WIENER TYPE THEOREMS FOR JACOBI SERIES WITH NONNEGATIVE COEFFICIENTS

H. N. MHASKAR AND S. TIKHONOV

ABSTRACT. This paper gives three theorems regarding functions integrable on $[-1,1]$ with respect to Jacobi weights, and having nonnegative coefficients in their (Fourier–)Jacobi expansions. We show that the L^p-integrability (with respect to the Jacobi weight) on an interval near 1 implies the L^p-integrability on the whole interval if p is an even integer. The Jacobi expansion of a function locally in L^∞ near 1 is shown to converge uniformly and absolutely on $[-1,1]$; in particular, such a function is shown to be continuous on $[-1,1]$. Similar results are obtained for functions in local Besov approximation spaces.

1. INTRODUCTION

A well-known theorem by Norbert Wiener states (see e.g. [4, pp. 242, 250]) that if f is a 2π–periodic function in $L^1(-\pi,\pi)$, $f \in L^2(-\delta,\delta)$ for some $\delta > 0$, and the Fourier coefficients $c_n(f) \ge 0$, $n \in \mathbb{Z}$, then $f \in L^2(-\pi,\pi)$. Since the Fourier coefficients of $|f|^2$ are given formally by $\sum_{k\in\mathbb{Z}} c_{n-k}(f)\overline{c_k(f)}$, it easy to deduce that a similar conclusion holds also when the L^2 norms are replaced by L^p norms, $p = 2, 4, 6, \cdots$. More concisely, for $1 \le p \le \infty$, let

$$L^p_{\text{loc},+} := \left\{ f \in L^1(-\pi,\pi) : f \sim \sum_{n\in\mathbb{Z}} c_n(f) e^{in\circ}, \right.$$
$$c_n(f) \ge 0 \text{ for every } n \in \mathbb{Z},$$
$$\left. \text{and there exists } \delta > 0 \text{ such that } f \in L^p(-\delta,\delta) \right\}.$$

Then Wiener's theorem states that for even, positive, integer values of p,

$$L^p_{\text{loc},+} \subset L^p(-\pi,\pi).$$

Wainger [15] and Shapiro [11] have given counterexamples to show that such an inclusion is not true if p is not an even, positive, integer.

Even though the inclusion $L^p_{\text{loc},+} \subset L^p(-\pi,\pi)$ does not hold in general, the variation stated in the following Theorem 1.1 was proved in [1]. A subspace

2000 *Mathematics Subject Classification.* Primary 33C45, 42C10; Secondary 46E30.
Key words and phrases. Fourier–Jacobi expansion, positive coefficients, Besov spaces.

$X \subset L^1(-\pi,\pi)$ is called *solid* if

$$f, g \in L^1(-\pi,\pi), \qquad |c_n(f)| \le c_n(g), \quad n \in \mathbb{Z},$$

and $g \in X$ imply that $f \in X$. (Some authors use the phrase "with upper majorant property" instead of solid.) For example, if $1 < p \le 2$, and p' is its conjugate exponent, then the Hausdorff–Young inequality [16, Chapter XII, (2.3)] implies that

$$L^p(-\pi,\pi) \subset \left\{ f \in L^1(-\pi,\pi) : \sum_{n\in\mathbb{Z}} |c_n(f)|^{p'} < \infty \right\} =: \ell^{p'}.$$

Similarly, a result of Hardy and Littlewood (cf. [16, Chapter XII, (3.19)]) shows that if $1 < p \le 2$ then

$$L^p(-\pi,\pi) \subset \left\{ f \in L^1(-\pi,\pi) : \sum_{n\in\mathbb{Z}} (|n|+1)^{p-2} |c_n(f)|^{p} < \infty \right\} =: HL^p.$$

Clearly, both $\ell^{p'}$ and HL^p are solid spaces. Other similar examples can be found in [1].

Theorem 1.1. *Let X be a solid space, $L^p(-\pi,\pi) \subseteq X$. Then*

$$L^p_{\mathrm{loc},+} \subseteq X. \tag{1.1}$$

In the case when p is an even, positive, integer, the space $L^p(-\pi,\pi)$ is itself a solid space, and hence, Theorem 1.1 is a generalization of Wiener's theorem.

In the case when $p=\infty$, Paley [10] observed that if $f \in L^\infty_{\mathrm{loc},+}$, and f is an even function, then f is continuous on $[-\pi,\pi]$ and its Fourier series converges uniformly and absolutely. In fact, certain smoothness properties of such functions can be *characterized* in terms of their Fourier coefficients. For example, a theorem of Lorentz [8, Section 4] implies immediately the following result: Let

$$f(x) \sim \sum_{n\ge 0} c_n \cos nx, \quad c_n \ge 0,$$

and assume further that $\{c_n\}$ is a monotone sequence. Let $E_n(f)$ be the best approximation of f (in the supremum norm) by trigonometric polynomials of order n, and $\gamma \in (0,1)$. Then

$$\limsup_{n\to\infty} n^\gamma E_n(f) < \infty$$

if and only if

$$\limsup_{n\to\infty} n^{\gamma+1} c_n < \infty.$$

Extensions of this result to the case of Besov and Besov–Nikolskii spaces have been given in [6, 2, 13], so as to include the case of L^p, $1 < p < \infty$ as well.

Our main goal in this paper is to present analogues of the above results for functions on $[-1,1]$ and their Jacobi polynomial expansions. In Section 2, we review some necessary facts regarding Jacobi polynomials, and introduce certain notations. In order to avoid unnecessarily complicated notations, the symbols used in this introduction will have different meanings in the rest of this paper. In Section 3, we state and prove Wiener-type results for the Jacobi polynomial expansions.

2. Jacobi polynomials

If $x \geq 0$, we will denote by Π_x the class of all polynomials of degree at most x. This notation is usually used with x an integer, but we find it convenient to extend it to other values of x rather than writing the more cumbersome notation $\Pi_{\lfloor x \rfloor}$. Let $\alpha, \beta \geq -1/2$, and

$$w_{\alpha,\beta}(x) := \begin{cases} (1-x)^\alpha (1+x)^\beta, & \text{if } -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $1 \leq p < \infty$ the space $L^p = L^p(\alpha, \beta)$ is defined as the space of (equivalence classes of) functions f with

$$\|f\|_{\alpha,\beta;p} := \left(\int_{-1}^{1} |f(x)|^p w_{\alpha,\beta}(x) dx \right)^{1/p} < \infty.$$

The space of all continuous real valued functions on $[-1, 1]$, equipped with the supremum norm, will be denoted by C, and the supremum norm of $f \in C$ will be denoted by $\|f\|_\infty$. The space of all infinitely often differentiable $f : [-1, 1] \to \mathbb{C}$ will be denoted by C^∞. We will denote the set of all nonnegative integers by $\mathbb{N}_0$.

There exists a unique system of (Jacobi) polynomials $\{{R_k}^{(\alpha,\beta)} \in \Pi_k\}_{k \in \mathbb{N}_0}$ such that for integer $k, \ell \in \mathbb{N}_0$, ${R_k}^{(\alpha,\beta)}(1) = 1$ and

$$\int_{-1}^{1} {R_k}^{(\alpha,\beta)}(x) {R_\ell}^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x) dx = \begin{cases} \rho_k, & \text{if } k = \ell, \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

where

$$\rho_k = \|R_k\|^2_{\alpha,\beta;2} = \frac{2^{\alpha+\beta+1} \Gamma(\alpha+1)^2}{2k+\alpha+\beta+1} \frac{\Gamma(k+1)\Gamma(k+\beta+1)}{\Gamma(k+\alpha+1)\Gamma(k+\alpha+\beta+1)}.$$

The uniqueness of the system implies that ${R_k}^{(\beta,\alpha)}(x) = {R_k}^{(\alpha,\beta)}(-x)/{R_k}^{(\alpha,\beta)}(-1)$, $x \in \mathbb{R}$, $k \in \mathbb{N}_0$. Therefore, we may assume in the sequel that $\alpha \geq \beta$. We will assume also that $\alpha \geq \beta \geq -1/2$.

For $f \in L^1$, we may define the Jacobi coefficients by

$$\hat{f}(k) := \hat{f}(\alpha, \beta; k) := \rho_k^{-1} \int_{-1}^{1} f(y) {R_k}^{(\alpha,\beta)}(y) w_{\alpha,\beta}(y) dy, \qquad k \in \mathbb{N}_0.$$

Then the formal Jacobi expansion of f has the form $\sum_{k=0}^{\infty} \hat{f}(k) R_k(x)$.

Next, we need formulas for functions $f * g$ such that

$$\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k), \qquad k \in \mathbb{N}_0,$$

and for the Jacobi coefficients of the product fg. Following Koornwinder [7], if $x, y \in [-1, 1]$, $r \in [0, 1]$, and $\psi \in [0, \pi]$, let

$$\begin{aligned} &Z(x, y; r, \psi) = \\ &\frac{1}{2}(1 + x)(1 + y) + \sqrt{1 - x^2}\sqrt{1 - y^2}\, r \cos\psi + \frac{1}{2}(1 - x)(1 - y)\, r^2 - 1. \end{aligned} \tag{2.2}$$

We observe that

$$\begin{aligned} Z(x, y; r, \psi) = \frac{1}{2}\left(\sqrt{(1 + x)(1 + y)} - r\sqrt{(1 - x)(1 - y)}\right)^2 \\ +r\sqrt{1 - x^2}\sqrt{1 - y^2}(1 + \cos\psi) - 1 \geq -1. \end{aligned}$$

If $\theta, \varphi \in [0, \pi]$ such that $x = \cos\theta$, $y = \cos\varphi$, then

$$\begin{aligned} Z(x, y; r, \psi) &= \frac{1}{2}(1 + x)(1 + y) + \frac{1}{2}(1 - x)(1 - y) + \sqrt{1 - x^2}\sqrt{1 - y^2} \\ &\quad + \frac{1}{2}(1 - x)(1 - y)(r^2 - 1) + \sqrt{1 - x^2}\sqrt{1 - y^2}(r\cos\psi - 1) - 1 \\ &= xy + \sqrt{1 - x^2}\sqrt{1 - y^2} + \frac{1}{2}(1 - x)(1 - y)(r^2 - 1) \\ &\quad +\sqrt{1 - x^2}\sqrt{1 - y^2}(r\cos\psi - 1), \end{aligned}$$

and we have

$$\begin{aligned} 1 - Z(x, y; r, \psi) &= 1 - (xy + \sqrt{1 - x^2}\sqrt{1 - y^2}) + \frac{1}{2}(1 - x)(1 - y)(1 - r^2) \\ &\quad +(1 - r\cos\psi)\sqrt{1 - x^2}\sqrt{1 - y^2} \\ &\geq 1 - (xy + \sqrt{1 - x^2}\sqrt{1 - y^2}) = 1 - \cos(\theta - \varphi) \geq 0. \end{aligned} \tag{2.3}$$

Thus, $Z(x, y; r, \psi) \in [-1, 1]$ for all $x, y \in [-1, 1]$, $r \in [0, 1]$ and $\psi \in [0, \pi]$.

For $\alpha \geq \beta \geq -1/2$, Koornwinder [7] has proved that there exists a probability measure $\nu^{(\alpha,\beta)}$ on $[0, 1] \times [0, \pi]$ such that we have

$$R_n^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(y) = \int_0^\pi \int_0^1 R_n^{(\alpha,\beta)}(Z(x, y; r, \psi))\, d\nu^{(\alpha,\beta)}(r, \psi), \quad n \in \mathbb{N}_0. \tag{2.4}$$

An interesting consequence of (2.4) is the following. For almost all $x, y \in [-1, 1]$, and $f \in L^1$, let

$$\mathcal{T}_y f(x) = \int_0^\pi \int_0^1 f(Z(x, y; r, \psi))\, d\nu^{(\alpha,\beta)}(r, \psi). \tag{2.5}$$

Then it is clear that for $x, y \in [-1, 1]$, $\mathcal{T}_y f(x) = \mathcal{T}_x f(y)$ and

$$\widehat{\mathcal{T}_y f}(k) = \hat{f}(k){R_k}^{(\alpha,\beta)}(y). \tag{2.6}$$

The corresponding convolution operator is defined by

$$(f * g)(x) := \int_{-1}^{1} f(y)\, \mathcal{T}_y g(x)\ w_{\alpha,\beta}(y)\, dy, \qquad f, g \in L^1.$$

We have

$$(f * g)\widehat{\ }(k) = \widehat{f}(k)\, \widehat{g}(k), \qquad k \in \mathbb{N}_0,\ f, g \in L^1. \tag{2.7}$$

Dual to the product formula (2.4) is a second product formula:

$$R_n^{(\alpha,\beta)}(x) R_m^{(\alpha,\beta)}(x) = \sum_{k=|n-m|}^{n+m} g^{(\alpha,\beta)}(n,m;k) R_k^{(\alpha,\beta)}(x) \tag{2.8}$$

for all $n, m \in \mathbb{N}_0, x \in [-1,1]$. The coefficients $g^{(\alpha,\beta)}(n,m;k)$ are non-negative for all $k, n, m \in \mathbb{N}_0$ and, moreover,

$$\sum_{k=|n-m|}^{n+m} g^{(\alpha,\beta)}(n,m;k) = 1. \tag{2.9}$$

Thus, we may think of $g^{(\alpha,\beta)}(n,m;\circ)$ as a probability distribution on a subset of $\mathbb{N}_0 \times \mathbb{N}_0$. We note also that

$$g^{(\alpha,\beta)}(n,m;k) = \rho_k^{-1} \int_{-1}^{1} {R_n}^{(\alpha,\beta)}(y) {R_m}^{(\alpha,\beta)}(y) {R_k}^{(\alpha,\beta)}(y) w_{\alpha,\beta}(y) dy.$$

In particular,

$$g^{(\alpha,\beta)}(k,0;k) = \rho_k^{-1} \int_{-1}^{1} \left\{ {R_k}^{(\alpha,\beta)}(y) \right\}^2 w_{\alpha,\beta}(y) dy = 1. \tag{2.10}$$

Just as (2.4) can be used via (2.5) to define a generalized convolution of two functions, (2.8) can be used to define a convolution of sequences. If $\mathbf{a} = \{a_k\}_{k=0}^{\infty}$ and $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$, we define formally

$$(\mathbf{a} * \mathbf{b})(k) = \sum_{n,m=0}^{\infty} g^{(\alpha,\beta)}(n,m;k) a_n b_m. \tag{2.11}$$

Analogous to (2.7) and the classical Cauchy formula for the products of power series, we have the following formula for products of formal Jacobi expansions:

$$\left(\sum_{n=0}^{\infty} a_n {R_n}^{(\alpha,\beta)}(x) \right) \left(\sum_{m=0}^{\infty} b_m {R_m}^{(\alpha,\beta)}(x) \right) = \sum_{k=0}^{\infty} (\mathbf{a} * \mathbf{b})(k) {R_k}^{(\alpha,\beta)}(x), \quad x \in [-1,1]. \tag{2.12}$$

3. Main results

In the sequel, we consider $\alpha \geq \beta \geq -1/2$ to be fixed parameters. Let $\mathbb{P}$ be the class of all $f \in L^1$ such that $\hat{f}(k) \geq 0$ for $k \in \mathbb{N}_0$. It is well known that in the theory of Jacobi expansions, the point 1 plays the role of 0 in the theory of Fourier series. Thus, if T is a trigonometric polynomial with nonnegative Fourier coefficients, then $\max_{x\in[-\pi,\pi]} |T(x)| = T(0)$. Similarly, in view of [12, Theorem 7.32.1],

$$\max_{x\in[-1,1]} |R_k{}^{(\alpha,\beta)}(x)| = R_k{}^{(\alpha,\beta)}(1) = 1. \tag{3.1}$$

Therefore, it is easy to verify that

$$\max_{x\in[-1,1]} |P(x)| = P(1) = \max_{x\in[a,1]} |P(x)|, \qquad P \in \mathbb{P} \cap (\cup_{n=0}^{\infty} \Pi_n),\ a \in [-1,1). \tag{3.2}$$

Accordingly, we define the analogues of local spaces as follows. If Y is any subspace of L^1, then Y_{loc} is the class of all $f \in L^1$ with the following property: there exists a nondegenerate interval $I \subseteq [-1,1]$ with $1 \in I$ such that for any $\phi \in C^\infty$ supported on I, $f\phi \in Y$. If $f, g \in L^1$, we will write $f \preceq g$ if $|\hat{f}(k)| \leq \hat{g}(k)$ for all $k \in \mathbb{N}_0$. A subspace $X \subset L^1$ is called *solid* if $f, g \in L^1$, $f \preceq g$, and $g \in X$ imply that $f \in X$. For example, using (2.12), (2.11), and the fact that $g^{(\alpha,\beta)}(n,m;k) \geq 0$ for all $n,m,k \in \mathbb{N}_0$, it is easy to conclude that when $f \preceq g$ then $|f|^2 \preceq |g|^2$. Hence, if p is an even positive integer, then L^p is a solid space. Thus, Theorem 3.1 below is the analogue of Wiener's theorem for Jacobi expansions.

Our first main result is the analogue of Theorem 1.1.

Theorem 3.1. *Let $X \subset L^1$ be a solid space. Then $X_{\mathrm{loc}} \cap \mathbb{P} = X \cap \mathbb{P}$. In particular, if $1 < p \leq \infty$ and $L^p \subseteq X$ then $L^p_{\mathrm{loc}} \cap \mathbb{P} \subseteq X$.*

In the case when p is not a positive, even integer, we note that using the counterexamples of Wainger and Shapiro, we can always construct a cosine trigonometric series with positive coefficients such that its sum is not p–th power integrable on $[0,\pi]$. Since $R_k{}^{(-1/2,-1/2)}$ are Chebyshev polynomials, these counterexamples also demonstrate that with $\alpha = \beta = -1/2$,

$$L^p_{\mathrm{loc}} \cap \mathbb{P} \not\subset L^p$$

when p is not a positive, even integer.

The proof of Theorem 3.1 relies upon the following two lemmas. If $f \in C$, we define

$$E_x(f) := \min_{P\in\Pi_x} \|f - P\|_\infty. \tag{3.3}$$

Lemma 3.2. *Let $f \in C^\infty$, and $x \in [-1,1]$. Then the function $x \mapsto \mathcal{T}_x f$ is in C^∞.*

Proof. The well-known direct theorem of approximation theory [14, Section 5.1.5, (22)] implies that for any integer $S \geq 1$, $E_n(f) \leq c_1(f,S)n^{-S}$

for some positive constant $c_1(f,S)$ independent of n. Let $\ell \ge 0$ be an integer. We choose $S > 2\ell + 2\alpha + 3$. In view of (2.1) and (3.1), we obtain for $k = 2, 3, \cdots$ and any $P \in \Pi_{k-1}$,

$$\begin{aligned} |\hat{f}(k)| = \rho_k^{-1}\left|\int_{-1}^{1} f(t)R_k(t)w_{\alpha,\beta}(t)dt\right| &= \rho_k^{-1}\left|\int_{-1}^{1}(f(t)-P(t))R_k(t)w_{\alpha,\beta}(t)dt\right| \\ &\le \rho_k^{-1}\rho_0\,\|f-P\|_\infty. \end{aligned}$$

Thus,

$$|\hat{f}(k)| \le c_2(S,\alpha,\beta,f)k^{2\alpha+1-S}.$$

Moreover, Markov's inequality [14, Section 4.8.62, (32)] implies that $\|{R_k}^{(\ell)}\|_\infty \le k^{2\ell}$. Since $S > 2\ell + 2\alpha + 3$, it follows that

$$\sum_{k=2}^{\infty} |\hat{f}(k)|\|R_k\|_\infty\|{R_k}^{(\ell)}\|_\infty \le c_2(S,\alpha,\beta,f)\sum_{k=2}^{\infty} k^{2\alpha+1-S+2\ell} < \infty.$$

This completes the proof. □

Lemma 3.3. *Let $\delta\in(0,1)$. There exists $\phi_\delta\colon[-1,1]\to[0,\infty)$ such that $\phi_\delta\in C^\infty$, $\phi_\delta(t)=0$ if $-1\le t\le 1-\delta$, $\phi_\delta\in\mathbb{P}$, and $\hat{\phi}_\delta(0)=1$. If $f\in\mathbb{P}$, then $f\preceq f\phi_\delta$.*

Proof. In this proof only, let $\theta_0 \in (0,\pi/4)$ be chosen so that $1-\delta = \cos(2\theta_0)$, and $g = g_\delta : [-1,1] \to [0,\infty)$ be a function in C^∞, supported on $[\cos\theta_0, 1]$, such that $\hat{g}(0) = 1$. Let

$$\begin{aligned} \phi_\delta(x) := (g*g)(x) &= \int_{-1}^{1} g(y)\mathcal{T}_y g(x)w_{\alpha,\beta}(y)dy \\ &= \int_{-1}^{1} g(y)\mathcal{T}_x g(y)w_{\alpha,\beta}(y)dy, \qquad x\in[-1,1]. \end{aligned}$$

In view of Lemma 3.2, $\phi_\delta : [-1,1] \to [0,\infty)$ is in C^∞. Also, in view of (2.7), $\hat{\phi}_\delta(k) = (\hat{g}(k))^2 \ge 0$, $k \in \mathbb{N}_0$, and in particular, $\hat{\phi}_\delta(0) = (\hat{g}(0))^2 = 1$.

If $|\theta-\varphi| \ge \theta_0$, then (2.3) implies that $Z(x,y;r,\psi) \le \cos(\theta-\varphi) \le \cos\theta_0$. Therefore, $g(Z(x,y;r,\psi)) = 0$ and hence, $\mathcal{T}_y g(x) = 0$. If J denotes the set $\{\cos\varphi : |\theta-\varphi| \le \theta_0\}$, then

$$\phi_\delta(x) = \int_J g(y)\mathcal{T}_y g(x)w_{\alpha,\beta}(y)dy.$$

If $\theta \ge 2\theta_0$ then for $y \in J$, $0 < \theta_0 \le \theta-\theta_0 \le \varphi$; i.e., $y = \cos\varphi \le \cos\theta_0$, and $g(y) = 0$. Thus, ϕ_δ is supported on $[\cos(2\theta_0), 1] = [1-\delta, 1]$.

Next, let $f \in \mathbb{P}$. Since $\hat{f}(n) \ge 0$ for all $n \in \mathbb{N}_0$, we obtain in view of (2.12), (2.11), (2.10) that for each $k \in \mathbb{N}_0$,

$$\begin{aligned} \widehat{f\phi_\delta}(k) &= \sum_{n,m=0}^{\infty} g^{(\alpha,\beta)}(n,m;k)\hat{f}(n)\hat{\phi}_\delta(m) \\ &\ge g^{(\alpha,\beta)}(k,0;k)\hat{f}(k)\hat{\phi}_\delta(0) = \hat{f}(k). \qquad (3.4) \end{aligned}$$

Thus, $f \preceq f\phi_\delta$. □

Proof of Theorem 3.1. Let $f \in X_{\text{loc}} \cap \mathbb{P}$, and $\delta \in (0,1)$ be found such that $f\phi \in X$ for every $\phi \in C^\infty$ supported on $[1-\delta, 1]$. We take $\phi = \phi_\delta$ as in Lemma 3.3. Since X is solid, $f \preceq f\phi_\delta$, and $f\phi_\delta \in X$, it follows that $f \in X$ as well. □

Next, we consider the case $p = \infty$. First, we state an analogue of the result of Paley.

Theorem 3.4. *If $f \in L^\infty_{\text{loc}} \cap \mathbb{P}$ then $\sum_{k=0}^\infty \hat{f}(k)|R_k(x)| < \infty$, $x \in [-1,1]$, with the series converging uniformly on $[-1,1]$. In particular, $L^\infty_{\text{loc}} \cap \mathbb{P} = C \cap \mathbb{P}$.*

The proof of this theorem depends upon the following lemma, which is observed by several authors; a proof can be found, for example, in [9, Lemma 4.2]. First, we define the de la Vallée-Poussin type operators. Let $h : [0,\infty) \to [0,1]$ be a C^∞, nonincreasing function, $= 1$ on $[0,1/2]$, and $= 0$ on $[1,\infty)$. For $f \in L^1$, $n \geq 0$, $x \in [-1,1]$, let

$$\sigma_n(h,f,x) = \sum_{k=0}^{n-1} h\left(\frac{k}{n}\right)\hat{f}(k)R_k(x), \tag{3.5}$$

and

$$\tau_n(h,f) = \sigma_{2n}(h,f) - \sigma_n(h,f). \tag{3.6}$$

Lemma 3.5. *Let $f \in C$. Then*

$$\|\sigma_n(h,f)\|_\infty \leq c\|f\|_\infty, \tag{3.7}$$

where c is a positive constant depending only on α, β, and h. Moreover, $\sigma_n(h,P) = P$ if $P \in \Pi_{n/2}$, and hence,

$$E_n(f) \leq \|f - \sigma_n(h,f)\|_\infty \leq (1+c)E_{n/2}(f). \tag{3.8}$$

Proof of Theorem 3.4. Let $f \in L^\infty_{\text{loc}} \cap \mathbb{P}$, and $\delta \in (0,1)$ be such that $f\phi \in L^\infty$ for every $\phi \in C^\infty$ supported on $[1-\delta,1]$. We choose $\phi = \phi_\delta$, where ϕ_δ is defined in Lemma 3.3. Let $n \geq 1$ be an integer. Since h is a nonnegative function, $f \in \mathbb{P}$, and $f \preceq f\phi_\delta$, we see using (3.1) that

$$h\left(\frac{k}{2n}\right)\hat{f}(k)\|R_k\|_\infty = h\left(\frac{k}{2n}\right)\hat{f}(k) \leq h\left(\frac{k}{2n}\right)\widehat{f\phi_\delta}(k), \qquad k \in \mathbb{N}_0.$$

Since $h(t) = 1$ for $0 \leq t \leq 1/2$, we obtain using (3.2) and (3.7) that for every integer $n \geq 1$,

$$\begin{aligned}\sum_{k=0}^n \hat{f}(k)\|R_k\|_\infty &\leq \sum_{k=0}^\infty h\left(\frac{k}{2n}\right)\hat{f}(k)\|R_k\|_\infty \leq \sum_{k=0}^\infty h\left(\frac{k}{2n}\right)\widehat{f\phi_\delta}(k)\\ &= \|\sigma_{2n}(h, f\phi_\delta)\|_\infty \leq c\|f\phi_\delta\|_\infty < \infty.\end{aligned}$$

This completes the proof. □

Finally, we discuss a variant for Besov spaces, which we now define. Let $0 < \rho \le \infty$, $\gamma > 0$, and $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers. We define a sequence space as follows

$$\|\mathbf{a}\|_{\rho,\gamma} := \begin{cases} \{\sum_{n=0}^{\infty} 2^{n\gamma\rho}|a_n|^\rho\}^{1/\rho}, & \text{if} \quad 0 < \rho < \infty, \\ \sup_{n\ge 0} 2^{n\gamma}|a_n|, & \text{if} \quad \rho = \infty \end{cases} \tag{3.9}$$

The space of sequences $\mathbf{a}$ for which $\|\mathbf{a}\|_{\rho,\gamma} < \infty$ will be denoted by $\mathsf{b}_{\rho,\gamma}$. For $0 < \rho \le \infty$, $\gamma > 0$, the Besov space $B_{\rho,\gamma}$ consists of functions $f \in L^\infty$ for which the sequence $\{E_{2^n}(f)\}_{n=0}^{\infty} \in \mathsf{b}_{\rho,\gamma}$. We have proved in [9, Theorem 2.1] that

$$f \in B_{\rho,\gamma} \quad \text{if and only if} \quad \{\|\tau_{2^n}(h,f)\|_\infty\} \in \mathsf{b}_{\rho,\gamma}.$$

The following theorem is a refinement of Theorem 3.4. Let $S_n(f)$ be the n-th partial sum of the Jacobi expansion of f, i.e., $S_n(f) = \sum_{k=0}^{n-1} \hat{f}(k) R_k$.

Theorem 3.6. *Let $0 < \rho \le \infty$, $\gamma > 0$. For $f \in \mathbb{P}$ the following conditions are equivalent:*

(a)
$$f \in (B_{\rho,\gamma})_{\text{loc}},$$

(b)
$$f \in B_{\rho,\gamma},$$

(c)
$$\left\{ \|S_{2^{n+1}} - S_{2^n}(f)\|_\infty = \sum_{k=2^n}^{2^{n+1}-1} \hat{f}(k) \right\} \in \mathsf{b}_{\rho,\gamma}.$$

Proof. To prove (a) $\Rightarrow$ (b), we need to show that $(B_{\rho,\gamma})_{\text{loc}} \cap \mathbb{P} \subset B_{\rho,\gamma}$. Let $f \in (B_{\rho,\gamma})_{\text{loc}} \cap \mathbb{P}$, and $\delta \in (0,1)$ be such that $f\phi_\delta \in B_{\rho,\gamma} \cap \mathbb{P}$, where ϕ_δ is as in Lemma 3.3. Then $\{\|\tau_{2^n}(h, f\phi_\delta)\|_\infty\} \in \mathsf{b}_{\rho,\gamma}$. Since h is nonincreasing, we have

$$g_{k,n} := h(k/2^{n+1}) - h(k/2^n) \ge 0, \qquad k \in \mathbb{N}_0, \ n = 1, 2, \cdots.$$

Since $f \preceq f\phi_\delta$, this implies that

$$0 \le g_{k,n}\hat{f}(k) \le g_{k,n}\widehat{f\phi_\delta}(k), \qquad k \in \mathbb{N}_0, \ n = 1, 2, \cdots.$$

So, using (3.2), we conclude that for $n = 1, 2, \cdots$,

$$\begin{aligned} \|\tau_{2^n}(h,f)\|_\infty &= \tau_{2^n}(h,f,1) = \sum_{k=0}^{\infty} g_{k,n}\hat{f}(k) \\ &\le \sum_{k=0}^{\infty} g_{k,n}\widehat{f\phi_\delta}(k) = \|\tau_{2^n}(h, f\phi_\delta)\|_\infty. \end{aligned}$$

Since $\{\|\tau_{2^n}(h, f\phi_\delta)\|_\infty\} \in \mathsf{b}_{\rho,\gamma}$, this implies that $\{\|\tau_{2^n}(h,f)\|_\infty\} \in \mathsf{b}_{\rho,\gamma}$; i.e., $f \in B_{\rho,\gamma}$.

The implication (b) $\Rightarrow$ (a) is clear.

We prove that (a) (alternatively (b)) $\Rightarrow$ (c). Since (a) implies in particular that $f \in L^\infty_{\text{loc}} \cap \mathbb{P}$, Theorem 3.4 implies that the series $\sum \hat{f}(k)R_k$ converges uniformly and absolutely to f. Then, by (3.1) and (3.8), we have

$$\begin{aligned} E_n(f) &\le \|f - S_n(f)\|_\infty \le \sum_{k=n}^{\infty} \hat{f}(k) \le \sum_{k=0}^{\infty}(1 - h(k/n))\hat{f}(k) = f(1) - \sigma_n(h, f, 1) \\ &\le \|f - \sigma_n(h, f)\|_\infty \le (1 + c)E_{n/2}(f). \end{aligned}$$

Thus, $f \in B_{\rho,\gamma}$ if and only if $\{\|f - S_{2^n}(f)\|_\infty\} \in \mathsf{b}_{\rho,\gamma}$ and secondly, $f \in B_{\rho,\gamma}$ if and only if $\{\sum_{k=2^n}^{\infty} \hat{f}(k)\} \in \mathsf{b}_{\rho,\gamma}$. In light of the discrete Hardy inequality [5, Lemma 3.4, p. 27], the latter is equivalent to

$$\left\{ \sum_{k=2^n}^{2^{n+1}-1} \hat{f}(k) \right\} \in \mathsf{b}_{\rho,\gamma}$$

for any positive ρ. □

Remark 1. Assuming additional conditions on the coefficients, we obtain an analogue of Lorentz' result: under conditions of Theorem 3.6, conditions (a) and (b) are equivalent to the condition $\{\hat{f}(2^n)\} \in \mathsf{b}_{\rho,\gamma}$ if $\{\hat{f}(n)\}$ is lacunary, or $\{\hat{f}(2^n)\} \in \mathsf{b}_{\rho,\gamma+1}$ if $\{\hat{f}(n)\}$ is monotone.

Remark 2. Let $a \ge b \ge -1/2$. In [3], Askey and Gasper have given sufficient conditions in order that the *connection coefficients* $\gamma_{j,k}$ in the expansion below are all nonnegative.

$$R_k^{(\alpha,\beta)}(x) = \sum_{j=0}^{k} \gamma_{j,k} R_j^{(a,b)}(x), \qquad k \in \mathbb{N}_0,\ x \in \mathbb{R}. \tag{3.10}$$

If $f \in C$, and $\hat{f}(\alpha, \beta; k) \ge 0$ for $k \in \mathbb{N}_0$, then (3.8) can be used to deduce easily that

$$\hat{f}(a, b; j) = \lim_{n\to\infty} \sum_{k=j}^{n} \gamma_{j,k} h(k/n) \hat{f}(\alpha, \beta; k) \ge 0.$$

Thus, Theorem 3.4 implies that for $f \in L^\infty_{\text{loc}} \cap \mathbb{P}$, the series

$$\sum |\hat{f}(a, b; k)| \|R_j^{(a,b)}\|_\infty$$

converges for all $a \ge b \ge -1/2$ for which the connection coefficients in (3.10) are all nonnegative. A similar conclusion can be made also about Theorem 3.6, and Remark 1 above.

Acknowledgements

The research of the first author was supported, in part, by grant DMS-0908037 from the National Science Foundation and grant W911NF-09-1-0465 from the

U.S. Army Research Office. The research of the second author was supported, in part, by grants MTM2008-05561-C02-02/MTM, RFFI 09-01-00175, and 2009 SGR 1303.

References

1. J. M. Ash, S. Tikhonov and J. Tung, *Wiener's positive Fourier coeffcients Theorem in variants of L^p spaces*, Michigan Math. J., **59(1)** (2010), 143–152.
2. R. Askey, *Smoothness conditions for Fourier series with monotone coefficients*, Acta Sci. Math. (Szeged) **28** (1967), 168–171.
3. R. A. Askey and G. Gasper, *Jacobi polynomial expansions of Jacobi polynomials with non-negative coefficients*, Math. Proc. Cambridge Philos. Soc. **70** (1971), 243–255.
4. R. P. Boas, Entire Functions, Academic Press, New York, 1954.
5. R. A. DeVore and G. G. Lorentz, Constructive approximation, Springer Verlag, Berlin, 1993.
6. A. A. Konyushkov, *Best approximations by trigonometric polynomials and Fourier coefficients (Russian)*, Mat. Sb. N.S., **44(86)** (1958), 53–84.
7. T. Koornwinder, *Jacobi polynomials II. An analytic proof of the product formula*, SIAM J. Math. Anal., **5** (1974), 125–137.
8. G. G. Lorentz, *Fourier–Koeffizienten und Funktionenklassen*, Math. Z. **51** (1948), 135–149.
9. H. N. Mhaskar, *Polynomial operators and local smoothness classes on the unit interval*, J. Approx. Theory, **131** (2004), 243–267.
10. R. E. A. C. Paley, *On Fourier series with positive coefficients,* J. London Math. Soc, **7** (1932), 205–208.
11. H. S. Shapiro, *Majorant problems for Fourier coefficients,* Quart. J. Math. Oxford, (2) **26** (1975), 9–18.
12. G. Szegő, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ. **23**, Amer. Math. Soc., Providence, 1975.
13. S. Tikhonov, *Characteristics of Besov Nikol'skii class of functions*, Elec. Trans. Numer. Anal., **19** (2005), 94–104.
14. A. F. Timan, Theory of approximation of functions of a real variable, English translation, Pergamon Press, 1963.
15. S. Wainger, *A problem of Wiener and the failure of a principle for Fourier series with positive coefficients,* Proc. Amer. Math. Soc., **20** (1969), 16–18.
16. A. Zygmund, Trigonometric Series, v.2, 2nd ed., 1958.

H. N. Mhaskar
Department of Mathematics
California State University
Los Angeles, California, 90032, USA
E-mail address: hmhaska@calstatela.edu

S. Tikhonov
ICREA and Centre de Recerca Matemàtica
Facultat de Ciències UAB
08193 Bellaterra
Barcelona,Spain
E-mail address: stikhonov@crm.cat