MODIFIED DIFFERENTIALS AND BASIC COHOMOLOGY FOR RIEMANNIAN FOLIATIONS

GEORGES HABIB AND KEN RICHARDSON

Abstract. We define a new version of the exterior derivative on the basic forms of a Riemannian foliation to obtain a new form of basic cohomology that satisfies Poincaré duality in the transversally orientable case. We use this twisted basic cohomology to show relationships between curvature, tautness, and vanishing of the basic Euler characteristic and basic signature.

1. Introduction

1.1. **Smooth foliations and basic forms.** Let $(M,\mathcal{F})$ be a smooth, closed manifold of dimension n endowed with a foliation $\mathcal{F}$ given by an integrable subbundle $L\subset TM$ of rank p, with $n=p+q$. The set $\mathcal{F}$ is a partition of M into immersed submanifolds (*leaves*) such that the transition functions for the local product neighborhoods (foliation charts) are smooth. The subbundle $L=T\mathcal{F}$ is the tangent bundle to the foliation; at each $p\in M$, $T_p\mathcal{F}=L_p$ is the tangent space to the leaf through p.

Basic forms are differential forms on M that locally depend only on the transverse variables in the foliation charts — that is, forms α satisfying $X\lrcorner\alpha = X\lrcorner d\alpha = 0$ for all $X\in\Gamma(L)$; the symbol "$\lrcorner$" stands for interior product. Let $\Omega(M,\mathcal{F})\subset\Omega(M)$ denote the space of basic forms. These differential forms are preserved by the exterior derivative and are used to define basic cohomology groups $H_d^*(M,\mathcal{F})$ given by

$$H_d^k(M,\mathcal{F})=\frac{\ker d_k}{\operatorname{image} d_{k-1}}$$

with

$$d_k=d:\Omega^k(M,\mathcal{F})\to\Omega^{k+1}(M,\mathcal{F}).$$

The basic cohomology can be infinite-dimensional, and it can be relatively trivial. We may also define basic cohomology with values in a foliated vector bundle; by doing this we gain more topological information about the leaf space.

Basic cohomology does not necessarily satisfy Poincaré duality, even if the foliation is transversally oriented (see [22], [41]). We emphasize that basic cohomology is a smooth foliation invariant and does not depend on the choice of metric or

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any transverse or leafwise geometric structure. In [12] the authors showed the topological invariance of basic cohomology.

1.2. **Riemannian foliations and bundle-like metrics.** We assume throughout the paper that the foliation is *Riemannian*; this means that there is a metric on the local space of leaves — a holonomy-invariant transverse metric g_Q on the normal bundle $Q = TM \diagup L$. The phrase *holonomy-invariant* means the transverse Lie derivative $\mathcal{L}_X g_Q$ is zero for all leafwise vector fields $X \in \Gamma(L)$. This condition is characterized by the existence of a unique metric and torsion-free connection ∇ on Q [30], [35], [41].

We often assume that the manifold is endowed with the additional structure of a *bundle-like metric* [35], i.e. the metric g on M induces the metric on $Q \simeq L^{\perp}$. Every Riemannian foliation admits bundle-like metrics that are compatible with a given $(M, \mathcal{F}, g_Q)$ structure. There are many choices, since one may freely choose the metric along the leaves and also the transverse subbundle Q. We note that a bundle-like metric on a smooth foliation is exactly a metric on the manifold such that the leaves of the foliation are locally equidistant. There are topological restrictions to the existence of bundle-like metrics (and thus Riemannian foliations). Important examples of requirements for the existence of a Riemannian foliations may be found in [23], [20], [30], [41], [42], [40]. One geometric requirement is that, for any metric on the manifold, the orthogonal projection

$$P : L^2(\Omega(M)) \to L^2(\Omega(M, \mathcal{F}))$$

must map the subspace of smooth forms onto the subspace of smooth basic forms ([33]).

1.3. **The basic Laplacian.** Many researchers have studied basic forms and the basic Laplacian on Riemannian foliations with bundle-like metrics (see [1], [23], [41]). The basic Laplacian Δ_b for a given bundle-like metric is a version of the Laplace operator that preserves the basic forms and that is essentially self-adjoint on the L^2-closure of the space of basic forms. The basic Laplacian Δ_b is defined to be

$$\Delta_b = d\delta_b + \delta_b d : \Omega(M, \mathcal{F}) \to \Omega(M, \mathcal{F}),$$

where δ_b is the L^2-adjoint of the restriction of d to basic forms: $\delta_b = P\delta$ is the ordinary adjoint of d followed by the orthogonal projection onto the space of basic forms.

The operator Δ_b and its spectrum depend on the choice of the bundle-like metric and provide invariants of that metric. See [19], [25], [26], [33], [36], [37] for results. One may think of this operator as the Laplacian on the space of leaves. This operator is the appropriate one for physical intuition. For example, the Laplacian is used in the heat equation, which determines the evolution of the temperature distribution over a manifold as a function of time. If we assume that the leaves of the foliation are perfect conductors of heat, then the basic Laplacian is the appropriate operator that allows one to solve the heat distribution problem in this situation.

It turns out that the basic Laplacian is the restriction to basic forms of a second order elliptic operator on all forms, and this operator is not necessarily symmetric ([33]). Only in special cases is this operator the same as the ordinary Laplacian. The basic Laplacian Δ_b is also not the same as the formal Laplacian defined on the local quotient manifolds of the foliation charts (or on a transversal). This transversal Laplacian is in general not symmetric on the space of basic forms, but it does preserve $\Omega(M,\mathcal{F})$. The basic heat flow asymptotics are more complicated than that of the standard heat kernel, but there is a fair amount known (see [33], [36], [37]).

1.4. **The basic adjoint of the exterior derivative and mean curvature.** We assume $(M,\mathcal{F},g_M)$ is a Riemannian foliation of dimension p and codimension q, with bundle-like metric g_M compatible with the Riemannian structure $(M,\mathcal{F},g_Q)$. Let

$$H=\sum_{i=1}^{p}\pi\left(\nabla^M_{f_i}f_i\right),$$

where $\pi:TM\to Q$ is the bundle projection and $(f_i)_{1\le i\le p}$ is a local orthonormal frame of $T\mathcal{F}$. This is the mean curvature vector field, and its dual one-form is $\kappa=H^\flat$. Let $\kappa_b=P\kappa$ be the (smooth) basic projection of this mean curvature one-form. It turns out that κ_b is a closed form whose cohomology class in $H^1_d(M,\mathcal{F})$ is independent of the choice of bundle-like metric (see [1]). Let $\kappa_b\lrcorner$ denote the (pointwise) adjoint of the operator $\kappa_b\wedge$. Clearly, $\kappa_b\lrcorner$ depends on the choice of bundle-like metric g_M, not simply on the transverse metric g_Q.

Recall the following expression for δ_b (see [41], [1], [33]):

$$\begin{aligned}\delta_b&=P\delta\\&=\pm\bar{*}d\bar{*}+\kappa_b\lrcorner\\&=\delta_T+\kappa_b\lrcorner,\end{aligned}$$

where

- δ_T is the formal adjoint (with respect to g_Q) of the exterior derivative on the transverse local quotients.
- the pointwise transversal Hodge star operator $\bar{*}$ is defined on all k-forms γ by
$$\bar{*}\gamma=(-1)^{p(q-k)}*(\gamma\wedge\chi_{\mathcal{F}}),$$
with $\chi_{\mathcal{F}}$ being the leafwise volume form, the characteristic form of the foliation, and $*$ being the ordinary Hodge star operator. Note that $\bar{*}^2=(-1)^{k(q-k)}$ on k-forms. All that is required for the formula above to be well-defined is that the Riemannian foliation is transversally oriented. The formula above is independent of the choice of orientation of the manifold (equivalently, of the leafwise tangent bundle $T\mathcal{F}$).

- The sign $\pm$ above only depends on dimensions and the degree of the basic form.

1.5. **Twisted duality for basic cohomology.** Even for transversally oriented Riemannian foliations, Poincaré duality does not necessarily hold for basic cohomology.

However, note that $d - \kappa_b\wedge$ is also a differential which defines a cohomology of basic forms. That is, since $d(\kappa_b) = 0$, it follows from the Leibniz rule that $(d - \kappa_b\wedge)^2 = 0$ as an operator on forms, and it maps basic forms to basic forms. On transversally oriented foliations, this differential also has the property that

$$\delta_b \overline{*} \alpha = (-1)^{k+1} \overline{*} (d - \kappa_b \wedge) \alpha$$

on every basic k-form α (see [33]). As a result of this equation and the basic cohomology version of the Hodge theorem ([23], [33]), the transversal Hodge star operator implements an isomorphism between different kinds of basic cohomology groups (see [22], [41]):

$$H_d^* (M, \mathcal{F}) \cong H_{d-\kappa_b\wedge}^{q-*} (M, \mathcal{F}) .$$

This is called *twisted Poincaré duality.*

1.6. **The basic Dirac operator and spectral rigidity.** We now discuss the construction of the basic Dirac operator (see [10], [15], [34], [6]), a construction which requires a choice of bundle-like metric. Let $(M, \mathcal{F})$ be a Riemannian manifold endowed with a Riemannian foliation. Let $E \to M$ be a foliated vector bundle (see [20]) that is a bundle of $\mathbb{C}\mathrm{l}(Q)$ Clifford modules with compatible connection ∇^E. The *transversal Dirac operator* D_{tr} is the composition of the maps

$$\Gamma(E) \xrightarrow{(\nabla^E)^{\mathrm{tr}}} \Gamma(Q^* \otimes E) \xrightarrow{\cong} \Gamma(Q \otimes E) \xrightarrow{\mathrm{Cliff}} \Gamma(E) ,$$

where the last map stands for Clifford multiplication, denoted by "$\cdot$", and the operator $(\nabla^E)^{\mathrm{tr}}$ is the projection of ∇^E. The transversal Dirac operator fixes the basic sections $\Gamma_b(E) \subset \Gamma(E)$ (i.e. $\Gamma_b(E) = \{s \in \Gamma(E) : \nabla^E_X s = 0$ for all $X \in \Gamma(L)\}$) but is not symmetric on this subspace. By modifying D_{tr} by a bundle map, we obtain a symmetric and essentially self-adjoint operator D_b on $\Gamma_b(E)$. We now define

$$D_{\mathrm{tr}}\ s = \sum_{i=1}^{q} e_i \cdot \nabla^E_{e_i} s\ ,$$

$$D_b s = \frac{1}{2}(D_{\mathrm{tr}} + D_{\mathrm{tr}}^*) s = \sum_{i=1}^{q} e_i \cdot \nabla^E_{e_i} s - \frac{1}{2}\kappa_b^\sharp \cdot s\ ,$$

where $\{e_i\}_{i=1,\cdots,q}$ is a local orthonormal frame of Q. A direct computation shows that D_b preserves the basic sections, is transversally elliptic, and thus has discrete spectrum ([15], [10], [11]).

An example of the basic Dirac operator is as follows. Using the bundle $\wedge^* Q$ as the Clifford bundle with Clifford action $e\cdot = e^* \wedge - e^* \lrcorner$ in analogy to the ordinary de Rham operator, we have

$$D_{\mathrm{tr}} = d + \delta_T = d + \delta_b - \kappa_b \lrcorner : \Omega^{\mathrm{even}}(M, \mathcal{F}) \to \Omega^{\mathrm{odd}}(M, \mathcal{F})$$
$$D_b = \frac{1}{2}(D_{\mathrm{tr}} + D_{\mathrm{tr}}^*)s = d - \frac{1}{2}\kappa_b \wedge + \delta_b - \frac{1}{2}\kappa_b \lrcorner.$$

One might have incorrectly guessed that $d + \delta_b$ is the basic de Rham operator in analogy to the ordinary de Rham operator, for this operator is essentially self-adjoint, and the associated basic Laplacian yields basic Hodge theory that can be used to compute the basic cohomology. The square D_b^2 of this operator and the basic Laplacian Δ_b do have the same principal symbol. In [16], we showed the invariance of the spectrum of D_b with respect to a change of metric on M in any way that leaves the transverse metric on the normal bundle intact (this includes modifying the subbundle $Q \subset TM$, as one must do in order to make the mean curvature basic, for example). That is,

Theorem 1.1. *(In* [16]*) Let $(M, \mathcal{F})$ be a compact Riemannian manifold endowed with a Riemannian foliation and basic Clifford bundle $E \to M$. The spectrum of the basic Dirac operator is the same for every possible choice of bundle-like metric that is associated to the transverse metric on the quotient bundle Q.*

We emphasize that the basic Dirac operator D_b depends on the choice of bundle-like metric, not merely on the Clifford structure and Riemannian foliation structure, since both projections $T^*M \to Q^*$ and P as well as $\kappa_b \lrcorner$ depend on the leafwise metric. It is well-known that the eigenvalues of the basic Laplacian Δ_b (closely related to D_b^2) depend on the choice of bundle-like metric; for example, in [37, Corollary 3.8], it is shown that the spectrum of the basic Laplacian on functions determines the L^2-norm of the mean curvature on a transversally oriented foliation of codimension one. This is one reason why the invariance of the spectrum of the basic Dirac operator is a surprise.

Corollary 1.2. *Let $(M, \mathcal{F})$ be a compact Riemannian manifold endowed with a Riemannian foliation and basic Clifford bundle $E \to M$. In calculating the spectrum of the basic Dirac operator, one may assume the bundle-like metric is chosen so that the mean curvature form is basic-harmonic.*

Proof. By Theorem 1.1, we may choose the bundle-like metric in any way that restricts to the given metric on Q. In [9] and [27, 29], the researchers showed that there exists such a metric such that the mean curvature is basic-harmonic. □

1.7. **Known tautness results.** A Riemannian foliation $(M, \mathcal{F})$ is called **minimalizable** or **geometrically taut** if there exists a Riemannian metric on M for which all leaves are minimal submanifolds. We will use the word **taut** for this property. In [14], E. Ghys proved that every Riemannian foliation on a simply connected manifold is taut. X. Masa showed in [28] that a transversally oriented

Riemannian foliation of codimension q is taut if and only if $H_d^q(M,\mathcal{F}) \neq \{0\}$ (see also [21], [22]). Moreover, J. Àlvarez-Lopez [1] characterized the tautness by the fact that the cohomology class of the basic mean curvature form vanishes in $H_d^1(M,\mathcal{F}) \subseteq H^1(M)$; therefore, every Riemannian foliation on a manifold with vanishing first Betti number is taut. In [22], the results of the F. W. Kamber and Ph. Tondeur suffice to prove that a foliation is taut if and only if the basic cohomology groups satisfy Poincaré duality. In [31] and [32], H. Nozawa showed that the Àlvarez class $[\kappa_b] \in H^1(M)$ is stable with respect to continuous perturbations of Riemannian foliations, and he also showed that the line integral of κ_b over a closed curve is always an algebraic integer if the fundamental group of M is polycyclic or has polynomial growth. Under such conditions, a family of Riemannian foliations consists entirely of taut foliations or of nontaut foliations. As a consequence, he proved that if $(M,\mathcal{F})$ is a codimension two Riemannian foliation with $\pi_1(M)$ of polynomial growth, then the foliation is taut.

Certain geometric conditions are also known to force the tautness condition. For example, J. Hebda showed in [17] that if the transversal Ricci curvature satisfies $\mathrm{Ric}(X,X) \geq a(q-1)|X|^2$ for some $a > 0$ and all vectors $X \in \Gamma(Q)$, then the foliation is taut. In a result by S. D. Jung that was later extended by the authors (see [18] and [16]), if the first eigenvalue λ of the basic spin Dirac operator satisfies equality in the general eigenvalue bound

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M \left(\mathrm{Scal}_M - \mathrm{Scal}_{\mathcal{F}} + |A_Q|^2 + |T_{\mathcal{F}}|^2\right),$$

then $\mathcal{F}$ is taut and there exists a transversal Killing spinor. Here, A_Q and $T_{\mathcal{F}}$ denote the O'Neill tensors of the foliation; see [16] for details.

1.8. **Main results and outline.** In this paper we introduce the new cohomology $\widetilde{H}^*(M,\mathcal{F})$ (called the **twisted basic cohomology**) of basic forms that uses $\widetilde{d} := d - \frac{1}{2}\kappa_b\wedge$ as a differential. Recall that the basic de Rham operator is $D_b = \widetilde{d} + \widetilde{\delta}$, where $\widetilde{\delta} := \delta_b - \frac{1}{2}\kappa_b\lrcorner$. We show in Section 2 that the corresponding Betti numbers and eigenvalues of the twisted basic Laplacian $\widetilde{\Delta} := \widetilde{d}\widetilde{\delta} + \widetilde{\delta}\widetilde{d}$ are independent of the choice of a bundle-like metric. In Theorem 3.1 we show that the twisted basic Laplacian commutes with the transversal Hodge star operator and thus the twisted basic cohomology satisfies Poincaré duality. As a corollary, we deduce that an odd codimension transversally oriented Riemannian foliation has zero basic Euler characteristic (Corollary 3.3). In Section 4, we prove that taut foliations give an isomorphism between the new cohomology group and the basic one. This lets us say that the tautness property is characterized by the fact that the top-dimensional twisted basic cohomology group is non-zero. In Section 5, we define the basic signature operator of a Riemannian foliation.

Using some computations with the Lie derivative, we establish in Section 6 a Weitzenböck-Bochner formula for the twisted basic Laplacian (see Proposition 6.7), which is more simple that the corresponding formula for the ordinary basic Laplacian. With the help of this formula, we deduce various corollaries relating transversal Ricci and sectional curvature to tautness and basic cohomology. In particular, we deduce direct proofs of known results of Hebda (see Theorem 6.16) and of El Kacimi and others (see Theorem 6.18).

We also study the case of codimension two Riemannian foliations. We prove in Proposition 6.20 that for nontaut foliations, the ordinary basic cohomology satisfies $H_d^0(M,\mathcal{F}) \cong H_d^1(M,\mathcal{F}) \cong \mathbb{R}$, $H_d^2(M,\mathcal{F}) = \{0\}$. Immediate consequences include for nontaut foliations:

- The twisted basic cohomology groups are all trivial.
- The basic Euler characteristic and basic signature are zero (Corollary 6.21).
- If $\pi_1(M)$ is polycyclic or has polynomial growth, then the basic Euler characteristic and basic signature are stable with respect to deformations of $(M,\mathcal{F})$ through continuous families of Riemannian foliations, and the dimensions of all basic cohomology groups are also stable. See Corollary 6.23.

To illustrate our results, we treat examples in Section 7.

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2. Modified differentials, Laplacians, and basic cohomology

Unlike the ordinary and well-studied basic Laplacian, the eigenvalues of $\widetilde{\Delta} = D_b^2$ are invariants of the Riemannian foliation structure alone and independent of the choice of compatible bundle-like metric. The operators $\widetilde{d}$ and $\widetilde{\delta}$ have the following interesting properties.

Lemma 2.1. *$\widetilde{\delta}$ is the formal adjoint of $\widetilde{d}$.*

Proof. We see

$$\left(\widetilde{d}\right)^* = \left(d - \frac{1}{2}\kappa_b\wedge\right)^* = \delta_b - \frac{1}{2}\left(\kappa_b\lrcorner\right) = \widetilde{\delta},$$

where the raised $*$ denotes formal L^2-adjoint on the space of basic forms (not the same as the adjoint on the space of all forms). □

Lemma 2.2. *The maps $\widetilde{d}$ and $\widetilde{\delta}$ are differentials; that is, $\widetilde{d}^2 = 0$, $\widetilde{\delta}^2 = 0$. As a result, $\widetilde{d}$ and $\widetilde{\delta}$ commute with $\widetilde{\Delta} = D_b^2$, and $\ker\left(\widetilde{d} + \widetilde{\delta}\right) = \ker\left(\widetilde{\Delta}\right)$.*

Proof. This follows from the fact that κ_b is a closed one-form [1]. □

Let $\Omega^k(M,\mathcal{F})$ denote the space of basic k-forms (either set of smooth forms or L^2-completion thereof), let $\widetilde{d}^k$ and $\widetilde{\delta_b}^k$ be the restrictions of $\widetilde{d}$ and $\widetilde{\delta_b}$ to k-forms, and let $\widetilde{\Delta}^k$ denote the restriction of D_b^2 to basic k-forms.

Proposition 2.3. *(Hodge decomposition) We have* $\Omega^k(M,\mathcal{F}) = \operatorname{image}\left(\widetilde{d}^{k-1}\right) \oplus \operatorname{image}\left(\widetilde{\delta_b}^{k+1}\right) \oplus \ker\left(\widetilde{\Delta}^k\right)$*, an* L^2*-orthogonal direct sum. Also,* $\ker\left(\widetilde{\Delta}^k\right)$ *is finite-dimensional and consists of smooth forms.*

Proof. The proof is very similar to the proof in [33] for the corresponding fact for the basic Laplacian and in [41], [23] for the basic mean curvature case. For that reason, we do not include it here. □

We call $\ker\left(\widetilde{\Delta}\right)$ the space of $\widetilde{\Delta}$-harmonic forms. In the remainder of this section, we assume that the foliation is transversally oriented so that the transversal Hodge $\overline{*}$ operator is well-defined.

Lemma 2.4. *(clear) The operator* $\overline{*}^2 = (-1)^{k(q-k)}$ *on* k*-forms, and the adjoint of* $\overline{*}$ *is* $(-1)^{k(q-k)}\,\overline{*}$*.*

Lemma 2.5. *(in* [33]*) The basic projection* P *commutes with* $\overline{*}$*.*

Lemma 2.6. *(in* [33]*) Given any* $\alpha \in (N\mathcal{F})^*$*,* $\alpha\lrcorner = (-1)^{q(k+1)}\,\overline{*}\,(\alpha\wedge)\,\overline{*}$ *as an operator on basic* k*-forms.*

Lemma 2.7. *(in* [33]*) If* β *is a basic* k*-form,* $\delta_b\beta = (-1)^{q(k+1)+1}\,\overline{*}\,(d-\kappa_b\wedge)\,\overline{*}\beta$*.*

Proposition 2.8. *We have the following identities for operators acting on* $\Omega^k(M,\mathcal{F})$*:*

(1) $(\kappa_b\lrcorner)\,\overline{*} = (-1)^k\,\overline{*}\,(\kappa_b\wedge)$
(2) $\overline{*}\,(\kappa_b\lrcorner) = (-1)^{k+1}\,(\kappa_b\wedge)\,\overline{*}$
(3) $\delta_b\overline{*} = (-1)^{k+1}\,\overline{*}\,(d-\kappa_b\wedge)$
(4) $\overline{*}\delta_b = (-1)^k\,(d-\kappa_b\wedge)\,\overline{*}$
(5) $\widetilde{\delta}\overline{*} = (-1)^{k+1}\,\overline{*}\widetilde{d}$
(6) $\overline{*}\widetilde{\delta} = (-1)^k\,\widetilde{d}\overline{*}$
(7) $\overline{*}\widetilde{d} = (-1)^{k+1}\,\widetilde{\delta}\overline{*}$
(8) $\widetilde{d}\overline{*} = (-1)^k\,\overline{*}\widetilde{\delta}$.

Proof. Acting on basic k-forms, we calculate each of the left sides of the identities above using the lemmas above:

$$\begin{aligned}(\kappa_b\lrcorner)\,\overline{*} &= (-1)^{q(q-k+1)}\,\overline{*}\,(\kappa_b\wedge)\,\overline{*}^2\\ &= (-1)^{q(q-k+1)+k(q-k)}\,\overline{*}\,(\kappa_b\wedge)\\ &= (-1)^k\,\overline{*}\,(\kappa_b\wedge)\,.\end{aligned}$$

$$\begin{aligned}\overline{*}\,(\kappa_b\lrcorner) &= (-1)^{q(k+1)}\,\overline{*}^2\,(\kappa_b\wedge)\,\overline{*}\\ &= (-1)^{q(k+1)+(q-k+1)(k-1)}\,(\kappa_b\wedge)\,\overline{*}\\ &= (-1)^{k+1}\,(\kappa_b\wedge)\,\overline{*}.\end{aligned}$$

$$\begin{aligned}\delta_b \overline{*} &= (-1)^{q(q-k+1)+1}\, \overline{*}\, (d - \kappa_b \wedge)\, \overline{*}^2 \\ &= (-1)^{q(q-k+1)+1+k(q-k)}\, \overline{*}\, (d - \kappa_b \wedge) \\ &= (-1)^{k+1}\, \overline{*}\, (d - \kappa_b \wedge) .\end{aligned}$$

$$\begin{aligned}\overline{*}\delta_b &= (-1)^{q(k+1)+1}\, \overline{*}^2\, (d - \kappa_b \wedge)\, \overline{*} \\ &= (-1)^{q(k+1)+1+(q-k+1)(k-1)}\, (d - \kappa_b \wedge)\, \overline{*} \\ &= (-1)^{k}\, (d - \kappa_b \wedge)\, \overline{*}.\end{aligned}$$

Putting the results above together, we have

$$\begin{aligned}\widetilde{\delta}\overline{*} &= \left(\delta_b - \frac{1}{2}\kappa_b \lrcorner\right) \overline{*} \\ &= (-1)^{k+1}\, \overline{*}\, (d - \kappa_b \wedge) - \frac{1}{2}\, (-1)^{k}\, \overline{*}\, (\kappa_b \wedge) \\ &= (-1)^{k+1}\, \overline{*} \left(d - \frac{1}{2}\kappa_b \wedge\right) \\ &= (-1)^{k+1}\, \overline{*}\widetilde{d},\end{aligned}$$

and

$$\begin{aligned}\overline{*}\widetilde{\delta} &= \overline{*} \left(\delta_b - \frac{1}{2}\kappa_b \lrcorner\right) \\ &= (-1)^{k}\, (d - \kappa_b \wedge)\, \overline{*} - \frac{1}{2}\, (-1)^{k+1}\, (\kappa_b \wedge)\, \overline{*} \\ &= (-1)^{k} \left(d - \frac{1}{2}\kappa_b \wedge\right) \overline{*} \\ &= (-1)^{k}\, \widetilde{d}\overline{*}.\end{aligned}$$

Switching sides of the equations in (5) and (6), we obtain

$$\begin{aligned}\overline{*}\widetilde{d} &= (-1)^{k+1}\, \widetilde{\delta}\overline{*} \\ \widetilde{d}\overline{*} &= (-1)^{k}\, \overline{*}\widetilde{\delta}.\end{aligned}$$

□

Definition 2.9. *We define the basic $\widetilde{d}$-cohomology $\widetilde{H}^*(M, \mathcal{F})$ by*

$$\widetilde{H}^k(M, \mathcal{F}) = \frac{\ker \widetilde{d}^k}{\text{image } \widetilde{d}^{k-1}}.$$

The following proposition follows from standard arguments and the Hodge theorem (Theorem 2.3).

Proposition 2.10. *The finite-dimensional vector spaces $\widetilde{H}^k(M, \mathcal{F})$ and $\ker \widetilde{\Delta}^k = \ker \left(\widetilde{d} + \widetilde{\delta}\right)^k$ are naturally isomorphic.*

We observe that for every choice of bundle-like metric, the differential $\widetilde{d}$ changes, and thus the cohomology groups change. However, note that κ_b is the only part that changes; for any two bundle-like metrics g_M, g_M' and associated κ_b, κ_b' compatible with $(M, \mathcal{F}, g_Q)$, we have $\kappa_b' = \kappa_b + dh$ for some basic function h (see [1]). In the proof of the main theorem in [16], we essentially showed that the the basic de Rham operator D_b is then transformed by $D_b' = e^{h/2} D_b e^{-h/2}$. Applying this to our situation, we see that the $(\ker D_b') = e^{h/2} \ker D_b$, and thus the cohomology groups are the same dimensions, independent of choices. To see this in our specific situation, note that if $\alpha \in \Omega^k(M, \mathcal{F})$ satisfies $\widetilde{d}\alpha = 0$, then

$$
\begin{aligned}
\left(\widetilde{d}\right)' \left(e^{h/2}\alpha\right) &= \left(d - \frac{1}{2}\kappa_b \wedge -\frac{1}{2} dh\wedge\right) \left(e^{h/2}\alpha\right) \\
&= e^{h/2} d\alpha + \frac{1}{2} e^{h/2} dh \wedge \alpha - \frac{e^{h/2}}{2}\kappa_b \wedge \alpha - \frac{e^{h/2}}{2} dh \wedge \alpha \\
&= e^{h/2} d\alpha - \frac{e^{h/2}}{2}\kappa_b \wedge \alpha = e^{h/2}\left(d - \frac{1}{2}\kappa_b\wedge\right)\alpha = e^{h/2}\widetilde{d}\alpha = 0.
\end{aligned}
$$

Similarly, as in [16] one may show $\ker\left(\widetilde{\delta}\right)' = e^{h/2}\ker\left(\widetilde{\delta}\right)$, through a slightly more difficult computation. Thus, we have

Theorem 2.11. *(Conformal invariance of cohomology groups) Given a Riemannian foliation $(M, \mathcal{F}, g_Q)$ and any two bundle-like metrics g_M and g_M' compatible with g_Q, the $\widetilde{d}$-cohomology groups $\widetilde{H}^k(M, \mathcal{F})$ are isomorphic, and that isomorphism is implemented by multiplication by a positive basic function. Further, the eigenvalues of the corresponding basic de Rham operators D_b and D_b' are identical, and the eigenspaces are isomorphic via multiplication by that same positive function.*

Corollary 2.12. *The dimensions of $\widetilde{H}^k(M, \mathcal{F})$ and the eigenvalues of D_b (and thus of $\widetilde{\Delta} = D_b^2$) are invariants of the Riemannian foliation structure $(M, \mathcal{F}, g_Q)$, independent of choice of compatible bundle-like metric g_M.*

Corollary 2.13. *The dimensions of $\widetilde{H}^k(M, \mathcal{F})$ are independent of the choice of the bundle-like metric and independent of the transverse Riemannian foliation structure.*

Proof. By [1], the basic components of the mean curvature forms for two different bundle-like metrics differ by an exact basic one-form $\kappa_b' = \kappa_b + dh$. Since

$$\left(\widetilde{d}\right)' = e^{h/2}\widetilde{d}e^{-h/2}$$

by the computation above, the twisted basic cohomology groups corresponding to the different metrics are conjugate. □

3. Poincaré duality and consequences

Theorem 3.1. *(Poincaré duality for $\widetilde{d}$-cohomology) Suppose that the Riemannian foliation $(M, \mathcal{F}, g_Q)$ is transversally oriented and is endowed with a bundle-like metric. For each k such that $0 \leq k \leq q$ and any compatible choice of bundle-like metric, the map $\overline{*} : \Omega^k(M, \mathcal{F}) \to \Omega^{q-k}(M, \mathcal{F})$ induces an isomorphism on the $\widetilde{d}$-cohomology. Moreover, $\overline{*}$ maps the $\ker \widetilde{\Delta}^k$ isomorphically onto $\ker \widetilde{\Delta}^{q-k}$, and it maps the λ-eigenspace of $\widetilde{\Delta}^k$ isomorphically onto the λ-eigenspace of $\widetilde{\Delta}^{q-k}$, for all $\lambda \geq 0$.*

Proof. Acting on basic forms of degree k, we use

$$\begin{aligned}
\overline{*}\widetilde{\Delta} &= \overline{*}\widetilde{d}\widetilde{\delta} + \overline{*}\widetilde{\delta}\widetilde{d} \\
&= (-1)^k \widetilde{\delta}\overline{*}\widetilde{\delta} + (-1)^{k+1} \widetilde{d}\overline{*}\widetilde{d} \\
&= (-1)^k \widetilde{\delta} (-1)^k \widetilde{d}\overline{*} + (-1)^{k+1} \widetilde{d} (-1)^{k+1} \widetilde{\delta}\overline{*} \\
&= \left(\widetilde{\delta}\widetilde{d} + \widetilde{d}\widetilde{\delta}\right) \overline{*} = \widetilde{\Delta}\overline{*}.
\end{aligned}$$

Since $\overline{*}$ commutes with $\widetilde{\Delta}$, it maps eigenspaces of $\widetilde{\Delta}$ to themselves. By the Hodge theorem, the result follows. □

This resolves the problem of the failure of Poincaré duality to hold for standard basic cohomology (see [22], [41]).

Corollary 3.2. *Let $(M, \mathcal{F})$ be a smooth transversally oriented foliation of odd codimension that admits a transverse Riemannian structure. Then the Euler characteristic associated to the $\widetilde{H}^*(M, \mathcal{F})$ vanishes.*

Corollary 3.3. *Let $(M, \mathcal{F})$ be a smooth transversally oriented foliation of odd codimension that admits a transverse Riemannian structure. Then the Euler characteristic associated to the ordinary basic cohomology $H_d^*(M, \mathcal{F})$ vanishes.*

Proof. The basic Euler characteristic is the basic index of the operator $D_0 = d + \delta_B : \Omega^{\text{even}}(M, \mathcal{F}) \to \Omega^{\text{odd}}(M, \mathcal{F})$. See [6], [10], [4], [11] for information on the basic index and basic Euler characteristic. The crucial property for us is that the basic index of D_0 is a Fredholm index and is invariant under perturbations of the operator through transversally elliptic operators that map the basic forms to themselves. In particular, the family of operators $D_t = d + \delta_b - \frac{t}{2}\kappa_b \lrcorner - \frac{t}{2}\kappa_b \wedge$ for $0 \leq t \leq 1$ meets that criteria, and $D_1 = D_b$ is the basic de Rham operator $D_b : \Omega^{\text{even}}(M, \mathcal{F}) \to \Omega^{\text{odd}}(M, \mathcal{F})$. Thus, the basic Euler characteristic of the basic cohomology complex is the same as the basic Euler characteristic of the $\widetilde{d}$-cohomology complex. The result follows from the previous corollary. □

Using this result, we give another proof for the well-known theorem [39]:

Theorem 3.4. *Let $\mathcal{F}$ be a transversally Riemannian oriented foliation of codimension 1 on a closed manifold M. Then $\mathcal{F}$ is taut.*

Proof. Since the codimension is odd, the basic Euler characteristic is zero. Thus, we get that $\dim H_B^1(M,\mathcal{F}) = 1$ and the foliation is taut by the result of Masa. □

4. Tautness Theorem

As we mentionned in the introduction, the author showed in [1] that the cohomology class $[\kappa_b] \in H_d^1(M,\mathcal{F})$ is an invariant of the Riemannian foliation structure and independent of the choice of bundle-like metric, and the foliation is taut if and only if $[\kappa_b] = 0$. If in addition the foliation is transversally oriented, this condition is equivalent to $H_d^q(M,\mathcal{F}) \neq 0$, which is true if and only if $H_d^*(M,\mathcal{F})$ satisfies Poincaré duality. We now prove the analogous result for our modified basic cohomology $\widetilde{H}^*(M,\mathcal{F})$.

First, we observe that the basic projection κ_b of the mean curvature one form is always $\widetilde{d}$-exact, because

$$\widetilde{d}\,(-2) = \left(d - \frac{1}{2}\kappa_b\wedge\right)(-2) = \kappa_b.$$

Also, we have the following:

Lemma 4.1. *A Riemannian foliation $(M,\mathcal{F},g_Q)$ of codimension q is taut, then $H_d^*(M,\mathcal{F}) \cong \widetilde{H}^*(M,\mathcal{F})$. The converse is true if the foliation is assumed to be transversally oriented.*

Proof. If the foliation is taut, by [1] $\kappa_b = df$ for some basic function f. Then $\widetilde{d} = d - \frac{1}{2}df\wedge = e^{\frac{f}{2}} \circ d \circ e^{-\frac{f}{2}}$. Thus $[\alpha] \mapsto \left[\exp\left(-\frac{f}{2}\right)\alpha\right]$ yields an isomorphism from $\widetilde{H}^*(M,\mathcal{F})$ to $H_d^*(M,\mathcal{F})$. Conversely, if $H_d^*(M,\mathcal{F}) \cong \widetilde{H}^*(M,\mathcal{F})$, then Poincaré duality is satisfied for the ordinary basic cohomology (from the fact that it is satisfied for our twisted cohomology), which means $H_d^q(M,\mathcal{F}) \neq 0$. This is equivalent to the tautness of the foliation if the foliation is assumed to be transversally oriented. □

Theorem 4.2. *A transversally oriented Riemannian foliation $(M,\mathcal{F},g_Q)$ of codimension q with bundle-like metric g_M is taut if and only if $\widetilde{H}^0(M,\mathcal{F}) \cong \widetilde{H}^q(M,\mathcal{F}) \neq 0$.*

Proof. If $(M,\mathcal{F},g_Q)$ is taut, by the above Lemma we conclude that

$$\widetilde{H}^0(M,\mathcal{F}) \cong \widetilde{H}^q(M,\mathcal{F}) \cong H_d^q(M,\mathcal{F}) \neq 0.$$

For the converse, we first assume that our metric is chosen so that the mean curvature is basic-harmonic, meaning that $\kappa = \kappa_b$ and

$$\delta_b\kappa = \left(-\overline{*}d\overline{*} + \kappa\lrcorner\right)\kappa = -\overline{*}d\overline{*}\kappa + |\kappa|^2 = 0.$$

As we said before, it is always possible to choose the bundle-like metric this way and our twisted basic cohomology groups are not affected. If $\widetilde{H}^0(M,\mathcal{F}) \neq 0$, there exists a nontrivial basic function h such that $\widetilde{d}h = 0$. Hence $dh = \frac{1}{2}h\kappa_b = \frac{1}{2}h\kappa$. Then

$$\begin{aligned}
\Delta_b h = \delta_b dh = \frac{1}{2}\delta_b(h\kappa) &= \frac{1}{2}\left(-\overline{*}d\overline{*} + \kappa\lrcorner\right)(h\kappa) \\
&= \frac{1}{2}\left(-\overline{*}\left(dh\wedge\right)\overline{*}\kappa - h\overline{*}d\overline{*}\kappa + h\left|\kappa\right|^2\right) \\
&= \frac{1}{2}\left(-\frac{1}{2}h\overline{*}\left(\kappa\wedge\left(\overline{*}\kappa\right)\right) - h\left|\kappa\right|^2 + h\left|\kappa\right|^2\right) \\
&= -\frac{1}{4}h\left|\kappa\right|^2.
\end{aligned}$$

The integral over M on both sides yields $|\kappa| = 0$. Thus, $(M, \mathcal{F})$ is taut. □

5. The basic signature operator

With notation as in Section 2, suppose that $(M, \mathcal{F}, g_Q)$ is a transversally oriented Riemannian foliation of even codimension q, and let g_M be a specific compatible bundle-like metric. Let

$$\bigstar = i^{k(k-1)+\frac{q}{2}}\overline{*}$$

as an operator on basic k-forms, analogous to the involution used to identify self-dual and anti-self-dual forms on a manifold. Note that this endomorphism is symmetric, and

$$\bigstar^2 = 1.$$

Proposition 5.1. *We have* $\bigstar\left(\widetilde{d} + \widetilde{\delta}\right) = -\left(\widetilde{d} + \widetilde{\delta}\right)\bigstar$. *In fact,* $\bigstar\widetilde{d} = -\widetilde{\delta}\bigstar$ *and* $\bigstar\widetilde{\delta} = -\widetilde{d}\bigstar$.

Proof. By Proposition 2.8, we have that, as an operator on basic k-forms,

$$\begin{aligned}
\bigstar\left(\widetilde{d} + \widetilde{\delta}\right) &= \bigstar\widetilde{d} + \bigstar\widetilde{\delta} \\
&= i^{(k+1)(k)+\frac{q}{2}}\overline{*}\widetilde{d} + i^{(k-1)(k-2)+\frac{q}{2}}\overline{*}\widetilde{\delta} \\
&= i^{(k+1)(k)+\frac{q}{2}}(-1)^{k+1}\widetilde{\delta}\overline{*} + i^{(k-1)(k-2)+\frac{q}{2}}(-1)^k\widetilde{d}\overline{*} \\
&= i^{k^2+k+2k+2+\frac{q}{2}}\widetilde{\delta}\overline{*} + i^{k^2-3k+2+2k+\frac{q}{2}}\widetilde{d}\overline{*} \\
&= i^{k(k-1)+4k+2+\frac{q}{2}}\widetilde{\delta}\overline{*} + i^{k(k-1)+2+\frac{q}{2}}\widetilde{d}\overline{*} \\
&= -\left(\widetilde{\delta} + \widetilde{d}\right)i^{k(k-1)+\frac{q}{2}}\overline{*} = -\left(\widetilde{d} + \widetilde{\delta}\right)\bigstar.
\end{aligned}$$

From the computation, we see that $\bigstar\widetilde{d} = -\widetilde{\delta}\bigstar$ and $\bigstar\widetilde{\delta} = -\widetilde{d}\bigstar$. □

Let $\Omega^+(M, \mathcal{F})$ denote the $+1$ eigenspace of $\bigstar$ in $\Omega^*(M, \mathcal{F})$, and let $\Omega^-(M, \mathcal{F})$ denote the -1 eigenspace of $\bigstar$ in $\Omega^*(M, \mathcal{F})$. By the proposition above, $D_b = \widetilde{d} + \widetilde{\delta}$

maps $\Omega^{\pm}(M,\mathcal{F})$ to $\Omega^{\mp}(M,\mathcal{F})$. Therefore, we may define the basic signature operator as follows.

Definition 5.2. *On a transversally oriented Riemannian foliation of even codimension, let the* ***basic signature operator*** *be the operator* $D_b\colon \Omega^{+}(M,\mathcal{F}) \to \Omega^{-}(M,\mathcal{F})$*. We define the* ***basic signature*** $\sigma(M,\mathcal{F})$ ***of the foliation*** *to be the index*

$$\sigma(M,\mathcal{F}) = \dim\ker\left(\left.\widetilde{\Delta}\right|_{\Omega^{+}(M,\mathcal{F})}\right) - \dim\ker\left(\left.\widetilde{\Delta}\right|_{\Omega^{-}(M,\mathcal{F})}\right).$$

We note that such a definition is not possible for the operator $d+\delta_b$, because the relationship in the proposition above does not hold for $d+\delta_b$.

6. The twisted basic Laplacian, curvature, and tautness

First, let $\chi_{\mathcal{F}}$ denote the characteristic form of any oriented foliation $\mathcal{F}$ on a Riemannian manifold; this is the leafwise volume form of the foliation, locally given by

$$\chi_{\mathcal{F}} = e_1^* \wedge \ldots \wedge e_p^*,$$

where $(e_1,\ldots,e_p,e_{p+1},\ldots,e_n)$ is a local orthonormal frame of the TM such that $(e_1,\ldots,e_p)$ is a local orthonormal frame of $T\mathcal{F}$. Then Rummler's formula (see [38]) gives

$$d\chi_{\mathcal{F}} = -\kappa\wedge\chi_{\mathcal{F}} + \varphi_0, \tag{6.1}$$

where φ_0 is a $(p+1)$-form on M with the property that $v_1\lrcorner v_2\lrcorner\cdots\lrcorner v_p\lrcorner\varphi_0 = 0$ if $v_j\in T_x\mathcal{F}$, $1\le j\le p$, are leafwise vectors.

We now determine relationships between the eigenvalues of the twisted basic Laplacian $\widetilde{\Delta}$ and curvature. First we do a few computations that will be useful later.

Lemma 6.1. *We have the following facts about the Lie derivative. If α is any form, V,W are vector fields, then*

(1) $\mathcal{L}_V = d\circ(V\lrcorner) + (V\lrcorner)\circ d$
(2) $\mathcal{L}_V^* = \left(V^\flat\wedge\right)\circ\delta + \delta\circ\left(V^\flat\wedge\right)$
(3) $\mathcal{L}_V\circ(\alpha\wedge) = (\mathcal{L}_V(\alpha)\wedge) + (\alpha\wedge)\circ\mathcal{L}_V$
(4) $\mathcal{L}_V\circ(W\lrcorner) = (\mathcal{L}_V(W)\lrcorner) + (W\lrcorner)\circ\mathcal{L}_V$
(5) $\mathcal{L}_V^*\circ(\alpha\wedge) = (\alpha\wedge)\circ\mathcal{L}_V^* - \left(\left(\mathcal{L}_V\left(\alpha^\#\right)\right)^\flat\wedge\right)$ *if α is a one-form.*

Proof. The first two facts follow from the Cartan formulas. The formulas after that are standard (up to taking adjoints) and can be found in [24]. □

Lemma 6.2. *The operator $(\mathcal{L}_V+\mathcal{L}_V^*)$ is zeroth order and thus commutes with multiplication by a function.*

Proof. One may easily compute the commutator $[(\mathcal{L}_V + \mathcal{L}_V^*), m_f]$, where m_f denotes multiplication by a function f, and the result is zero. $\square$

Lemma 6.3. *If α is a one-form, β is any form, and V is a vector field, then $(\mathcal{L}_V + \mathcal{L}_V^*)(\alpha \wedge \beta) = \alpha \wedge (\mathcal{L}_V + \mathcal{L}_V^*)\beta + \gamma \wedge \beta$, where*

$$\begin{aligned}\gamma &= \mathcal{L}_V(\alpha) - \left(\mathcal{L}_V\left(\alpha^{\#}\right)\right)^{\flat} \\ &= (\mathcal{L}_V g)\left(\alpha^{\#}, \bullet\right) \\ &= \nabla_{\alpha^{\#}} V^{\flat} + \alpha\left(\nabla_{\bullet} V\right),\end{aligned}$$

where $\mathcal{L}_V g$ is the Lie derivative of the metric tensor.

Proof. By Lemma 6.1, part (3), we have

$$\mathcal{L}_V(\alpha \wedge \beta) = \mathcal{L}_V(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_V(\beta).$$

Now using the part (5) of the same lemma,

$$\mathcal{L}_V^*(\alpha \wedge \beta) = \alpha \wedge \mathcal{L}_V^*(\beta) - \left(\mathcal{L}_V\left(\alpha^{\#}\right)\right)^{\flat} \wedge \beta.$$

The result with the first expression for γ follows from adding the two equations. Also, for any vector field Z,

$$\begin{aligned}\gamma(Z) &= \mathcal{L}_V(\alpha)(Z) - \left(\mathcal{L}_V\left(\alpha^{\#}\right), Z\right) \\ &= V\left(\alpha^{\#}, Z\right) - \left(\alpha^{\#}, \mathcal{L}_V Z\right) - \left(\mathcal{L}_V\left(\alpha^{\#}\right), Z\right) \\ &= (\mathcal{L}_V g)\left(\alpha^{\#}, Z\right),\end{aligned}$$

and also from the second line above we have

$$\begin{aligned}\gamma(Z) &= \left(\nabla_V \alpha^{\#}, Z\right) + \left(\alpha^{\#}, \nabla_V Z\right) - \left(\alpha^{\#}, \mathcal{L}_V Z\right) - \left(\mathcal{L}_V\left(\alpha^{\#}\right), Z\right) \\ &= \left(\nabla_{\alpha^{\#}} V, Z\right) + \left(\alpha^{\#}, \nabla_Z V\right).\end{aligned}$$ $\square$

Corollary 6.4. *If α is a one-form and V is a vector field, then $(\mathcal{L}_V + \mathcal{L}_V^*)\alpha = \left(\delta V^{\flat}\right)\alpha + \gamma$, with γ given in the previous Lemma, and we also have $((\mathcal{L}_V + \mathcal{L}_V^*)\alpha, \alpha) = \left(\delta V^{\flat}\right)(\alpha, \alpha) + 2\left(\nabla_{\alpha^{\#}} V, \alpha^{\#}\right)$.*

Proof. It follows from $(\mathcal{L}_V + \mathcal{L}_V^*)(1) = \delta V^{\flat}$. $\square$

Corollary 6.5. *Let $\alpha = \sum \alpha_\tau e^\tau$ be any form, with $\tau = (\tau_1, ..., \tau_k)$ a multi-index and $e^\tau = e^{\tau_1} \wedge ... \wedge e^{\tau_k}$. Then for any vector field V,*

$$(\mathcal{L}_V + \mathcal{L}_V^*)(\alpha) = \left(\delta V^{\flat}\right)\alpha + \sum_{j=1}^{k} \alpha_\tau e^{\tau_1} \wedge ... \wedge \gamma^j \wedge ... \wedge e^{\tau_k},$$

with $\gamma^j = (\mathcal{L}_V g)\left(e_{\tau_j}, \bullet\right)$ replacing the e^{τ_j}.

Lemma 6.6. *If V is any vector field such that $dV^{\flat} = 0$, then*

$$\mathcal{L}_V^* = \mathcal{L}_V - 2\nabla_V - \left(\delta V^{\flat}\right)$$

as operators on forms.

Proof. Choose a local orthonormal frame (e_j) of TM, and let (e^j) be the dual coframe. We assume that the frame is chosen so that at the point in question, all $\nabla_{e_j} e_k$ vanish. From Lemma 6.1, parts (1) and (2), we have, using the Einstein summation convention,

$$\begin{aligned}\mathcal{L}_V - \mathcal{L}_V^* &= e^j \wedge \nabla_{e_j} (V \lrcorner) + (V \lrcorner)\, e^j \wedge \nabla_{e_j} + e_j \lrcorner \nabla_{e_j} \left(V^\flat \wedge\right) + \left(V^\flat \wedge\right) e_j \lrcorner \nabla_{e_j} \\ &= e^j \wedge \left(\nabla_{e_j} V\right) \lrcorner + e^j \wedge (V \lrcorner)\, \nabla_{e_j} + (V \lrcorner)\, e^j \wedge \nabla_{e_j} \\ &\quad + e_j \lrcorner \left(\nabla_{e_j} V^\flat\right) \wedge + e_j \lrcorner \left(V^\flat \wedge\right) \nabla_{e_j} + \left(V^\flat \wedge\right) e_j \lrcorner \nabla_{e_j} \ .\end{aligned}$$

Writing $V = V_k e_k$, we get

$$\begin{aligned}\mathcal{L}_V - \mathcal{L}_V^* &= e_j (V_k) \left(e^j \wedge\right) (e_k \lrcorner) + V_k \left(e^j \wedge\right) (e_k \lrcorner) \nabla_{e_j} + V_k (e_k \lrcorner) \left(e^j \wedge\right) \nabla_{e_j} \\ &\quad + e_j (V_k) (e_j \lrcorner) \left(e^k \wedge\right) + V_k (e_j \lrcorner) \left(e^k \wedge\right) \nabla_{e_j} + V_k \left(e^k \wedge\right) (e_j \lrcorner) \nabla_{e_j} \ .\end{aligned}$$

Since $dV^\flat = 0$, therefore $e_j (V_k) = e_k (V_j)$. Also we have that $(e^j \wedge)(e_k \lrcorner) + (e_k \lrcorner)(e^j \wedge) = \delta^j_k$. Thus, we find

$$\begin{aligned}\mathcal{L}_V - \mathcal{L}_V^* &= e_j (V_j) + 2 V_j \nabla_{e_j} \\ &= \delta V^\flat + 2 \nabla_V \ .\end{aligned}$$

□

Now we will use the above computations to establish the Weitzenböck-Bochner formula:

Proposition 6.7. ***(Weitzenböck-Bochner formula for the twisted basic Laplacian)*** *Let the bundle-like metric of a Riemannian foliation $(M, \mathcal{F})$ be chosen so that the mean curvature form κ is basic-harmonic. Then for any basic form $\alpha \in \Omega(M, \mathcal{F})$,*

$$\widetilde{\Delta} \alpha = \nabla^* \nabla \alpha + \rho(\alpha) + \frac{1}{4} |\kappa|^2 \alpha,$$

where $\rho(\alpha) = \sum_{i,j} e^j \wedge e_i \lrcorner R(e_i, e_j)\, \alpha$, with R the transversal Riemann curvature operator, and the sum is over a local orthonormal frame $\{e_j\}_{j=1,\cdots,q}$ of Q. In a particular case where α is a one form,

$$\left\langle \widetilde{\Delta} \alpha, \alpha \right\rangle = \langle \nabla^* \nabla \alpha, \alpha \rangle + \mathrm{Ric}\left(\alpha^\#, \alpha^\#\right) + \frac{1}{4} |\kappa|^2 |\alpha|^2,$$

where Ric *is the transversal Ricci curvature.*

Proof. Since κ is basic-harmonic, from [33, Proposition 2.4], we get

$$\begin{aligned} 0 = \delta_b \kappa &= P \delta \kappa \\ &= \delta \kappa + (P\kappa - \kappa) \lrcorner \kappa - \varphi_0 \lrcorner (\chi_{\mathcal{F}} \wedge \kappa) \\ &= \delta \kappa, \end{aligned}$$

with φ_0 and $\chi_{\mathcal{F}}$ from Rummler's Formula (6.1). Next, we write $\delta_b = \delta_T + \kappa \lrcorner = -e_j \lrcorner \nabla_{e_j} + \kappa \lrcorner$, where we use the Einstein summation convention. We have

$$\Delta_b = (d \delta_T + \delta_T d) + \mathcal{L}_H.$$

Let (e^j) be the dual coframe corresponding to (e_j). Suppose that we have chosen the frame so that $\nabla_{e_i} e_j = 0$ at the point in question. For any basic form α,

$$\begin{aligned}(d\delta_T + \delta_T d)\,\alpha &= -e^j \wedge \nabla_{e_j}(e_i \lrcorner \nabla_{e_i}\alpha) - e_i \lrcorner \nabla_{e_i}\left(e^j \wedge \nabla_{e_j}\alpha\right) \\ &= -e^j \wedge e_i \lrcorner \nabla_{e_j}\nabla_{e_i}\alpha + e^j \wedge e_i \lrcorner \nabla_{e_i}\nabla_{e_j}\alpha - \nabla_{e_i}\nabla_{e_i}\alpha \\ &= e^j \wedge e_i \lrcorner R\,(e_i, e_j)\,\alpha + \nabla^*\nabla\alpha - \nabla_H\alpha \\ &= \rho\,(\alpha) + \nabla^*\nabla\alpha - \nabla_H\alpha.\end{aligned}$$

Here we have used the fact that for the adapted frame,

$$\nabla^*\nabla = \nabla^*_{e_j}\nabla_{e_j} = \left(-\nabla_{e_j} + (H, e_j)\right)\nabla_{e_j} = -\nabla_{e_i}\nabla_{e_i} + \nabla_H \ .$$

Then

$$\Delta_b\alpha = \rho\,(\alpha) + \nabla^*\nabla\alpha + \mathcal{L}_H\alpha - \nabla_H\alpha,$$

and

$$\begin{aligned}\widetilde{\Delta}\alpha &= \Delta_b\alpha - \frac{1}{2}\left(\mathcal{L}_H + \mathcal{L}_H^*\right)\alpha + \frac{1}{4}|\kappa|^2\alpha \\ &= \nabla^*\nabla\alpha + \rho\,(\alpha) + \frac{1}{2}\left(\mathcal{L}_H - \mathcal{L}_H^*\right)\alpha + \frac{1}{4}|\kappa|^2\alpha - \nabla_H\alpha.\end{aligned}$$

With the help of Lemma 6.6, we have

$$\frac{1}{2}\left(\mathcal{L}_H - \mathcal{L}_H^*\right) - \nabla_H = \delta\kappa = 0.$$

The result follows. □

Corollary 6.8. *Suppose that $(M, \mathcal{F})$ is a Riemannian foliation on a connected manifold M. Suppose that the bundle-like metric is chosen so that κ is basic-harmonic. If the operator $\rho + \frac{1}{4}|\kappa|^2$ on r-forms is strictly positive, then the twisted basic cohomology group $\widetilde{H}^r(M, \mathcal{F})$ is trivial.*

Since the basic Euler characteristic and basic signature may be computed using the dimensions of $\ker\widetilde{\Delta}$, we have the following result.

Corollary 6.9. *Under the same hypothesis as in Corollary 6.8 for all r such that $0 \le r \le q$, the basic Euler characteristic and basic signature are zero. Thus the foliation is nontaut.*

Remark 6.10. *This fact for the basic Euler characteristic could be deduced from the Hopf index theorem for Riemannian foliations ([4]), but only in the case where the basic mean curvature never vanishes. In that case, the dual vector field is a basic normal vector field to the foliation that never vanishes and thus yields $\chi(M, \mathcal{F}) = 0$.*

Corollary 6.11. *Suppose that the transversal Ricci curvature satisfies $Ric\,(X, X) \ge 0$ for all vectors $X \in \Gamma(Q)$. If M is nontaut, then $\widetilde{H}^1(M, \mathcal{F}) \cong \{0\}$.*

Corollary 6.12. *Suppose that the transversal Ricci curvature satisfies* $Ric\,(X,X) \geq 0$ *for all vectors* X *orthogonal to the Riemannian foliation* $(M,\mathcal{F})$ *and* $Ric\,(\bullet,\bullet) > 0$ *for at least one point of* M*. Then* $\widetilde{H}^1\,(M,\mathcal{F}) \cong \{0\}$*.*

Corollary 6.13. *Suppose that the transversal sectional curvatures of* $(M,\mathcal{F})$ *are nonnegative. If the foliation is nontaut, then the twisted basic cohomology is identically zero, and thus the basic Euler characteristic and signature are zero.*

Corollary 6.14. *Suppose that the transversal sectional curvatures are nonnegative and are all positive for at least one point of* M*. If the foliation is nontaut, then* $\widetilde{H}^r\,(M,\mathcal{F}) \cong \{0\}$ *for* $1 < r < q$*.*

Remark 6.15. *Note that the curvature bounds above are weaker than those required by previous results in* [17]*,* [18]*, etc.*

Using the Weitzenböck-Bochner formula, we can give a direct proof of Hebda's result [17].

Theorem 6.16. *Let* M *be a compact Riemannian manifold endowed with a Riemannian foliation. If the transversal Ricci curvature is positive, then* $H^1_B(M,\mathcal{F}) = 0$*.*

Proof. Since the basic cohomology groups are independent of the choice of the bundle-like metric, we may assume that the mean curvature κ is basic-harmonic. Let α be a basic one-form closed and coclosed, i.e. $d\alpha = 0$ and $\delta_b\alpha = 0$. Then we find $\widetilde{d}\alpha = -\frac{1}{2}\kappa\wedge\alpha$ and $\widetilde{\delta}_b(\alpha) = -\frac{1}{2}\kappa\lrcorner\alpha$. Thus $|\widetilde{d}\alpha|^2 + |\widetilde{\delta}_b(\alpha)|^2 = \frac{1}{4}|\kappa|^2|\alpha|^2$. With the use of the Weitzenböck formula, we have

$$\int_M (\widetilde{\Delta}\alpha,\alpha) = \frac{1}{4}\int_M |\kappa|^2|\alpha|^2 = \int_M |\nabla\alpha|^2 + \int_M \mathrm{Ric}(\alpha,\alpha) + \frac{1}{4}\int_M |\kappa|^2|\alpha|^2.$$

Under the curvature assumption, we deduce that $\alpha = 0$. □

Corollary 6.17. *Let* $(M,\mathcal{F})$ *be a Riemannian foliation of a compact manifold and suppose that the transversal Ricci curvature satisfies* $\mathrm{Ric}(X,X) \geq 0$ *for all* $X \in \Gamma Q$ *and* $\mathrm{Ric}(X_p,X_p) > 0$ *for all nonzero* $X_p \in \Gamma_p Q$ *at one point* $p \in M$*. Then* $H^1_B(M,\mathcal{F}) = 0$*.*

Proof. With the weaker hypothesis, $\mathrm{Ric}(X,X) > 0$ for all unit normal vectors X to the foliation on a neighborhood of p. If α is a closed and coclosed basic one-form, by the previous proof α is zero on that neighborhood. Since $(d+\delta_b)\,\alpha = 0$, by [33, Proposition 2.4] we have

$$(d+P\delta)\,\alpha = (d+\delta-\varphi_0\lrcorner\chi_{\mathcal{F}}\wedge)\,\alpha = (d+\delta)\,\alpha = 0,$$

where $P : L^2\,(\Omega\,(M)) \to L^2\,(\Omega\,(M,\mathcal{F}))$ is the orthogonal projection, δ is the ordinary L^2 adjoint of d on all forms, and φ_0, $\chi_{\mathcal{F}}$ are from Rummler's formula (6.1). The operator $d+\delta$ is a linear, first order elliptic operator that satisfies the weak unique continuation property (see [5], [3], [2], [8]). This means that since α is zero on an open set, it is identically zero on all of M. □

We also find a direct proof for the following theorem established in [13] (see also [21], [22]).

Theorem 6.18. *Let M be a compact, connected manifold endowed with a Riemannian foliation $\mathcal{F}$. The top-dimensional basic cohomology is either isomorphic to* 0 *or* $\mathbb{R}$.

Proof. Let α be a basic q-form closed and coclosed. Since $\alpha = f\nu$ where ν is the transverse volume form of the foliation and f a basic real-valued function on M, the term $(\rho(\alpha), \alpha) = f^2(\rho(\nu), \nu)$ is equal to zero by the fact that ν is parallel and f is a function. Now applying the Weitzenböck-Bochner formula to α gives that α is parallel, which means that f is constant. If f is always equal to zero, the basic cohomology is zero; otherwise it is isomorphic to $\mathbb{R}$. □

In the following, we prove that the spectrum of the basic Laplacian is the same as the twisted one under a curvature assumption.

Proposition 6.19. *Let M be a compact manifold endowed with a Riemannian foliation $\mathcal{F}$ with strictly positive transversal curvature. Then* $\mathrm{spec}(\widetilde{\Delta}) = \mathrm{spec}(\Delta_b)$.

Proof. By the Mason result [29], any bundle-like metric can be dilated to another one $\bar{g}$ with basic-harmonic mean curvature $\bar{\kappa}$ and with the same basic Laplacian. Since the transversal curvature is positive, the first cohomology group is zero and hence $\bar{\kappa} = 0$. In this case, the operator $\Delta_b^{\bar{g}} = \widetilde{\Delta}^{\bar{g}}$. Using the fact that the spectrum of $\widetilde{\Delta}^{\bar{g}}$ remains the same for any possible change of the bundle-like, we deduce the proof of the proposition. □

Proposition 6.20. *Let $(M, \mathcal{F})$ be a Riemannian foliation of codimension* 2 *on a connected manifold. If the foliation is nontaut, the basic cohomology groups satisfy $H_d^0(M, \mathcal{F}) = H_d^1(M, \mathcal{F}) = \mathbb{R}$, $H_d^2(M, \mathcal{F}) = \{0\}$. In all other cases, the foliation is taut. Also, the twisted basic cohomology $\widetilde{H}^*(M, \mathcal{F})$ is identically zero if and only if the foliation is nontaut.*

Proof. Suppose that the foliation is nontaut, so that $H_d^0(M, \mathcal{F}) \cong \mathbb{R}$, $H_d^2(M, \mathcal{F}) = \{0\}$. Let κ be chosen to be basic harmonic as in Corollary 1.2. Since $(d + \delta_b)\kappa = 0$, by the same argument as in Corollary 6.17, the weak unique continuation property implies that if κ were zero on an open set, it would be identically zero on M. Since $[\kappa] \in H_d^1(M, \mathcal{F})$ is nontrivial, the set on which κ is nonzero is open and dense. Furthermore, by [3] we know that the zero set of κ is codimension two or more. By Rummler's Theorem ([38]), not all leaves are closed. Since M is connected, the principal stratum of the foliation must be saturated by noncompact leaves. In the case where there are no compact leaves, the zero set of κ is one or less, so that κ would have to be nonzero. Then, by the basic Hopf index theorem [4], the basic Euler characteristic would have to be zero, so that $H_d^1(M, \mathcal{F}) \cong \mathbb{R}$.

The only remaining case is where κ is zero at a finite number of isolated closed leaves. If we use $H = \kappa^\#$ as the basic normal vector field in the basic Hopf index theorem [4], it remains to calculate the sign of the determinant of the matrix $(a_{ij}) = (\nabla_{e_i} H, e_j)$, i.e. the type of singularity of H at each singular point. Since the leaf closures near $\kappa = 0$ are codimension one, the space of leaf closures looks locally like concentric circles around the origin in $\mathbb{R}^2$. Because κ is basic, it must be either a source or a sink, which in both cases implies the index of H at the singular leaf is 1. Since no orientation issues occur, we see that the basic Euler characteristic is the number of singular leaves, a positive number. On the other hand,

$$\begin{aligned}\chi(M, \mathcal{F}) &= \dim H_d^0(M, \mathcal{F}) - \dim H_d^1(M, \mathcal{F}) + \dim H_d^2(M, \mathcal{F}) \\ &= 1 - \dim H_d^1(M, \mathcal{F}) \leq 0.\end{aligned}$$

We conclude that this last case cannot occur, and the basic Euler characteristic of the foliation must be zero. The result follows. □

Corollary 6.21. *The basic Euler characteristic and basic signature of a nontaut Riemannian foliation of codimension two are zero.*

Remark 6.22. *In Section 7, we show that it is possible to construct nontaut Riemannian foliations of higher codimension with nonzero twisted basic cohomology and nonzero twisted basic Euler characteristics.*

Recall that a group G is **polycyclic** if there exists a finite sequence of nested subgroups $1 \lhd G_1 \lhd \ldots \lhd G_k = G$ such that all factor groups are cyclic.

Corollary 6.23. *Suppose that $(M, \mathcal{F})$ is a nontaut Riemannian foliation of codimension two, and $\pi_1(M)$ is polycyclic or has polynomial growth. Then the basic Euler characteristic and basic signature are stable with respect to deformations of $(M, \mathcal{F})$ through continuous families of Riemannian foliations, and in fact the dimensions of all basic cohomology groups are also stable.*

Proof. In [31], Nozawa showed that nontautness is preserved in families of Riemannian foliations on such manifolds. The two previous corollaries imply the result. □

Remark 6.24. *Note that in general the dimensions of basic cohomology groups are not stable under such deformations; see* [32, Example 7.0.4] *for a simple example. However,* $\dim H_d^0(M, \mathcal{F})$ *and* $\dim H_d^q(M, \mathcal{F})$ *are stable with respect to deformations if $\pi_1(M)$ is polycyclic or has polynomial growth, as implied by the discussion above.*

Remark 6.25. *Because the twisted basic cohomology and ordinary basic cohomology groups are independent of the choices of bundle-like metric and transverse Riemannian structure (see Corollary 2.13), we note that the vanishing theorems in this section may be restated in terms of the existence of bundle-like metrics with the required properties.*

7. Examples

7.1. **The Carrière example.** We will compute the cohomology groups of the Carrière example from [7] in the 3-dimensional case. Let A be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ of trace strictly greater than 2. We denote respectively by V_1 and V_2 the eigenvectors associated with the eigenvalues λ and $\frac{1}{\lambda}$ of A with $\lambda > 1$ irrational. Let the hyperbolic torus $\mathbb{T}^3_A$ be the quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the equivalence relation which identifies (m, t) to $(A(m), t+1)$. The flow generated by the vector field V_2 is a transversally Lie foliation of the affine group. We denote by K the holonomy subgroup. The affine group is the Lie group $\mathbb{R}^2$ with multiplication $(t, s).(t', s') = (t + t', \lambda^t s' + s)$, and the subgroup K is

$$K = \{(n, s), n \in \mathbb{Z}, s \in \mathbb{R}\}.$$

We choose the bundle-like metric (letting (x, s, t) denote the local coordinates in the V_2 direction, V_1 direction, and $\mathbb{R}$ direction, respectively) as

$$g = \lambda^{-2t} dx^2 + \lambda^{2t} ds^2 + dt^2.$$

We will show that the twisted cohomology groups all vanish. First, we notice that the mean curvature of the flow is $\kappa = \kappa_b = \log(\lambda)\, dt$, since $\chi_{\mathcal{F}} = \lambda^{-t} dx$ is the characteristic form and $d\chi_{\mathcal{F}} = -\log(\lambda)\,\lambda^{-t} dt \wedge dx = -\kappa \wedge \chi_{\mathcal{F}}$. Since the flow is nontaut, we have $\widetilde{H}^0(M, \mathcal{F}) \cong \widetilde{H}^2(M, \mathcal{F}) = 0$ by Theorem 4.2. We now show directly that $\widetilde{H}^1(M, \mathcal{F}) = 0$ (allthough this is guaranted by Proposition 6.20). The 1-forms $\alpha = dt$ and $\beta = \lambda^t ds$ are left invariant. Every K-invariant 1-form ω can be written as $\omega = f(t)\alpha + g(t)\beta$, where f and g are periodic functions. For any $\widetilde{d}$-closed basic 1-form ω, we have

$$\begin{aligned} d\omega &= (g'(t) + g\log(\lambda))\,\alpha \wedge \beta \\ &= \frac{1}{2} g \log(\lambda)\,\alpha \wedge \beta = \frac{1}{2}\kappa_b \wedge \omega. \end{aligned}$$

We then deduce that $g' = -\frac{1}{2}\log(\lambda)\, g$, or $g = c\lambda^{-\frac{t}{2}}$ for some $c \in \mathbb{R}$. Since g is periodic, it is zero. If ω is also $\widetilde{\delta}$-coclosed,

$$\begin{aligned} \delta_b \omega &= \delta_b(f\alpha) = -\alpha(f) + f\delta_b(\alpha) \\ &= -f'(t) + f\log(\lambda) = \frac{1}{2} f \log(\lambda) = \frac{1}{2}\kappa_b \lrcorner (f\alpha). \end{aligned}$$

The solution is again reduced to zero for periodic functions f. Thus, the first twisted cohomology group is zero. □

7.2. **Nontautness and nontrivial twisted cohomology.** In this example, the Riemannian foliation is nontaut, and the twisted basic cohomology and basic Euler characteristic are nontrivial.

First, let $S^1 = \mathbb{R}/\mathbb{Z}$, and let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, with flat metrics to be chosen later. Consider the manifold $X = \mathbb{R} \times_\varphi T^2$, a suspension of T^2 and a T^2 bundle over

S^1, constructed using the identification:

$$\varphi\left(\widetilde{x},(a,b)\right)=\left(\widetilde{x}+1,(-a,-b)\right)$$

for all $\widetilde{x}\in\mathbb{R}$, $(a,b)\in T^2=\mathbb{R}^2\diagup\mathbb{Z}^2$. We now exhibit a Riemannian foliation of X, constructed as follows. First, observe that φ is an orientation-preserving isometry of T^2, for any given flat metric. Observe that the lines in T^2 with slope $\frac{3+\sqrt{5}}{2}$ (parallel to one eigenvector of the matrix $\begin{pmatrix}1&1\\1&2\end{pmatrix}$) are preserved by these isometries. For $b_0\in\mathbb{R}\diagup\mathbb{Z}$, the sets of the form

$$\widetilde{L_{b_0}}=\left\{\left(\widetilde{x},(a,b)\right):\widetilde{x}\in\mathbb{R},a\in\mathbb{R}\diagup\mathbb{Z},b=\frac{3+\sqrt{5}}{2}a+b_0\right\}\subset\widetilde{X}=\mathbb{R}\times T^2$$

form a Riemannian foliation $\widetilde{\mathcal{F}_X}$. Then the sets

$$\begin{gathered}L_{b_0}:=\widetilde{L_{b_0}}\diagup\sim\\ \left(\widetilde{x},(a,b)\right)\sim\varphi\left(\widetilde{x},(a,b)\right)\end{gathered}$$

form a Riemannian foliation $\mathcal{F}_X$ of the quotient $X=\mathbb{R}\times_\varphi T^2$ that is not transversally oriented, again for any flat metrics. Note that $L_{b_0}=L_b$ for any b in the orbit of b_0 via the action generated by $b\mapsto\frac{3+\sqrt{5}}{2}+b$, $b\mapsto -b$. Note that this Riemannian foliation $\mathcal{F}_X$ is dense in X, and that it admits no basic vector fields or basic one-forms.

Next, let Y be a surface of genus 2 with universal cover $\widetilde{Y}=\mathbb{H}$. Then $\pi_1(Y)$ is a group with presentation $\langle A,B,C,D:ABCDA^{-1}B^{-1}C^{-1}D^{-1}=1\rangle$. We define the homomorphism

$$\widetilde{\psi}:\pi_1(Y)\to Diff\left(\widetilde{X},\widetilde{\mathcal{F}_X}\right)$$

from $\pi_1(Y)$ to the group of foliated diffeomorphisms of $\left(\widetilde{X},\widetilde{\mathcal{F}_X}\right)$ defined by

$$\begin{aligned}\widetilde{\psi}(A)\left(\widetilde{x},(a,b)\right)&=\left(\widetilde{x},(a+b,a+2b)\right),\\ \psi(B)=\psi(C)&=\psi(D)=\mathbf{1}.\end{aligned}$$

Since $\widetilde{L_{b_0}}$ consists of lines parallel to one eigenvector of $\begin{pmatrix}1&1\\1&2\end{pmatrix}$, $\widetilde{\psi}(A)$ maps leaves of $\widetilde{\mathcal{F}_X}$ to themselves and commutes with the action φ, and thus it descends to a homomorphism

$$\begin{gathered}\psi\colon\pi_1(Y)\to Diff\left(X,\mathcal{F}_X\right),\\ \psi(g)\left[\left(\widetilde{x},(a,b)\right)\right]:=\left[\widetilde{\psi}(g)\left(\widetilde{x},(a,b)\right)\right],\ g\in\pi_1(Y).\end{gathered}$$

Now we form the suspension

$$M=\widetilde{Y}\times_\psi X$$

$$= \left\{ [(\widetilde{y}', x') : (\widetilde{y}', x') = (g\widetilde{y}, \psi(g)\, x) \text{ for some } g \in \pi_1(Y)] : (\widetilde{y}, x) \in \widetilde{Y} \times X \right\},$$

which is naturally endowed with the foliation $\mathcal{F}_M$ whose leaves are equivalence classes $[(\widetilde{y}, L_X)] \in \widetilde{Y} \times_\psi \mathcal{F}_X$ with $L_X \in \mathcal{F}_X$ (not unique in L_X). This foliation will be a Riemannian foliation if we pull back any metric on Y via $\widetilde{Y} \times_\psi X \to Y$ and choose a metric on each fiber ($\cong X$) that is flat. Note that we will need to modify the fiberwise metric as a function of $y \in Y$ so that $\psi(g)$ acts by transverse isometries on the fibers. Specifically, let $\lambda = \frac{3+\sqrt{5}}{2}$ be one eigenvalue corresponding to the eigenvector $V_1 = \left(1, \frac{1+\sqrt{5}}{2}\right)^T$ of $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, and let V_2 denote the other eigenvector corresponding to eigenvalue λ^{-1}. Choose a specific smooth closed curve $u \to \gamma_A(u) \in Y$ corresponding to $A \in \pi_1(Y)$, with $\gamma_A(0) = \gamma_A(1)$. Letting t_1 and t_2 denote the (lifted) coordinates of T^2 corresponding to directions V_1 and V_2 respectively, x the coordinate on S^1, choose

$$ds^2 = du^2 + dx^2 + \lambda^{2u} dt_1^2 + \lambda^{-2u} dt_2^2$$

to be the metric on the submanifold $\pi^{-1}(\gamma_A)$, where $\pi : M = \widetilde{Y} \times_\psi X \to Y$ is the projection. Similarly choose metrics along paths γ_B, γ_C, γ_D, but this time guaranteeing that the torus metrics on (t_1, t_2) agree after traversing the circle (as well as on intersections coming from the other curves). Then we extend the metric to a metric on M in any way so that it is fiberwise flat and that the metrics on the horizontal submanifolds ($\widetilde{Y}$ parameter slices) are pullbacks of metrics on Y. The resulting metric will be a bundle-like metric for $(M, \mathcal{F}_M)$. The metric along the leaves may then be modified so that the mean curvature form κ is basic-harmonic and is thus a harmonic one-form, and it is the pullback of a one-form on Y. By doing a line integral along γ_A we see that κ determines a nontrivial class in $H^1(M)$, in fact in $H^1(Y)$. Thus $(M, \mathcal{F}_M)$ is nontaut.

By construction there are no basic forms except constants on X, and thus every basic form on the codimension three foliation $(M, \mathcal{F}_M)$ is an element of $\Omega^*(Y)$. Thus, the ordinary basic cohomology groups are

$$\dim H_d^0(M, \mathcal{F}_M) = 1, \ \dim H_d^1(M, \mathcal{F}_M) = 4$$
$$\dim H_d^2(M, \mathcal{F}_M) = 1, \ \dim H_d^3(M, \mathcal{F}_M) = 0.$$

Then the basic Euler characteristic satisfies $\chi(M, \mathcal{F}_M) = -2$, so that the twisted basic cohomology groups are

$$\dim \widetilde{H}^0(M, \mathcal{F}_M) = 0, \ \dim \widetilde{H}^1(M, \mathcal{F}_M) = \widetilde{h}^1$$
$$\dim \widetilde{H}^2(M, \mathcal{F}_M) = \widetilde{h}^2, \ \dim \widetilde{H}^3(M, \mathcal{F}_M) = 0,$$

where $\widetilde{h}^2 - \widetilde{h}^1 = -2$. Note that Poincaré duality is not satisfied, because $(M, \mathcal{F}_M)$ is not transversally oriented. We have $\widetilde{h}^2 \geq 0$, $\widetilde{h}^1 \geq 2$.

7.3. **A transversally oriented example.** We now modify the previous example to produce a transversally oriented Riemannian foliation that is nontaut and has nontrivial twisted basic cohomology.

First, let N be the connected sum of two copies of $S^1 \times S^2$, which has the property that $\pi_1(N) = \mathbb{Z} * \mathbb{Z}$, the free group on two generators α_1 and α_2. Let $\widetilde{N}$ be the universal cover of N, on which $\pi_1(N)$ acts by deck transformations. Let $T^3 = \mathbb{R}^3 \diagup \mathbb{Z}^3$, with a flat metric to be specified later. Choose $\eta \in \mathbb{R} \setminus \mathbb{Q}$. Consider the manifold

$$X = \widetilde{N} \times_{\varphi} T^3 = \widetilde{N} \times T^3 \diagup \sim$$
$$(\widetilde{x}, (a, b, c)) \sim (\beta \widetilde{x}, \varphi(\beta)(a, b, c)), \ \beta \in \pi_1(N),$$

a suspension of T^3 and a T^3 bundle over N, constructed using the homomorphism $\varphi : \pi_1(N) \to \mathrm{Isom}(T^3)$ generated by

$$\varphi(\alpha_1)(a, b, c) = (-a, -b, -c)$$
$$\varphi(\alpha_2)(a, b, c) = (a, b, c + \eta)$$

for all $(a, b, c) \in T^3 = \mathbb{R}^3 \diagup \mathbb{Z}^3$. We now exhibit a Riemannian foliation of X, constructed as follows. First, observe that each $\varphi(\beta)$ is an isometry of T^3, for any given flat metric. Observe that the lines in T^3 parallel to $\left(1, \frac{1+\sqrt{5}}{2}, 0\right)^T$, one eigenvector of the matrix $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, are preserved by these isometries. For $(b_0, c_0) \in \mathbb{R}^2 \diagup \mathbb{Z}^2$, the sets of the form

$$\widetilde{L_{b_0,c_0}} = \left\{ (\widetilde{x}, (a, b_0, c_0)) : \widetilde{x} \in \widetilde{N}, a \in \mathbb{R} \diagup \mathbb{Z}, b = \frac{3+\sqrt{5}}{2} a + b_0 \right\} \subset X' = \widetilde{N} \times T^3$$

form a Riemannian foliation $\widetilde{\mathcal{F}_X}$. Then the sets

$$L_{b_0,c_0} := \widetilde{L_{b_0,c_0}} \diagup \sim$$

form a Riemannian foliation $\mathcal{F}_X$ of the quotient $X = \widetilde{N} \times_{\varphi} T^3$ that is transversally oriented, again for any flat metrics. (We see that $\varphi(\alpha_1)$, although orientation-reversing as a map from T^3 to itself, is transversally orientation preserving.) Note that $L_{b_0,c_0} = L_{b,c}$ for any (b, c) in the orbit of (b_0, c_0) via the action generated by $b \mapsto \frac{3+\sqrt{5}}{2} + b$, $(b, c) \mapsto (-b, -c)$, $c \mapsto c + \eta$. Note that this codimension two Riemannian foliation $\mathcal{F}_X$ is dense in X, and that it admits no basic vector fields or basic one-forms. The only basic forms for this foliation are the constant functions and constant multiples of the transverse volume form.

Next, let Y be a surface of genus 2 with universal cover $\widetilde{Y} = \mathbb{H}$. Then $\pi_1(Y)$ is a group with presentation $\langle A, B, C, D : ABCDA^{-1}B^{-1}C^{-1}D^{-1} = 1 \rangle$. We define the homomorphism

$$\widetilde{\psi} : \pi_1 (Y) \to Diff \left(X', \widetilde{\mathcal{F}_X} \right)$$

from $\pi_1 (Y)$ to the group of foliated diffeomorphisms of $\left(X', \widetilde{\mathcal{F}_X} \right)$ defined by

$$\widetilde{\psi} (A) (\widetilde{x}, (a, b, c)) = (\widetilde{x}, (a + b, a + 2b, c)) ,$$
$$\psi (B) = \psi (C) = \psi (D) = \mathbf{1}.$$

Since $\widetilde{L_{b_0,c_0}}$ consists of lines parallel to one eigenvector of the matrix B, $\widetilde{\psi} (A)$ maps leaves of $\widetilde{\mathcal{F}_X}$ to themselves and commutes with the action φ, and thus it descends to a homomorphism

$$\psi \colon \pi_1 (Y) \to Diff (X, \mathcal{F}_X) ,$$
$$\psi (g) [(\widetilde{x}, (a, b, c))] := \left[\widetilde{\psi} (g) (\widetilde{x}, (a, b, c)) \right] , \ g \in \pi_1 (Y) .$$

Now we form the suspension

$$\begin{aligned} M &= \widetilde{Y} \times_\psi X \\ &= \left\{ [(\widetilde{y}', x') : (\widetilde{y}', x') = (g\widetilde{y}, \psi (g) x) \text{ for some } g \in \pi_1 (Y)] : (\widetilde{y}, x) \in \widetilde{Y} \times X \right\} , \end{aligned}$$

which is naturally endowed with the foliation $\mathcal{F}_M$ whose leaves are equivalence classes $[(\widetilde{y}, L_X)] \in \widetilde{Y} \times_\psi \mathcal{F}_X$ with $L_X \in \mathcal{F}_X$ (not unique in L_X). This foliation will be a Riemannian foliation if we pull back any metric on Y via $\widetilde{Y} \times_\psi X \to Y$ and choose a metric on each fiber ($\cong X$) that is transversally flat. Note that we will need to modify the fiberwise metric as a function of $y \in Y$ so that $\psi (g)$ acts by transverse isometries on the fibers. Specifically, let $\lambda = \frac{3+\sqrt{5}}{2}$ be one eigenvalue corresponding to the eigenvector $V_1 = \left(1, \frac{1+\sqrt{5}}{2}, 0 \right)^T$ of B, let V_2 denote the other eigenvector corresponding to eigenvalue λ^{-1}, and let $e_3 = (0, 0, 1)^T$. Choose a specific smooth closed curve $u \to \gamma_A (u) \in Y$ corresponding to $A \in \pi_1 (Y)$, with $\gamma_A (0) = \gamma_A (1)$. Letting t_1, t_2, t_3 denote the (lifted) coordinates of T^3 corresponding to directions V_1, V_2, e_3 respectively, x the coordinate on N, choose

$$ds^2 = du^2 + dx^2 + \lambda^{2u} dt_1^2 + \lambda^{-2u} dt_2^2 + dt_3^2$$

to be the metric on the submanifold $\pi^{-1} (\gamma_A)$, where $\pi : M = \widetilde{Y} \times_\psi X \to Y$ is the projection. Similarly choose metrics along paths γ_B, γ_C, γ_D, but this time guaranteeing that the torus metrics on (t_1, t_2, t_3) agree after traversing the circle (as well as on intersections coming from the other curves. Then we extend the metric to a metric on M in any way so that it is fiberwise flat and that the metrics on the horizontal submanifolds ($\widetilde{Y}$ parameter slices) are pullbacks of metrics on Y. The resulting metric will be a bundle-like metric for $(M, \mathcal{F}_M)$. The metric along the leaves may then be modified so that the mean curvature form κ is basic-harmonic and is thus a harmonic one-form, and it is the pullback of a one-form on Y. By doing a line integral along γ_A we see that κ determines a nontrivial class in $H^1(M)$, in fact in $H^1(Y)$. Thus $(M, \mathcal{F}_M)$ is nontaut.

By construction there are no basic forms except constants and constant multiples of the transverse volume form ν_X on X, and thus every basic form on the codimension four foliation $(M, \mathcal{F}_M)$ is of the form $\omega_1 + \omega_2 \wedge \nu_X$ with $\omega_1, \omega_2 \in \Omega^*(Y)$. Thus, the ordinary basic cohomology groups are

$$\begin{aligned} &\dim H^0_d(M, \mathcal{F}_M) = 1,\ \dim H^1_d(M, \mathcal{F}_M) = 4 \\ &\dim H^2_d(M, \mathcal{F}_M) = 2,\ \dim H^3_d(M, \mathcal{F}_M) = 4,\ \dim H^4_d(M, \mathcal{F}_M) = 1. \end{aligned}$$

Then the basic Euler characteristic satisfies $\chi(M, \mathcal{F}_M) = -4$, so that the twisted basic cohomology groups are

$$\begin{aligned} &\dim \widetilde{H}^0(M, \mathcal{F}_M) = 0,\ \dim \widetilde{H}^1(M, \mathcal{F}_M) = \widetilde{h}^1 \\ &\dim \widetilde{H}^2(M, \mathcal{F}_M) = \widetilde{h}^2,\ \dim \widetilde{H}^3(M, \mathcal{F}_M) = \widetilde{h}^3 = \widetilde{h}^1, \dim \widetilde{H}^4(M, \mathcal{F}_M) = 0, \end{aligned}$$

where $\widetilde{h}^2 - 2\widetilde{h}^1 = -4$. Since the mean curvature form as a form on Y agrees with the mean curvature form in the previous example and because the basic one-forms are the same, we must have $\widetilde{h}^3 = \widetilde{h}^1 \geq 2$, $\widetilde{h}^2 \geq 0$.

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Georges Habib
Lebanese University
Faculty of Sciences II
Department of Mathematics
P.O. Box 90656 Fanar-Matn
Lebanon
E-mail address, G. Habib: `ghabib@ul.edu.lb`

Ken Richardson
Department of Mathematics
Texas Christian University
Fort Worth, Texas 76129, USA
E-mail address, K. Richardson: `k.richardson@tcu.edu`