution is bounded as claimed.

Equation (261) for $i = 2$ is solved analogously. We rewrite it as

$$u = \mathcal{G}^u(\tilde{m}(\eta_2^u + u)), \tag{269}$$

and observe that, again by inequalities (120) and (257),

$$\|\tilde{m}\eta_2^u\|_5 \leq O(\varepsilon h A_\delta^N) + O(\varepsilon h^{1+\sigma} |\ln h|^5). \tag{270}$$

Hence, by Lemma 9.2,

$$\|\mathcal{G}^u(\tilde{m}\eta_2^u)\|_9 \leq O(\varepsilon\frac{1}{h}A_\delta^N|\ln h|) + O(\varepsilon h^{\sigma-1}|\ln h|^6). \tag{271}$$

Moreover, the map defined by the right hand side of (269), as a map from $\mathcal{Z}_9$ to itself, has Lipschitz constant bounded again by

$$\frac{1}{h^2}|\ln h|\|\tilde{m}\|_{-4} \leq O(\varepsilon h^{-4}|\ln h|^{-7}(A_\delta^N + h^\sigma)). \qquad \square$$

9.2. **The Wronskian.** We define the functions $\tilde{\nu}_i = \eta_i^u + u_i$, $i = 1, 2$. They are solutions of equation (123). It remains to prove that they are independent. Once this fact is checked, we will modify them in order to obtain two solutions such that their Wronskian is 1.

Lemma 9.4. *Under the hypothesis of Theorem 2.20, if $\sigma > 11$ then for any $N \geq 0$ there exists a constant $\rho_N^2 > 0$ such that if $A_\delta^N < \rho_N^2 h^{12}$, the functions $\tilde{\nu}_i$, $i = 1, 2$, satisfy*

$$\|W_h(\tilde{\nu}_1, \tilde{\nu}_2) - 1\|_0 \leq O(\varepsilon\frac{A_\delta^N}{h^{11}|\ln h|^{11}}) + O(\varepsilon\frac{h^{\sigma-11}}{|\ln h|^6}), \tag{272}$$

and

$$\|W_h(\tilde{\nu}_1, \tilde{\nu}_2) - 1\|_7 \leq O(\varepsilon\frac{A_\delta^N}{h^4|\ln h|^4}) + O(\varepsilon h^{\sigma-4}|\ln h|). \tag{273}$$

Proof. First notice that if $u \in \mathcal{Z}_{k_1}$ and $v \in \mathcal{Z}_{k_2}$, $k_1, k_2 > 0$, then their Wronskian verifies

$$|W_h(u, v)(t)| \leq \frac{4}{|\cosh t|^{k_1+k_2}}\|u\|_{k_1}\|v\|_{k_2}$$

Then, since $\tilde{\nu}_i = \eta_i^u + u_i$, $i = 1, 2$, and using that $W_h(\eta_1, \eta_2) = 1$, we have that

$$W_h(\tilde{\nu}_1, \tilde{\nu}_2) - 1 = W_h(u_1, \eta_2^u) + W_h(\eta_1, u_2) + W_h(u_1, u_2).$$

Combining Corollary 2.21 and Lemma 9.3, we obtain, for $t \in \mathcal{R}_\sigma$,

$$\begin{aligned} |W_h(u_1, \eta_2^u)(t)| &\leq \frac{4}{|\cosh t|^{11}}\|u_1\|_9\|\eta_2^u\|_2 \\ &\leq \frac{1}{|\cosh t|^{11}}\big(O(\varepsilon A_\delta^N|\ln h| + O(\varepsilon h^\sigma|\ln h|^6)\big), \end{aligned}$$

which is smaller than 1 in $\mathcal{R}_\sigma$. The same bound is obtained for $W_h(u_2, \eta_1^u)$. In the same way, we have that

$$|W_h(u_1, u_2)(t)| \leq \frac{1}{|\cosh t|^{18}}\big(O(\varepsilon A_\delta^N|\ln h|) + O(\varepsilon h^\sigma|\ln h|^6)\big)^2.$$

Since this last bound is smaller than the square of the previous one, we have that

$$|W_h(\tilde{\nu}_1, \tilde{\nu}_2)(t) - 1| \leq \frac{1}{|\cosh t|^{11}}\big(O(\varepsilon A_\delta^N|\ln h|) + O(\varepsilon h^\sigma|\ln h|^6)\big).$$

In particular,

$$\|W_h(\tilde{\nu}_1, \tilde{\nu}_2) - 1\|_0 \leq O(\varepsilon \frac{A_\delta^N}{h^{11}|\ln h|^{10}}) + O(\varepsilon \frac{h^{\sigma-11}}{|\ln h|^5}),$$

and

$$\|W_h(\tilde{\nu}_1, \tilde{\nu}_2) - 1\|_7 \leq O(\varepsilon \frac{A_\delta^N}{h^4|\ln h|^3}) + O(\varepsilon h^{\sigma-4}|\ln h|^2). \qquad \square$$

Now we define $\omega = W_h(\tilde{\nu}_1, \tilde{\nu}_2)^{1/2}$. By the previous Lemma, ω is analytic in $\mathcal{R}_\sigma$, and it is real analytic and h-periodic. Moreover, since

$$\omega^{-1} - 1 = \frac{1 - W_h(\tilde{\nu}_1, \tilde{\nu}_2)}{\omega(\omega + 1)},$$

we have that

(274)
$$\|\omega^{-1} - 1\|_0 \leq O(1)\|W_h(\tilde{\nu}_1, \tilde{\nu}_2) - 1\|_0 \quad \text{and} \quad \|\omega^{-1} - 1\|_7 \leq O(1)\|W_h(\tilde{\nu}_1, \tilde{\nu}_2) - 1\|_7.$$

We finally introduce $\nu_i = \omega^{-1}\tilde{\nu}_i$. These functions satisfy all the properties claimed in Theorem 3.1. Indeed, they are real analytic. Moreover, since ω is h-periodic, they are solutions of equation (123). Their Wronskian satisfies

$$\begin{aligned} W_h(\nu_1, \nu_2) &= W_h(\omega^{-1}\tilde{\nu}_1, \omega^{-1}\tilde{\nu}_2) \\ &= \omega^{-2} W_h(\tilde{\nu}_1, \tilde{\nu}_2) \\ &= 1. \end{aligned}$$

Finally, from (272), (273) and (264), we have that

(275)
$$\begin{aligned} \|\nu_1 - \eta_1^u\|_9 &= \|\omega^{-1}\tilde{\nu}_1 - \eta_1^u\|_9 \\ &\leq \|\omega^{-1}(\tilde{\nu}_1 - \eta_1^u)\|_9 + \|(\omega^{-1} - 1)\eta_1^u\|_9 \\ &\leq \|\omega^{-1}\|_0 \|\tilde{\nu}_1 - \eta_1^u\|_9 + \|\omega^{-1} - 1\|_7 \|\eta_1^u\|_2 \\ &\leq O(1)\Big(O(\varepsilon h^2 A_\delta^N |\ln h|) + O(\varepsilon h^{2+\sigma}|\ln h|^6)\Big) \\ &\quad + \Big(O(\varepsilon \frac{A_\delta^N}{h^4|\ln h|^3}) + O(\varepsilon h^{\sigma-4}|\ln h|^2)\Big) O(h) \\ &\leq O\Big(\varepsilon \frac{A_\delta^N}{h^3|\ln h|^3}\Big) + O(\varepsilon h^{\sigma-3}|\ln h|^2). \end{aligned}$$

Finally, using again (272), (273) and (265), we can bound

$$
\begin{aligned}
\|\nu_2 - \eta_2^u\|_9 &= \|\omega^{-1}\tilde{\nu}_2 - \eta_2^u\|_9 \\
&\le \|\omega^{-1}(\tilde{\nu}_2 - \eta_2^u)\|_9 + \|(\omega^{-1} - 1)\eta_2^u\|_9 \\
&\le \|\omega^{-1}\|_0\|\tilde{\nu}_2 - \eta_2^u\|_9 + \|\omega^{-1} - 1\|_7\|\eta_2^u\|_2 \\
&\le O(1)\Big(O\Big(\frac{\varepsilon}{h}A_\delta^N|\ln h|\Big) + O(\varepsilon h^{\sigma-1}|\ln h|^6)\Big) \\
&\quad + \Big(O\Big(\varepsilon\frac{A_\delta^N}{h^4|\ln h|^3}\Big) + O(\varepsilon h^{\sigma-4}|\ln h|^2)\Big)O(\frac{1}{h^2}) \\
&\le O\Big(\varepsilon\frac{A_\delta^N}{h^6|\ln h|^3}\Big) + O(\varepsilon h^{\sigma-6}|\ln h|^2),
\end{aligned}
\tag{276}
$$

which proves inequalities (128) and (129).

Inequalities (130) and (131) follow from the relation $\eta_2^{u,i\pi/2} = \eta_2 + A\eta_1^u$ (cf. Theorem 2.4) and from inequalities (116), (117), (275) and (276). □

9.3. **Proof of Lemma 9.2.** Since η_1^u and η_2^u satisfy $\mathcal{L}^u(\eta_i^u) = 0$, a solution of the equation $\mathcal{L}^u u = v$ is given by

$$u = \eta_1^u\Delta_h^{-1}(\eta_2^u v) - \eta_2^u\Delta_h^{-1}(\eta_1^u v)$$

where Δ_h^{-1} is a right inverse of the operator Δ_h. As we pointed out in Section 4, it is easy to define such an inverse for functions defined in some large domains. However, in the case we are now dealing, the problem is much more difficult, and it was solved by Lazutkin. Here we quote a result by Gelfreich in [Gel99], page 210, which provides us with a right inverse.

Lemma 9.5. *Let $\mu > 0$. There is a linear operator $\Delta_h^{-1}\colon \mathcal{Z}_\mu \to \mathcal{Z}_\mu$ such that, for any $v \in \mathcal{Z}_\mu$ the function $u = \Delta_h^{-1}(v)$ is a solution of the equation $\Delta_h u = v$ and*

$$\|\Delta_h^{-1}\| \le C\frac{1}{h}|\ln h|.$$

The constant C does not depend on σ. Moreover, if v is an analytic continuation of a real analytic function defined on the intersection of $\mathcal{R}_\sigma$ with the real axis, the same is true about $\Delta_h^{-1}(v)$.

We remark that, by Corollary 2.21, $\|\eta_1^u\|_2 \le O(h)$ and $\|\eta_2^u\|_2 \le O(h^{-2})$. Hence, by Lemma 9.5, if $v \in \mathcal{Z}_\mu$, we have that $\|\eta_2^u v\|_{\mu+2} \le O(h^{-2})\|v\|_\mu$, which implies that

$$\|\Delta_h^{-1}(\eta_2^u v)\|_{\mu+2} \le O(\frac{1}{h^3}|\ln h|)\|v\|_\mu, \tag{277}$$

and, on the other hand, $\|\eta_1^u v\|_{\mu+2} \le O(h)\|v\|_\mu$, from which we deduce

$$\|\Delta_h^{-1}(\eta_1^u v)\|_{\mu+2} \le O(|\ln h|)\|v\|_\mu. \tag{278}$$

Finally, from (277) and (278), if $v \in \mathcal{Z}_\mu$ we have that

$$\begin{aligned}\|\eta_1^u \Delta_h^{-1}(\eta_2^u v) - \eta_2^u \Delta_h^{-1}(\eta_1^u v)\|_{\mu+4} &\leq \|\eta_1^u \Delta_h^{-1}(\eta_2^u v)\|_{\mu+4} + \|\eta_2^u \Delta_h^{-1}(\eta_1^u v)\|_{\mu+4} \\ &\leq \|\eta_1^u\|_2 \|\Delta_h^{-1}(\eta_2^u v)\|_{\mu+2} + \|\eta_2^u\|_2 \|\Delta_h^{-1}(\eta_1^u v)\|_{\mu+2} \\ &\leq O\Big(\frac{1}{h^2}|\ln h|\Big)\|v\|_\mu. \qquad \square\end{aligned}$$

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