A SCHOENFLIES EXTENSION THEOREM FOR A CLASS OF LOCALLY BI-LIPSCHITZ HOMEOMORPHISMS

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Abstract. In this paper we prove a new version of the Schoenflies extension theorem for collared domains Ω and Ω' in $\mathbb{R}^n$: for $p \in [1, n)$, locally bi-Lipschitz homeomorphisms from Ω to Ω' with locally p-integrable, second-order weak derivatives admit homeomorphic extensions of the same regularity.

Moreover, the theorem is essentially sharp. The existence of exotic 7-spheres shows that such extension theorems cannot hold, for $p > n = 7$.

1. Introduction

1.1. **Embeddings of Collars.** In point-set topology, the Schoenflies Theorem [Wil79, Thm III.5.9] is a stronger form of the well-known Jordan Curve Theorem: it states that *every simple closed curve separates the sphere* $\mathbb{S}^2$ *into two domains, each of which is homeomorphic to* $\mathbb{B}^2$*, the open unit disc.* The same statement does not hold in higher dimensions, since the Alexander horned sphere [Ale24] provides a counter-example in $\mathbb{R}^3$. Despite this, Brown [Bro60] proved that for each $n \in \mathbb{N}$, every embedding of $\mathbb{S}^{n-1} \times (-\epsilon, \epsilon)$ into $\mathbb{R}^n$ extends to an embedding of $\mathbb{B}^n$ into $\mathbb{R}^n$.

Similar extension problems arise by varying the regularity of the embeddings. To this end, we prove a Schoenflies-type theorem for a new class of homeomorphisms. Their regularity is given in terms of Sobolev spaces and Lipschitz continuity.

To begin, recall that a homeomorphism $f : \Omega \to \Omega'$ is *locally bi-Lipschitz* if for each $z \in \Omega$, there is a neighborhood O of z and $L \geq 1$ so that the inequality

$$L^{-1} |x - y| \;\leq\; |f(x) - f(y)| \;\leq\; L\,|x - y| \tag{1.1}$$

holds for all $x, y \in O$. Recall also that for $p \geq 1$ and $k \in \mathbb{N}$, the Sobolev space $W^{k,p}_{\mathrm{loc}}(\Omega; \Omega')$ consists of maps $f : \Omega \to \Omega'$, where each component f_i lies in $L^p_{\mathrm{loc}}(\Omega)$ and has weak derivatives of orders up to k in $L^p_{\mathrm{loc}}(\Omega)$.

Definition 1.1. Let $f : \Omega \to \Omega'$ be a locally bi-Lipschitz homeomorphism. For $p \in [1, \infty)$, we say that f is of *class* LW^p_2 if $f \in W^{2,p}_{\mathrm{loc}}(\Omega; \Omega')$ and $f^{-1} \in W^{2,p}_{\mathrm{loc}}(\Omega'; \Omega)$. If K and K' are closed sets, a homeomorphism $f : K \to K'$ is of class LW^p_2 if the restriction of f to the interior of K is of class LW^p_2.

Instead of product sets of the form $\mathbb{S}^{n-1} \times (-\epsilon, \epsilon)$, we will consider domains in $\mathbb{R}^n$ of a similar topological type.

Definition 1.2. A bounded domain D in $\mathbb{R}^n_*$ is *Jordan* if its boundary ∂D is homeomorphic to $\mathbb{S}^{n-1}$. A *collared domain* (or *collar*) is a domain in $\mathbb{R}^n$ of the form $D_2 \setminus \bar{D}_1$, where D_1 and D_2 are Jordan domains with $\bar{D}_1 \subset D_2$.

We now state the extension theorem for homeomorphisms of class LW^p_2 between collared domains.

Theorem 1.3. *Let D_1 and D_2 be Jordan domains in $\mathbb{R}^n$ so that $\bar{D}_1 \subset D_2$, let B_1 and B_2 be balls so that $\bar{B}_1 \subset B_2$, and let $p \in [1, n)$.*

If $f : \bar{D}_2 \setminus D_1 \to \bar{B}_2 \setminus B_1$ is a homeomorphism of class LW^p_2 so that $f(\partial D_i) = \partial B_i$ holds, for $i = 1, 2$, then there exists a homeomorphism $F : \bar{D}_2 \to \bar{B}_2$ of class LW^p_2 and a neighborhood N of ∂D_2 so that
$F|(N \cap \bar{D}_2) = f|(N \cap \bar{D}_2)$.

The proof is an adaptation of Gehring's argument [Geh67, Thm 2'] from the class of quasiconformal homeomorphisms to the class LW^p_2. For the locally bi-Lipschitz class, the extension theorem was known to Sullivan [Sul75] and later proved by Tukia and Väisälä [TV81, Thm 5.10]. For more about quasiconformal homeomorphisms, see [Väi71].

As in Gehring's case, Theorem 1.3 is not quantitative. His extension depends on the distortion (resp. Lipschitz constants) of g as well as the configurations of the collars $D_2 \setminus \bar{D}_1$ and $B_2 \setminus \bar{B}_1$. In addition, our modification of his extension also depends explicitly on the parameters p and n.

1.2. **Motivations, Smoothness, and Sharpness.** The motivation for Theorem 1.3 comes from the study of Lipschitz manifolds.

Specifically, Heinonen and Keith have recently shown that *if an n-dimensional Lipschitz manifold, for $n \neq 4$, admits an atlas with coordinate charts in the Sobolev class $W^{2,2}_{loc}(\mathbb{R}^n; \mathbb{R}^n)$, then it admits a smooth structure* [HK09].

On the other hand, there are 10-dimensional Lipschitz manifolds without smooth structures [Ker60]. This leads to the following question:

Question 1.4. For $n \neq 4$, does there exist $p \in [1, 2)$ so that every n-dimensional Lipschitz manifold admits an atlas of charts in $W^{2,p}_{loc}(\mathbb{R}^n; \mathbb{R}^n)$?

Sullivan has shown that *every n-dimensional topological manifold, for $n \neq 4$, admits a Lipschitz structure* [Sul75]. A key step in the proof is to show that bi-Lipschitz homeomorphisms satisfy a Schoenflies-type extension theorem. One may inquire whether this direction of proof would also lead to the desired Sobolev regularity. Theorem 1.3 would be a first step in this direction. For more about Lipschitz structures on manifolds, see [LV77].

It is worth noting that Theorem 1.3 is not generally true for $p > n$. Recall that for any domain Ω in $\mathbb{R}^n$, Morrey's inequality [EG92, Thm 4.5.3.3] gives $W^{2,p}(\Omega) \hookrightarrow C^{1,1-n/p}(\Omega)$, so homeomorphisms of class LW^p_2 are necessarily C^1-diffeomorphisms.

Indeed, every C^∞-diffeomorphism $\varphi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ admits a radial extension

$$\bar{\varphi}(x) \;:=\; |x|\, \varphi\Big(\frac{x}{|x|}\Big)$$

that is, a C^∞-diffeomorphism between round annuli. The validity of Theorem 1.3, for $p > n$, would therefore imply that every such φ extends to a C^1-diffeomorphism of $\bar{\mathbb{B}}^n$ onto itself. However, for $n = 7$ this conclusion is impossible.

Recall that every such φ also determines a C^∞-smooth, n-dimensional manifold M^n_φ that is homeomorphic to $\mathbb{S}^n$ [Mil56, Construction (C)]. Indeed, M^n_φ is the quotient of two copies of $\mathbb{R}^n$ under the relation $x \sim \varphi^*(x)$ on $\mathbb{R}^n \setminus \{0\}$, where

$$\varphi^*(x) \;:=\; \frac{1}{|x|}\, \varphi\Big(\frac{x}{|x|}\Big). \tag{1.2}$$

If φ is the identity map on $\mathbb{S}^{n-1}$, then φ^* is the inversion map $x \mapsto |x|^{-2}x$, and M^n_φ is precisely $\mathbb{S}^n$. By using invariants from differential topology, Milnor proved the following theorem about such manifolds [Mil56, Thm 3].

Theorem 1.5 (Milnor, 1956)**.** *There exist C^∞-smooth manifolds of the form M^7_φ that are homeomorphic, but not C^∞-diffeomorphic, to $\mathbb{S}^7$.*

Such manifolds are better known as *exotic spheres*. The next lemma is an analogue of [Hir94, Thm 8.2.1]; it relates exotic spheres to extension theorems.

Lemma 1.6. *Let $\varphi\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be a C^∞-diffeomorphism and let $\bar{\varphi}\colon \bar{\mathbb{B}}^n \setminus \{0\} \to \bar{\mathbb{B}}^n \setminus \{0\}$ be its radial (diffeomorphic) extension. If there exists a C^1-diffeomorphism $\Phi\colon \bar{\mathbb{B}}^n \to \bar{\mathbb{B}}^n$ that agrees with $\bar{\varphi}$ on a neighborhood of $\mathbb{S}^{n-1}$ in $\bar{\mathbb{B}}^n$, then M^n_φ is C^1-diffeomorphic to $\mathbb{S}^n$.*

Proof of Lemma 1.6. Let φ^* be the diffeomorphism defined in Equation (1.2). By construction, there is an atlas of charts $\{M_i\}_{i=1}^2$ for M^n_φ with homeomorphisms $\psi_i : M_i \to \mathbb{R}^n$ that satisfy $\psi_1 \circ \psi_2^{-1} = \varphi^*$.

Let $\pi_1, \pi_2 : \mathbb{R}^n \to \mathbb{S}^n$ be stereographic projections relative to the "north" and "south" poles on $\mathbb{S}^n$, respectively, so
$\pi_2^{-1} \circ \pi_1 = \mathrm{id}^* = (\mathrm{id}^*)^{-1}$.
Observe that

$$((\mathrm{id}^*)^{-1} \circ \varphi^*)(x) \;=\; \frac{\varphi^*(x)}{|\varphi^*(x)|^2} \;=\; |x|\, \varphi\Big(\frac{x}{|x|}\Big) \;=\; \bar{\varphi}(x)$$

holds for all $x \in \mathbb{R}^n \setminus \{0\}$. It follows that

$$x \;\mapsto\; \begin{cases} (\pi_1^{-1} \circ \psi_1)(x), & \text{if } x \in M_1 \\ (\pi_2^{-1} \circ \Phi \circ \psi_2)(x), & \text{if } x \in M_2 \end{cases}$$

is a C^1-diffeomorphism of M^n_φ onto $\mathbb{S}^n$. □

By [Hir94, Thm 2.2.10], if two C^∞-smooth manifolds are C^1-diffeomorphic, then they are C^∞-diffeomorphic. It follows that there exist C^1-diffeomorphisms of collars in $\mathbb{R}^7$ that do not admit diffeomorphic extensions of class LW_2^p, for any $p > 7$.

The next result follows from the inclusion $W^{2,p}_{loc}(\Omega;\Omega') \subseteq W^{2,q}_{loc}(\Omega;\Omega')$, for $q \leq p$.

Corollary 1.7. *Let $n = 7$. For $p > n$, there exist collars Ω, Ω' in $\mathbb{R}^n$ and homeomorphisms $\varphi : \Omega \to \Omega'$ of class LW_2^p that admit homeomorphic extensions of class LW_2^q, for every $1 \leq q < n$, but not of class LW_2^p.*

Since the above discussion relies crucially on Sobolev embedding theorems, it leaves open the borderline case $p = n$.

Question 1.8. Is Theorem 1.3 true for the case $p = n$?

For $p > n$, the main obstruction to an extension theorem is the existence of exotic n-spheres. It is known that no exotic spheres exist for $n = 1, 2, 3, 5, 6$ [KM63], and the case $n = 1$ can be done by hand.

It would be interesting to determine whether other geometric obstructions arise.

Question 1.9. For $n = 2, 3, 5, 6$, is Theorem 1.3 true for all $p \geq 1$?

The outline of the paper is as follows. In Section 2 we review basic facts about Lipschitz mappings, Sobolev spaces, and the class LW_2^p. In Section 3 we prove extension theorems in the setting of doubly-punctured domains. Section 4 addresses the case of homeomorphisms between collars, by employing suitable generalizations of inversion maps and reducing to previous cases.

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2. Notation and Basic Facts

For $A \subset \mathbb{R}^n$, we write A^c for the complement of A in $\mathbb{R}^n$. The open unit ball in $\mathbb{R}^n$ is denoted $\mathbb{B}^n$; if the dimension is understood, we will write $\mathbb{B}$ for $\mathbb{B}^n$.

We write $A \lesssim B$ for inequalities of the form $A \leq kB$, where k is a fixed dimensional constant and does not depend on A or B.

For domains Ω and Ω' in $\mathbb{R}^n$, recall that a map $f : \Omega \to \Omega'$ is *Lipschitz* whenever

$$L(f) \;:=\; \sup\left\{ \frac{|f(x) - f(y)|}{|x - y|} \;:\; x, y \in \Omega,\; x \neq y \right\} \;<\; \infty.$$

The map f is *locally Lipschitz* if every point in Ω has a neighborhood on which f is Lipschitz. A homeomorphism $f : \Omega \to \Omega'$ is *bi-Lipschitz* (resp. *locally bi-Lipschitz*) if f and f^{-1} are both Lipschitz (resp. locally Lipschitz); compare Equation (1.1).

The following lemmas about bi-Lipschitz maps are used in Section 2. The first is a special case of [TV81, Lemma 2.17]; the second one is elementary, so we omit the proof.

Lemma 2.1 (Tukia-Väisälä)**.** *Let O and O' be open, connected sets in $\mathbb{R}^n$ and let K be a compact subset of O. If $f : O \to O'$ is locally bi-Lipschitz, then $f|K$ is bi-Lipschitz, where $L((f|K)^{-1})$ depends only on O, K, and $L(f)$.*

Lemma 2.2. *For $i = 1, 2$, let $h_i : \Omega_i \to \mathbb{R}^n$ be locally bi-Lipschitz embeddings so that $h_1(\Omega_1 \setminus \Omega_2) \cap h_2(\Omega_2 \setminus \Omega_1) = \emptyset$. If $h_1 = h_2$ holds on all of $\Omega_1 \cap \Omega_2$, then*

$$h(x) = \begin{cases} h_1(x), & \text{if } x \in \Omega_1 \\ h_2(x), & \text{if } x \in \Omega_2 \setminus \Omega_1 \end{cases}$$

is also a locally bi-Lipschitz embedding.

For $f \in W^{2,p}(\Omega; \Omega')$, we will use the Hilbert-Schmidt norm for the weak derivatives $Df(x) := [\partial_j f_i(x)]_{i,1=1}^n$ and $D^2 f(x) := [\partial_k \partial_j f_i(x)]_{i,j,k=1}^n$. That is,

$$|Df(x)| := \Big[\sum_{i,j=1}^n |\partial_j f_i(x)|^2 \Big]^{1/2}, \quad |D^2 f(x)| := \Big[\sum_{i,j,k=1}^n |\partial_k \partial_j f_i(x)|^2 \Big]^{1/2}.$$

In what follows, we will use basic facts about Sobolev spaces, such as the change of variables formula [Zie89, Thm 2.2.2] and that Lipschitz functions on Ω are characterized by the class $W^{1,\infty}(\Omega)$ [EG92, Thm 4.2.3.5]. The lemma below gives a gluing procedure for Sobolev functions.

Lemma 2.3. *For $i = 1, 2$, let O_i be a domain in $\mathbb{R}^n$ and let $f_i \in W^{1,p}_{loc}(O_i)$. If $f_1 = f_2$ holds a.e. on $O_1 \cap O_2$, then $\chi_{O_1} f_1 + \chi_{O_2 \setminus O_1} f_2 \in W^{1,p}_{loc}(O_1 \cup O_2)$.*

Proof. Let O be a bounded domain in $\mathbb{R}^n$ so that $\bar{O} \subset O_1 \cup O_2$. For each $x \in O$, there exists $r > 0$ so that $B(x, r)$ lies entirely in O_1 or in O_2. Since $\bar{O}$ is compact, there exists $N \in \mathbb{N}$ and a collection of balls $\{B(x_i, r_i)\}_{i=1}^N$ whose union covers O.

Let $\{\varphi_i\}_{i=1}^N$ be a smooth partition of unity that is subordinate to the cover $\{B(x_i, r_i)\}_{i=1}^N$. For each $i = 1, 2, \dots, N$, one of $f_1 \varphi_i$ or $f_2 \varphi_i$ is well-defined and lies in $W^{1,p}(O)$; call it ψ_i. We now observe that $\psi := \sum_{i=1}^N \psi_i$ also lies in $W^{1,p}(O)$ and by construction, it agrees with $\chi_{O_1} f_1 + \chi_{O_2 \setminus O_1} f_2$. □

It is a fact that the class LW_2^p is preserved under composition. This is stated as a lemma below, and it follows directly from the product rule [EG92, Thm 4.2.2.4] and the change of variables formula [Zie89, Thm 2.2.2].

Lemma 2.4. *Let $p \geq 1$. If $f : \Omega \to \Omega'$ and $g : \Omega' \to \Omega''$ are homeomorphisms of class LW_2^p, then so is $h := g \circ f$.*

In addition, for a.e. $x \in \Omega$ and for all $i, j, k \in \{1, \cdots n\}$, the weak derivatives satisfy

$$\begin{cases} \partial_j h_i(x) &= \sum_{l=1}^{n} \partial_l g_i(f(x)) \partial_j f_l(x) \\ \partial_{kj}^2 h_i(x) &= \sum_{l=1}^{n} \Big[\partial_l g_i(f(x)) \partial_{kj}^2 f_l(x) + \sum_{m=1}^{n} \partial_{ml}^2 g_i(f(x)) \partial_k f_m(x) \partial_j f_l(x)\Big]. \end{cases} \tag{2.1}$$

Remark 2.5. Linear maps (homeomorphisms) such as dilation and translation, are clearly of class LW_2^p. So if $g : \Omega \to \Omega'$ is any homeomorphism of class LW_2^p, then by Lemma 2.4, its composition with such linear maps is also of class LW_2^p. In what follows, we will implicitly use this fact to obtain convenient geometrical configurations.

3. Extensions for Homeomorphisms of Class LW_2^p between Doubly-Punctured Domains

First we formulate the extension theorem in a different geometric configuration.

Theorem 3.1. *Let $p \geq 1$, let E_1 and E_2 be Jordan domains so that $\overline{E_1} \cap \overline{E_2} = \emptyset$, and let B_1 and B_2 be balls so that $\overline{B_1} \cap \overline{B_2} = \emptyset$.*

If $g \colon (E_2 \cup E_1)^c \to (B_1 \cup B_2)^c$ is a homeomorphism of class LW_2^p so that $g(\partial E_i) = \partial B_i$ holds, for $i = 1, 2$, then there exists a homeomorphism $G : E_2^c \to B_2^c$ of class LW_2^p and a neighborhood N of ∂E_2 so that $g|(N \cap E_2^c) = G|(N \cap E_2^c)$.

Following the outline of [Geh67, Sect 3], we begin with a special case.

Lemma 3.2. *Theorem 3.1 holds under the additional assumption that*

$$g|B^c \;=\; \mathrm{id}\,|B^c \tag{3.1}$$

where B is an open ball that contains $\bar{E}_1$ and $\bar{E}_2$.

Proof. Step 1. By composing with linear maps, we may assume that $B = \mathbb{B}$, and that there exist $a, b \in \mathbb{R}$ so that $a < b$ and $\bar{B}_1 \subset \{x_n < a\}$ and $\bar{B}_2 \subset \{x_n > b\}$.

Put $c = (b - a)/2$. Define an odd, $C^{1,1}$-smooth function $s_0 : \mathbb{R} \to [-1, 1]$ by

$$s_0(t) \;:=\; \begin{cases} 1 - (t - c)^2/c^2, & \text{if } 0 \leq t \leq c \\ 1, & \text{if } t > c \end{cases}$$

and using the auxiliary function $s : \mathbb{R} \to [0, 3]$, given by

$$s(t) \;:=\; \frac{3}{2}\Big(s_0\Big(t - \frac{a+b}{2}\Big) + 1\Big)$$

we define a bi-Lipschitz homeomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$ by

$$S(x) \;=\; x - s(x_n)\, e_1. \tag{3.2}$$

It is clear that S is of class LW^p and satisfies the a.e. estimate

$$|D^2S| \;\le\; 2c^{-2}. \tag{3.3}$$

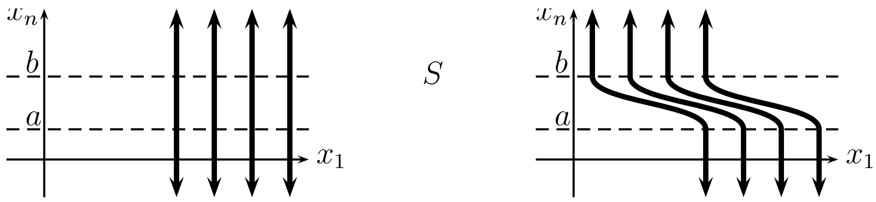

FIGURE 1. For $\mathbb{R}^2$, level curves for the map S.

Step 2.
For $k \in \mathbb{Z}$, put $\tau_k(x) = x + 3ke_1$ and consider the sets

$$\Omega \;:=\; \Big(\bigcup_{k=0}^{\infty} \tau_k(E_1) \cup \tau_k(E_2)\Big)^c \text{ and } \Omega' \;:=\; \Big(\bigcup_{k=0}^{\infty} \tau_k(B_1) \cup \tau_k(B_2)\Big)^c.$$

We now modify g into a new homeomorphism $g_* : \Omega \to \Omega'$, as follows:

$$g_*(x) \;:=\; \begin{cases} (\tau_k \circ g \circ \tau_{-k})(x), & \text{if } x \in \Omega \cap \tau_k(\mathbb{B}), \text{ for some } k \ge 0 \\ x, & \text{if } x \in \Omega \setminus \bigcup_{k=0}^{\infty} \tau_k(\mathbb{B}). \end{cases} \tag{3.4}$$

By our hypotheses, there exists $r \in (0,1)$ so that $E_1 \cup E_2 \subset B(0,r)$ and so that $g|\mathbb{B} \setminus B(0,r) = \mathrm{id}$. Putting $\Omega_1 := \tau_k(\mathbb{B}) \cap \Omega$ and $\Omega_2 := \Omega \setminus \bigcup_{l=0}^{\infty} \tau_k(\overline{B(0,r)})$ for each $k \in \mathbb{N}$, Lemma 2.2 implies that g_* is locally bi-Lipschitz.

Similarly, for any bounded domain O in Ω that meets $\tau_k(\partial\mathbb{B})$, put $O_1 := O \cap \Omega$ and $O_2 := O \setminus \tau_k(\overline{B(0,r)})$. For $f_1 := D(\tau_k \circ g \circ \tau_{-k})$ and $f_2 := D(\mathrm{id})$, Lemma 2.3 implies that $g_* \in W^{2,p}(O)$ and therefore $g_* \in W^{2,p}_{loc}(\Omega;\Omega')$. By symmetry, the same is true of g_*^{-1}, so g_* is of class LW_2^p.

Step 3. Consider the bi-Lipschitz homeomorphism given by

$$G_* \;:=\; \tau_1 \circ g_*^{-1} \circ S \circ g_*. \tag{3.5}$$

By Lemma 2.4, it is also of class LW_2^p. We now define $G : E_2^c \to B_2^c$ as

$$G(x) \;:=\; \begin{cases} G_*(x), & \text{if } x \in \Omega \\ \tau_1(x), & \text{if } x \in \bigcup_{k=0}^{\infty} \tau_k(E_1) \\ x, & \text{if } x \in \bigcup_{k=1}^{\infty} \tau_k(E_2). \end{cases} \tag{3.6}$$

By the same argument as [Geh67, pp. 153-4], the map G is a homeomorphism. We also note that G is "periodic" in the sense that, for each $k \in \mathbb{N}$,

$$\big(\tau_k \circ G \circ \tau_{-k}\big)|\tau_k(\bar{\mathbb{B}} \setminus E_2) \;=\; G|\tau_k(\bar{\mathbb{B}} \setminus E_2). \tag{3.7}$$

To see that G extends g, consider the set $\sigma_{ab} := g_*^{-1}\big(\{a \le x_n \le b\}\big)$. Its complement $\mathbb{R}^n \setminus \sigma_{ab}$ consists of two (connected) components. Let σ_b be the component containing the vector e_n, let σ_a be the component containing $-e_n$, and consider

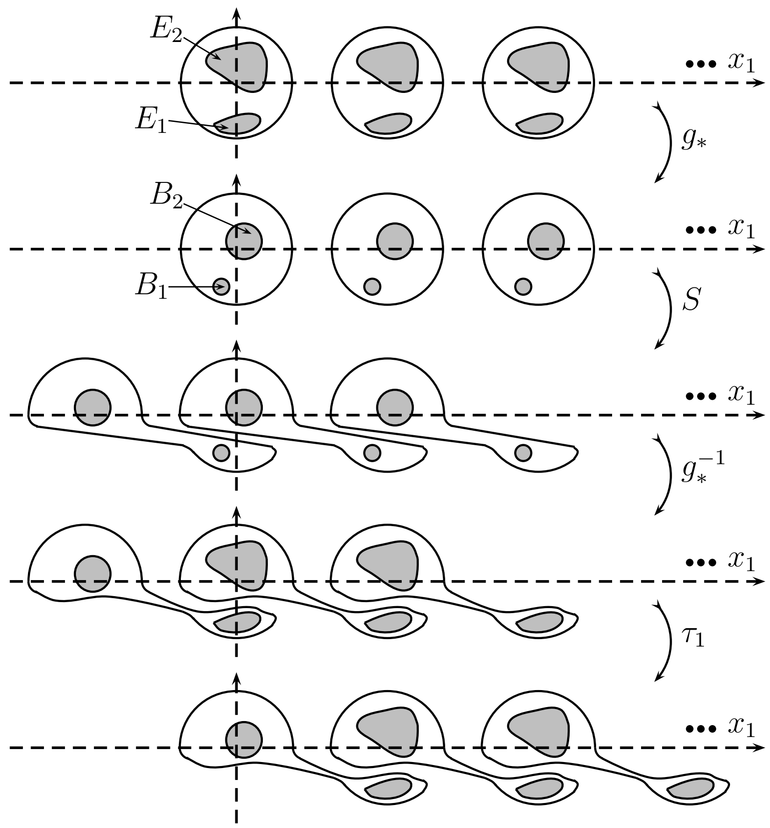

FIGURE 2. A schematic of the mapping G_*.

the open set $N := \mathbb{B} \cap \sigma_b$. By assumption, $\bar{B}_2$ lies in $\mathbb{B} \cap \{x_n > b\}$, so $\bar{E}_2$ lies in N. From before, we have $g_* = g$ on $\mathbb{B}$ and $S = \tau_{-1}$ on $\{x_n > b\}$, which imply that

$$(S \circ g_*)(N) \;=\; (\tau_{-1} \circ g)(N) \;=\; \tau_{-1}(\mathbb{B} \cap \{x_n > b\}) \;\subset\; \tau_{-1}(\mathbb{B}).$$

By hypothesis we have $g_*^{-1} = \mathrm{id}$ on $\tau_{-1}(\mathbb{B})$ and hence on $(S \circ g)(N)$. It follows that

$$G|N \;=\; G_*|N \;=\; (\tau_1 \circ g_*^{-1} \circ S \circ g_*)|N \;=\; (\tau_1 \circ \mathrm{id} \circ \tau_{-1} \circ g)|N \;=\; g|N.$$

As a result, G agrees with g on $N \cap E_2^c$.

Lastly, $G = \mathrm{id}$ holds on $\sigma_b \setminus E_2$ and $G = \tau_1$ holds on σ_a. Using these domains for Ω_1 and $\mathbb{R}^n \setminus \bigcup_{k=0}^{\infty} \tau_k(\bar{\mathbb{B}})$ for Ω_2, Lemma 2.2 implies that G is locally bi-Lipschitz. With the same choice of domains, Lemma 2.3 further implies that $G \in W^{2,p}_{loc}(E_2^c; B_2^c)$.

For the case of G^{-1}, note that the inverse is given by

$$G^{-1}(x) \;=\; \begin{cases} G_*^{-1}(x), & \text{if } x \in \Omega \setminus \tau_{-1}(B_2) \\ \tau_{-1}(x), & \text{if } x \in \bigcup_{k=0}^{\infty} \tau_k(E_1) \\ x, & \text{if } x \in \bigcup_{k=1}^{\infty} \tau_k(E_2). \end{cases} \tag{3.8}$$

Arguing similarly with $g_*(N)$ for N, it follows that $G^{-1} \in W^{2,p}_{loc}(B_2^c; E_2^c)$, which proves the lemma. $\square$

We now observe that Lemma 3.2 holds true even when B_1 and B_2 are not balls. In the preceding proof it is enough that, up to rotation, there is a slab $\{c_1 < x_n < c_2\}$ that separates B_1 from B_2. This result, stated below, is used in Section 4.

Lemma 3.3. *Let $p \geq 1$ and let E_1, E_2, C_1, and C_2 be Jordan domains so that $\overline{E_1} \cap \overline{E_2} = \emptyset$ and $\overline{C_1} \cap \overline{C_2} = \emptyset$. If $g : (E_1 \cup E_2)^c \to (C_1 \cup C_2)^c$ is a homeomorphism of class LW_2^p so that*

1. *$g(\partial E_i) = \partial B_i$ holds, for $i = 1, 2$,*
2. *there exists a ball B containing $\bar{E}_1$ and $\bar{E}_2$ so that $g|B^c = \mathrm{id}\,|B^c$,*
3. *there exist a rotation $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ and numbers $c_1, c_2 \in \mathbb{R}$, with $c_1 < c_2$, so that $\Theta(C_1) \subset \{x_n < c_1\}$ and $\Theta(C_2) \subset \{x_n > c_2\}$,*

then there is a homeomorphism $G : E_2^c \to C_2^c$ of class LW_2^p and a neighborhood N of ∂E_2 so that $g|(N \cap E_2^c) = G|(N \cap E_2^c)$.

Though the regularity of the extension G is local in nature, it nonetheless enjoys certain uniform properties. We summarize them in the next lemma.

Lemma 3.4. *Let E_1, E_2, C_1, C_2, B, and g be as in Lemma 3.3. If G is the extension of g as defined in Equation (3.6), then*

1. *$DG \in L^\infty(E_2^c)$ and $DG^{-1} \in L^\infty(C_2^c)$;*
2. *the restriction $G|B^c$ is a bi-Lipschitz homeomorphism.*

Proof. From Lemma 3.3, the map G is already locally bi-Lipschitz. To prove item (1), we will give a uniform bound for $L(G|K)$ over all compact subsets K of B^c. Let $B = \mathbb{B}$ and let S and g_* be as defined in the proof of Lemma 3.2.

Again, let $\sigma_{ab} := g_*^{-1}\big(\{a \leq x_n \leq b\}\big)$ and let σ_b and σ_a be the (connected) components of $\mathbb{R}^n \setminus \sigma_{ab}$ containing the vectors e_n and $-e_n$, respectively.

By Equation (3.2), we have $S|\{x_n < a\} = \mathrm{id}$ and $S|\{x_n > b\} = \tau_{-1}$, which imply, respectively, the bounds $L(G|\mathbb{B}^c \cap \sigma_a) \leq 1$ and $L(G|\mathbb{B}^c \cap \sigma_b) \leq 1$.

It remains to estimate $L(G|\mathbb{B}^c \cap \sigma_{ab})$. For each $k \in \mathbb{N}$, the set $\sigma_{ab}^k := \sigma_{ab} \cap \tau_k(\bar{\mathbb{B}})$ is compact, so by Lemma 2.1, the restriction $G|\sigma_{ab}^k$ is bi-Lipschitz. Equation (3.7) then implies that

$L(G|\sigma_{ab}^k) = L(G|\sigma_{ab}^1)$ holds for eack $k \in \mathbb{N}$.

The remaining set $\sigma_{ab} \setminus \bigcup_{k=0}^\infty \tau_k(\mathbb{B})$ consists of infinitely many components, one of which is an unbounded subset U of $\{x_1 < 0\}$ and the others are translates of a compact subset K_0 of $\sigma_{ab} \cap B(0,3)$. Since $g|U = \mathrm{id}$, it follows that

$$G|\sigma \;=\; (\tau_1 \circ g_*^{-1} \circ S \circ g_*)|\sigma \;=\; (\tau_1 \circ S)|\sigma$$

from which $L(G|\sigma) \leq L(S)$ follows. By the 'periodicity' of G (Equation (3.7)), for all $k \in \mathbb{N}$ we also have $L(G|\tau_k(K_0) = L(G|K_0)$. Item (1) of the lemma follows from [EG92, Thm 4.2.3.5] and from the above estimates, where

$$\|DG\|_{L^\infty(E_2^c)} \;\leq\; \max\left\{1, L(G|K_0), L(G|\sigma^1_{ab}), L(S)\right\}.$$

Using the explicit formula in Equation (3.8), the case of G^{-1} follows similarly.

To prove item (2), let ℓ be any line segment that does not intersect $\mathbb{B}$. The restriction $G|\ell$ is bi-Lipschitz with $L(G|\ell) \leq C$. Since $\partial\mathbb{B}$ is compact, it follows from Lemma 2.1 that the restriction $G|\partial\mathbb{B}$ is bi-Lipschitz.

Let x_1 and x_2 be arbitrary points in $\mathbb{B}^c$ and let ℓ be the line segment in $\mathbb{R}^n$ which joins x_1 to x_2. If ℓ crosses through $\mathbb{B}$, then let y_1 and y_2 be points on $\ell \cap \partial\mathbb{B}$, where $|x_1 - y_1| < |x_1 - y_2|$. Since ℓ is a geodesic, we have the identity

$$|x_1 - x_2| \;=\; |x_1 - y_1| + |y_1 - y_2| + |y_2 - x_2|.$$

The Triangle inequality then implies that

$$\begin{aligned} |G(x_1) - G(x_2)| &\leq |G(x_1) - G(y_1)| + |G(y_1) - G(y_2)| + |G(y_2) - G(x_2)| \\ &\leq C\,(|x_1 - y_1| + |y_2 - x_2|) + L(G|\partial\mathbb{B})\,|y_1 - y_2| \\ &\leq \big(C + L(G|\partial\mathbb{B})\big)\,(|x_1 - y_1| + |y_1 - y_2| + |y_2 - x_2|) \\ &= \big(C + L(G|\partial\mathbb{B})\big)\,|x_1 - x_2|. \end{aligned}$$

Again, the argument is symmetric for G^{-1}, so this proves the lemma. □

Theorem 3.1 now follows easily from Lemma 3.2, and a more general version of the theorem follows from Lemma 3.3. As in [Geh67, Lemma 2], one takes compositions with the extension, its inverse, and a radial stretch map.

Proof of Theorem 3.1. By composing g with linear maps, we may assume that E_1, E_2, B_1 and B_2 are subsets of $\mathbb{B}$, that $0 \in E_2$, and that $\mathbb{B}^c \subset g(\mathbb{B}^c)$.

Choose $r_1, r_2 \in (0,1)$ so that $B(0, r_1) \subset E_2$ and that $E_1 \cup E_2 \subset B(0, r_2)$.

Let $\rho\colon [0,\infty) \to [0,\infty)$ be a smooth increasing function so that $\rho\big([0, r_1]\big) = [0, r_2]$ and $\rho\big([1,\infty)\big) = [1,\infty)$. Define a homemorphism $R : \mathbb{R}^n \to \mathbb{R}^n$ by

$$R(x) \;:=\; \begin{cases} \rho(|x|) \cdot |x|^{-1}x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases} \tag{3.9}$$

Clearly, R is of class LW^p_2 and bi-Lipschitz, and maps $B(0, r_1)$ onto $B(0, r_2)$.

Putting $E_1' := (g \circ R)(E_1)$ and $E_2' := \big((g \circ R)(E_2^c)\big)^c$, Lemma 2.4 implies that

$$h := g \circ R \circ g^{-1} : (E_1' \cup E_2')^c \to (B_1 \cup B_2)^c$$

is also a homeomorphism of class LW^p_2. Since $R|\mathbb{B}^c = \mathrm{id}\,|\mathbb{B}^c$, we further obtain

$$h|\mathbb{B}^c \;=\; (g \circ R \circ g^{-1})|\mathbb{B}^c \;=\; \mathrm{id}\,|\mathbb{B}^c. \tag{3.10}$$

So with E_1' and E_2' in place of E_1 and E_2, respectively, h satisfies Equation (3.1) and the other hypotheses of Lemma 3.2. As a result, there exists a homeomorphism $H : (E_2')^c \to B_2^c$ of class LW^p_2 and a neighborhood N' of $\partial E_2'$ so that

$$h|(N' \cap (E_2')^c) \;=\; H|(N' \cap (E_2')^c).$$

Let $G := H \circ g \circ R^{-1}$. The open set

$$N \ := \ (R \circ g^{-1})(N' \setminus (\bar{B}_1 \cup \bar{B}_2))$$

contains ∂E_2, and by Lemma 2.4, the map G is of class LW_2^p. Moreover, for each $x \in N \setminus E_2$, there is a $y \in N' \setminus D_2'$ so that $x = (R \circ g^{-1})(y)$ and therefore

$$\begin{aligned} G(x) = (H \circ g \circ R^{-1})\big((R \circ g^{-1})(y)\big) \ &= \ H(y) \\ = h(y) \ = \ (g \circ R \circ g^{-1})\big((g \circ R^{-1})(x)\big) \ &= \ g(x). \end{aligned}$$

We thereby obtain $g = G$ on $N \cap E_2^c$, as desired. □

4. Extensions of Homeomorphisms of Class LW_2^p between Collars

4.1. **Generalized Inversions.** To pass to the configurations of domains in Theorem 1.3, we will use *generalized inversions.* For fixed $a, r > 0$, these are homeomorphisms $I_{a,r} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ of the form

$$I_{a,r}(x) \ := \ r^{a+1}|x|^{-(a+1)}x.$$

Indeed, the inverse map satisfies $(I_{a,r})^{-1} = I_{1/a,\,r}$, as well as the estimate

$$|x|^{a+1} \ = \ \big(r^{1/a+1}|I_{a,r}(x)|^{-1/a}\big)^{a+1} \ \approx \ |I_{a,r}(x)|^{-(1/a+1)}. \tag{4.1}$$

For derivatives of $I_{a,r}$, an elementary computation gives

$$|D^k I_{a,r}(x)| \ \lesssim \ r^{a+1}|x|^{-(a+k)} \tag{4.2}$$

and similarly, for the Jacobian determinant $JI_{a,r} := |\det(DI_{a,r})|$ we have

$$JI_{a,r}(x) \ \leq \ n\, r^{n(a+1)}|x|^{-n(a+1)} \ \approx \ |I_{a,r}(x)|^{n(a+1)/a}. \tag{4.3}$$

If $a = 1$, then $I_{1,r}$ is conformal and maps spheres to spheres. In general, the map $I_{a,r}$ possesses weaker properties which are sufficient for our purposes. For instance, it preserves radial rays, or sets of the form $\{\lambda x : \lambda > 0\}$ for some $x \in \mathbb{R}^n \setminus \{0\}$.

Another property, stated below, is used in the proof of Theorem 1.3 under the following hypotheses. To begin, write $B_1 = B(t, r_1)$ and $B_2 = B(z, r_2)$, where $\bar{B}_1 \subset B_2$. By composing with linear maps, we may assume that

(H1) The x_n-coordinate axis crosses through the points t and z, with $t_n \leq z_n \leq 0$. As a result, the 'south poles' $\tau := t - r_1\vec{e}_n$ on $\bar{B}_1$ and $\zeta := z - r_2\vec{e}_n$ on $\bar{B}_2$ satisfy $\zeta_n < \tau_n$ and $|\zeta - \tau| = \operatorname{dist}(\bar{B}_1, B_2^c)$.

(H2) There exists $r \in (0, r_2)$ so that the sphere $\partial B(0, r)$ is tangent to both ∂B_1 and ∂B_2, with $B(0, r) \subset B_2 \setminus B_1$. In particular, this gives $r_1 < |t_n|$.

Lemma 4.1. *Let $a \in (0, 1)$. If B_1 and B_2 are balls in $\mathbb{R}^n$ with $\bar{B}_1 \subset B_2$ and which satisfy hypotheses (H1) and (H2), then there exist real numbers $c_1 < c_2$ so that $I_{a,r}(B_1) \subset \{x_n < c_1\}$ and $I_{a,r}(B_2^c) \subset \{x_n > c_2\}$.*

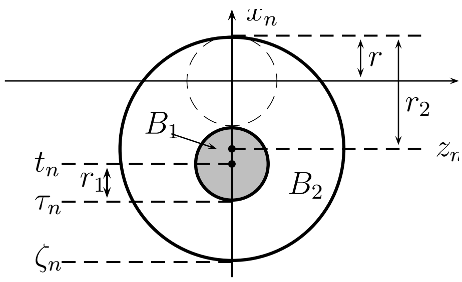

FIGURE 3. A possible configuration for B_1, B_2, and $B(0,r)$.

The proof is a computation, and the basic idea is simple. Though the bounded domains $I_{a,r}(B_1)$ and $I_{a,r}(B_2^c)$ may not be balls, the distance between them is still attained by the images of the 'north' and 'south' poles of B_1 and B_2, respectively.

Proof. Once again, let τ and ζ be the "south poles" of B_1 and B_2, respectively. From Hypotheses (H1) and (H2), we have

$$\zeta_n \;=\; -|\zeta| \;<\; -|\tau| \;=\; \tau_n.$$

and putting $I_: = I_{a,r}$, the image points $\tau' := I(\tau)$ and $\zeta' := I(\zeta)$ therefore satisfy

$$\tau'_n \;=\; -|\tau'| \;<\; -|\zeta'| \;=\; \zeta'_n. \tag{4.4}$$

Claim 4.2. For all $y' \in I(B_1)$, we have $y'_n < \tau'_n$.

Supposing otherwise, there exists $y \in \partial B_1$ with $y \neq \tau$ and so that y' has the same nth coordinate as τ'. Let θ be the angle between the x_n-axis and the line crossing through y' and 0. By our hypotheses, we have $t_n \leq 0$ and $0 < \theta < \frac{\pi}{2}$ and therefore $0 < \cos\theta < 1$. From $|\tau| = r_1 - t_n$, we obtain

$$|y'| \;=\; \frac{|\tau'|}{\cos\theta} \;=\; \frac{r^{a+1}|\tau|^{-a}}{\cos\theta} \;=\; \frac{r^{a+1}}{(r_1 - t_n)^a\cos\theta}$$

so from $|y'| = r^{a+1}|y|^{-a}$ and the above identity, we further obtain

$$|y| \;=\; r^{(a+1)/a}\left[\frac{r^{a+1}}{(r_1 - t_n)^a\cos\theta}\right]^{-1/a} \;=\; (\cos\theta)^{1/a}(r_1 - t_n). \tag{4.5}$$

On the other hand, I preserves radial rays and hence angles between radial rays. As a result, $y \in \partial B_1$ (and the Law of Cosines) imply that

$$r_1^2 = |y|^2 + t_n^2 - 2|y|t_n\cos\theta,$$

$$\text{so } |y| = -t_n\cos\theta \;+\; \sqrt{r_1^2 - t_n^2\sin^2\theta}.$$

From Hypothesis (H2) once again, we obtain $r_1 < |\tau_n|$ and hence

$$|y| \;<\; -t_n\cos\theta \;+\; \sqrt{r_1^2 - r_1^2\sin^2\theta} \;=\; (r_1 - t_n)\cos\theta.$$

This is in contradiction with Equation (4.5), since the inequality $\cos\theta \le (\cos\theta)^{1/a}$ follows from $a \ge 1$. The claim follows.

Claim 4.3. For all $w' \in I(B_2^c)$, we have $\zeta_n' < w_n'$.

Suppose there exists $w \in \partial B_2$ so that $w \neq \zeta$ and $w_n' = \zeta_n'$. If α is the angle between w and the x_n-axis, then a similar computation as above gives

$$(2r_2 - r)\cos^{1/a}\alpha \;=\; |w| \;=\; (r_2 - r)\cos\alpha + \sqrt{r_2^2 - (r_2 - r)^2\sin^2\theta}$$

Computing further, we obtain $\psi(a) = r_2^2$, where $\psi : (0,\infty) \to (0,\infty)$ is given by

$$\psi(a) \;:=\; \big((2r_2 - r)\cos^{1/a}\alpha - (r_2 - r)\cos\alpha\big)^2 + (r_2 - r)^2\sin^2\alpha$$

Clearly ψ is smooth and an elementary computation shows that it attains a minimum at a unique point in $(0,1)$. We observe that

$$\psi(1) \;=\; r_2^2\cos^2\alpha \;+\; (r_2 - r)^2\sin^2\alpha \;<\; r_2^2.$$

Since $0 < \cos\alpha < 1$, we see that $\cos^{1/a}\alpha \to 0$ as $a \to 0$. It follows that

$$\lim_{a\to 0}\psi(a) \;=\; \big(0 + (r_2 - r)\cos\alpha\big)^2 + (r_2 - r)^2\sin^2\alpha \;=\; (r_2 - r)^2 \;<\; r_2^2$$

and therefore $\psi(a) < r_2^2$ holds for all $(0,1)$. This is a contradiction, which proves Claim 4.3. Combining both claims and Equation (4.4), the lemma follows. □

4.2. **From Doubly-Punctured Domains to Collars.** We now prove Theorem 1.3. The argument requires several lemmas.

Lemma 4.4. *Let $a > 0$ and let D_1, D_2, B_1, B_2, and f be given as in Theorem 1.3. If there exists $r > 0$ so that $\bar{B}(0,r) \subset D_2 \setminus D_1$ and $\bar{B}(0,r) \subset B_2 \setminus B_1$, and if $f(0) = 0$, then $I_{a,r} \circ f \circ I_{a,r}^{-1}$ is a homeomorphism of class LW_2^p.*

Proof. Since $\Omega := I_{a,r}(D_2 \setminus (\bar{D}_1 \cup \{0\}))$ and $I_{a,r}(B_2 \setminus (\bar{B}_1 \cup \{0\}))$ lie in $\mathbb{R}^n \setminus B(0,\epsilon)$, for some $\epsilon > 0$, the restricted maps $I_{a,r}^{-1}|\Omega$ and $I_{a,r}|\Omega'$ are diffeomorphisms. By Lemma 2.4, it follows that $g := I_{a,r} \circ f \circ I_{a,r}^{-1} : \Omega \to \Omega'$ is of class LW_2^p. □

Lemma 4.5. *Let E_1, E_2, C_1, C_2, B, and g be given as in Lemma 3.3, and let G be given as in Equation* (3.6)*. If $0 \in E_2$, if $0 \in C_2$, and if there exists $r > 0$ so that $B = B(0,r)$, then for each $a > 0$, the map*

$$F(x) \;:=\; \begin{cases} \big(I^{-1} \circ G \circ I\big)(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is a locally bi-Lipschitz homeomorphism.

Proof. Without loss of generality, let $r = 1$ and put $I = I_{a,r}$ and $b = 1/a$. By Equation (3.6), we have $|G(x)| \to \infty$ as $|x| \to \infty$, so F is a well-defined homeomorphism. For each $\epsilon > 0$, put $B_\epsilon := B(0,\epsilon)$. The restrictions $I|B_\epsilon^c$ and $I^{-1}|B_\epsilon^c$ are diffeomorphisms, so $F|B_\epsilon^c$ is already locally bi-Lipschitz for each $\epsilon > 0$.

To show that $F|B_\epsilon$ is bi-Lipschitz, recall that $DG \in L^\infty(E_2^c)$ follows from Lemma 3.4. So from Equations (2.1), (4.1), and (4.2), it follows that, for a.e. $x \in I^{-1}(E_2^c)$,

$$\begin{aligned} |DF(x)| &\leq |DI^{-1}\big((G\circ I)(x)\big)|\,|DG(I(x))|\,|DI(x)| \\ &\lesssim \frac{\|DG\|_\infty}{|(G\circ I)(x)|^{b+1}\,|x|^{a+1}} \approx \frac{\|DG\|_\infty\,|I(x)|^{b+1}}{|(G\circ I)(x)|^{b+1}}. \end{aligned}$$

Now fix $y_0 \in E_2^c$. Putting $L := L(G^{-1}|B^c)$, for all $x \in B_\epsilon$ we have

$$|G(I(x)) - G(y_0)| \;\geq\; L^{-1}\big(|I(x) - y_0|\big) \;\geq\; L^{-1}\big(|I(x)| - |y_0|\big).$$

Applying the triangle inequality to the right-hand side, we obtain

$$|G(I(x))| \;\geq\; L^{-1}\big(|I(x)| - |y_0|\big) \;-\; |G(y_0)|$$

and taking reciprocals, we further obtain

$$\text{(4.6)} \qquad \left\{\begin{aligned} \frac{|I(x)|}{|(G\circ I)(x)|} &\leq \frac{L\,|I(x)|}{|I(x)| - |y_0| - L\,|G(y_0)|} \\ &= \frac{L\,r^{a+1}}{r^{a+1} - |x|^a\,|y_0| - |x|^a\,L\,|G(y_0)|} \;\to\; L \end{aligned}\right.$$

as $x \to 0$. Combining the previous estimates, for sufficiently small $\epsilon > 0$

$$|DF(x)| \;\lesssim\; \frac{\|DG\|_\infty\,|I(x)|^{b+1}}{|(G\circ I)(x)|^{b+1}} \;\lesssim\; (2L)^{b+1}\|DG\|_\infty \;<\; \infty$$

holds for a.e. $x \in B_\epsilon$, and therefore $|DF| \in L^\infty_{\mathrm{loc}}(I^{-1}(E_2^c))$. By [EG92, Thm 4.2.3.5], it follows that F is locally Lipschitz on $B(0,\epsilon)$. By symmetry, the same holds for F^{-1}, so F is locally bi-Lipschitz on all of $I^{-1}(E_2^c)$. □

In the remaining proofs, we will require explicit forms of the extensions from Lemma 3.2 and from Theorem 3.1.

Lemma 4.6. *Let E_1, E_2, C_1, C_2, g, and $B = B(0,r)$ be given as in Lemma 4.5, let G be given as in Equation* (3.6)*, and let $p \in [1,n)$. If $a < n/p - 1$, then the homeomorphism $I_{a,r}^{-1} \circ G \circ I_{a,r}$ is of class LW_2^p.*

Proof. For convenience, we reuse the notation from the proof of Lemma 4.5. As before, $I|B_\epsilon^c$ and $I^{-1}|B_\epsilon^c$ are diffeomorphisms, so by Lemma 2.4, the map $F|B_\epsilon^c$ is of class LW_2^p. It suffices to show that $F \in W^{2,p}_{loc}(B_\epsilon;\mathbb{R}^n)$ and $F^{-1} \in W^{2,p}_{loc}(F(B_\epsilon);B_\epsilon)$, for each $\epsilon > 0$.

To estimate second derivatives, we use Equations (2.1), (4.1), (4.2), and (4.6) once again. As a shorthand, put $y := I(x)$ and $z := (G \circ I)(x)$. We then obtain

$$
\left\{\begin{aligned}
|D^2F(x)| &= |D^2(I^{-1} \circ G \circ I)(x))| \\
&\leq |D^2I^{-1}(z)|\,|DG(y)|^2|DI(x)|^2 \\
&\quad + |DI^{-1}(z)|\Big(|D^2G(y)|\,|DI(x)|^2 + |DG(y)|\,|D^2I(x)|\Big) \\
&\lesssim \frac{\|DG\|_\infty^2}{|z|^{b+2}|x|^{2(a+1)}} + \frac{1}{|z|^{b+1}}\left(\frac{|D^2G(y)|}{|x|^{2(a+1)}} + \frac{\|DG\|_\infty}{|x|^{a+2}}\right) \\
&\lesssim \frac{|I(x)|^{2(b+1)}}{|G(I(x))|^{b+2}} + \frac{|I(x)|^{2(b+1)}|D^2G(I(x))|}{|G(I(x))|^{b+1}} + \frac{|I(x)|^{b+1}}{|G(I(x))|^{b+1}|x|} \\
&\lesssim |I(x)|^b + |I(x)|^{b+1}|D^2G(I(x))| + |x|^{-1}
\end{aligned}\right. \tag{4.7}
$$

for a.e. $x \in B_\epsilon$. Since $p < n$ and $b = 1/a$, the function $x \mapsto |I(x)|^b = |x|^{-1}$ lies in $L^p(B_\epsilon)$. For the remaining term, Equations (4.1) and (4.3) imply that

$$1 \;=\; JI^{-1}\big(I(x)\big)JI(x) \;\lesssim\; |I^{-1}(I(x))|^{n(a+1)}JI(x) \;=\; |I(x)|^{-n(b+1)}JI(x)$$

so by a change of variables [Zie89, Thm 2.2.2] and Equation (4.3), we have

$$
\left\{\begin{aligned}
\int_{B_\epsilon} |I(x)|^{p(b+1)}|D^2G(I(x))|^p\,dx &\lesssim \int_{B_\epsilon} \frac{|D^2G(I(x))|^p JI(x)}{|I(x)|^{(n-p)(b+1)}}\,dx \\
&= \int_{\mathbb{B}^c} \frac{|D^2G(y)|^p}{|y|^{(n-p)(b+1)}}\,dy.
\end{aligned}\right. \tag{4.8}
$$

For each $k \in \mathbb{N}$, Equation (3.6) implies that $G|\tau_k(E_2) = \mathrm{id}$ and $G|\tau_k(E_1) = \tau_1$, and therefore $D^2G|\tau_k(E_1 \cup E_2) = 0$. The rightmost integral in Equation (4.8) can therefore be restricted to the subset

$$\Omega \;:=\; \mathbb{B}^c \setminus \bigcup_{k=1}^{\infty} \tau_k(E_1 \cup E_2).$$

As defined in the proof of Lemma 3.2 the maps g_*, G_*, and G satisfy

$$|D^2G(y)| \;\lesssim\; |D^2g_*^{-1}((S \circ g_*)(y))| + |D^2S(g_*(y))| + |D^2g_*(y)| \tag{4.9}$$

for a.e. $y \in I^{-1}(E_2^c)$, and where $\lesssim$ includes the constants $L(g_*)$, $L(g_*^{-1})$, $L(S)$, and $L(\tau_1)$. Using the second derivative bound for S (Equation (3.3)), we obtain

$$\int_\Omega \frac{|D^2S(g_*(y))|^p}{|y|^{(n-p)(b+1)}}\,dy \;\leq\; \int_\Omega \frac{2c^{2p}}{|y|^{(n-p)(b+1)}}\,dy \;\lesssim\; \int_1^\infty \frac{\rho^{n-1}}{\rho^{(n-p)(b+1)}}\,d\rho.$$

The rightmost integral is finite, since $a < n/p - 1$ implies that $b > p/(n-p)$ and

$$(n-1) - (n-p)(b+1) \;<\; (n-1) - (n-p)\Big(\frac{p}{n-p} - 1\Big) \;=\; -1.$$

For the other terms of Equation (4.9), Equation (3.4) implies that $D^2g_*^{-1}(z) = 0$ for a.e. $z \notin \bigcup_{k=1}^{\infty} \tau_k(\mathbb{B})$. Since $S \circ g_*$ is locally bi-Lipschitz, we estimate

$$\int_{\Omega} \frac{|D^2g_*^{-1}\big((S\circ g_*)(y)\big)|^p}{|y|^{(n-p)(b+1)}}\,dy = \sum_{k=1}^{\infty} \int_{\tau_k((S\circ g_*)^{-1}(\mathbb{B}))\cap\Omega} \frac{|D^2g_*^{-1}\big((S\circ g_*)(y)\big)|^p}{|y|^{(n-p)(b+1)}}\,dy$$
$$\approx \sum_{k=1}^{\infty} \int_{g_*^{-1}(\Omega)\cap\tau_k(\mathbb{B})} \frac{|D^2g_*^{-1}(z)|^p\,dz}{|(S\circ g_*)^{-1}(z)|^{(n-p)(b+1)}}$$

Equation (3.2) implies that $|S^{-1}(y)| \geq |y|$ holds, for each $y \in \mathbb{R}^n$, and therefore

$$|(S\circ g_*)^{-1}(z)| \;\geq\; 3k-1 \;>\; k$$

holds, for each $z \in \tau_k(\mathbb{B})$ and each $k \in \mathbb{N}$. From the above inequalities and another change of variables, we further estimate

$$\int_{g_*^{-1}(\Omega)\cap\tau_k(\mathbb{B})} \frac{|D^2g_*^{-1}(z)|^p}{|(S\circ g_*)^{-1}(z)|^{(n-p)(b+1)}}\,dz \lesssim \int_{g_*^{-1}(\Omega)\cap\tau_k(\mathbb{B})} \frac{|D^2g_*^{-1}(z)|^p}{k^{(n-p)(b+1)}}\,dz$$
$$\leq \frac{\int_{\mathbb{B}\setminus(C_1\cup C_2)} |D^2g^{-1}(z)|^p\,dz}{k^{(n-p)(b+1)}},$$
$$\text{so } \int_{\Omega} \frac{|D^2g_*^{-1}\big((S\circ g_*)(y)\big)|^p}{|y|^{(n-p)(b+1)}}\,dy \lesssim \sum_{k=1}^{\infty} \frac{\|D^2g^{-1}\|_{L^p(\mathbb{B}\setminus(C_1\cup C_2))}}{k^{(n-p)(b+1)}}.$$

The rightmost sum is finite, since $(n-p)(b+1) > 1$ follows from the hypothesis that $a < n/p - 1$. A similar estimate gives $|y|^{(p-n)(b+1)}|D^2g_*(y)| \in L^p(B_\epsilon)$, so by Equations (4.7)-(4.9), we obtain $|D^2F| \in L^p(B_\epsilon)$, as desired.

The same argument, with G^{-1} for G, shows that the map $F^{-1} = I^{-1}\circ G^{-1}\circ I$ also lies in $W^{2,p}_{loc}(F(B_\epsilon); B_\epsilon)$. This proves the lemma. □

Using the previous lemmas, we now prove the main theorem.

Proof of Theorem 1.3. Let $a < n/p - 1$ be given.

By post-composing f with linear maps, we may assume that the balls B_1 and B_2 satisfy hypotheses (H1) and (H2) from Section 4.1, so in particular we have $B(0,r) \subset B_2 \setminus \bar{B}_1$. We further assume that $B(0,r) \subset D_2 \setminus \bar{D}_1$ and $f(0) = 0$.

By Lemma 4.1, there exist $c_1 < c_2$ so that $B_1 \subset \{x_n < c_1\}$ and $B_2 \subset \{x_n > c_2\}$. For $I := I_{a,r}$ and $g := I \circ f \circ I^{-1}$, Lemma 4.4 implies that g is of class LW^p_2.

Put $E_1 = I(D_1)$, $E_2 := I(D_2^c)^c$, $C_1 := I(B_1)$, and $C_2 := I((B_2)^c)^c$. By Lemma 3.3 and the proof of Theorem 3.1, there exists a homeomorphism $G\colon E_2^c \to C_2^c$ of class LW^p_2 and a neighborhood N' of ∂E_2 so that

$$g|(N'\cap E_2^c) \;=\; G|(N'\cap E_2^c).$$

As a result, the homeomorphism F, as defined in Lemma 4.5, and the open set $N := I^{-1}(N')$, a neighborhood of ∂D_2, therefore satisfy the identity

$$f|(N\cap \bar{D}_2) \;=\; F|(N\cap \bar{D}_2).$$

Recalling the proof of Theorem 3.1, we have $G = H \circ g \circ R^{-1}$, where

(H3) R is a diffeomorphism that agrees with the identity map on $\mathbb{B}^c$;

(H4) H is a homeomorphism of class LW_2^p, as given from Lemma 3.3, that agrees with $h = g \circ R \circ g^{-1}$ on the open set $(g \circ R)(N')$.

Putting $H_* := I^{-1} \circ H \circ I$ and $R_* := I^{-1} \circ R \circ I$, we rewrite

$$F \;=\; I^{-1} \circ (H \circ g \circ R^{-1}) \circ I \;=\; H_* \circ f \circ R_*^{-1}.$$

From property (H3) and properties of I and I^{-1}, we see that R_*^{-1} is a diffeomorphism from $\mathbb{R}^n \setminus \{0\}$ onto itself. In particular, for each $r > 0$ the restriction $R_*^{-1}|B(0,r)^c$ is bi-Lipschitz. On the other hand, for sufficiently small $r > 0$ we have $R^{-1} \circ I = I$ on $B(0,r)$. Letting Id_n be the $n \times n$ identity matrix,

$$DR_*^{-1}|B(0,r) = D(I^{-1} \circ R^{-1} \circ I)|B(0,r) \;=\; D(I^{-1} \circ I)|B(0,r) \;=\; \mathrm{Id}_n$$
$$D^2R_*^{-1}|B(0,r) = D^2(I^{-1} \circ R^{-1} \circ I)|B(0,r) \;=\; D^2(I^{-1} \circ I)|B(0,r) \;=\; 0.$$

This implies that $R_*^{-1} \in W^{2,p}_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ and by Lemma 2.2, that R_*^{-1} is bi-Lipschitz. By symmetry the same holds for $R_* = I^{-1} \circ R \circ I$, so R_*^{-1} is of class LW_2^p.

Property (H4) and Lemma 4.6 imply that H_* is of class LW_2^p. By hypothesis, f is of class LW_2^p, so by Lemma 2.4, F is of class LW_2^p. The theorem follows. □

References

[Ale24] J.W. Alexander. An example of a simply connected surface bounding a region which is not simply connected. *Proc. Nat. Acad. Sci. U.S.A.*, 10:8–10, 1924.

[Bro60] M. Brown. A proof of the generalized Schoenflies theorem. *Bull. Amer. Math. Soc.*, 66:74–76, 1960.

[EG92] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[Geh67] F. W. Gehring. Extension theorems for quasiconformal mappings in n-space. *J. Analyse Math.*, 19, 1967.

[Hir94] M. W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[HK09] Juha Heinonen and Stephen Keith. Flat forms, bi-lipschitz parametrizations, and smoothability of manifolds. preprint, 2009.

[Ker60] Michel A. Kervaire. A manifold which does not admit any differentiable structure. *Comment. Math. Helv.*, 34:257–270, 1960.

[KM63] M. A. Kervaire and J. W. Milnor. Groups of homotopy spheres. I. *Ann. of Math. (2)*, 77:504–537, 1963.

[LV77] J. Luukkainen and J. Väisälä. Elements of Lipschitz topology. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 3(1):85–122, 1977.

[Mil56] John Milnor. On manifolds homeomorphic to the 7-sphere. *Ann. of Math. (2)*, 64:399–405, 1956.

[Sul75] D. Sullivan. Hyperbolic geometry and homeomorphisms. In *Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977)*, pages 543–555, 1975.

[TV81] P. Tukia and J. Väisälä. Lipschitz and quasiconformal approximation and extension. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 6(2):303–342 (1982), 1981.

[Väi71] Jussi Väisälä. *Lectures on n-dimensional quasiconformal mappings*. Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin, 1971.

[Wil79] Raymond Louis Wilder. *Topology of manifolds*, volume 32 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, R.I., 1979. Reprint of 1963 edition.

[Zie89] W. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989.

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