NOTE ON COMPANION FORMS OF LOW WEIGHT ON $GSp_4(\mathbb{Q})$

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1. Introduction

In a recent paper, Gee and Geraghty [1] have shown the existence of companion forms, in the p-ordinary case, for genus two Siegel forms with cohomological weights, thus proving several conjectures in [3]. In this note, we show how to apply their result to construct companion forms, in the p-ordinary case, for a non-cohomological weight. Namely, under a certain decomposability assumption for the Galois representation associated to an ordinary p-adic cusp form f_α of weight $(2,2)$ on GSp_4, we find another weight $(2,2)$ p-adic form f_β, companion to f_α.

Let us recall a theorem due to V. Pilloni [6], improving upon [11]. Let $A/\mathbb{Q}$ be a simple, principally polarized, semistable abelian surface with good ordinary reduction at a prime p. Let ϕ be the crystalline Frobenius on the covariant crystalline module of $T_p(A)$. We label the roots of its characteristic polynomial $\alpha, \beta, \gamma, \delta$ in such a way that α, β,$\frac{\gamma}{p}$ and $\frac{\delta}{p}$ are p-adic units.

Let ρ_A, resp. $\overline{\rho}_A$, be the Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the p-adic Tate module T_pA, resp on $A[p](\overline{\mathbb{Q}})$; we assume that ρ_A is congruent modulo p to the Galois representation ρ_f associated to an holomorphic Siegel cusp form of weight (k,ℓ) with $k \geq \ell \geq 3$ of prime to p level N. Note that this implies that $k \equiv \ell \equiv 2 \pmod{p-1}$, hence the motivic weight $k+\ell-3$ cannot be less than $p-1$. We also assume that f is p-ordinary and the conductor of f is equal to $\mathrm{Cond}(A)$. Let $F \subset \overline{\mathbb{Q}}_p$ be a p-adic field containing the eigenvalues of f, the roots $\alpha, \beta, \gamma, \delta$, and over which the representation $\overline{\rho}_{f,p}$ is defined; let $\mathcal{O}$ be its valuation ring. Let ϖ be a uniformizing parameter of $\mathcal{O}$ and κ its residue field.

Theorem 1. *Let us assume*

(1) $\mathrm{Im}\overline{\rho}_A = GSp_4(\mathbb{Z}/p\mathbb{Z})$,
(2) $\overline{\rho}_A$ *is "bien ramifiée" (in the sense of* [2] *Définition 2.2.2) at each prime* ℓ *dividing* $\mathrm{Cond}(A)$,
(3) *the p-adic units* $\alpha, \beta, \frac{\gamma}{p}, \frac{\delta}{p}$ *are mutually distinct modulo* ϖ.

then there exists an overconvergent p-adic cuspform f_α of weight $(2,2)$, with level N or Np with at most Iwahori level at p, such that

$$\rho_{f_\alpha} = \rho_A$$

Moreover, f_α is p-ordinary; more precisely, we have $f_\alpha | U_{p,1} = \alpha f_\alpha$ and $f_\alpha | U_{p,2} = \alpha\beta f_\alpha$.

A consequence of this theorem is that ρ_{f_α} is geometric at p. According to the modular version of the Fontaine-Mazur conjecture, this should imply that f_α is classical (possibly of level Np). In order to prove this conjecture, we propose, following Buzzard-Taylor's method for the Hilbert modular case, to construct a companion form f_β of weight $(2,2)$ for f_α (in a sense specified below) and prove the analytic continuation of f_α to the whole Siegel variety (of level Np, with Iwahori level at p). In this note we explain the construction of the companion form f_β. In the sequel, we write $\widetilde{\alpha} = \alpha, \widetilde{\beta} = \beta, \widetilde{\gamma} = \frac{\gamma}{p}, \widetilde{\delta} = \frac{\delta}{p}$ the p-adic units associated to $\alpha, \beta, \gamma, \delta$.

The main point is to note that we can apply Theorem 7.6.6 of [1] to our situation as follows. Let $\overline{\rho} = \overline{\rho}_f \pmod{\varpi}$. Recall that f has level prime to p. Assumptions (1) and (2) of Th.7.6.6 are satisfied.

Let $\chi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$ be the global p-adic cyclotomic character; we still denote by χ its restriction to the decomposition group $G_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ at p. Similarly let ω be its reduction modulo p, as a global or local character. For any p-adic unit $x \in \mathcal{O}^\times$, resp. $\overline{x} \in \kappa^\times$, we denote by $x_g\colon G_p \to \mathcal{O}^\times$, resp. $\overline{x}_g$, the unramified character sending an arithmetic Frobenius to x, resp. to $\overline{x}$. Let us recall that

$$\rho_A|_{G_p} \sim \begin{pmatrix} \widetilde{\delta}_g\chi & 0 & * & * \\ 0 & \widetilde{\gamma}_g\chi & * & * \\ 0 & 0 & \widetilde{\beta}_g & 0 \\ 0 & 0 & 0 & \widetilde{\alpha}_g \end{pmatrix}.$$

It follows from this that $\overline{\rho}$ satisfies the following partial decomposability condition:

$$\overline{\rho}|_{I_p} \sim \begin{pmatrix} \omega^{k+\ell-3} & 0 & * & * \\ 0 & \omega^{k-1} & * & * \\ 0 & 0 & \omega^{\ell-2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

recall that $k+\ell-3 \equiv k-1 \equiv 0 \pmod{p-1}$ and $\ell-2 \equiv 0 \pmod{p-1}$.

Let (k',ℓ') with $k' \equiv k$ and $\ell' \equiv \ell$ mod. $p-1$ with $k' > \ell' > 3$. Let $n=0$. We will check in the following section that Assumption (3) of Th.7.6.6 [1] is satisfied for weight (k',ℓ'). More precisely, let $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $w = \begin{pmatrix} s & 0_2 \\ 0 & s \end{pmatrix} \in$

$GSp_4(\mathbb{Z})$. The G_p-representation $\overline{\rho}_p^w = w \circ \overline{\rho}|_{G_p} \circ w$ is of the form

$$\overline{\rho}_p^w = \begin{pmatrix} \widetilde{\gamma}_g\omega & 0 & * & * \\ 0 & \widetilde{\delta}_g\omega & * & * \\ 0 & 0 & \widetilde{\beta}_g & 0 \\ 0 & 0 & 0 & \widetilde{\alpha}_g \end{pmatrix}$$

where, for obvious notational reasons, we denoted used the same notation for $\widetilde{\gamma}$ and its reduction modulo ϖ (and similarly for $\widetilde{\delta}$, $\widetilde{\alpha}$ and $\widetilde{\beta}$). Then we have

Proposition 1. *There exists an ordinary crystalline representation $\rho\colon G_p \to GSp_4(\mathcal{O})$ which lifts the restriction of $\overline{\rho}_p^w$ to G_p, and whose restriction to the inertia subgroup I_p is given by*

$$\rho|_{I_p} \sim \begin{pmatrix} \chi^{k'+\ell'-3} & 0 & * & * \\ 0 & \chi^{k'-1} & * & * \\ 0 & 0 & \chi^{\ell'-2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark: It follows from Proposition 1 that there exist $\alpha', \beta', \gamma', \delta'$ in $\mathcal{O}$ with respective p-adic valuation $\ell'-2$, 0, $k'+\ell'-3$, $k'-1$, (in this order), which are the eigenvalues of the crystalline Frobenius on $D_{cr}(\rho)$, such that such that the p-adic units $\widetilde{\beta}' = \beta', \widetilde{\alpha}' = \frac{\alpha'}{p^{\ell'-2}}, \widetilde{\delta}' = \frac{\delta'}{p^{k'-1}}, \widetilde{\gamma}' = \frac{\gamma'}{p^{k'+\ell'-3}}$ are congruent modulo ϖ to $\widetilde{\beta}, \widetilde{\alpha}, \widetilde{\delta}, \widetilde{\gamma}$ respectively.

We obtain the following

Theorem 2. *There exists a p-ordinary cusp eigenform g of weight (k', ℓ'), same level as f such that $\overline{\rho}_g = \overline{\rho}_f$ and $g|T_{p,1} = b_{g,1}g$ and $g|T_{p,2} = b_{g,2}g$ with $b_{g,1} \equiv \beta \pmod{\varpi}$ and $b_{g,2} \equiv \alpha\beta \pmod{\varpi}$.*

From this theorem, we deduce by applying Theorem 1 to g instead of f the

Corollary 1. *Under the same assumptions as in Theorem 1, there exists another overconvergent p-adic cuspform f_β of weight $(2,2)$, with prime-to-p level N and with at most Iwahori level at p, such that*

$$\rho_{f_\beta} = \rho_A$$

Moreover, f_β is p-ordinary; more precisely, we have $f_\beta|U_{p,1} = \beta f_\beta$ and $f_\beta|U_{p,2} = \alpha\beta f_\beta$.

En posant $(k', \ell') = (k+p-1, 4-\ell+p-1)$, on obtient un poids cohomologique: $k' > \ell' > 3$.

2. Local lifting

We prove

Proposition 2. *Let $a, b, c, d \in \kappa^\times$ four mutually distinct elements. Let $\overline{\rho}_p \colon G_p \to GSp_4(\kappa)$ be a representation of the form*

$$\overline{\rho}_p = \begin{pmatrix} d_g\omega & 0 & * & * \\ 0 & c_g\omega & * & * \\ 0 & 0 & b_g & 0 \\ 0 & 0 & 0 & a_g \end{pmatrix}.$$

Let (k', ℓ') with $k' \geq \ell' \geq 3$, $k' \equiv \ell' \equiv 2 \pmod{p-1}$. Then for any choice of numbers $A, B, C, D \in \mathcal{O}^\times$ such that $AD = BC$ whose reductions modulo ϖ are respectively a, b, c, d, the representation $\overline{\rho}_p$ admits a symplectic lift ρ of the form

$$\rho = \begin{pmatrix} D_g\chi^{k'+\ell'-3} & 0 & * & * \\ 0 & C_g\chi^{k'-1} & * & * \\ 0 & 0 & B_g\chi^{\ell'-2} & 0 \\ 0 & 0 & 0 & A_g \end{pmatrix},$$

Moreover, if $k' > \ell' > 3$, any such lifting ρ is crystalline.

Proof: The representation $\overline{\rho}_p$ defines an element of $\mathrm{Ext}^1_{G_p,symp}(a_g \oplus b_g, c_g\omega \oplus d_g\omega)$. This κ-vector space can be written as

$$H^1(G_p, (ca^{-1})_g\omega) \oplus H^1(G_p, (da^{-1})_g\omega) \oplus H^1(G_p, (cb^{-1})_g\omega).$$

Let us fix numbers $A, B, C, D \in \mathcal{O}^\times$ as in the statement. The set of liftings ρ is the inverse image of $\overline{\rho}_p$ in the $\mathcal{O}$-module $\mathrm{Ext}^1_{G_p,symp}(A_g \oplus B_g\chi^{\ell'-2}, C_g\chi^{k'-1} \oplus D_g\chi^{k'+\ell'-3})$, which can be rewritten as

$$H^1(G_p, (CA^{-1})_g\chi^{k'-1}) \oplus H^1(G_p, (DA^{-1})_g\chi^{k'+\ell'-3}) \oplus H^1(G_p, (CB^{-1})_g\chi^{k'-\ell'+1}).$$

We are therefore led to study the surjectivity of the reduction maps

- $H^1(G_p, (CA^{-1})_g\chi^{k'-1}) \to H^1(G_p, (ca^{-1})_g\omega)$,
- $H^1(G_p, (DA^{-1})_g\chi^{k'+\ell'-3}) \to H^1(G_p, (da^{-1})_g\omega)$ and
- $H^1(G_p, (CB^{-1})_g\chi^{k'-\ell'+1}) \to H^1(G_p, (cb^{-1})_g\omega)$.

In general, let $E \in \mathcal{O}^\times$ with $E \not\equiv 1 \pmod{\varpi}$, let $e \in \kappa$ be its reduction modulo ϖ; let $n \equiv 1 \pmod{p-1}$. In order to show the surjectivity of

$$H^1(G_p, E_g\chi^n) \to H^1(G_p, e\omega)$$

it is enough to show the vanishing of $H^2(G_p, E_g\chi^n)$. By local Tate's duality, this amounts to show $H^0(G_p, E^{-1}\chi^{1-n} \otimes F/\mathcal{O}) = 0$. For this, it is enough to show $H^0(G_p, \kappa(e_g^{-1})) = 0$. This is indeed the case since $e \neq 1$.

In cas $k' > \ell' > 3$, we can apply a result of Perrin-Riou [8] to conclude that any such lifting ρ is crystalline.

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